SIX FUNCTOR FORMALISM FOR SOLID QUASI-COHERENT SHEAVES ON RIGID SPACES

In this talk we shall discuss the construction of the six functor formalism of solid quasi-coherent sheaves on (derived) rigid spaces over a complete non-archimedean field K. First, we briefly discuss the abstract theory of six functor formalisms following [Man22]. Then, we define derived rigid spaces and show that the functor $X \mapsto \mathscr{D}_{\square}(X)$ of solid quasi-coherent sheaves can be promoted to a six functor formalism.

1. Definition of six functor formalisms

The idea behind a six functor formalism is the following: let $\mathscr{C}$ be a category (more generally an ∞-category) of geometric objects. Attached to $X \in \mathscr{C}$ we have defined a category $\mathscr{D}(X)$ of coefficients, together with the six functors $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$ and $f^!$, where $\otimes$ is a symmetric monoidal structure on $\mathscr{D}(X)$ and $\underline{\mathrm{Hom}}$ is its right adjoint, where the last four functors are attached to a morphism $f : Y \to X$ in $\mathscr{C}$, and where the !-functors are defined for a possibly smaller class of maps E.

The six functors are subject to certain compatibilities: the formation of f^* and $f_!$ is contravariant and covariant respectively, and they have right adjoints f_* and $f^!$. In particular all the structure of a six functor formalism is totally encoded in the functors $\otimes$, f^* and $f_!$. The $*$ and !-functors are compatible with composition and (proper) base change, and there is a projection formula involving all three functors $\otimes$, f^* and $f_!$.

In practice, constructing the functors $\otimes$ and f^* is not hard. For example, if $\mathscr{C}$ is the category of derived schemes, the functor $\mathscr{D} : \mathscr{C} \to \mathrm{Cat}_{\infty}^{\otimes}$ of quasi-coherent sheaves is such an example. The difficulty in the construction of six functors relies in the functor $f_!$ and all the compatibilities with respect to the other two. The problem is that features such as projection formula and proper base change are not properties attached to $f_!$ but additional structure that needs to be coherently defined. We can encode all such compatibilities in the following construction:

Definition 1.1. [[Man22, Definition A.5.2] and [Sch23, Definition 2.3]] A geometric set up is a pair $(\mathscr{C}, E)$ consisting on an ∞-category $\mathscr{C}$ with finite limits and E a family of arrows in $\mathscr{C}$ containing all the equivalences and stable under pullbacks and compositions. We let $\mathrm{Corr}(\mathscr{C}, E)$ be the ∞-category of correspondences of $(\mathscr{C}, E)$.

To give some intuition of what the category $\mathrm{Corr}(\mathscr{C}, E)$ is let us describe its homotopy category. The objects of $\mathrm{Corr}(\mathscr{C}, E)$ are the same objects of $\mathscr{C}$. A morphism from X to Y is given by a correspondence

$$X \xleftarrow{\ f\ } V \xrightarrow{\ g\ } Y$$

where $g \in E$. Finally, a composite of two morphisms $X \leftarrow V \to Y$ and $Y \leftarrow W \to Z$ is given by the outer correspondence of the pullback

$$\begin{array}{ccccccc} & & & V \times_Y W & & & \\ & & \swarrow & & \searrow & & \\ & V & & & & W & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X & & & Y & & & Z. \end{array}$$

Note that, for the composition of correspondences to be well defined, we need the class E to be stable under pullbacks and compositions.

The category $\mathrm{Cat}(\mathscr{C}, E)$ is moreover symmetric monoidal (here we use that $\mathscr{C}$ has finite products, see [Man22, Remark A.5.5]) with symmetric monoidal structure given by the cartesian product in $\mathscr{C}$. We can then define a six functor formalism as follows:

Definition 1.2. Let $(\mathscr{C}, E)$ be a geometric set up as before. A three functor formalism on $(\mathscr{C}, E)$ is a lax symmetric monoidal functor

$$\mathscr{D} : \mathrm{Corr}(\mathscr{C}, E) \to \mathrm{Cat}_\infty$$

where Cat_∞ is endowed with the cartesian symmetric monoidal structure. A six functor formalism is a three functor formalism as before for which f^*, $f_!$ and $\otimes$ have right adjoints.

We have a natural functor $\mathscr{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathscr{C}, E)$ by mapping $X \mapsto X$ and a morphism $f : Y \to X$ to $X \leftarrow Y = Y$. The composite with the six functor formalism produces a lax symmetric monoidal functor $\mathscr{D} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which is the same datum as a functor $\mathscr{D} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}$. This encodes the functors $\otimes$ and f^*. The functors $f_!$ are encoded in the right vertical arrows of the correspondences in such a way that a morphism $X \xleftarrow{f} V \xrightarrow{g} Y$ is mapped to the morphism

$$g_! f^* : \mathscr{D}(X) \to \mathscr{D}(Y)$$

in Cat_∞. Additional structures such as base change and projection formula are also encoded in the category $\mathrm{Corr}(\mathscr{C}, E)$, see the discussion after [Sch23, Definition 3.12].

