$\mathrm{Nuc}(X)$ IS DUALIZABLE

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In this talk, we want to define the category $\mathrm{Nuc}(X)$ of nuclear sheaves on an analytic adic space X and prove the following result of Andreychev.

Theorem 1 ([And23, Satz 4.6])**.** *Let X be a qcqs analytic adic space. Then $\mathrm{Nuc}(X)$ is a dualizable category.*

Using Theorem 1 and the Efimov K-theory introduced in Talk 4, we can finally define the K-theory of qcqs analytic adic spaces.

Definition 2. Let X be a qcqs analytic adic space. Then the *(continuous) K-theory* of X is the presheaf valued in spectra given by

$$\mathbf{K}_{\mathrm{cont}} \colon \{\text{qc open } U \subseteq X\} \longrightarrow \mathrm{Sp}, \quad U \mapsto \mathbf{K}^{\mathrm{Ef}}(\mathrm{Nuc}(U)).$$

1. Dualizability for affinoid analytic adic spaces

We first study Theorem 1 for affinoid adic spaces subject to the following condition, which was first appeared in [Har67, § 2].

Definition 3. A complete Huber ring A is called *weakly proregular* if there exists a pair of definition (A_0, I) and a sequence $a_1, \dots, a_r \in A_0$ such that $I = (a_1, \dots, a_r)$ and the system of Koszul complexes $K(A_0; a_1^n, \dots, a_r^n)$ is protrivial; that is, for all $q > 0$ and all $n \geq 0$, there exists $m \geq n$ such that the transition maps $\mathrm{H}_q\big(K(A_0; a_1^m, \dots, a_r^m)\big) \to \mathrm{H}_q\big(K(A_0; a_1^n, \dots, a_r^n)\big)$ induced by multiplication with a_i^{m-n} are 0. A complete Huber pair (A, A^+) is weakly proregular if A is so.

It is easy to check that if the condition on the progregularity of the ideal $I \subset A_0$ holds for one pair of definition (A_0, I), then it holds for all pairs of definition; see [And23, Lem. 3.5].

Example 4 ([Har67, Lem. 2.5])**.** If A admits a noetherian ring of definition A_0, then A is weakly proregular.

Example 5. Any complete Tate ring A is weakly proregular: Let $\varpi \in A^{\circ\circ}$ be a pseudouniformizer. Then the topology of a ring of definition $A_0 \subset A$ containing ϖ is the (ϖ)-adic one. Since ϖ is a unit in A, the subring A_0 is ϖ-torsionfree and hence $\mathrm{H}_1\big(K(A_0; \varpi^n)\big) = 0$ for all n.

Example 5 shows that the analytic adic spaces relevant to this talk always satisfy the weak proregularity condition locally. Though not strictly necessary to set up our theory for analytic adic spaces, we find conceptually pleasing to generalize this example to all affinoid analytic adic spaces:

Example 6. Let A be an analytic complete Huber ring, that is, a complete Huber ring whose topologically nilpotent elements generate the unit ideal; these are exactly the Huber rings which underlie affinoid analytic adic spaces. We claim that A is weakly proregular. To see this, pick topologically nilpotent $\varpi_1, \dots, \varpi_r \in A$ that generate the unit ideal and a ring of definition $A_0 \subset A$ which contains the ϖ_i and has the $(\varpi_1, \dots, \varpi_r)$-adic topology. It suffices to prove that $\mathrm{colim}_n \mathrm{Ext}^q(K(A_0; \varpi_1^n, \dots, \varpi_r^n), I) = 0$ for all $q > 0$ and all injective A_0-modules I [Har67, Lem. 2.4].

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Since $A_0 \subset A$ is an open subring, the localizations $A_0[1/\varpi_i] \to A[1/\varpi_i]$ are isomorphisms for $1 \leq i \leq r$. This implies that $\operatorname{Spec} A \to \operatorname{Spec} A_0$ is an open immersion whose complement is given by $V(\varpi_1, \ldots, \varpi_r)$. The long exact sequence for local cohomology and the vanishing $\mathrm{H}^{>0}(\operatorname{Spec} A, \widetilde{I}) = \mathrm{H}^{>0}(\operatorname{Spec} A_0, \widetilde{I}) = 0$ then show that $\operatorname{colim}_n \operatorname{Ext}^q(K(A_0; \varpi_1^n, \ldots, \varpi_r^n), I) = 0$ for $q > 1$ and that it suffices to prove that $I \to I \otimes_{A_0} A$ is surjective.

Let $s \in I \otimes_{A_0} A$. Since $\operatorname{coker}(I \to I \otimes_{A_0} A)$ is $(\varpi_1, \ldots, \varpi_r)$-power torsion, there exists an integer $m > 0$ such that $\varpi_i^m \cdot s$ can be lifted to $\tilde{s}_i \in I$ for all $1 \leq i \leq r$. The ϖ_i^m still generate the unit ideal in A, so we can pick $a_1, \ldots, a_r \in A$ such that $a_1 \varpi_1^m + \cdots + a_r \varpi_r^m = 1$. Again, there exists an integer $n > 0$ with $\varpi_j^n a_i \in A_0$ for all $1 \leq i, j \leq r$. Since I is injective, we can find $t_{ij} \in I$ for $1 \leq i, j \leq r$ that induce the dashed map rendering the following diagram commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & A_0 & \xhookrightarrow{(\cdot \varpi_j^n)} & A_0^{\oplus r} \\ & & \downarrow{\scriptstyle (\tilde{s}_i)} & \swarrow{\scriptstyle (t_{ij})} & \\ & & I^{\oplus r} & & \end{array}$$

In particular, $s = \sum_{i=1}^r a_i \varpi_i^m \cdot s = \sum_{i=1}^r \tilde{s}_i \otimes a_i = \sum_{i,j=1}^r (\varpi_j^n \cdot t_{ij}) \otimes a_i = \sum_{i,j=1}^r (\varpi_j^n a_i) \cdot (t_{ij} \otimes 1)$ lies in $\operatorname{im}(I \to I \otimes_{A_0} A)$ because $\varpi_j^n a_i \in A_0$. This proves the claim.

Now we can state a version of Theorem 1 in the affinoid case.

Theorem 7 ([And23, Satz 3.17])**.** *Let (A, A^+) be a weakly proregular complete Huber pair. Denote by $(\mathscr{A}, \mathscr{M}) := (A, A^+)_\square$ the associated analytic ring and by $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A}, \mathscr{M})$ the ∞-category of nuclear modules over $(\mathscr{A}, \mathscr{M})$. Then $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A}, \mathscr{M})$ is a dualizable category.*

Let us review some of the notions used in the statement.

Recollection 8. (i) We saw in Talk 7 that the analytic ring $(\mathscr{A}, \mathscr{M}) := (A, A^+)_\square$ attached to a complete Huber pair (A, A^+) is given by $\mathscr{M}[S] := \underline{A}[S] \otimes_{A^+_{\mathrm{disc}}} (A^+_{\mathrm{disc}})_\square$ for all profinite S [And21, § 3.3].[1] Here, A^+_{disc} denotes the ring A^+ equipped with the discrete topology and $(A^+_{\mathrm{disc}})[S] := \operatorname{colim}_{A' \subseteq A} A'_\square[S]$, where the colimit runs over all finitely generated $\mathbf{Z}$-subalgebras of A'.

(ii) We saw in Talk 6 that an object X of a closed symmetric monoidal ∞-category $\mathscr{C}$ is called *nuclear* if for all compact objects $Y \in \mathscr{C}$, the natural map $(Y^\vee \otimes X)(*) \to \operatorname{Hom}_{\mathscr{C}}(Y, X)$ is an equivalence. When $\mathscr{C} = \mathscr{D}(\mathscr{A}, \mathscr{M})$, nuclearity amounts to the condition that the natural map

$$(\underline{C(S,A)} \otimes X)(*) \simeq (\mathscr{M}[S]^\vee \otimes X)(*) \to \operatorname{Hom}_{(\mathscr{A},\mathscr{M})}(\mathscr{M}[S], X) \simeq X(S)$$

is an equivalence in $\mathscr{D}(\mathrm{Ab})$ for all extremally disconnected S. By [And21, Prop. 5.35], this is equivalent to the "inner nuclearity" condition that the natural map

$$\underline{C(S,A)} \otimes X \to \mathscr{H}om_{(\mathscr{A},\mathscr{M})}(\mathscr{M}[S], X)$$

is an equivalence in $\mathscr{D}(\mathscr{A}, \mathscr{M})$. We recall that the full subcategory of nuclear objects is closed under colimits (immediate from the definition) and an object of $\mathscr{D}(\mathscr{A}, \mathscr{M})$ is dualizable if and only if it is compact and nuclear [And21, Prop. 5.37].

