Analytic adic spaces and descent.

Reminder: We are trying to define K-theory of, e.g. a rigid space X, as the Efimov K-theory of a dualiseable category $\mathrm{Nuc}(X)$ of nuclear sheaves on X.

Saw last time: For (A, A^+) discrete Huber pair, $(A, A^+) \mapsto (A, A^+)_{\blacksquare}$ analytic ring. $\mathrm{Nuc}(D(A, A^+)_{\blacksquare}) = D(A)$.

TODAY:

Goal 1: A complete f-adic ring, A^+ an open subring of integral elements.

For any complete Huber pair (A, A^+) define a 0-truncated* analytic ring $(A, A^+)_{\blacksquare}$ with underlying condensed ring $\underline{A}$

*: i.e. $(A, A^+)_{\blacksquare}[S]$ is in $\mathrm{Mod}_{\underline{A}} \subseteq D_{\geq 0}(\underline{A})$.

Idea: M in $\mathrm{Mod}_{\underline{A}}$ is complete if null-sequences are summable (non-arch / $\mathbb{Z}_{\blacksquare}$) and if for any f in A^+ ($|f| \leq 1$ at all points of $\mathrm{Spa}(A, A^+)$), any $m_i \to 0$ in M, $k_i \in \mathbb{Z}$, $f^{k_i} m_i$ is still a nullsequence.

Formally (Clausen-Scholze): Colimit of induced analytic ring structures from $R_{\blacksquare}$. $R \subseteq A^+$ f.g. $\mathbb{Z}$-algebra.

Andreychev: $(A, A^+)_{\blacksquare}[S]$ can be made explicit (in particular 0-truncated).

Goal 2:

For any complete, analytic, sheafy (A, A^+), establish descent properties for modules.

- Analytic — top. nilp. units generate unit ideal in A.

  E.g. $(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ but not $(\mathbb{Z}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ or discrete pairs.

  (i.e. really living in world of non-arch analytic geometry).

- Sheafy ~ $X = \mathrm{Spa}(A, A^+)$ = cont. valuations $|\cdot|$ with $|a| \le 1$ for a in A^+.

  Basis of opens $U = X\left(\frac{f_1 \dots f_n}{g}\right)$

  $|f_i| \le |g|$, $(f_1, \dots, f_n) = A$.

  $$\mathcal{O} : U \mapsto A_U = A\langle t_1, \dots, t_n\rangle \big/ \overline{(gt_1 - f_1, gt_2 - f_2, \dots)}$$

  Sheafy if $\mathcal{O}$ is a sheaf (in which case closure is redundant)

Fine in rigid analytic geometry, affinoid perfectoid, sousperfectoid..

(nonsheafy examples ~ Buzzard Verberkmoes).

Remark – Topology of modules, possible failure of flatness for $A \to A_U$, completed tensor products, all make classical approach to a "quasi coherent" sheaf theory a nonstarter on $\mathrm{Spa}(A, A^+)$.

$$M \to \tilde{M}???$$

Coherent sheaves OK for rigid analytic spaces (Kiehl), generalized to pseudocoherent for general analytic adic spaces by Kedlaya–Liu

Want to show:

① $U \mapsto D((A_U, A_U^+)_\blacksquare)$ for U rational open defines a sheaf of ∞-categories on $|\mathrm{Spa}(A, A^+)|$,

Very similar to proof for discrete Huber pairs.

② That nuclear, compact (together $\Rightarrow$ dualizable) can be checked locally. Allows to globalize.

③ Nuclear + pseudocoherent $\Rightarrow$ discrete $\left(\begin{matrix}\text{comes from}\\ D(A)\end{matrix}\right)$

since compactness preserved by condensification

$$\Longrightarrow D((A, A^+)_\blacksquare)^{\text{dualizable}} = \mathrm{Perf}(A)$$

Thus $U \to \mathrm{Perf}(A_U)$ is also a sheaf.

$(A, A^+)_\blacksquare$ $\leftarrow$ (A, A^+) any complete Huber pair.

Lemma: For any f.g. $\mathbb{Z}$-algebra $R \subseteq A^+$, $\underline{A}$ is an $R_\blacksquare$-module.

Proof: $R \subseteq A_0$ for A_0 ring of def. Take $I \subset A_0$ ideal of def. $A = \lim \underbrace{A/I^n}_{R\text{-module.}}$ so $\underline{A} = \lim \underline{A/I^n}$

$\uparrow$ $R_\blacksquare$-module since discrete, Limits preserve $R_\blacksquare$-modules.