2. Construction of six functor formalisms

Let $(\mathscr{C}, E)$ be a geometric set up. When $\mathscr{C}$ is an ∞-category (and not just a category) it is difficult to construct a six functor formalism due to all the higher coherences that one needs to keep track in the correspondence category. To remedy this problem one can construct six functor formalisms out from a minimal amount of datum that heuristically corresponds to "compactifying" the maps in E.

Definition 2.1. Let $(\mathscr{C}, E)$ be a geometric set up. A suitable decomposition of E is a pair (I, P) consisting on families of arrows I and P in E satisfying the following conditions:

(1) Any arrow in E can be written as $p \circ j$ with $p \in P$ and $j \in I$.
(2) An arrow $f \in I \cap P$ is n-truncated for some $n \geq -2$ (which might depend in f).
(3) The families I and P contain all the isomorphisms, are stable under pullbacks and compositions.
(4) Given two composable arrows $f : Z \to Y$ and $g : Y \to X$ in $\mathscr{C}$, if $g \circ f$ and g are in I (resp. P), then $f \in I$ (resp. P).

Proposition 2.2 ([Man22, Proposition A.5.10])**.** *Let $(\mathscr{C}, E)$ be a geometric set up and (I, P) a suitable decomposition. Let $\mathscr{D} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}$ be a functor. Suppose that the following conditions are satisfied:*

(a) *For every $j : U \to X$ in I the following is true:*
  - *The functor j^* admits a left adjoint $j_!$.*
  - *$j_!$ satisfies proper base change and the projection formula.*

(b) *For every $f : Y \to X$ in P the following is true:*
  - *The functor f^* admits a right adjoint f_*.*
  - *The functor f_* satisfies proper base change and the projection formula.*

(c) *For every cartesian diagram*

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

with $j \in I$ and $f \in P$ the natural map $j_! f'_* \xrightarrow{\sim} f_* j'_!$ is an isomorphism.

Then $\mathscr{D}$ can be extended to a three functor formalism

$$\mathscr{D} : \mathrm{Corr}(\mathscr{C}, E) \to \mathrm{Cat}_\infty$$

such that for $j \in I$ the functor $j_!$ is the left adjoint of $j^!$ and for $f \in P$ the functor $f_!$ is the right adjoint of f^*.

Remark 2.3. The map $j_! f'_* \to f_* j'_!$ in the condition (c) of Theorem 2.2 is adjoint to a map

$$f^* j_! f'_* \cong j'_* f^{',*} f'_* \to j'_*$$

where in the left isomorphism we use proper base change of condition (a) and the right map follows from the adjunction between $f^{',*}$ and f'_*. There is another way to construct the map $j_! f'_* \to f_* j'_!$ which is by taking the adjoint of the map

$$f'_* \to f'_* j^{',*} j'_! \cong j^* f_* j'_!$$

where in the right isomorphism we use condition (b) and the left map follows from the adjunction between j'_* and $j^{',*}$. The fact that both adjuctions produce the same map follows from an inductive argument and the truncated condition (2) in Definition 2.1, see [Sch23, Constructions 4.3-4.5].

Given a six functor formalism $\mathscr{D}$ in a geometric set up $(\mathscr{C}, E)$, one would like to extend the category of objects $\mathscr{C}$ to stacks and the class of morphisms E to a class of "stacky !-able maps". The key statements that make this possible are [Man22, Lemma A.5.11 and Propositions A.5.12, A.5.14 and A.5.16]. A more geometric point of view is explained in [Sch23, Appendix of Lecture IV].

3. Pseudo-coherent modules and derived Tate algebras

In the second part of this talk we shall construct the six functor formalism for solid quasi-coherent sheaves of rigid spaces. In the definition of a six functor formalism we assumed that the geometry set up $(\mathscr{C}, E)$ was such that $\mathscr{C}$ has finite limits. In order to get the correct finite limits for rigid spaces we need to jump to the world of derived algebraic geometry. We shall follow the analogue discussion of [CS22, Lecture XI]

Definition 3.1. Let R be a ring, a module $M \in \mathscr{D}_+(R)$ is said pseudo-coherent if $\mathrm{Hom}_R(M, -)$ commutes with filtered colimits of diagrams in $\mathscr{D}_{\leq 0}(R)$ (equivalently arbitrary direct sums). We let $\mathscr{D}(R)^{\mathrm{pscoh}}$ denote the full subcategory of pseudo-coherent modules.

