K-THEORY OF ADIC SPACES

Time: Fridays 2:30-4:30
Location: Fine Hall 314
Organizers: Greg Andreychev (griga@ias.edu), Vadim Vologodsky (vologod@gmail.com), and Bogdan Zavyalov (bogd.zavyalov@gmail.com)

This semester we plan to discuss the new approach to non-archimedean analytic geometry due to Clausen–Scholze. The ultimate goal of the seminar is to state and prove a version of the Grothendieck–Riemann–Roch theorem for analytic adic spaces using the machinery of condensed mathematics. Along the way, we will see how Clausen–Scholze's notion of an analytic ring allows one to define a "category of quasi-coherent sheaves" on adic spaces which, moreover, comes with a natural 6-functor formalism. This will lead to a natural definition of K-theory using recent results of Efimov and will allow us to adapt the classical approach of Thomason to the K-theory of schemes to our setting.

Plan

1. Introductory Talk by Greg Andreychev (January 26)

2. Recollections on K-theory (February 2)
Define the Grothendieck group of a commutative ring. Then define K_1 and K_2 and state Matsumoto's theorem. Explain the definitions of higher K-theory in terms of the $+$-construction and in terms of the ∞-group completion of groupoid of finite projective modules. Define K_0 of a stable ∞-category and sketch the Quillen Q-construction and the Waldhausen S-construction. State the Gillet–Waldhausen resolution theorem for the K-theory of rings. Finally, explain the characterization of K-theory as the universal additive invariant of stable ∞-categories.

References: [Wei13, Chapters 2-4], [Heb21, Chapter IV], [BGT13].

3. Dualizable Categories (February 9)
Introduce the notion of stable compactly generated ∞-category and discuss its properties and some examples. In particular, prove that the ∞-category of small idempotent complete stable ∞-categories is equivalent to the ∞-category of compactly generated stable ∞-categories with strongly continuous functors. After that, recall the definition of dualizable object in a symmetric monoidal (∞-)category. Briefly review the Lurie tensor product of (stable) presentable ∞-categories and then discuss the notion of dualizable stable ∞-categories. Prove some of its equivalent characterizations: as retracts of compactly generated stable ∞-categories, in terms of the double left adjoint to the Yoneda embedding, and in terms of compact maps. If time permits, prove that the ∞-category of sheaves of spectra on a locally compact Hausdorff topological space is dualizable ([Hoy18, Theorem 15]).

References: [Lur09, Chapter 5.3], [Lur18, Chapters 21.1.12 and D.7 (see especially D.7.3 and D.7.7)], [And21, Section 5.3], [And23, Kapitel 2], [Cal+21, Section A.2], [Cla23, Lecture 1], [Hoy18].

4. Efimov K-Theory (February 16)

Introduce the notion of Verdier sequences of presentable stable ∞-categories and then discuss the behavior of dualizable ∞-categories in Verdier sequence; in particular, discuss Thomason's trick (see [And23, Kapitel 2], especially [And23, Lemma 2.12, Lemma 2.13]). Introduce the notion of localizing invariants of small stable ∞-categories and then illustrate it by non-connective K-theory. In particular, define negative K-groups for commutative rings by classical means (see [Wei13, Chapter III.4]) and then explain its relation to the definition of non-connective K-theory as the universal localizing Sp-valued invariant under core. Following Efimov, prove that localizing invariants extend uniquely to all dualizable stable ∞-categories.

References: [Hoy18], [Cla23, Lecture 1], [And23, Kapitel 2].

5. Analytic Animated Rings (February 23)

Recall the notion of condensed analytic ring, then introduce its animated variant and discuss its properties. Explain the construction of induced analytic structures on animated condensed rings (see [Sch20, Proposition 12.8]). Introduce the notion of steady maps of analytic rings and discuss its properties and some examples. For a discrete Huber pair (A, A^+), explain the construction of the analytic ring $(A, A^+)_{\blacksquare}$.

References: [Sch19, Lecture 7] [Sch20, Lectures 11 and 12], [And21, Sections 2, 3.2, and 3.3], [Man22, Sections 2.3 and 2.9].