So, for any such R, have induced analytic ring $(A, R)_\blacksquare$ with underlying condensed ring $\underline{A}$ and measure spaces

$$(A, R)_\blacksquare[S] := \underline{A}[S] \overset{L}{\underset{\underline{R}}{\otimes}} R_\blacksquare$$
$$= \left(\underline{A} \overset{L}{\underset{\underline{R}}{\otimes}} \underline{R}[S]\right) \overset{L}{\underset{\underline{R}}{\otimes}} R_\blacksquare$$
$$= \underline{A} \overset{L}{\underset{R_\blacksquare}{\otimes}} R_\blacksquare[S].$$

$$(A, A^+)_\blacksquare := \underset{\substack{R \subseteq A^+ \\ \text{f.g.}}}{\operatorname{colim}} (A, R)_\blacksquare$$

(colimit happens just on measure spaces)

Note — makes sense for <u>any</u> subring of $\underline{A}^{\circ}$ powerbdd elements

Theorem (Andreychev)

Let A be a complete f-adic ring. Let $A^+ \subseteq A^{\circ}$ be any subring.

Let S be a profinite set with $\mathbb{Z}[S]_{\blacksquare} = \prod_J \mathbb{Z}$.

$$(A, A^+)_{\square}(S) = \operatorname*{colim}_{\substack{R \subseteq A^+ \text{ f.g.} \\ M \subseteq A \text{ q.f.g.}/R}} \prod_J M$$

(q.f.g. = quasi-finitely generated — for (A_0, I), $B \subseteq A_0$, M/I^n f.g. B-mod and $M = \varprojlim M/I^n$)

In particular, $(A, A^+)_{\square}$ is 0-truncated.

Corollary: The following are equivalent for $M \in D(\underline{A})$

- $M \in D((A, A^+)_{\square})$
- $M \in D(A^+_{\square})$
- $M \in D(R_{\square})$ for all $R \subseteq A^+$ f.g.
- $M \in D(\mathbb{Z}[T]_{\square})$ for all $T \mapsto a$, $a \in A^+$
- $M \in D(\mathbb{Z}_{\square})$ and $M \in D(\mathbb{Z}[T]_{\square})$ for $T \to a$, a running over rep's for $(A^+/A^{\circ\circ} \cap A^+)^{(\times)}$

• $M \in D((A, \bar{A}^+)_\square)$ for $\bar{A}^+$ the smallest integrally closed open subring of A° containing A^+.

Idea: Suppose $R \subseteq A^\circ$ f.g. subalg. if $a \in A^{\circ\circ}$ or a integral over R, then $R[a]$ is a f.g. R-module so colimit defining measures are equal.

__Example:__ Over $(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$,

$$\mathrm{Spa}(A, A^+) \underset{\substack{\text{canonical} \\ \text{compactification}}}{\rightsquigarrow} \mathrm{Spa}(A, A^{min}), \quad A^{min} = \mathcal{O}_{\mathbb{C}_p} + A^{\circ\circ}.$$

$M \in D(A)$ is in $D((A, A^{min})_\blacksquare) \iff M$ is solid over $\mathbb{Z}$!

(Over (K, K^+) always get $\iff$ solid over K^+).

Geometric example

$\mathbb{D} = \mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$ disk $|T| \le 1$.

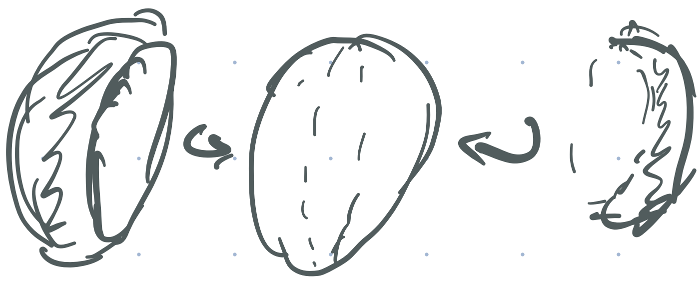
$\mathbb{D} \xhookrightarrow{i^\circ} \mathbb{P}^1$ ← closure in $\mathbb{P}^1$.

$\overline{\mathbb{D}} = \mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}\mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$.

canonical compactification of $\mathbb{D}$

$\partial\overline{\mathbb{D}} = \overline{\mathbb{D}} \setminus \mathbb{D}$ = unique rk 2 pt. where $|T| > 1$

$\overline{\mathbb{D}} \xhookrightarrow{i} \mathbb{P}^1 \xhookleftarrow{j} \mathbb{P}^1 \setminus \overline{\mathbb{D}}$ (open disk $\bigcup_n |T| > |p^{-1/n}|$)



$i^* j_* \mathcal{O}$ supported at $\partial\overline{\mathbb{D}}$ $\quad (i^{\circ *} j_* \mathcal{O} = \{0\})$.