Lemma 3.2. *Let R be a noetherian ring. Then $M \in \mathscr{D}_+(R)$ is pseudo-coherent if and only if $H^i(M)$ is of finite type for all $i \in \mathbb{Z}$.*

Proof. This is [Sta22, Tag 066E]. □

Definition 3.3. Let $R = (R^{\triangleright}, \mathscr{D}(R))$ be an analytic ring, consider the fully faithful embedding $\mathscr{D}(R^{\triangleright}(*)) \to \mathscr{D}(R)$ arising from the fact that $R\mathrm{Hom}_R(R^{\triangleright}, R^{\triangleright}) = R^{\triangleright}(*)$. We let $\mathscr{D}(R)^{\mathrm{dis}}$ denote its essential image and call it the full subcategory of relative discrete R-modules. We also let $\mathscr{D}(R)^{\mathrm{dis,pscoh}}$ be the essential image of $\mathscr{D}(R^{\triangleright}(*))^{\mathrm{pscoh}}$.

Remark 3.4. The terminology of discrete modules in Definition 3.3 might be confusing. If the condensed ring $R^{\triangleright}$ has a non-trivial topology (eg. a Banach ring) then $R^{\triangleright}$ itself is not a discrete condensed ring. However, relative discrete modules are obtained by taking "discrete" colimits of the ring $R^{\triangleright}$ (namely, colimits without any kind of further completion), this justifies the name of "relative discrete modules".

Lemma 3.5. *Let R be a noetherian Banach algebra. Then $\mathscr{D}(R(*)) \to \mathscr{D}^{\mathrm{cond}}(R)$ is t-exact.*

Proof. Let $I(*) \subset R(*)$ be an ideal, since R is noetherian we have a resolution $M^{\bullet}$ of I by finite free R-modules. By the open Banach theorem, the resolution is strictly exact which is equivalent to being exact as condensed R-modules. This shows that $I(*) \otimes^L_{R(*)} R = I$ is in degree 0, proving what we wanted. ☐

Theorem 3.6 (Andreychev)**.** *Let (A, A^+) be an analytic Huber pair, then pseudo-coherent modules satisfy analytic descent. Moreover, if A is strongly noetherian then the t-structure of $\mathscr{D}(A)^{\mathrm{dis,pscoh}}$ satisfies analytic descent.*

Proof. Descent of pseudo-coherent complexes on analytic adic spaces is [And21, Theorem 5.44]. Flatness of rational localizations of rigid spaces and Lemma 3.5 show that we also have descent for the t-structures in pseudo-coherent sheaves. ☐

From now on we fix a complete non-archimedean field (K, K^+) and work with adic spaces over (K, K^+).

Definition 3.7. We let $\mathcal{T}_{n,K} := K\langle T_1, \dots, T_n\rangle$ denote the Tate algebra over K in n-variables. An animated solid K-algebra A is a derived Tate algebra if there is a morphism of K-algebras

$$\mathcal{T}_{n,K} \to A$$

such that A is a pseudo-coherent $\mathcal{T}_{n,K}$-module.

Proposition 3.8. *Let A be a solid K-algebra. The following are equivalent.*

1. *A is a derived Tate algebra.*
2. *$\pi_0(A)$ is a classical Tate algebra of finite type and $\pi_i(A)$ is a coherent $\pi_0(A)$-module for all $i \geq 0$.*

Moreover, under the previous equivalent assumptions we can find a morphism $\mathcal{T}_{n,K} \to A$ that is surjective on π_0.

Proof. Suppose that (1) holds, then $\pi_0(A)$ is a finite $\mathcal{T}_{n,K}$-module and so a classical Tate algebra of finite type. Since the $\mathcal{T}_{n,K}$-modules $\pi_i(A)$ are finite for all $i \geq 0$, then so are as A modules proving (2) since Tate algebra of finite type are noetherian. Conversely, suppose that (2) holds. Let $\mathcal{T}_{n,K} \to \pi_0(A)$ be a surjection of derived Tate algebras, and take any lift $f : K[T_1, \dots, T_n] \to A$. We claim that f localizes to $\mathcal{T}_{n,K} \to A$. Indeed, the morphism $K[T_1, \dots, T_n] \to \mathcal{T}_{n,K}$ is idempotent as solid K-algebras. Then, since $\pi_*(A)$ is a module over $\pi_0(A)$ and so is over $\mathcal{T}_{n,K}$, the map f will naturally extend to $\mathcal{T}_{n,K}$ since $\mathrm{Mod}_{\mathcal{T}_{n,K}}(\mathscr{D}_{\square}(K[T_1, \dots, T_n]))$ is stable under small limits and colimits. Note that the proof of "(2) implies (1)" also constructs the map $\mathcal{T}_{n,K} \to A$ that is surjective in π_0. ☐

Lemma 3.9. *The category of derived Tate K-algebras is stable under finite colimits.*