(iii) Denote by $\mathrm{Pr}^L_{\mathrm{st}}$ the ∞-category of stable presentable ∞-categories. The Lurie tensor product endows $\mathrm{Pr}^L_{\mathrm{st}}$ with a symmetric monoidal structure whose unit is the ∞-category Sp of spectra. We saw in Talk 3 that a category $\mathscr{C} \in \mathrm{Pr}^L_{\mathrm{st}}$ is *dualizable* (with respect to this symmetric monoidal structure) if and only if it is a retract in $\mathrm{Pr}^L_{\mathrm{st}}$ of a compactly generated category $\mathscr{D} \in \mathrm{Pr}^L_{\mathrm{st}}$; that is, there exists a fully faithful (colimit preserving) functor $\mathscr{C} \to \mathscr{D}$ in $\mathrm{Pr}^L_{\mathrm{st}}$ which has a (colimit preserving) right adjoint $\mathscr{D} \to \mathscr{C}$ in $\mathrm{Pr}^L_{\mathrm{st}}$.

[1] In this talk, we will omit the derived decorations for all our functors, so for example $\otimes$ stands for the derived tensor product, / for the derived quotient, and $\mathscr{H}om$ for the derived internal hom.

The idea for the proof of Theorem 7 is now simple: Since $\mathscr{D}(\mathscr{A},\mathscr{M})$ is compactly generated by the $\mathscr{M}[S]$ for S extremally disconnected [Sch19, Prop. 7.5], it suffices to construct a colimit preserving right adjoint to the fully faithful inclusion $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M}) \hookrightarrow \mathscr{D}(\mathscr{A},\mathscr{M})$. We will see soon that such an adjoint is given by the trace functor $(-)^{\mathrm{tr}}$. But first, we note that the machinery of dualizable categories and Efimov K-theory seems really necessary if one wants to obtain a reasonable notion of K-theory in the adic context.

Warning 9. (i) Even though the category $\mathscr{D}(\mathscr{A},\mathscr{M})$ is compactly generated, it has too many compact objects to support a reasonable notion of K-theory. For example, $\mathbf{N}\cup\{\infty\}$ with the profinite topology gives rise to a short exact sequence in $\mathscr{D}(\mathscr{A},\mathscr{M})$

$$0 \to \mathscr{M}[\{0\}] \to \mathscr{M}[\mathbf{N}\cup\{\infty\}] \to \mathscr{M}[\mathbf{N}_{>0}\cup\{\infty\}] \to 0$$

Since translation by 1 induces an isomorphism $\mathbf{N}_{>0}\cup\{\infty\} \xrightarrow{\sim} \mathbf{N}\cup\{\infty\}$ and thus $\mathscr{M}[\mathbf{N}_{>0}\cup\{\infty\}] \xrightarrow{\sim} \mathscr{M}[\mathbf{N}\cup\{\infty\}]$, we deduce that the class of $\mathscr{A} \simeq \mathscr{M}[\{0\}]$ in $K_0(\mathscr{D}(\mathscr{A},\mathscr{M})^{\omega})$ must be 0.

(ii) The stable ∞-category $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})$ is not compactly generated. To see this, consider the chain of colimit preserving inclusions in the upper row of

$$\begin{array}{ccccc} \mathscr{D}(A) & \hookrightarrow & \mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M}) & \hookrightarrow & \mathscr{D}(\mathscr{A},\mathscr{M}) \\ \cup & & \cup & & \cup \\ \mathrm{Perf}(A) \simeq \mathscr{D}(A)^{\omega} & \overset{\sim}{\dashrightarrow} & \mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})^{\omega} & \dashrightarrow & \mathscr{D}(\mathscr{A},\mathscr{M})^{\omega}. \end{array}$$

Since the inclusion $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M}) \hookrightarrow \mathscr{D}(\mathscr{A},\mathscr{M})$ has a colimit preserving right adjoint (namely the trace functor to be defined later), it preserves compact objects, so we obtain the right dashed arrow. On the other hand, compact and nuclear objects in $\mathscr{D}(\mathscr{A},\mathscr{M})$ are dualizable and thus discrete when $(\mathscr{A},\mathscr{M})$ comes from a sheafy analytic complete Huber pair (A,A^+) [And21, Cor. 5.51.1]; therefore, we obtain the left dashed arrow and see that it is in fact an equivalence of full subcategories. However, when $\mathscr{A}$ is not discrete, $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})$ contains nondiscrete objects and can thus not be generated by $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})^{\omega}$.

We return to the construction of the trace functor. Contemplating the definition of nuclear objects, one arrives at

Definition 10. Let $(\mathscr{A},\mathscr{M})$ be an analytic ring and $\mathscr{C} := \mathscr{D}(\mathscr{A},\mathscr{M})$. Since $\mathscr{C}$ is compactly generated, it is given by the ∞-category of exact functors $\mathscr{C} \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathscr{C}^{\omega})^{\mathrm{op}},\mathrm{Sp})$. Via this equivalence, the trace functor is described by the endofunctor

$$(-)^{\mathrm{tr}} \colon \mathscr{C} \to \mathscr{C} \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathscr{C}^{\omega})^{\mathrm{op}},\mathrm{Sp}), \quad X \mapsto ((-)^{\vee} \otimes_{(\mathscr{A},\mathscr{M})} X)(*).$$

It follows directly from Recollection 8.(ii) that there is a natural transformation of endofunctors $(-)^{\mathrm{tr}} \to \mathrm{id}_{\mathscr{C}}$ given on an object $X \in \mathscr{C}$ by $X^{\mathrm{tr}} = ((-)^{\vee} \otimes_{(\mathscr{A},\mathscr{M})} X)(*) \to \mathrm{Hom}_{\mathscr{C}}(-,X) = X$ and that $X \in \mathscr{C}$ is nuclear if and only if $X^{\mathrm{tr}} \to X$ is an equivalence. Moreover, one checks easily that $(-)^{\mathrm{tr}}$ preserves small colimits. Following the idea set forth above, the proof of Theorem 7 therefore amounts to showing that $(-)^{\mathrm{tr}}$ factors through $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M}) \hookrightarrow \mathscr{D}(\mathscr{A},\mathscr{M})$.

To see this, we use that since $\mathscr{C}$ is compactly generated, we can alternatively describe the trace functor as

$$X^{\mathrm{tr}} \simeq \operatorname*{colim}_{\substack{Q\in\mathscr{C}^{\omega},\\ Q\to X^{\mathrm{tr}}}} Q \simeq \operatorname*{colim}_{\substack{Q\in\mathscr{C}^{\omega},\\ \mathscr{A}\to(Q^{\vee}\otimes X)}} Q \simeq \operatorname*{colim}_{\substack{P,Q\in\mathscr{C}^{\omega},\\ P\to Q^{\vee},\\ \mathscr{A}\to(P\otimes X)}} Q \simeq \operatorname*{colim}_{\substack{P,Q\in\mathscr{C}^{\omega},\\ Q\to P^{\vee},\\ \mathscr{A}\to(P\otimes X)}} Q \simeq \operatorname*{colim}_{\substack{P\in\mathscr{C}^{\omega},\\ \mathscr{A}\to(P\otimes X)}} P^{\vee}; \tag{1}$$

here, the fourth equivalence uses that

$$\mathrm{Hom}(P,Q^{\vee}) = \mathrm{Hom}(P,\mathscr{H}om(Q,\mathscr{A})) \simeq \mathrm{Hom}(P\otimes Q,\mathscr{A}) \simeq \mathrm{Hom}(Q,\mathscr{H}om(P,\mathscr{A})) \simeq \mathrm{Hom}(Q,P^{\vee}).$$

Moreoever, we need the following key statement:

Lemma 11. *Let (A,A^+) be a weakly proregular complete Huber pair. Denote by $(\mathscr{A},\mathscr{M}) := (A,A^+)_\square$ the associated analytic ring. Then for any profinite S, the dual $\mathscr{M}[S]^\vee \simeq \underline{C(S,A)}$ is nuclear.*