So $M = H^0(\overline{\mathbb{D}}, i^* j_* \mathcal{O})$ should be a $\mathbb{C}_p\langle T\rangle$ module which is a solid $\mathbb{Z}$-module but whose solidification as a $\mathbb{Z}[T]$-module is 0.

Can see this: e.g. $\frac{1}{T} + \frac{1}{T^2} + \frac{1}{T^3} + \dots$ is in M, so $\left(\frac{1}{T}, \frac{1}{T^2}, \ldots\right)$ is a nullsequence in M.

But if solidify as a $\mathbb{Z}[T]$-module, then

$$\left(T\left(\frac{1}{T}\right), T^2\left(\frac{1}{T^2}\right), T^3\left(\frac{1}{T^3}\right), \ldots\right)$$

$$= (1, 1, 1, \ldots)$$ is a nullsequence

So $1 = 0$.

Prop $(X \mapsto X)$

$(A, A^+) \mapsto (A, A^+)_\blacksquare$ is fully faithful.

(complete Huber pairs $\to$ analytic rings)

In the talk we only sort-of covered the next two pages, and nothing after it.

$D((A,A^+)_\blacksquare)$ is a sheaf of ∞-categories (§4 Andreychev)
for (A,A^+) analytic + sheafy.

Theorem 4.1. *Let X be an analytic adic space and let U denote an arbitrary affinoid subspace of X. Then the association $U \mapsto \mathcal{D}((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare)$ defines a sheaf of ∞-categories on X.*

Idea: Same strategy as discrete case,
but use coverings as in Kedlaya-Liu pseudocoh arguments

Proposition 4.13 ([CM], Proposition 10.5). *Let X be a spectral topological space equipped with a basis B of quasi-compact open subsets stable under intersections. Let $\mathcal{C}$ be a stable ∞-category and $U \mapsto \mathcal{C}_U \subset C$ a covariant functor from B to full subcategories of $\mathcal{C}$ such that every inclusion map $\mathcal{C}_U \hookrightarrow \mathcal{C}$ admits a left adjoint L_U. Furthermore, assume that*

(i) for every pair $U, V \in B$, we have $\mathcal{C}_{U \cap V} = \mathcal{C}_U \cap \mathcal{C}_V$ and $L_{U \cap V} = L_U \circ L_V = L_V \circ L_U$;

(ii) let U be an element of B and suppose that $\{U_i\}_{i=1}^n, U_i \in B$ form a covering of U; for every $M \in \mathcal{C}_X$ such that $L_{U_i}(M) = 0$ for every $i = 1, \ldots, n$, we have $M = 0$.

Then the covariant functor $U \mapsto \mathcal{C}_U$, with the left adjoints L_U as the restriction functors, defines a sheaf of ∞-categories.

Definition 4.2 ([Kdl], Definition 1.6.6). Let X be an affinoid adic space and f an element of $\mathcal{O}_X(X)$. The open covering $\{X(\frac{f}{1}), X(\frac{1}{f})\}$ of X is called *the simple Laurent covering defined by* f and the open covering $\{X(\frac{1}{f}), X(\frac{1}{1-f})\}$ is called *the simple balanced covering defined by* f. A covering is called *nice*, if it is a composition of coverings, each of which is either a simple Laurent covering or a simple balanced covering. A rational open subspace is called *nice* if it is an element of some nice covering.

Proposition 4.3 ([Kdl], Lemma 1.6.13 and Lemma 1.9.14). *Let X be an analytic affinoid adic space. Then the family of its nice rational subsets forms a basis of quasi-compact open subsets, stable under intersections. Furthermore, every open covering of X can be refined by a nice covering.*

Proposition 4.11. *Let (A, A^+) be a complete Huber pair and f, g elements of A that generate an open ideal. Assume that the map*

$$A\langle U\rangle \xrightarrow{gU-f} A\langle U\rangle$$

is a closed embedding (this is true, for example, if A is discrete or analytic and sheafy). Endow the condensed ring $A[g^{-1}] \overset{\text{def}}{=} A \overset{L}{\underset{\mathbb{Z}[T]}{\otimes}} \mathbb{Z}[T, T^{-1}]$, where the $\mathbb{Z}[T]$-module structure on A is given by $T \mapsto g$, with the $\mathbb{Z}[U]$-module structure given by $U \mapsto \frac{f}{g}$. Then we have the following isomorphism of analytic rings:

$$(A(\tfrac{f}{g}), A^+(\tfrac{f}{g}))_\blacksquare \cong ((A, A^+)_\blacksquare \overset{L}{\underset{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}{\otimes}} (\mathbb{Z}[T, T^{-1}], \mathbb{Z})_\blacksquare) \overset{L}{\underset{(\mathbb{Z}[U],\mathbb{Z})_\blacksquare}{\otimes}} \mathbb{Z}[U]_\blacksquare.$$

Furthermore, the map

$$(A, A^+)_\blacksquare \to (A(\tfrac{f}{g}), A^+(\tfrac{f}{g}))_\blacksquare$$

is a steady localization.