Proof. Since $\mathrm{AffRing}_K$ has an inital object it suffices to show that it is stable under pushouts. Let $B \leftarrow A \to C$ be a diagram in $\mathrm{AffRing}_K$. We can compute

$$B \otimes_A C = (B \otimes C) \otimes_{A \otimes A} A.$$

Thus, we can either assume that $A = K$ or that $A \to B$ is surjective on π_0. If $A = K$ consider surjections $\mathcal{T}_{n,K} \to A$ and $\mathcal{T}_{m,K}$, then $\mathcal{T}_{n+m,K} \to A \otimes B$ is a surjective map and a standard spectral sequence argument shows that $\pi_i(A \otimes B)$ is a finite $\mathcal{T}_{n+m,K}$-module proving that $A \otimes B$ is $\mathcal{T}_{n+m,K}$-pseudo-coherent by Lemma 3.2. Finally, suppose that $A \to B$ is a surjection on π_0, then $\pi_i(B)$ is a finite $\pi_0(A)$-module for all $i \in \mathbb{Z}$ and the standard spectral sequence argument shows that $\pi_i(B \otimes_A C)$ is a finite $\pi_0(C)$-module for all $i \in \mathbb{Z}$. Since $\pi_0(C) \to \pi_0(B \otimes_A C)$ is surjective, the second term is a classical Tate algebra and so $B \otimes_A C$ is a derived Tate algebra by Proposition 3.8. ☐

4. Derived affinoid rings

Having introduced derived Tate algebras we can define derived affinoid rings. For this we adopt an extension of Huber's theory of affinoid pairs. First recall an homotopical invariance of analytic ring structures.

Proposition 4.1 ([CS20, Proposition 12.22])**.** *Let $R^{\triangleright}$ be a condensed animated ring. Then the map sending an (uncompleted) analytic ring structure R over $R^{\triangleright}$ to $\pi_0(R) := \pi_0(R^{\triangleright})_{R/} = R \otimes_{R^{\triangleright}} \pi_0(R^{\triangleright})$ induces a bijection between (uncompleted) analytic ring structures over $R^{\triangleright}$ and $\pi_0(R^{\triangleright})$. Under this equivalence, an $R^{\triangleright}$-module M is R-complete if and only if $\pi_i(M)$ is $\pi_0(R)$-complete for all $i \in \mathbb{Z}$.*

In other words, the previous proposition says that (uncompleted) analytic ring structures over a fixed condensed animated ring $R^{\triangleright}$ only depend on the abelian heart of complete modules. This allows the construction of analytic rings just by declaring what complete modules are for static rings.

Definition 4.2. A derived affinoid pair over K is a pair (A, A^+) consisting on a derived Tate K-algebra A and a power bounded, open and integrally closed subring $A^+ \subset \pi_0(A)(*)$. A morphism $(A, A^+) \to (B, B^+)$ of derived affinoid pairs is a map $A \to B$ that sends A^+ to B^+. The adic spectrum of (A, A^+) is defined as $|\operatorname{Spa}(A, A^+)| := |\operatorname{Spa}(\pi_0(A), A^+)|$. We let $\mathrm{AffRing}_K$ denote the ∞-category of derived affinoid rings over K.

Recall that for a static $\mathbb{Z}$-algebra of finite type R there is the solid ring $R_{\square}$ whose measures at a profinite set S written as a limit of finite sets $S = \varprojlim_i S_i$ are given by

$$R_{\square}[S] = \varprojlim_i R[S_i].$$

More generally, for a discrete static ring R one defines $R_{\square} = \varinjlim_{B \subset R} B_{\square}$ where B runs over all the subalgebras of R of finite type over $\mathbb{Z}$.

Definition 4.3. Let (A, A^+) be a derived affinoid pair. We define the analytic ring $(A, A^+)_{\square}$ to be the analytic ring structure over A associated to $\pi_0(A)_{A^+_{\square}/}$ under Proposition 4.1. In other words, an A module M is $(A, A^+)_{\square}$ complete if and only if $\pi_i(M)$ is $A^+_{\square}$-complete for all $i \in \mathbb{Z}$.

Theorem 4.4 (Andreychev)**.** *The functor $(A, A^+) \mapsto (A, A^+)_{\square}$ is a right exact fully faithful embedding from the category of derived affinoid pairs to the category of analytic K-algebras.*

Proof. [And21, Proposition 3.34] shows that the functor $(A, A^+) \mapsto (A, A^+)_{\square}$ is fully faithful for static analytic Huber pairs. To deduce the theorem for general derived Tate algebras, note that A is $(A, A^+)_{\square}$-complete and so it is the unit in $\mathscr{D}((A, A^+)_{\square})$. Moreover, we can recover A^+ as $\pi_0 \operatorname{Map}_{\mathbb{Z}}(\mathbb{Z}[T]_{\square}, (A, A^+)_{\square}$ thanks to [And21, Proposition 3.34]. Thus, the mapping space

$\mathrm{Map}_K((A,A^+)_{\square},(B,B^+)_{\square})$ is the subspace of $\mathrm{Map}_K(A,B)$ that sends A^+ to B^+, which is precisely the mapping space in affinoid pairs.