6. Trace Class Maps and Nuclearity (March 1)

Introduce the notion of nuclear object in a symmetric monoidal stable (∞-)category and discuss its properties. In particular, prove that if the monoidal unit is compact, then an object is dualizable if and only if it is compact and nuclear. Introduce the notions of trace class maps and basic nuclear objects. Prove that the full subcategory of nuclear objects is generated under colimits by basic nuclear objects. Discuss the notion of inner nuclearity following [And21, Section 5.3] Prove that for a discrete Huber pair (A, A^+), the full subcategory of nuclear objects in $\mathcal{D}((A, A^+)_{\blacksquare})$ is precisely the derived category $\mathcal{D}(A)$ (see [Man22, Theorem 2.9.7]).

References: [Sch20, Lecture 13], [And21, Section 5.3], [Man22, Sections 2.3 and 2.9].

7. Analytic Adic Spaces and Descent (March 22)

For a general complete Huber pair (A, A^+), define the analytic ring $(A, A^+)_{\blacksquare}$. Prove, following [And21, Section 4, Section 5.3] that the associations $\operatorname{Spa}(A, A^+) \mapsto \mathcal{D}((A, A^+)_{\blacksquare})$, $\operatorname{Spa}(A, A^+) \mapsto \mathcal{D}^{\omega}((A, A^+)_{\blacksquare})$, and $\operatorname{Spa}(A, A^+) \mapsto \mathcal{D}^{\mathrm{nuc}}((A, A^+)_{\blacksquare})$ satisfy Zariski descent. Deduce from that the association $\operatorname{Spa}(A, A^+) \mapsto \operatorname{Perf} A$ satisfies Zariski descent as well.

References: [And21, Sections 3.3, 4 and 5.3].

8. 6-functors (March 29)

Define the category of solid quasi-coherent sheaves $\mathcal{D}_{\blacksquare}(X)$ on a rigid-analytic variety X. Show that this assignment comes with a natural 6-functor formalism. Finally, prove the Grothendieck–Serre duality in the rigid-analytic context.

9. $\mathrm{Nuc}(X)$ **is dualizable (April 5)**
Prove that for a qcqs analytic adic space X, the category of nuclear sheaves on X is dualizable in two steps. First, establish the result locally over affinoid Tate adic spaces (see [And23, Kapitel 2]) using the notion of weak proregularity and then prove [And23, Satz 2.14] concerning gluing of dualizable categories. Define (continuous) K-theory of adic spaces.
References: [And23, Kapitel 2-4], [Yek21, Sections 2 and 3].

10. K**-theory satisfies Nisnevich descent (April 12)** Review the notion of cd-structure for a Grothendieck topology and explain how it helps check the sheaf conditions (see [AHW17, Section 3.2]). Introduce and discuss the Nisnevich topology for schemes and analytic adic spaces (see [BH17, Appendix A], [Lur18, Chapter B], and [And23, Appendix A]); in particular, discuss their equivalent characterizations and describe the points of the corresponding topoi. Prove that the Nisnevich topology in both situations comes with a cd-structure. Using it, prove that nuclear sheaves satisfies Nisnevich descent on qcqs analytic adic spaces and deduce from it Nisnevich descent for localizing invariants. Sketch the proof of the fact that étale descent is Nisnevich+Galois (see [Lur18, Chapter B.7]) and prove that nuclear sheaves satisfy étale descent (see [And23, Korollar 4.10]).
References: [AHW17, Section 3.2], [BH17, Appendix A], [Lur18, Chapter B], [And23, Kapitel 4 and Appendix A].