Remark. Let A be a general (i.e., not necessarily sheafy) analytic complete Huber ring and let f, g be elements in A that generate an open ideal (equivalently, the unit ideal). Then the proof shows that

$$(A \overset{L}{\underset{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}{\otimes}} (\mathbb{Z}[T,T^{-1}],\mathbb{Z})_\blacksquare) \overset{L}{\underset{(\mathbb{Z}[U],\mathbb{Z})_\blacksquare}{\otimes}} \mathbb{Z}[U]_\blacksquare$$

is quasi-isomorphic to the complex

$$\underline{A\langle T\rangle}/^{\mathbb{L}}(gT - f) := [\underline{A\langle T\rangle} \xrightarrow{gT-f} \underline{A\langle T\rangle}].$$

Steady: plays nice with base change.

Lemma 4.7. *Let A be a complete Huber ring. Then we have the following isomorphism of analytic rings:*

$$(A\langle T\rangle, A^+\langle T\rangle)_\blacksquare \cong (A, A^+)_\blacksquare \overset{L}{\underset{\mathbb{Z}_\blacksquare}{\otimes}} \mathbb{Z}[T]_\blacksquare.$$

In particular, for any profinite set Q, we have the following isomorphism in $\mathcal{D}(\underline{A}[T])$:

$$(A\langle T\rangle, A^+\langle T\rangle)_\blacksquare[Q] \cong (A, A^+)_\blacksquare[Q] \overset{L}{\underset{\mathbb{Z}_\blacksquare}{\otimes}} \mathbb{Z}[T]_\blacksquare.$$

Furthermore, the map

$$(A, A^+)_\blacksquare \to (A\langle T\rangle, A^+\langle T\rangle)_\blacksquare$$

is steady.

Proof. As the map $\mathbb{Z}_\blacksquare \to \mathbb{Z}[T]_\blacksquare$ is steady, both the second and the last parts of the statement follow from the first. Applying Proposition 3.14, we obtain an isomorphism

$$\underline{A\langle T\rangle} = \underline{A} \overset{L}{\underset{\mathbb{Z}_\blacksquare}{\otimes}} \mathbb{Z}[T]_\blacksquare.$$

To prove the lemma, it suffices to show that the categories $\mathcal{D}((A, A^+)_\blacksquare \overset{L}{\underset{\mathbb{Z}_\blacksquare}{\otimes}} \mathbb{Z}[T]_\blacksquare)$ and $\mathcal{D}((A\langle T\rangle, A^+\langle T\rangle)_\blacksquare)$, both of which are full subcategories of $\mathcal{D}(\underline{A\langle T\rangle})$ by the above isomorphism, are equal. Indeed, the desired claim will follow as the completion functors in both cases are given by the left adjoints of the inclusion functors. By Propositions 2.18 and 3.29, an object $M \in \mathcal{D}(\underline{A}\langle T\rangle)$ lies in $\mathcal{D}((A, A^+)_\blacksquare \overset{L}{\underset{\mathbb{Z}_\blacksquare}{\otimes}} \mathbb{Z}[T]_\blacksquare)$ if and only if it is $(A^+)_\blacksquare$- and $\mathbb{Z}[T]$-complete. By Proposition 3.32 this is equivalent to being $A^+\langle T\rangle_\blacksquare$-complete; thus, we obtain the desired statement applying Proposition 3.29 one more time. $\square$

Locality of subcategories (§5.3 of Andreychev).

Some properties of objects

1. $D(A) \to D(\underline{A})$ factors through $D((A, A^+)_\blacksquare)$ (since $\underline{A}$ is complete, closed under colimits).
   Fully faithful, image called <u>discrete</u>.
2. Nuclearity, compactness, dualizeable as before (Nuclear + compact $\iff$ dualizeable)
3. M pseudocoherent if quasi-iso to a bounded above complex with terms $(A, A^+)_\blacksquare[S]$ for profinite S.