Finally, to show that the functor is right exact it suffices to see that it sends push-outs of derived affinoid rings to pushouts of analytic rings. The pushout of a diagram $(B,B^+) \leftarrow (A,A^+) \to (C,C^+)$ $\mathrm{AffRing}_K$ is given by the affinoid ring (D,D^+) where $D = B \otimes^L_{(A,\mathbb{Z})_{\square}} C$ is a derived Tate algebra by Lemma 3.9, and D^+ is the open integral closure of the image of B^+ and C^+ in $\pi_0(D)$. On the other hand, the pushout of $(B,B^+)_{\square} \leftarrow (A,A^+)_{\square} \to (C,C^+)_{\square}$ is the analytic ring (E,E^+) obtained as the completion of the analytic ring structure on D with respect to the solid variables in B^+ and C^+. But D being a Banach K-algebra (and so nuclear) is already $B^+_{\square}$ and $C^+_{\square}$-complete (see [Man22, Proposition 2.3.22 (ii)]). Then $E = D$ and by Proposition 4.1 and [And21, Propositions 3.32 and 3.34] the ring E^+ is the open integral closure of the images of B^+ and C^+ in $\pi_0(D)$. This proves the theorem. $\square$

5. Derived rigid spaces

Next we defined derived rigid spaces. Roughly speaking these are constructed from derived affinoid pairs by gluing in the analytic topology. However, in order to make the gluing compatible with the theory of analytic rings it is convenient to discuss the notion of smash spectrum and open and closed immersions in stable symmetric monoidal categories.

Definition 5.1. Let $\mathscr{C}$ be a presentably symmetric monoidal stable ∞-category. The smash spectrum of $\mathscr{C}$ is the poset $\mathscr{S}(\mathscr{C})$ of idempotent algebras in $\mathscr{C}$. In other words, $\mathscr{S}(\mathscr{C})$ is the category of objects A in $\mathscr{C}$ endowed with an arrow $1_{\mathscr{C}} \xrightarrow{\mu} A$ such that the natural map

$$A \xrightarrow{\mu\otimes \mathrm{id}_A} A \otimes A$$

is an equivalence.

From a categorical point of view idempotent algebras behave as closed subspaces in a topological space.

Proposition 5.2 ([CS22, Proposition 5.3])**.** *The poset $\mathscr{S}(\mathscr{C})$ is a locale whose closed subspaces $Z \in \mathscr{S}(\mathscr{C})$ correspond to idempotent algebras A. Moreover, the following is satisfied:*

1. *The whole space is given by $A = 1_{\mathscr{C}}$.*
2. *The empty space is given by $A = 0$.*
3. *The intersection $Z(A) \cap Z(B)$ is given by $Z(A \otimes B)$.*
4. *More generally, given a diagram $\{Z(A_i)\}_{i\in I}$ in the locale, its intersection is given by $Z(\varinjlim_i A_i)$.*
5. *The union $Z(A) \cup Z(B)$ is given by the idempotent algebra $\mathrm{fib}(A \oplus B \to A \otimes B)$ where the unit*
$$1_{\mathscr{C}} \to \mathrm{fib}(A \oplus B \to A \otimes B)$$
is induced by $1_{\mathscr{C}} \xrightarrow{(\mu_A,-\mu_B)} A \oplus B$.

The formalism of the smash spectrum allows the construction of "closed" and "open" localizations of the category $\mathscr{C}$ from a six functors point of view.

Definition 5.3. Let $Z \in \mathscr{S}(\mathscr{C})$ be a closed subspace corresponding to the idempotent algebra $A = A(Z)$, we write U for its formal open complement. We define $\mathscr{C}(Z) := \mathrm{Mod}_A(\mathscr{C})$ and $\mathscr{C}(U) = \mathscr{C}/\mathscr{C}_A$. Moreover, we define the following functors:

1. $\iota_Z^* : \mathscr{C} \to \mathscr{C}(Z)$ to be the natural base change and $j_U^* : \mathscr{C} \to \mathscr{C}(U)$ to be the natural localization functor.
2. $\iota_{*,Z}$ and $j_{*,U}$ to be the (fully faithful) right adjoints of ι_Z^* and j_U^* respectively.
3. $\iota_{!,Z} = \iota_{*,Z}$ and $j_{!,U}$ to be the (fully faithful) left adjoint of j_U^*.
4. $j_U^! = j_U^*$ and $\iota_Z^!$ to be the right adjoint of $\iota_{*,Z}$.