Assuming Lemma 11, we can now give the

Proof of Theorem 7. We show that $(-)^{\mathrm{tr}}$ factors through $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M}) \hookrightarrow \mathscr{D}(\mathscr{A},\mathscr{M})$ and $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})$ is thus a retract of a compactly generated category in $\mathrm{Pr}^L_{\mathrm{st}}$. First, note that any compact $P \in \mathscr{D}(\mathscr{A},\mathscr{M})$ is the retract of a finite complex whose terms are all of the form $\bigoplus_{i=1}^n \mathscr{M}[S_i]$ for some profinite S_i and some $n \geq 0$ [SP24, Tag 094B]. Therefore, $P^\vee$ is nuclear by Lemma 11. Since $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})$ is closed under colimits, we can then conclude from (1) that X^{tr} is nuclear for all $X \in \mathscr{D}(\mathscr{A},\mathscr{M})$. □

Remark 12. Since $\underline{C(S,A)}$ does not depend on the choice of subring of integral elements A^+, the formula (1) and Lemma 11 show in particular that the fully faithful embedding $\mathscr{D}((A,A^+)_\square) \hookrightarrow \mathscr{D}((A,\mathbf{Z})_\square)$ induces an equivalence

$$\mathscr{D}^{\mathrm{nuc}}((A,A^+)_\square) \xrightarrow{\sim} \mathscr{D}^{\mathrm{nuc}}((A,\mathbf{Z})_\square)$$

upon applying $(-)^{\mathrm{tr}}$. In fact, it follows from the isomorphism (2) below that this equivalence is even symmetric monoidal. Therefore, we can suppress the A^+ from the notation and simply set

$$\mathrm{Nuc}(A) := \mathscr{D}^{\mathrm{nuc}}((A,A^+)_\square).$$

We proceed with the proof of Lemma 11. Note that by Recollection 8.(ii), the (inner) nuclearity of $\underline{C(S,A)}$ amounts to showing that for any profinite S', the natural map

$$\underline{C(S,A)} \otimes_{(\mathscr{A},\mathscr{M})} \underline{C(S',A)} \to \mathscr{H}om_{(\mathscr{A},\mathscr{M})}(\mathscr{M}[S], \underline{C(S',A)}) \simeq \underline{C(S\times S',A)} \tag{2}$$

is an isomorphism. This will be proven in two steps:

(i) in the special case when $(A,A^+) = (R,R)$, where $R := \mathbf{Z}[[t_1,\dots,t_r]]$ equipped with the $\mathfrak{m}$-adic topology for $\mathfrak{m} := (t_1,\dots,t_r) \subset R$
(ii) for general (A,A^+), using the statement for (R,R).

Remark 13. Since $t_i \in R$ is topologically nilpotent, we have

$$(\mathbf{Z}[[t_1,\dots,t_r]], \mathbf{Z}[[t_1,\dots,t_r]])_\square \simeq (\mathbf{Z}[[t_1,\dots,t_r]], \mathbf{Z}[t_1,\dots,t_r])_\square \simeq (\mathbf{Z}[[t_1,\dots,t_r]], \mathbf{Z})_\square$$

by [And21, Prop. 3.32, Lem. 3.31]. This often simplifies computations because it eliminates the need to take colimits over finitely generated subalgebras in Recollection 8.(i).

Proof of Lemma 11 for $(A,A^+) = (R,R)$ ([And23, Lem. 3.7]). We verify (2). Since $R/(t_1^n,\dots,t_r^n)$ is discrete for all $n \in \mathbf{N}$ and the module of continuous functions on a profinite set S with values in a discrete module is free by a result of Nöbeling (see [Sch19, Thm. 5.4]),[2] we have

$$\underline{C(S,R)} \simeq \lim_n C(S, R/(t_1^n,\dots,t_r^n)) \simeq \lim_n \bigoplus_J R/(t_1^n,\dots,t_r^n)$$

for some index set J. On the other hand, the ω_1-small sequential limit over $n \in \mathbf{N}$ commutes with ω_1-filtered colimits, hence

$$\lim_n \bigoplus_J R/(t_1^n,\dots,t_r^n) \simeq \operatorname*{colim}_{\substack{\tilde{J}\subseteq J \\ \text{countable}}} \lim_n \bigoplus_{\tilde{J}} R/(t_1^n,\dots,t_r^n) \simeq \operatorname*{colim}_{\substack{\tilde{J}\subseteq J \\ \text{countable}}} \lim_n \bigoplus_{\mathbf{N}} R/(t_1^n,\dots,t_r^n).$$

[2]Strictly speaking, *loc. cit.* only proves this statements for $C(S,\mathbf{Z})$. However, since a continuous function from a profinite set to a discrete module X has finite image, the natural map $C(S,\mathbf{Z}) \otimes_{\mathbf{Z}} X \to C(S,X)$ is an isomorphism.

The last term $\lim_n \bigoplus_{\mathbf{N}} R/(t_1^n,\ldots,t_r^n)$ is the "Tate algebra" over R in r variables. The presentation of Tate algebras in terms of convergent power series shows that

$$\lim_n \bigoplus_{\mathbf{N}} R/(t_1^n,\ldots,t_r^n) \simeq \operatorname*{colim}_{\substack{f\colon \mathbf{N}\to\mathbf{N}\\ f(m)\to\infty}} \prod_{m\in\mathbf{N}} (t_1^{f(m)},\ldots,t_r^{f(m)})\cdot R$$

(where the last colimit runs over the set of nonnegative sequences that tend toward ∞). Thus, for any profinite sets S, S', we have

$$\begin{aligned}&\underline{C(S,R)} \otimes_{R_\square} \underline{C(S',R)}\\ &\qquad \simeq \operatorname*{colim}_{\substack{\tilde J\subseteq J,\, \tilde J'\subseteq J'\\ \text{countable}}} \operatorname*{colim}_{\substack{f,f'\colon \mathbf{N}\to\mathbf{N}\\ f(m),\,f'(m')\to\infty}} \prod_{m\in\mathbf{N}} (t_1^{f(m)},\ldots,t_r^{f(m)})\cdot R \otimes_{R_\square} \prod_{m'\in\mathbf{N}} (t_1^{f'(m')},\ldots,t_r^{f'(m')})\cdot R.\end{aligned}$$

Since $(t_1^{f(m)},\ldots,t_r^{f(m)})$ has a finite (Koszul) resolution by finite free modules and $\prod_I \underline{R}\otimes_{R_\square}\prod_{I'}\underline{R} \simeq \prod_{I\times I'}\underline{R}$ for all index sets I, I', we obtain

$$\prod_{m\in\mathbf{N}} (t_1^{f(m)},\ldots,t_r^{f(m)})\cdot R \otimes_{R_\square} \prod_{m'\in\mathbf{N}} (t_1^{f'(m')},\ldots,t_r^{f'(m')})\cdot R \simeq \prod_{m,m'\in\mathbf{N}} (t_i^{f(m)}\cdot t_j^{f'(m')})_{1\le i,j\le r}\cdot R.$$

Any function $g\colon \mathbf{N}^2\to\mathbf{N}$ with $g(m,m')\to\infty$ as $(m,m')\to(\infty,\infty)$ can be dominated by a function of the form $g(m,m')\le \max\{f(m),f'(m')\}$ with $f(m),f'(m')\to\infty$ as $m,m'\to\infty$: one may simply set $f(m) := \max\{g(\ell,\ell')\}_{1\le\ell,\ell'\le m}$ and likewise for $f'(m')$. Therefore,

$$\operatorname*{colim}_{\substack{f,f'\colon \mathbf{N}\to\mathbf{N}\\ f(m),\,f'(m')\to\infty}} \prod_{m,m'\in\mathbf{N}} (t_i^{f(m)}\cdot t_j^{f'(m')})_{1\le i,j\le r}\cdot R \simeq \operatorname*{colim}_{\substack{g\colon \mathbf{N}^2\to\mathbf{N}\\ g(m,m')\to\infty}} \prod_{m,m'\in\mathbf{N}} (t_1^{g(m,m')},\ldots,t_r^{g(m,m')})\cdot R.$$