Theorem:

1) Nuclearity, compactness, and dualizeability can all be checked locally on $\mathrm{Spa}(A, A^+)$
2) Pseudocoherence can be checked locally on $\mathrm{Spa}(A, A^+)$
3) a) Discrete objects are nuclear.
   b) If $M \in D((A, A^+)_\blacksquare)$ is nuclear & pseudocoherent, it is discrete.

Corollary $U \to \mathrm{Perf}(A_U)$ is a sheaf on $\mathrm{Spa}(A, A^+)$

Pf $\mathrm{Perf}(A) =$ Dualizable in $D((A, A^+)_\blacksquare)$

because dualizable $\Leftrightarrow$ nuclear + compact $\Rightarrow$ nuclear + pseudocoh

$\Rightarrow$ discrete. Compact in $D(A) \Leftrightarrow$ compact in $D((A, A^+)_\blacksquare)$ since functor fully faithful, preserves colimits.

For theorem, talk about 3b)

Theorem 5.50. *Let M be a nuclear pseudocoherent object of $\mathcal{D}^-(\mathcal{A}, \mathcal{M})$. Then M is discrete. In particular, any dualizable object is discrete.*

Proof. Without loss of generality, we assume that M can be written as

$$\ldots \to \mathcal{M}[S_2] \to \mathcal{M}[S_1] \to \mathcal{M}[S_0] \to 0,$$

where $\{S_i\}_{i \geq 0}$ are profinite sets. Since M is nuclear, the canonical map $f' : \mathcal{M}[S_0] \to M$ can be factored as the composition

$$\mathcal{M}[S_0] \xrightarrow{1 \otimes g'} \mathcal{M}[S_0] \underset{(\mathcal{A},\mathcal{M})}{\overset{L}{\otimes}} \mathcal{M}[S_0]^\vee \underset{(\mathcal{A},\mathcal{M})}{\overset{L}{\otimes}} M \xrightarrow{ev_{\mathcal{M}[S_0]} \otimes 1} M$$

for some $g' : \mathcal{A} \to \mathcal{M}[S_0]^\vee \underset{(\mathcal{A},\mathcal{M})}{\overset{L}{\otimes}} M$. As $\mathcal{A}$ is projective, g' can in turn be factored as the composition

$$\mathcal{A} \xrightarrow{g} \mathcal{M}[S_0]^\vee \underset{(\mathcal{A},\mathcal{M})}{\overset{L}{\otimes}} \mathcal{M}[S_0] \xrightarrow{1 \otimes f'} \mathcal{M}[S_0]^\vee \underset{(\mathcal{A},\mathcal{M})}{\overset{L}{\otimes}} M.$$

$= 1 \otimes g$ followed by evaluation,

Thus, we have constructed a trace class map $f : \mathcal{M}[S_0] \to \mathcal{M}[S_0]$ such that the composition $f' \circ f$ is equal to f'. Therefore, the composition $f' \circ (1 - f)$ is trivial, and, hence, we obtain the following morphism of triangles:

$$\begin{tikzcd}
\mathcal{M}[S_0] \arrow[r, "1-f"] \arrow[d] & \mathcal{M}[S_0] \arrow[r] \arrow[d, "f'"] & \mathrm{cone}(1-f) \arrow[r] \arrow[d, "\phi"] & \mathcal{M}[S_0][1] \arrow[d] \\
0 \arrow[r] & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r] & 0
\end{tikzcd}$$

It can be easily checked that the induced map $H^0(\operatorname{cone}(1-f)) \to H^0(M)$ is surjective. Applying Lemma 5.51, we conclude that $\operatorname{cone}(1-f)$ is a perfect complex over A. Consequently, $\operatorname{cone}\phi$ is pseudocoherent and nuclear (by Lemma 5.40). It is straightforward to check that the cohomology group $H^0(\operatorname{cone}(\phi))$ is trivial. Thus, we have the triangle

$$\operatorname{cone}\phi[-1] \longrightarrow \operatorname{cone}(1-f) \xrightarrow{\ \phi\ } M \longrightarrow \operatorname{cone}\phi,$$

whose left hand side is pseudocoherent and nuclear. Arguing by induction, one shows that M can be written as a colimit of discrete objects, and, hence, is discrete itself. $\square$

Lemma 5.51. *Let S be a profinite set and $f : \mathcal{M}[S] \to \mathcal{M}[S]$ a trace-class morphism. Then* $\operatorname{cone}(f-1)$ *is quasi-isomorphic to a complex of the form* $\underline{A}^n[-1] \to \underline{A}^n[0]$.

(Fredholm theory)