Lemma 5.4. *We have semi-orthogonal decompositions*

$$\mathscr{C}(Z) \xrightarrow{\iota_{Z,*}} \mathscr{C} \xrightarrow{j_U^*} \mathscr{C}(U) \text{ and } \mathscr{C}(U) \xrightarrow{j_{!,U}} \mathscr{C} \xrightarrow{\iota_Z^*} \mathscr{C}(Z).$$

Proof. This follows by unravelling the definitions. For example, we have that

$$\iota_{Z,*}\iota_Z^* M = A \otimes M,$$
$$\iota_{Z,*}\iota_Z^! M = \underline{\mathrm{Hom}}(A, M),$$
$$j_{U,!}j_U^* M = \mathrm{fib}(1_{\mathscr{C}} \to A) \otimes M$$

and

$$j_{U,*}j_U^* M = \underline{\mathrm{Hom}}(\mathrm{fib}(1_{\mathscr{C}} \to A), M).$$

□

Definition 5.5. A morphism $f^* : \mathscr{C} \to \mathscr{D}$ of presentably symmetric monoidal stable ∞-categories is a closed (resp. an open) immersion if it is of the form ι_Z^* (resp. j_U^*) for an idempotent algebra $A(Z) \in \mathscr{S}(\mathscr{C})$.

The following proposition characterizes open and closed immersions in terms of six functors.

Proposition 5.6. *Let $f^* : \mathscr{C} \to \mathscr{D}$ be a morphism of presentably symmetric monoidal stable ∞-categories.*

(1) *f is a closed immersion if and only if f_* is a colimit preserving fully faithful functor satisfying the projection formula: for $N \in \mathscr{C}$ and $M \in \mathscr{D}$ the natural map*
$$f_*M \otimes N \to f_*(M \otimes f^*N)$$
is an equivalence.

(2) *f is an open immersion if and only if it admits a fully faithful left adjoint $f_!$ satisfying the projection formula: for $N \in \mathscr{C}$ and $M \in \mathscr{D}$ the natural map*
$$f_!(M \otimes f^*N) \to f_!M \otimes N$$
is a equivalence.

Moreover, the adjunctions are preserved under base change in presentably symmetric monoidal stable ∞-categories.

The formal descent result in the framework of locales is the following theorem.

Theorem 5.7 ([CS22, Theorem 6.7])**.** *Let* $\mathrm{Sym} := \mathrm{CAlg}(\mathrm{Pr}^{L,\mathrm{ex}})$ *be the ∞ category of presentably symmetric monoidal stable ∞-categories with colimit preserving pullbacks.*

(1) *There is a Grothendieck topology on* $\mathrm{Sym}^{\mathrm{op}}$ *where the covering sieves over a given C are those which contain some set of open immersions whose corresponding open subsets cover $\mathscr{S}(C)$.*
(2) *The identity functor* $(\mathrm{Sym}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Sym}$ *is a sheaf with respect to this Grothendieck topology.*
(3) *The poset of open (resp. closed) immersions also satisfy descent for this Grothendieck topology.*

Remark 5.8. In the set up of Theorem 5.7, one could ask for a more refined Grothendieck topology, namely the one generated by disjoint unions of locally closed immersions $C \to D$ that in addition satisfy !-descent (for the !-functors defined as in Definition 5.3). For example, a cover of C by finitely many closed subspaces would be a cover in this topology (this amounts to a descendable map of algebras in the sense of [Mat16]). Then the theorem will hold for this Grothendieck topology.

More generally, consider Lurie's tensor product in Sym which induces a cartesian structure in $\mathrm{Sym}^{\mathrm{op}}$. It is possible to construct a whole six functor formalism in $\mathrm{Sym}^{\mathrm{op}}$ by taking I to be the category of open immersions and P the morphisms of algebras $f^* : C \to D$ where f_* is a colimit preserving functor satisfying projection formula. Equivalently, the maps P are those f for which

$D = \mathrm{Mod}_A(C)$ for some algebra object $A \in C$. Then, within this six functor formalism one can define the !-topology by declaring $C \to D$ to be a !-cover if the category satisfies !-descent. This condition is strong enough to guarantee usual $*$-descent and Theorem 5.7 will hold for this very strong Grothendieck topology. The topology used in the theory of analytic stacks is the pullback to analytic rings of the !-topology in $\mathrm{Sym}^{\mathrm{op}}$.

Proposition 5.9. *The following morphisms of analytic rings induce open immersions of symmetric monoidal categories.*

(1) $(\mathbb{Z}[T], \mathbb{Z})_\square \to (\mathbb{Z}[T], \mathbb{Z}[T])_\square$.
(2) $(\mathbb{Z}[T], \mathbb{Z})_\square \to (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_\square$.