Similarly, any countable subset $\tilde J''\subseteq J\times J'$ is contained in a countable subset of the form $\tilde J\times\tilde J'$ by setting $\tilde J := \{j\in J \mid (j,j')\in \tilde J'' \text{ for some } j'\in J'\}$ and likewise for $\tilde J'$. Hence,

$$\begin{aligned}&\operatorname*{colim}_{\substack{\tilde J\subseteq J,\, \tilde J'\subseteq J'\\ \text{countable}}} \operatorname*{colim}_{\substack{g\colon \mathbf{N}^2\to\mathbf{N}\\ g(m,m')\to\infty}} \prod_{m,m'\in\mathbf{N}} (t_1^{g(m,m')},\ldots,t_r^{g(m,m')})\cdot R\\ &\simeq \operatorname*{colim}_{\substack{\tilde J''\subseteq J\times J'\\ \text{countable}}} \operatorname*{colim}_{\substack{g\colon \mathbf{N}^2\to\mathbf{N}\\ g(m,m')\to\infty}} \prod_{m,m'\in\mathbf{N}} (t_1^{g(m,m')},\ldots,t_r^{g(m,m')})\cdot R \simeq \lim_n \bigoplus_{J\times J'} R/(t_1^n,\ldots,t_r^n) \simeq \underline{C(S\times S')}.\end{aligned}$$

Combining everything, we obtain (2). ◻

Before we move on to the proof of Lemma 11 for arbitrary weakly proregular complete Huber pairs (A,A^+), we mention a more general statement for $(A,A^+)=(R,R)$ that will be useful later on. Keeping in mind our convention that all functors are derived, recall

Definition 14. An object $X\in\mathscr{D}(R_\square)$ is called *derived* $\mathfrak{m}$*-adically complete* (or $\mathfrak{m}$*-complete* in short) if the natural map $X\to\lim_n X/(t_1^n,\ldots,t_r^n)$ is an equivalence. We denote by $\mathscr{D}(R_\square)^\wedge_\mathfrak{m}\hookrightarrow\mathscr{D}(R_\square)$ the resulting full subcategory of $\mathfrak{m}$-complete objects. Its left adjoint $\mathscr{D}(R_\square)\to\mathscr{D}(R_\square)^\wedge_\mathfrak{m}$ which is on objects given by $X\mapsto X^\wedge_\mathfrak{m} := \lim_n X/(t_1^n,\ldots,t_r^n)$ is called the $\mathfrak{m}$*-adic completion functor*.

The more general statement is now

Proposition 15. *If* $X,Y\in\mathscr{D}^-(R_\square)^\wedge_\mathfrak{m}$*, then* $X\otimes_{R_\square}Y$ *is again* $\mathfrak{m}$*-adically complete.*

Proof. This is a special case of [Man22, Prop. 2.12.10], applied to $A=R$ and $A^+=\mathbf{Z}[t_1,\ldots,t_r]_{\mathrm{disc}}$, keeping in mind Remark 13. ◻

When $X=\underline{C(S,R)}$ and $Y=\underline{C(S',R)}$, Proposition 15 recovers Lemma 11 for (R,R) because the natural map $\underline{C(S,R)}\otimes_{R_\square}\underline{C(S',R)}\to\underline{C(S\times S',R)}$ is an isomorphism modulo $\mathfrak{m}^n$ for all $n\in\mathbf{N}$ as observed in the proof above. In fact, the proof of [Man22, Prop. 2.12.10] (or rather that of [Man22, Lem. 2.12.9] on which it relies) essentially reduces to this case.

We can finally deal with the proof of Lemma 11 for an arbitrary weakly proregular complete Huber pair (A, A^+). Let (A_0, I) be a pair of definition. By Definition 3, we can find a sequence $a_1, \dots, a_r \in A$ such that $I = (a_1, \dots, a_r)$ and the natural map $A_0/(a_1^n, \dots, a_r^n) \to \pi_0(A_0/(a_1^n, \dots, a_r^n))$ is a pro-isomorphism of pro-systems in n. The idea is now to consider (A, A^+) as an (R, R)-algebra via $t_i \mapsto a_i$ and bootstrap to the case of (R, R). We will need a series of three lemmas:

Lemma 16. *Let S be a profinite set. The condensed module $\underline{C(S, A_0)}$ is the I-adic completion of $\underline{C(S, (A_0)_{\mathrm{disc}})}$.*

Proof. Since A_0 is classically I-adically complete and the module of continuous functions on S with values in a discrete module is free, we have

$$\underline{C(S, A_0)} \simeq \lim_n C(S, \pi_0(A_0/(a_1^n, \dots, a_r^n))) \simeq \lim_n \pi_0(C(S, (A_0)_{\mathrm{disc}})/(a_1^n, \dots, a_r^n)).$$

Now (A, I) is weakly proregular, so $A_0/(a_1^n, \dots, a_r^n) \to \pi_0(A_0/(a_1^n, \dots, a_r^n))$ is a pro-isomorphism of pro-systems in n and remains a pro-isomorphism after applying any A_0-linear functor such as (infinite) direct sums. Thus,

$$C(S, (A_0)_{\mathrm{disc}})_I^\wedge := \lim_n C(S, (A_0)_{\mathrm{disc}})/(a_1^n, \dots, a_r^n) \to \lim_n \pi_0(C(S, (A_0)_{\mathrm{disc}})/(a_1^n, \dots, a_r^n))$$

is still an isomorphism. Combined with the previous display equation, this finishes the proof. □

Lemma 17 ([And23, Lem. 3.12])**.** *If X is an $\mathfrak{m}^\infty$-torsion module over R, then $X \otimes_{R_{\mathrm{disc}}} R_\square \simeq X$. In particular, X is nuclear over (R, R).*

Proof. Using Koszul resolutions, one can see that the canonical map

$$R/(t_1^n, \dots, t_r^n) \otimes_{R_{\mathrm{disc}}} R_\square \to R/(t_1^n, \dots, t_r^n)$$

is an isomorphism for all $n \in \mathbf{N}$. The first statement is now a consequence of the fact that $- \otimes_{R_{\mathrm{disc}}} R_\square$ preserves small colimits. The second statement then follows from the fact that all discrete modules are nuclear [And23, Lem. 5.45]. □

Lemma 18 ([And23, Lem. 3.12])**.** *Let A be a weakly proregular complete Huber ring. Then $\underline{A}$ is nuclear as a condensed module over $R_\square$.*

Proof. Consider the short exact sequence of condensed R-modules

$$0 \to \underline{A_0} \to \underline{A} \to A/A_0 \to 0.$$

Since A/A_0 is $\mathfrak{m}^\infty$-torsion, Lemma 17 shows that A/A_0 is nuclear. As the category of nuclear modules is closed under extensions (and indeed all colimits), it suffices to show that $\underline{A_0}$ is nuclear, that is, that for all profinite sets S, the natural map

$$\underline{C(S, R)} \otimes_{R_\square} \underline{A_0} \longrightarrow \underline{C(S, A_0)}$$

is an equivalence. The target is $\mathfrak{m}$-complete by Lemma 16 and the source is $\mathfrak{m}$-complete by Proposition 15. The statement therefore follows again from the fact that

$$C(S, R/(t_1^n, \dots, t_r^n)) \otimes_{R/(t_1^n, \dots, t_r^n)} A_0/(a_1^n, \dots, a_r^n) \longrightarrow C(S, A_0/(a_1^n, \dots, a_r^n))$$

is an equivalence for all n and all profinite S. □

Proof of Lemma 11 for general (A, A^+) ([And23, Kor. 3.14]). We verify again (2), which now follows directly from Lemma 18 and the case $(A, A^+) = (R, R)$:

$$\underline{C(S, A)} \otimes_{(A, A^+)_\square} \underline{C(S', A)} \simeq \underline{A} \otimes_{R_\square} \underline{C(S, R)} \otimes_{R_\square} \underline{C(S', R)} \simeq \underline{A} \otimes_{R_\square} \underline{C(S \times S', R)} \simeq \underline{C(S \times S', A)}. \ \square$$

We conclude our discussion of $\mathrm{Nuc}(A)$ for weakly proregular complete Huber rings A by explaining an additional structural property that it enjoys. Recall the following definitions.

Definition 19. A morphism $X \to Y$ in a dualizable category $\mathscr{C}$ is called *compact* if for any cofiltered system $\{Z_i\}_{i\in I}$ of $\mathscr{C}$ and any morphism $Y \to \operatorname{colim}_{i\in I} Z_i$, the composition $X \to Y \to \operatorname{colim}_{i\in I} Z_i$ factors through Z_j for some $j \in I$.

Example 20. If X is a compact object of a dualizable category, then any morphism $X \to Y$ is compact.

Definition 21. A morphism $X \to Y$ in a closed symmetric monoidal ∞-category $\mathscr{C}$ is called *trace-class* if it lies in the image of the natural map $(Y^\vee \otimes X)(*) \to \operatorname{Hom}_{\mathscr{C}}(X, Y)$.

