CONDENSED MATHEMATICS SEMINAR

Time: Fridays 2:30-4:30
Location: Fine Hall 314 (except 10/13, when it will be at Fine Hall 401)
Organizers: Vadim Vologodsky (vologod@gmail.com) and Bogdan Zavyalov (bogd.zavyalov@gmail.com)

The goal of the seminar is to understand the notion of solid A-module. The main application that we plan to discuss is to the formalism of 6-functors in the coherent setting. We will follow lecture notes by Peter Scholze available electronically at https://www.math.uni-bonn.de/people/scholze/Condensed.pdf

Plan

1. [September 29] Introductory talk by Bogdan Zavyalov.

2. [October 6] Recall profinite sets (both as the pro-category and as totally disconnected compact spaces), Stone-Cech compactification, and extremally disconnected profinite sets. Give the definition of condensed sets and condensed groups [using either the site of compact Hausdorff spaces, profinite sets, or extremally disconnected profinite sets]. Discuss the example of $\mathbf{R}^{\mathrm{disc}} \to \mathbf{R}$ in Cond(Ab). Show that condensed sets have enough projectives. Prove that condensed abelian groups satisfy the AB6 axioms.

References:[Sta23, Tag 08ZW], [Sta23, Tag 08YH], [Sch19, Lectures 1 and 2]

3. [October 13] Discuss the "Yoneda" embedding from compactly generated (weakly) Hausdorff spaces into condensed sets. Briefly mention coherent topoi, quasi-compact and quasi-separated objects in coherent topoi. Then classify qcqs and qs condensed sets. Give an example when the "Yoneda" embedding does not commute with filtered colimits.

References: [Sch19, Lecture 1, Appendix to Lecture 2], [CS22, §1], [Cas23, §3.1], [Lur, Appendix C]

4. [October 20] Discuss cohomology of (extremally disconnected) profinite sets. Show that sheaf cohomology with $\mathbf{Z}$-coefficients coincide with the condensed cohomology with $\mathbf{Z}$-coefficients. Then prove that higher condensed cohomology with $\mathbf{R}$-coefficients vanish (also, adapt the proof to work for any Banach space).

[If time permits, discuss [Hai22] which significantly generalizes the isomorphism $R\Gamma_{\mathrm{cond}}(X, \mathbf{Z}) = R\Gamma_{\mathrm{sheaf}}(X, \mathbf{Z})$]

References: [Sch19, Lecture 3], [Hai22]. Also see [CC23, Ch.2, Condensed abelian groups] for the proof that $\mathrm{H}^i(S, V) = 0$ for a Banach space V and an integer $i > 0$.

5. [October 27] Recall the classification of locally compact abelian topological groups and prove that $D^b(\mathrm{LCA})$ embeds fully faithfully into $D(\mathrm{Cond}(\mathrm{Ab}))$.

[If time permits, discuss the Deligne-Breen resolution]

References: [Sch19, Lecture 4, Appendix to §4], [Deg, §3, §4]

6. [November 3] Define the notion of analytic rings, maps of analytic rings, and the (derived) category of modules over analytic rings. Then show that a map of analytic maps induces a pullback functor on the category of modules (In particular, explain [Sch19, Lemma 5.10]). Then explain the forthcoming examples of $\mathbf{Z}_\square$, $\mathbf{Z}_{p,\square}$, $(A, A)_\square$, and $(A, \mathbf{Z})_\square$ for a finitely generated $\mathbf{Z}$-algebra A (and mention that the base change functor $M \mapsto M \otimes_{\mathbf{Z}_\square} \mathbf{Z}[1/p]_\square$ is different from the tensor product $M \mapsto M \otimes_{\mathbf{Z}_\square} \mathbf{Z}[1/p]$).

References: [Sch19, Lectures 5, 7, and Appendix to Lecture 7], [CS22, Proposition 12.20]

7. [November 10] Define and describe $\mathbf{Z}[S]$ and $\mathbf{Z}[S]_\square$ for a profinite set S. Then define solid abelian groups. Explain the computation of $R\operatorname{Hom}(\prod_I \mathbf{Z}, \mathbf{Z})$ in condensed abelian groups. Sketch the proof that $\mathbf{Z}_\square$ is an analytic ring. Then explain [Sch19, Corollary 6.1, Proposition 6.3, and Example 6.4].

References: [Sch19, Lectures 5 and 6], [CS22, Lecture 2].

8. [December 1] Construct analytic rings $(\mathbf{Z}[T], \mathbf{Z}[T])_\square$ and $(\mathbf{Z}[T], \mathbf{Z})_\square$. Also, construct the lower shriek functor $f_! : D_\square(\mathbf{Z}[T]) \to D_\square(\mathbf{Z})$. Then sketch the construction of $(A, R)_\square$ for finitely generated A and R.

References: [Sch19, Lecture 8 and Appendix to Lecture 8]

9. [December 8] Very briefly mention discrete adic spaces. Globalize the definition of solid modules on arbitrary discrete adic space. In particular, prove analytic descent. Mention that open immersions are not t-exact in the solid world (with a particular example).

[If time permits: Deduce faithfully flat descent from the 6-functor formalism that is going to be constructed in the next lecture]

References: [Sch19, Lectures 9 and 10]. See [Mik22] for the descent statement (it can be simplified in our situation)

10. [December 15] Construct 6-functors. Finally, explain the claim from Talk 6 that the functors $M \mapsto M \otimes_{\mathbf{Z}_\square} \mathbf{Z}[1/p]_\square$ and $M \mapsto M \otimes_{\mathbf{Z}_\square} \mathbf{Z}[1/p]$ are different. In particular, prove Serre-Grothendieck duality and discuss a new local proof that Rf_* preserves coherent sheaves for a proper morphism of finite type $\mathbf{Z}$-schemes.

References: [Sch19, Lecture 11].

References

[Cas23] A. Castano. *A Berkovich Approach to Perfectoid Spaces*. https://arxiv.org/abs/2304.09266. 2023.

[CC23] *Crash Course Condensed Mathematics*. https://math.commelin.net/2023/condensed/. 2023.

[CS22] D. Clausen and P. Scholze. *Lectures on Analytic Geometry*. https://www.math.uni-bonn.de/people/scholze/Analytic.pdf. 2022.

[Deg] F. Deglise. *Condensed and Locally Compact Abelian Groups*. https://deglise.perso.math.cnrs.fr/docs/2020/condensed.pdf.

[Hai22] P. Haine. *Descent for sheaves on compact Hausdorff spaces*. https://arxiv.org/abs/2210.00186. 2022.

[Lur] J. Lurie. *Ultracategories*. https://www.math.ias.edu/~lurie/papers/Conceptual.pdf.

[Mik22] Y. Mikami. *Faithfully flat descent of quasi-coherent complexes on rigid analytic varieties via condensed mathematics*. https://arxiv.org/abs/2206.14795. 2022.

[Sch19] P. Scholze. *Lectures on Condensed Mathematics*. http://people.mpim-bonn.mpg.de/scholze/Condensed.pdf. 2019.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.