Proof. The map in (1) is the complement of the idempotent $(\mathbb{Z}[T], \mathbb{Z})_\square$-algebra $\mathbb{Z}((T^{-1}))$ while the map in (2) is the complement of the algebra $\mathbb{Z}[[T]]$. □

Theorem 5.10 (Andreychev)**.** *Let (A, A^+) be a classical affinoid ring over K. Then for $U \subset \mathrm{Spa}(A, A^+)$ an open affinoid subspace the map of analytic rings*

$$(A, A^+)_\square \to (\mathscr{O}(U), \mathscr{O}^+(U))_\square \tag{5.1}$$

is an open immersion and defines a sheaf on ∞-categories. In other words, we have a morphism of locales

$$\mathscr{S}(\mathscr{D}((A, A^+)_\square)) \to \mathrm{Spa}(A, A^+).$$

Proof. This is essentially [And21, Theorem]. The only part to justify is that (5.1) induces an open immersion of symmetric monoidal categories. For this, by covering U with rational localizations, it suffices to show that $(A, A^+)_\square \to (B, B^+)_\square$ is an open immersion, where

$$(B, B^+) = (A, A^+)(\frac{f_1, \dots, f_d}{g})$$

with $f_d = \pi$ a pseudo-uniformizer. But any rational localization in an analytic Huber ring is an iteration of localizations of the form $(A, A^+)(\frac{1}{f})$ and $(A, A^+)(\frac{f}{1})$ for $f \in A$. Finally, by [And21, Proposition 4.11] we have

$$(A(\frac{1}{f}), A(\frac{1}{f})^+)_\square = (A, A^+)_\square \otimes_{(\mathbb{Z}[T], \mathbb{Z})} (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_\square$$

and

$$(A(\frac{f}{1}), A(\frac{f}{1})^+)_\square = (A, A^+)_\square \otimes_{(\mathbb{Z}[T], \mathbb{Z})} (\mathbb{Z}[T], \mathbb{Z}[T])_\square$$

where $T \mapsto f$. The theorem follows from Proposition 5.9. □

Corollary 5.11. *Let (A, A^+) be a derived affinoid pair over K. Then the map of locales*

$$\mathscr{S}(\mathscr{D}((\pi_0(A), A^+)_\square)) \to \mathrm{Spa}(A, A^+)$$

of Theorem 5.10 lifts uniquely to a map of locales $\mathscr{S}(\mathscr{D}((A, A^+)_\square)) \to \mathrm{Spa}(A, A^+)$.

Proof. By Theorems 5.10 and 3.6 we have a sheaf $\pi_* \mathscr{O}_X$ on the classical adic space $X = \mathrm{Spa}(\pi_0(A), A^+)$ with $\pi_i(\mathscr{O}_X)$ a coherent $\pi_0(\mathscr{O}_X)$-module. Thus, by Proposition 4.1, given an open affinoid $U \subset X$ we have a morphism of analytic rings

$$(A, A^+)_\square \to (\mathscr{O}(U), \mathscr{O}^+(U))_\square \tag{5.2}$$

that is equivalent to a morphism of derived affinoid rings by Theorem 4.4. Then, to prove the corollary it suffices to show that (5.2) is an open immersion. By taking a covering of U we can assume without loss of generality that it is a rational localization of X, and by induction we can even assume that it is of the form $X(\frac{1}{f})$ or $X(\frac{f}{1})$. But then we have that

$$(\mathscr{O}(U), \mathscr{O}^+(U))_\square = (A, A^+)_\square \otimes_{(\mathbb{Z}[T], \mathbb{Z})} (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_\square$$

or

$$(\mathscr{O}(U), \mathscr{O}^+(U))_{\square} = (A, A^+)_{\square} \otimes_{(\mathbb{Z}[T],\mathbb{Z})} (\mathbb{Z}[T], \mathbb{Z}[T])_{\square}$$

for any lift $\mathbb{Z}[T] \to A$ of $\mathbb{Z}[T] \to \pi_0(A)$ mapping f to A. One deduces that (5.2) is an open immersion by Proposition 5.9. $\square$

After the analytic localization of Corollary 5.11 we can define derived adic spaces of finite type over K.

Definition 5.12. Let $\mathrm{AffRing}_K$ be the category of derived affinoid K-algebras and let Aff_K be its opposite category. We let $\mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}_K)$ be the category of sheaves on anima of Aff_K for the analytic topology. A derived adic space of finite type over K is a sheaf $X \in \mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}_K)$ that is locally in the analytic topology representable by an object in Aff_K. We let $\mathrm{Spa}(A, A^+)$ be the sheaf correpresented by (A, A^+) and call it its adic spectrum.

6. Six functors for solid quasi-coherent sheaves

We finally put all the pieces together to construct the six functor formalism for solid quasi-coherent sheaves on rigid spaces.

Definition 6.1. A map $X \to Y$ of derived adic spaces of finite type over K is locally compactifiable if, locally in the analytic topology of X and Y, the map $X \to X^{/Y}$ from X into its relative compactification is an open immersion.

Theorem 6.2. *Let $\mathscr{C}$ be the ∞-category of derived adic spaces of finite type over K. Then the functor $X \mapsto \mathscr{D}(X)$ of solid quasi-coherent sheaves on X can be naturally extended to a six functor formalism*

$$\mathscr{D} : \mathrm{Corr}(\mathscr{C}, E) \to \mathrm{Pr}^L$$

with E the family of locally compactifiable maps. In particular, E includes all the arrows of classical rigid spaces over K.