A diagram chase shows that trace-class morphisms form a two-sided ideal in all morphisms of $\mathscr{C}$; that is, for any composable morphisms f, g and h of $\mathscr{C}$, the composition $h \circ g \circ f$ is trace-class if g is trace-class. Moreover, it is easy to see that all trace-class morphisms are compact. The additional structure on $\operatorname{Nuc}(A)$ concerns the reverse implication.

Definition 22 (Gaitsgory–Rozenblyum)**.** A symmetric monoidal dualizable ∞-category $\mathscr{C} \in \operatorname{CAlg}(\operatorname{Pr}^L_{\mathrm{st}})$ is called *rigid* if all compact morphisms of $\mathscr{C}$ are trace-class.

Notice the connection to nuclearity: an object X of a closed symmetric monoidal ∞-category $\mathscr{C}$ is nuclear exactly when any map $X \to Y$ to a compact object $Y \in \mathscr{C}^\omega$ is trace-class.

Proposition 23. *Let A be a weakly proregular complete Huber ring. Then the stable ∞-category $\operatorname{Nuc}(A)$ is rigid.*

Proof. We have already seen in Theorem 7 that $\operatorname{Nuc}(A)$ is dualizable. Thus, it remains to show that all compact morphisms of $\operatorname{Nuc}(A)$ are trace-class. In fact, by the two-sided ideal property of trace-class morphisms, it suffices to shows that any compact morphism factors through a trace-class morphism.

Pick a subring of integral elements $A^+ \subset A$ and denote by $(\mathscr{A}, \mathscr{M}) := (A, A^+)_\square$ the associated analytic ring. We will identify $\operatorname{Nuc}(A)$ as the full subcategory $\mathscr{D}^{\mathrm{nuc}}(\mathscr{A}, \mathscr{M}) \subseteq \mathscr{D}(\mathscr{A}, \mathscr{M})$. Let $X \to Y$ be a compact morphism of $\operatorname{Nuc}(A)$. Since $\mathscr{D}(\mathscr{A}, \mathscr{M})$ is compactly generated, we can write Y as the colimit of a cofiltered system $\{Z_i\}_{i\in I}$ of compact objects of $\mathscr{D}(\mathscr{A}, \mathscr{M})$. Moreover, the trace functor $(-)^{\mathrm{tr}}$ from Definition 10 preserves nuclear objects and preserves small colimits, so $Y \simeq \operatorname{colim}_{i\in I} Z_i^{\mathrm{tr}}$. By the defining property of compact morphisms, we can now pick $j \in I$ together with a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y \simeq \operatorname{colim}_{i\in I} Z_i^{\mathrm{tr}} & \xrightarrow{\sim} & \operatorname{colim}_{i\in I} Z_i \\ & \searrow & \uparrow & & \uparrow \\ & & Z_i^{\mathrm{tr}} & \longrightarrow & Z_i \end{array}$$

in which the right horizontal arrows are given by the counit of adjunction.

We claim that the map $Z_i^{\mathrm{tr}} \to Y$ is trace-class; as observed before, this will finish the proof. To prove the claim, we need to chase through the diagram

$$\begin{array}{ccccc} \big((\mathscr{H}om_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i, \mathscr{A}))^{\mathrm{tr}} \otimes Y\big)(*) & \longrightarrow & \big(\mathscr{H}om_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i, \mathscr{A}) \otimes Y\big)(*) & \longrightarrow & \operatorname{Hom}_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i, Y) \\ \downarrow & & \downarrow & & \downarrow \\ \big((\mathscr{H}om_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i^{\mathrm{tr}}, \mathscr{A}))^{\mathrm{tr}} \otimes Y\big)(*) & \longrightarrow & \big(\mathscr{H}om_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i^{\mathrm{tr}}, \mathscr{A}) \otimes Y\big)(*) & \longrightarrow & \operatorname{Hom}_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i^{\mathrm{tr}}, Y) \\ \| & & & & \| \\ \big(\mathscr{H}om_{\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})}(Z_i^{\mathrm{tr}}, \mathscr{A}) \otimes Y\big)(*) & & \longrightarrow & & \operatorname{Hom}_{\mathscr{D}^{\mathrm{nuc}}(\mathscr{A},\mathscr{M})}(Z_i^{\mathrm{tr}}, Y). \end{array}$$

By construction, the map $Z^{\mathrm{tr}} \to Y$ (considered as an element of the bottom right) can be lifted to a map $Z_i \to Y$ (an element of the top right). Since Z_i is compact and Y is nuclear, it follows directly from the definitions that the natural map $Z_i \to Y$ is trace-class in $\mathscr{D}(\mathscr{A}, \mathscr{M})$. In other words, the element of the top right can be lifted to the top middle. Since Z_i is compact, Lemma 11 guarantees that $Z_i^\vee = \mathscr{H}om_{\mathscr{D}(\mathscr{A},\mathscr{M})}(Z_i, \mathscr{A})$ is nuclear, so the top left horizontal arrow is an isomorphism and the element of the top middle can be lifted to the top left. Taken together, this shows that the element of the bottomw right can be lifted to the bottom left, as claimed. □

2. Dualizability for general qcqs analytic adic spaces

In this section, we prove Theorem 1 in the generality of qcqc analytic adic spaces. To do so, we first need to make sense of the category of nuclear sheaves on such a gadget. Recall that we already defined the category of nuclear sheaves on spaces of the form $X = \mathrm{Spa}(A, A^+)$ for a complete analytic Huber pair (A, A^+). In Remark 12, we observed that this category of nuclear sheaves on X does not depend on the choice of the ring of integral elements A^+ and therefore denoted it by $\mathrm{Nuc}(A)$. Moreover, we saw in Talk 7 that the functor

$$\{\text{affinoid open } U \subseteq X\}^{\mathrm{op}} \to \mathrm{Pr}^L_{\mathrm{st}}, \qquad \begin{array}{c} U \mapsto \mathrm{Nuc}(\mathscr{O}_X(U)) \\ (U \overset{j}{\hookrightarrow} V) \mapsto j^*\colon \mathrm{Nuc}(\mathscr{O}_X(V)) \to \mathrm{Nuc}(\mathscr{O}_X(U)) \end{array}$$

is a sheaf for the analytic topology on X [And21, Thm. 5.42]; here, the pullback $j^*\colon \mathrm{Nuc}(\mathscr{O}_X(V)) \to \mathrm{Nuc}(\mathscr{O}_X(U))$ is induced by the base change functor

$$- \otimes_{(\mathscr{O}_X(V),\mathscr{O}_X^+(V))_\square} (\mathscr{O}_X(U), \mathscr{O}_X^+(U))_\square \colon \mathscr{D}((\mathscr{O}_X(V), \mathscr{O}_X^+(V))_\square) \to \mathscr{D}((\mathscr{O}_X(U), \mathscr{O}_X^+(U))_\square),$$

which was shown in the proof of [And21, Thm. 5.42] to preserve nuclear objects. Using this analytic descent result, we can formally extend Nuc to all open subsets of X.

Definition 24 ([And23, Def. 4.3])**.** Let X be an analytic adic space. The ∞-category of *nuclear sheaves on* X is

$$\mathrm{Nuc}(X) := \lim_{\substack{U \subseteq X \\ \text{affinoid open}}} \mathrm{Nuc}(\mathscr{O}_X(U)).$$

Since the pullback functors between affinoid opens preserve all small colimits, $\mathrm{Nuc}(X) \in \mathrm{Pr}^L_{\mathrm{st}}$; see [Lur09, Prop. 5.5.3.13].

To make $\mathrm{Nuc}(-)$ into a sheaf on X, one can define pullback functors $j^*\colon \mathrm{Nuc}(V) \to \mathrm{Nuc}(U)$ for any open immersion $U \overset{j}{\hookrightarrow} V$ of open subspaces of X by gluing the pullback functors on affinoid open subsets. In fact, the j^* are commonly functors in $\mathrm{Pr}^L_{\mathrm{st}}$ and their right adjoints are as good as could be hoped:

Lemma 25 ([And23, Lem. 4.4, Lem. 4.5])**.** *Let $U \overset{j}{\hookrightarrow} V \hookrightarrow X$ be open subsets. Assume that U is quasicompact and that V is quasiseparated. Then $j^*\colon \mathrm{Nuc}(V) \to \mathrm{Nuc}(U)$ has a fully faithful right adjoint $j_*\colon \mathrm{Nuc}(U) \to \mathrm{Nuc}(V)$ which preserves small colimits.*

Actually, the proof we will only need that U is a retrocompact open subset of V.