11. Étale Hyperdescent (April 19)
Introduce the notion of hypersheaf and discuss its basic properties and equivalent characterizations. In particular, discuss the notions of homotopy and cohomological dimensions and their relation to hyperdescent. Sketch the proof of the fact for finite dimensional qcqs schemes and analytic adic space, the corresponding Nisnevich topos has finite homotopy dimension. Explain the counterexample [CM21, Example 4.15] in the étale case. Sketch the proof of the hypercompleteness criterion in terms of topos theoretic points (see [CM21, Theorem 4.36] and its adic analog [And23, Satz B.31]) and use it to prove that for finite dimensional analytic adic spaces, localizing invariants satisfy étale hyperdescent after chromatic localization (see [And23, Satz 5.14]) using the algebraic analog as input (see [CM21, Theorem 7.14]). If time permits, explain the counterexample due to Clausen for étale descent of K-theory in general: https://mathoverflow.net/questions/239393/simplest-example-of-failure-of-finite-galois-descent-in-algebraic-k-theory.
References: [Lur18, Chapter 1.1.3], [CM21, Sections 2, 4, and 5], [And23, Kapitel 5 and Appendix B].

12. Grothendieck–Riemann–Roch, Part I (April 26) Explain the formulation of the Grothendieck–Riemann–Roch theorem for analytic adic spaces: go through [And23, pp. 32-38] and define all relevant objects and maps. Before explaining the construction of the Chern class map, define the sheaf $KU_p^\wedge$ on analytic adic spaces and identify it with $L_{K(1)}K(-)$ as in [And23, Satz 5.15] (see also the preceding discussion therein and [BCM20, Section 3]); in particular, state the Efimov continuity theorem. Finish by constructing the Chern class map using [And23, Satz 6.12] (do not prove it!).

References: [And23, Kapitel 5 and 6], [CS22, Lectures 14 and 15].

13. Grothendieck–Riemann–Roch, Part II (May 6)

Prove the Grothendieck–Riemann–Roch theorem: first prove [And23, Satz 6.12] and then explain the sketch of the proof on [And23, p. 38] by proving the relevant statements from the second half of [CS22, Lecture 15].

References: [And23, Kapitel 6], [CS22, Lecture 15].

References

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt. "Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles". In: *Duke Mathematical Journal* 166.10 (2017), pp. 1923–1953.

[And21] Grigory Andreychev. *Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces*. arXiv: 2105.12591. 2021.

[And23] Grigory Andreychev. *K-Theorie adischer Räume*. arXiv: 2311.04394. 2023.

[BCM20] Bhargav Bhatt, Dustin Clausen, and Akhil Mathew. "Remarks on $K(1)$-local K-theory". In: *Selecta Mathematica* 26 (2020).

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. "A universal characterization of higher algebraic K–theory". In: *Geometry & Topology* 17 (2013), pp. 733–838.

[BH17] Tom Bachmann and Marc Hoyois. "Norms in motivic homotopy theory". In: *Astérisque* 425 (2017).

[Cal+21] Baptiste Calmès et al. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity*. arXiv: 2009.07224. 2021.

[Cla23] Dustin Clausen. *Efimov K-Theory*. Lecture 1, Lecture 2, Lecture 3. 2023.

[CM21] Dustin Clausen and Akhil Mathew. "Hyperdescent and étale K-theory". In: *Inventiones mathematicae* 225 (2021), pp. 981–1076.

[CS22] Dustin Clausen and Peter Scholze. *Condensed Mathematics and Complex Geometry*. 2022. URL: `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`.

[Heb21] Fabian Hebestreit. *Lecture Notes for Algebraic and Hermitian K-Theory*. 2021. URL: `https://florianadler.github.io/AlgebraBonn/KTheory.pdf`.

[Hoy18] Marc Hoyois. *K-Theory of Dualizable Categories (After A. Efimov)*. 2018. URL: `https://hoyois.app.uni-regensburg.de/papers/efimov.pdf`.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, 2009.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry*. 2018. URL: `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. arXiv: 2206.02022. 2022.

[Sch19] Peter Scholze. *Lectures on Condensed Mathematics*. 2019. URL: `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[Sch20] Peter Scholze. *Lectures on Analytic Geometry*. 2020. URL: `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`.

[Wei13] Charles A. Weibel. *The K-book: An Introduction to Algebraic K-theory.* Vol. 145. Graduate Studies in Mathematics. American Mathematical Society, 2013.

[Yek21] Amnon Yekutieli. "Weak Proregularity, Derived Completion, Adic Flatness, and Prisms". In: *Journal of Algebra* 583 (2021), pp. 126–152.