In order to construct the six functors of Theorem 6.2 we need to start with some basic input:

Definition 6.3. Let I be the class of open immersions $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ in $\mathrm{AffRing}_K$. Let P be the class of arrows $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ with B^+ being the open integral closure of A^+ in $\pi_0(B)$.

Proposition 6.4. *The pair (I, P) is a suitable decomposition of (Aff_K, E). Moreover, the tuple $(\mathrm{Aff}_K, E, I, P)$ satisfies the conditions of Proposition 2.2 and there is a six functor formalism*

$$\mathscr{D} : \mathrm{Corr}(\mathrm{Aff}_K, E) \to \mathrm{Pr}^L$$

extending $\mathrm{Spa}(A, A^+) \mapsto \mathscr{D}((A, A^+)_{\square})$ such that the arrows in I are open immersions and the arrows in P are proper.

Sketch of the proof. We first prove that (I, P) is a suitable decomposition of (Aff_K, E), see Definition 2.1. We need to show that any map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of derived affinoid K-algebras that is locally compactifiable factors as a composite of I and P. At the level of rings this correspond to

$$(A, A^+)_{\square} \to (B, A^+)_{\square} \to (B, B^+)_{\square}$$

where the first arrow is in P since has the induced structure, and the second arrow is an open immersion by assumption of being (locally) compactifiable (for example a map weakly of $+$-finite type). This shows (1).

Open immersions are (-1)-truncated, this shows (2). It is clear that the arrows I and P contain all the isomorphisms. P is stable under compositions and pullbacks being just the induced analytic ring structure. The class I is stable under pullbacks and composition by the criterion of Proposition 5.6. This shows (3).

Finally, one easily verifies (4) using that the arrows in P have the induced analytic structure and Proposition 5.6 for the arrows in I.

To finish the proof of the proposition, we need to show that the conditions (a)-(c) in Proposition 2.2 hold. Condition (a) is immediate from Proposition 5.6 and the definition of I as open immersions. Condition (b) can be proven easily using that the arrows in P have the induced analytic structure (and so the statement becomes the classical affine base change and projection formula in algebra). Condition (c) is also easy to verify by a computation that we left to the reader (hint: recall that open immersions are complements of idempotent algebras). See [RC24, Lemma 3.2.5] for more details. ☐

The key proposition that allows the extension of six functors from affinoid rings to arbitrary rigid spaces is the following !-descent result:

Proposition 6.5. *The functor* $\mathrm{Spa}(A, A^+) \mapsto \mathscr{D}((A, A^+)_\square)$ *satisfies* !*-descent for the analytic topology.*

Proof. This is a special case of smooth descent [Sch23, Proposition 6.18]. See also Theorem 5.7 and Remark 5.8. ☐

Remark 6.6. Instead of giving a formal proof of Theorem 6.2 that would involve applying several times [Man22, Propositions A.5.12, A.5.14 and A.5.16], let us explain how the six functors are computed. Let $f : Y \to X$ be a locally compactifiable map of derived adic spaces of finite type over K. Theorem 6.2 provides a natural and functorial way to describe the functor $f_!$ as follows:

- First, by replacing X with open affinoid subspaces, we can assume that $X = \mathrm{Spa}(A, A^+)$.
- If Y is affinoid then the functor $f_!$ is constructed as in Proposition 6.4 by composing Y with an open immersion and a partially proper map (a suitable decomposition).
- If $Y = \bigsqcup_i \mathrm{Spa}(B, B_i)$ is a disjoint union of affinoid spaces we then have $f_! = \bigoplus f_{i,!}$.
- If f is separated, let $\mathscr{U} = \{\mathrm{Spa}(B_i, B_i^+)\}_i$ be an open cover of Y by open affinoid spaces and set $Z = \bigsqcup_i \mathrm{Spa}(B_i, B_i^+)$. Then $g : Z \to Y$ is a locally compactifiable morphism of derived rigid spaces represented in Aff_K, and so it is !-able (i.e. it admits !-functors). Then, we have
$$f_! = \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (f \circ g_n)_! \circ g_n^!$$
where $g_n : Z^{\times_Y n+1} \to Y$ is the Čech nerve, and the map $f \circ g_n$ is a disjoint union of affinoid maps (so !-able).
- For general f, take $\mathscr{U}$ an affinoid cover of Y and let Z be as before. Then the map $g : Z \to Y$ is separated and locally compactifiable so it is !-able. Since g has !-descent the map f is !-able and we can compute
$$f_! = \varinjlim_{[n]\in\Delta^{\mathrm{op}}} (f \circ g_n)_! \circ g_n^!$$
where g_n being the Čech nerve as before, and the map $f \circ g_n$ is separated and locally compactifiable (so !-able).

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