Proof. We first assume that U and V are affinoid opens of X. Set $(A, A^+) := (\mathscr{O}_X(V), \mathscr{O}_X^+(V))$ and $(B, B^+) := (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$. Restriction of scalars along the map of analytic complete Huber pairs $(A, A^+) \to (B, B^+)$ defines a functor $\mathscr{D}((B, B^+)_\square) \to \mathscr{D}((A, A^+)_\square)$. It is right adjoint to the base change functor $- \otimes_{(A,A^+)_\square} (B, B^+)_\square$ and fully faithful by [And21, Prop. 4.12.(i), Def. 4.8]. We claim that it moreover preserves nuclear objects.

To see this, we now show that for any $M \in \mathscr{D}((B, B^+)_\square)$, we have $M \in \mathrm{Nuc}(B)$ if and only if the restriction of scalars $M \in \mathrm{Nuc}(A)$. Fix topologically nilpotent elements $\varpi_1, \dots, \varpi_r \in A$ which generate the unit ideal; they form a weakly proregular sequence for A and B by Example 6 and make them into algebras over $R := \mathbf{Z}[\![t_1, \dots, t_r]\!]$ via $t_i \mapsto \varpi_i$. We obtain a commutative diagram

$$\begin{array}{ccc}
\big(\underline{C(S,B)} \otimes_{(B,B^+)_\square} M\big)(*) & \searrow & \\
\wr\Vert & & \\
\big(\underline{C(S,R)} \otimes_{R_\square} \underline{B} \otimes_{(B,B^+)_\square} M\big)(*) & & \\
\wr\Vert & & \\
\big(\underline{C(S,R)} \otimes_{R_\square} M\big)(*) & & M(S) \\
\wr\Vert & & \\
\big(\underline{C(S,R)} \otimes_{R_\square} \underline{A} \otimes_{(A,A^+)_\square} M\big)(*) & & \\
\wr\Vert & & \\
\big(\underline{C(S,A)} \otimes_{(A,A^+)_\square} M\big)(*) & \nearrow &
\end{array}$$

in which the first and last vertical equivalence of the first column come from the nuclearity of $\underline{B}$ and $\underline{A}$ over $R_\square$ from Lemma 18, the second and third vertical equivalence come from the fact that $M \in \mathscr{D}((B,B^+)_\square)$ and $M \in \mathscr{D}((A,A^+)_\square)$, and the bent arrows are the morphisms testing the nuclearity of M over $(B,B^+)_\square$ and $(A,A^+)_\square$ from Recollection 8.(ii), respectively. Since one of the bent arrows is an equivalence if and only if the other one is, the desired statement about M follows. This finishes the construction of $j_* \colon \operatorname{Nuc}(U) \to \operatorname{Nuc}(V)$ in case U and V are affinoid opens of X because $\operatorname{Nuc}(A) \subseteq \mathscr{D}((A,A^+)_\square)$ and $\operatorname{Nuc}(B) \subseteq \mathscr{D}((B,B^+)_\square)$ are closed under colimits.

Next, let $V \hookrightarrow X$ be an affinoid open and $U \overset{j}{\hookrightarrow} V$ be a general quasicompact open subset. Choose a finite cover $U = \bigcup_{i=1}^n U_i$ by rational open subspaces U_i of V; then the intersections $U_I := \bigcap_{i\in I} U_i \overset{j_I}{\hookrightarrow} V$ are still affinoid for all $I \subseteq \{1,\dots,n\}$. The first part of the proof produced pushforward functors $j_{I,*} \colon \operatorname{Nuc}(U_I) \to \operatorname{Nuc}(V)$, which are compatible with another by [And21, Prop. 4.12.(iii)]. Thanks to the sheaf property of $\operatorname{Nuc}(-)$, we can thus define

$$j_* := \lim_{I \subseteq \{1,\dots,n\}} j_{I,*} \colon \operatorname{Nuc}(U) \simeq \lim_{I\subseteq\{1,\dots,n\}} \operatorname{Nuc}(U_I) \to \operatorname{Nuc}(V).$$

Since the $j_{I,*}$ are right adjoint to j_I^* and fully faithful, j_* is still right adjoint to j^* and fully faithful. Moreover, j_* commutes with colimits because the limit is taken over a *finite* index set.

Lastly, let $U \overset{j}{\hookrightarrow} V \hookrightarrow X$ be general open subsets such that U is quasicompact and V is quasiseparated. Choose a cover by affinoid open subspaces $V = \bigcup_{j\in J} V_j$ and set $V_{\tilde J} := \bigcap_{j \in \tilde J} V_j$ for all $\tilde J \subseteq J$. Since V is quasiseparated, the intersections $U_{\tilde J} := U \cap V_{\tilde J} \overset{j_{\tilde J}}{\hookrightarrow} U_{\tilde J}$ are quasicompact. The previous paragraph produced pushforward functors $j_{\tilde J,*} \colon \operatorname{Nuc}(U_{\tilde J}) \to \operatorname{Nuc}(V_{\tilde J})$ which are again compatible with another. We can therefore again exploit the sheaf property of $\operatorname{Nuc}(-)$ to define a fully faithful right adjoint

$$j_* := \lim_{\tilde J \subseteq J} j_{\tilde J,*} \colon \operatorname{Nuc}(U) \simeq \lim_{\tilde J\subseteq J} \operatorname{Nuc}(U_{\tilde J}) \to \lim_{\tilde J \subseteq J} \operatorname{Nuc}(V_{\tilde J}) \simeq \operatorname{Nuc}(V)$$

to j^*. The assertion that j_* commutes with colimits can be checked locally on V and hence reduces to the previous paragraph, finishing the proof. $\square$

Thanks to the dual of [Lur09, Cor. 5.2.2.5], the pushforward functors j_* from Lemma 25 can all be made compatible with another. We can now promote $\operatorname{Nuc}(-)$ to a $\mathrm{Pr}^L_{\mathrm{st}}$-valued sheaf for the analytic topology on X:

Definition 26. The *nuclear sheaves* on an analytic adic space X are given by the functor

$$\operatorname{Nuc} \colon \{\text{qcqs open } U \subseteq X\}^{\mathrm{op}} \to \mathrm{Pr}^L_{\mathrm{st}}, \qquad \begin{array}{c} U \mapsto \operatorname{Nuc}(U) \\ (U \overset{j}{\hookrightarrow} V) \mapsto j^* \colon \operatorname{Nuc}(V) \to \operatorname{Nuc}(U). \end{array}$$

Theorem 1 in the generality of qcqs analytic adic spaces now follows readily from the next, general result about sheaves of dualizable categories, whose proof in the case of compactly generated categories goes back to Bondal–van den Bergh.

Theorem 27 ([BvdB03, Proof of Thm. 3.1.1], [And23, Satz 2.14])**.** *Let X be a qcqs topological space and $\mathscr{B}$ be a basis of quasicompact open subsets of X. Let*

$$\mathscr{C}\colon \{qc\ open\ U \subseteq X\} \longrightarrow \mathrm{Pr}^L_{\mathrm{st}}, \quad U \mapsto \mathscr{C}_U$$

be a sheaf of presentable stable ∞-categories. Assume:

(a) For all $V \in \mathscr{B}$, the category $\mathscr{C}_V$ is compactly generated (resp. dualizable).
(b) For all $V \in \mathscr{B}$ and all quasicompact open $V' \subseteq V$, the restriction functor $\mathscr{C}_V \to \mathscr{C}_{V'}$ is a localization[3] whose fully faithful right adjoint preserves small colimits and whose fiber is compactly generated (resp. dualizable).

Then the category $\mathscr{C}_X$ is compactly generated (resp. dualizable).

Remark 28. In the dualizable case of assumption (b), the requirement that the fiber of $\mathscr{C}_V \to \mathscr{C}_{V'}$ be dualizable is automatic: If $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$ is a sequence in $\mathrm{Pr}^L_{\mathrm{st}}$ with $G \circ F \simeq 0$, then G has a fully faithful right adjoint that preserves small colimits if and only if F is fully faithful and has a right adjoint that preserves small colimits; cf. e.g. the proof of [CDH+20, Lem. A.2.5]. Therefore, the assumption that $\mathscr{C}_V \to \mathscr{C}_{V'}$ is a localization whose fully faithful right adjoint preserves small colimits already guarantees that the fiber is a retract in $\mathrm{Pr}^L_{\mathrm{st}}$ of the dualizable category $\mathscr{C}_V$ and thus itself dualizable.

Assuming Theorem 27 for now, we can finally give the

Proof of Theorem 1. The underlying qcqs topological space $|X|$ has a basis of opens $\mathscr{B}$ whose constituents are the underlying spaces of affinoid opens of X. Since X is analytic, Theorem 7 (and Example 6) show that $\mathrm{Nuc}(V)$ is a dualizable category for all $V \in \mathscr{B}$. Lemma 25 (and Remark 28) show that for any $V \in \mathscr{B}$ and any quasicompact open $V' \overset{j}{\hookrightarrow} V$, the pullback functor $j^*\colon \mathrm{Nuc}(V) \to \mathrm{Nuc}(V')$ has a colimit preserving fully faithful right adjoint $j_*\colon \mathrm{Nuc}(V') \to \mathrm{Nuc}(V)$ and its fiber is again dualizable. The assertion therefore follows from Theorem 27 applied to the $\mathrm{Pr}^L_{\mathrm{st}}$-valued sheaf $\mathrm{Nuc}(-)$ on $|X|$. ☐

In the proof of Theorem 27, we will need the classical *Neeman–Thomason localization theorem*:

Proposition 29 ([TT90, Lem. 5.5.1, Prop. 5.5.4], [Nee92, Thm. 2.1, Cor. 0.9])**.** *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a localization functor of stable ∞-categories whose fully faithful right adjoint preserves small colimits.*

(i) If $\mathscr{C}$ is compactly generated, then $\mathscr{D}$ is also compactly generated and $\mathscr{D}^\omega$ is the idempotent completion of the essential image of $\mathscr{C}^\omega$ in $\mathscr{D}$. More generally, given any compact object $X \in \mathscr{D}^\omega$, any object $Y \in \mathscr{C}$ and any morphism $f\colon X \to F(Y)$, there exists a compact object $\tilde{X} \in \mathscr{C}$ and a morphism $\tilde{f}\colon \tilde{X} \to Y$ such that $F(\tilde{f})$ factors as

$$F(\tilde{X}) \xrightarrow{\sim} X \oplus X' \xrightarrow{\mathrm{pr}_X} X \xrightarrow{f} F(Y)$$

for some compact object $X' \in \mathscr{D}^\omega$.

(ii) If $\mathscr{C}$ and the fiber of F are compactly generated, then a compact object of $\mathscr{D}$ lies in the essential image of F if and only if its class in $\mathrm{K}_0(\mathscr{D})$ lies in the image of $\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathscr{C}) \to \mathrm{K}_0(\mathscr{D})$.

For completeness, we present (part of) a proof of Proposition 29, following the presentation in [And23, Satz 2.13].

[3] in the sense of [Lur09, Def. 5.2.7.2], i.e., it has a fully faithful right adjoint

Proof. (i). Let us prove the last assertion; for the rest, we refer the reader to [Lur09, Cor. 5.5.7.3]. Let G be the right adjoint to F. Let $f\colon X \to F(Y)$ be a morphism as in the statement. Since $\mathscr{C}$ is compactly generated, we have $G(X) \simeq \operatorname{colim} X_i$ for some compact objects $X_i \in \mathscr{C}^\omega$, and since F is a localization, $X \simeq FG(X) \simeq \operatorname{colim} F(X_i)$. The last isomorphism must factor through one of the $F(X_i)$ because X is compact, making X into a retract of $F(X_i)$. The natural units of adjunction and the splitting $f_i\colon F(X_i) \to X$ then give the nondashed arrows in the following diagram:

$$\begin{array}{ccccccc}
\tilde{X} & & & \overset{\tilde{f}}{\dashrightarrow} & & & Y \\
\downarrow{\scriptstyle f'} & & & & & & \downarrow \\
X_i & \longrightarrow & GF(X_i) & \xrightarrow{G(f_i)} & G(X) & \xrightarrow{G(f)} & GF(Y) \\
\downarrow{\scriptstyle g} & & & & & & \downarrow \\
Z_j & & & \dashrightarrow & & & \operatorname{cofib}\big(Y \to GF(Y)\big).
\end{array} \tag{3}$$

Next, we fill in the dashed arrows of (3). Writing $\operatorname{cofib}\big(Y \to GF(Y)\big) \in \mathscr{C}$ as a colimit of compact objects $Z_j \in \mathscr{C}^\omega$, the map from the compact X_i must factor through some $g\colon X_i \to Z_j$, giving the lower rectangle. Since $F\big(\operatorname{cofib}(Y \to GF(Y))\big) \simeq \operatorname{cofib}\big(F(Y) \to FGF(Y)\big) \simeq 0$, we may additionally assume that $F(g) = 0$. Set $\tilde{X} := \operatorname{fib}(X_i \to Z_j)$. The additional assumption that $F(g) = 0$ guarantees that $F(\tilde{X}) \simeq F(X_i) \oplus F(Z_j[-1])$ and the induced map $F(\tilde{X}) \to F(X_i) \to FG(X) \simeq X$ is a retraction. Since $\tilde{X} \to \operatorname{cofib}\big(Y \to GF(Y)\big)$ factors through $\tilde{X} \to Z_j$ and is hence 0, we obtain a morphism $\tilde{f}\colon \tilde{X} \to Y$, completing the upper rectangle of (3) and the proof of (i).

(ii). Let $Z \in \mathscr{D}^\omega$ and $Y' \in \mathscr{C}^\omega$ be compact objects such that $[Z] = \big[F(Y')\big]$ in $\mathrm{K}_0(\mathscr{D})$. First, we claim that one can find compact objects $X', Y \in \mathscr{C}^\omega$ together with an isomorphism $Z \oplus F(X') \simeq F(Y)$ in $\mathscr{D}$. To see this, note that the definition of $\mathrm{K}_0(\mathscr{D})$ guarantees the existence of exact triangles $A_1 \to A_2 \to A_3$ and $B_1 \to B_2 \to B_3$ such that

$$Z \oplus A_1 \oplus A_3 \oplus B_2 \simeq F(Y') \oplus A_2 \oplus B_1 \oplus B_3. \tag{4}$$

By (i), there exist $A_1', A_3' \in \mathscr{D}^\omega$ such that $A_1 \oplus A_1'$ and $A_3 \oplus A_3'$ lie in the essential image of F. Analogously for $B_1', B_3' \in \mathscr{D}^\omega$. Since the essential image of F is closed under extensions, it must also contain $A_2 \oplus A_1' \oplus A_3'$ and $B_2 \oplus B_1' \oplus B_3'$. Thus, there are $X', Y'' \in \mathscr{C}^\omega$ such that

$$F(X') \simeq A_1 \oplus A_1' \oplus A_3 \oplus A_3' \oplus B_2 \oplus B_1' \oplus B_3' \quad \text{and} \quad F(Y'') \simeq A_2 \oplus A_1' \oplus A_3' \oplus B_1 \oplus B_1' \oplus B_3 \oplus B_3'.$$

Set $Y := Y' \oplus Y''$. Taking the direct sum of (4) with $A_1' \oplus A_3' \oplus B_1' \oplus B_3'$, we obtain the claimed isomorphism $Z \oplus F(X') \simeq F(Y)$.

In order to prove the statement of (ii), it now suffices to show that we can find a compact object $\tilde{X} \in \mathscr{C}^\omega$ and maps $X' \xleftarrow{f'} \tilde{X} \xrightarrow{\tilde{f}} Y$ such that $F(f')$ is an equivalence and the diagram

$$\begin{array}{ccccc}
 & \overset{F(f')}{\swarrow\sim} & F(\tilde{X}) & \overset{F(\tilde{f})}{\searrow} & \\
F(X') & \xrightarrow{0 \oplus \mathrm{id}} & Z \oplus F(X') & \xrightarrow{\sim} & F(Y)
\end{array}$$

commutes; then $\operatorname{cofib}(\tilde{f})$ will be the desired lift of Z. To verify this claim, we can slightly modify the proof of (i) for $X := F(X')$: First, replace the retraction $F(X_i) \to X$ by the identity map $F(X') \xrightarrow{=} X$. Second, since $\operatorname{cofib}\big(Y \to GF(Y)\big)$ is contained in the fiber of F, which we assume to be compactly generated, we may also pick the Z_j to be in the fiber of F. Taken together, this means that $F(\tilde{X}) \simeq F(X') \oplus F(Z_j[-1]) \xrightarrow{\sim} F(X')$ as claimed. □

Proof of Theorem 27. For simplicity, we only consider the case where $X = U \cup V$ for some $U, V \in \mathscr{B}$. The general statement then follows by an induction on the minimal number of opens in $\mathscr{B}$ needed to cover X; this number is always finite by the quasicompactness of X. First, we prove the version of the statement for compactly generated categories. The version for dualizable for dualizable categories will reduce to this.

Pick $M \in \mathscr{C}_X$ such that any map $N \to M$ from a compact object $N \in \mathscr{C}_X^\omega$ is trivial. To show that $\mathscr{C}_X$ is compactly generated, we need to see that $M \simeq 0$. By way of contradiction, suppose that $M \not\simeq 0$. Since $\mathscr{C}$ is a sheaf, we obtain a fiber square

$$\begin{array}{ccc} \mathscr{C}_X & \longrightarrow & \mathscr{C}_U \\ \downarrow & & \downarrow \\ \mathscr{C}_V & \longrightarrow & \mathscr{C}_{U\cap V}, \end{array}$$

so either $M|_U \not\simeq 0$ or $M|_V \not\simeq 0$. Without loss of generality, we may assume that $M|_U \not\simeq 0$.

By assumption, $\mathscr{C}_U$ is compactly generated, so there exists a compact $N_U \in \mathscr{C}_U^\omega$ and a nontrivial map $N_U \xrightarrow{f_U} M|_U$. Moreover, the open $U \cap V$ is still quasicompact because X is quasiseparated, so the arrows $\mathscr{C}_U \to \mathscr{C}_{U\cap V}$ and $\mathscr{C}_V \to \mathscr{C}_{U\cap V}$ are localizations whose fully faithful right adjoints preserve small colimits and whose fiber is compactly generated. Thus, by Proposition 29.(i), there exists compact objects $N_V \in \mathscr{C}_V^\omega$ and $N' \in \mathscr{C}_{U\cap V}^\omega$, a map $f_V \colon N \to M|_V$, and a factorization

$$\begin{array}{ccc} N_V|_{U\cap V} & \xrightarrow{f_V|_{U\cap V}} & M|_{U\cap V} \\ \wr\!\mid & & \uparrow f_U|_{U\cap V} \\ N_U|_{U\cap V} \oplus N' & \xrightarrow{\mathrm{pr}_{N_U|_{U\cap V}}} & N_U|_{U\cap V}. \end{array}$$

The direct sum $N' \oplus N'[1] \in \mathscr{C}_{U\cap V}^\omega$ has class $[N' \oplus N'[1]] = 0$ in $\mathrm{K}_0(\mathscr{C}_{U\cap V})$, so by Proposition 29.(ii), it has a lift $N'' \in \mathscr{C}_U^\omega$. Then the two morphisms

$$N_V \oplus N_V[1] \xrightarrow{\mathrm{pr}_{N_V}} N_V \xrightarrow{f_V} M|_V \quad \text{and} \quad N_U \oplus N_U[1] \oplus N'' \xrightarrow{\mathrm{pr}_{N_U}} N_U \xrightarrow{f_U} M|_U$$

agree on $U \cap V$ and thus glue to a map $f \colon N \to M$. Since N is compact (this can be checked locally on U and V) and f is nontrivial (it is nontrivial on U), this produces the desired contradiction.

Now, we prove the version of the statement for dualizable categories. Let $\star \in \{X, U, V, U \cap V\}$. As we saw in Talk 3, the "colimit functor" $\operatorname{colim}\colon \operatorname{Ind}(\mathscr{C}_\star) \to \mathscr{C}_\star$ given by the Ind-extension of $\mathrm{id}_{\mathscr{C}_\star}$ admits a left adjoint $\hat{y}\colon \mathscr{C}_\star \to \operatorname{Ind}(\mathscr{C}_\star)$; see [Lur18, Thm. 21.1.2.10.(3)].[4] In fact, it follows from the proof of [Lur09, Prop. 5.4.6.6] and the characterization of dualizable categories as fibers of localizations of compactly generated stable ∞-categories that $\mathscr{C}_\star$ is ω_1-compactly generated, so $\hat{y}$ factors through the fully faithful subcategory $\operatorname{Ind}(\mathscr{C}_\star^{\omega_1}) \subset \operatorname{Ind}(\mathscr{C}_\star)$.

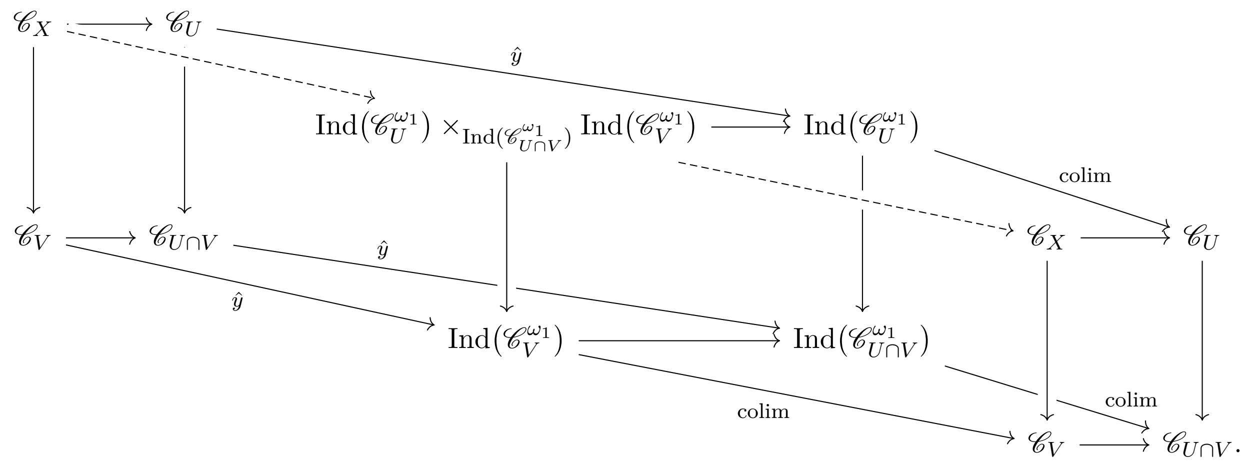
In particular, we may write $\mathscr{C}_\star$ canonically as a retract of the compactly generated stable ∞-category $\operatorname{Ind}(\mathscr{C}_\star^{\omega_1})$; cf. the proof of [Lur18, Cor. 21.1.2.18]. The functors $\mathscr{C}_\star \to \mathscr{C}_{U\cap V}$ for $\star \in \{U, V\}$ carry ω_1-compact objects to ω_1-compact objects [Lur09, Prop. 5.5.7.2]. The induced functors $\operatorname{Ind}(\mathscr{C}_\star^{\omega_1}) \to \operatorname{Ind}(\mathscr{C}_{U\cap V}^{\omega_1})$ are still localizations whose fully faithful right adjoints preserve small colimits. Moreover, the functors

$$\operatorname{Ind}(\operatorname{fib}(\mathscr{C}_\star^{\omega_1} \to \mathscr{C}_{U\cap V}^{\omega_1})) \to \operatorname{fib}(\operatorname{Ind}(\mathscr{C}_\star^{\omega_1}) \to \operatorname{Ind}(\mathscr{C}_{U\cap V}^{\omega_1}))$$

are equivalences by [NS18, Prop. I.3.5]. Thus, the first part of the proof for compactly generated categories shows that $\operatorname{Ind}(\mathscr{C}_U^{\omega_1}) \times_{\operatorname{Ind}(\mathscr{C}_{U\cap V}^{\omega_1})} \operatorname{Ind}(\mathscr{C}_V^{\omega_1})$ is compactly generated.

[4]Beware that this is not the Yoneda embedding, which is right adjoint to colim.

We now show that $\mathscr{C}_X$ is dualizable by expressing it as a retract of $\operatorname{Ind}(\mathscr{C}_U^{\omega_1}) \times_{\operatorname{Ind}(\mathscr{C}_{U\cap V}^{\omega_1})} \operatorname{Ind}(\mathscr{C}_V^{\omega_1})$. For this, consider the diagram



We have $\operatorname{colim} \circ \hat{y} = \operatorname{id}$. Using the universal property of $\operatorname{Ind}(-)$, it is easy to see that the bottom and back squares commute (in the upper box, one first has to pass to the adjoint squares). Hence, the universal properties of the fiber squares in the three "slices" produce the colimit preserving dashed functors and show that their composition is homotopic to $\operatorname{id}_{\mathscr{C}_X}$. This finishes the proof. □

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