Trace class maps & nuclearity

§ Abstract context

Fix $\mathcal{C}$ an arbitrary closed symm. monoidal ∞-category

<u>Notation</u>: $X, Y \in \mathcal{C}$, internal hom Y^X.

$X^\vee := 1^X$ (where 1 is the unit)

& $X(*) := \mathrm{Hom}(1, X) \in \mathcal{S}$. (mapping space)

<u>Remark</u>: There is a natural map

$X^\vee \otimes Y \to Y^X$

given via adjunction from

$$ev \otimes id_Y : X^\vee \otimes X \otimes Y \to Y$$

Evaluating at $\mathrm{Hom}(1, -)$, we obtain a natural map

$$X^\vee \otimes Y(*) \to \mathrm{Hom}(X, Y) \quad \text{of anima.}$$

<u>Def</u>: Let $f : X \to Y$ be a map in $\mathcal{C}$. We say f is <u>trace-class</u> if it lies in the image of the map $X^\vee \otimes Y(*) \to \mathrm{Hom}(X, Y)$.

<u>Lemma</u>: Let $\mathcal{C}$ be as above

1) If $f: X \to Y$ trace class and $g: Y \to Y'$, $h: X' \to X$ arbitrary, then $g \circ f \circ h$ is trace class.

2) If $f: X \to Y$ & $f': X' \to Y'$ are trace class, then $f \otimes f': X \otimes X' \to Y \otimes Y'$ is trace class.

3) If $f: X \to Y$ is trace-class & $c \in \mathcal{C}$ is arbitrary, then there is a diagonal map

$X^c \to c^\vee \otimes Y$

$$\begin{tikzcd} c^\vee \otimes X \arrow[r] \arrow[d] & c^\vee \otimes Y \arrow[d] \\ X^c \arrow[r] \arrow[ur, dashed, "\exists"] & Y^c \end{tikzcd}$$

making both triangles commute.

just in case...

Let's now assume that

1) $\mathcal{C}$ is stable & compactly generated
2) the unit $1 \in \mathcal{C}$ is a compact object.

<u>Def</u>. Let $\mathcal{C}$ be as above. The

1) we say $X \in \mathcal{C}$ is nuclear if $\forall$ compact object $c \in \mathcal{C}$, the natural map
$$(c^\vee \otimes X)(*) \to \mathrm{Hom}(c, X)$$
is an equivalence.

2) $X \in \mathcal{C}$ is <u>basic nuclear</u> if it is isomorphic to the colimit $X_0 \to X_1 \to \cdots$ of a sequence of trace class maps, and X_i are compact.

<u>Rmk</u>: The objects x_i in 2) above don't need to be assumed compact, but then one can prove that $x_0 \to x_1 \to x_2 \cdots$ can be refined to a diagram $c_0 \to c_1 \to c_2 \ldots$ all compact.

<u>Remark</u>: We can replace mapping anima with mapping spectra; condition 1) is insensitive to such a choice.

<u>Lemma</u>: Basic nuclear objects are themselves nuclear.

<u>Pf</u>: Fix $X = \underrightarrow{\mathrm{colim}}\, x_n$, basic nuclear,

Choose maps $1 \to x_i^\vee \otimes x_{i+1}$ corresponding to trace class maps $x_i \to x_{i+1}$

Let c be any compact object; we want to show that

$$(c^\vee \otimes X)(*) \to \mathrm{Hom}_{\mathcal{C}}(c, X) \text{ is an equivalence.}$$

Now, both sides commute with colimits in $\underset{\smile}{\mathcal{C}}$ and there are natural backwards maps:

$\underline{\mathrm{Hom}}(c, x_i) \otimes \underline{\mathrm{Hom}}(x_i, 1) \to \underline{\mathrm{Hom}}(c, 1)$

$$\mathrm{Hom}(c, x_i) \to \left(\underline{\mathrm{Hom}}(c, x_i) \otimes x_i^\vee \otimes x_{i+1}\right)(*) \to \left(c^\vee \otimes x_{i+1}\right)(*)$$

Thus, we obtain equivalent colimits, and so, X is nuclear.

Now let's study the category of nuclear objects.

<u>Theorem</u>: Let $\mathcal{C}$ satisfy standing assumptions. Then

1) the full subcategory Nuc of nuclear objects in $\mathcal{C}$ is stable, closed under colimits, & closed under tensor products,

2) The stable ∞-category Nuc is $\aleph_1$-compactly generated, and the $\aleph_1$-compact objects are exactly the basic nuclear objects.

Proof: Stability & cocompleteness of Nuc follows since the (spectrum-valued) functors

$$(c^\vee \otimes -)(*) \quad \text{and} \quad \mathrm{Hom}(c, -)$$

commute with all colimits (again, c is compact here...)

Closure under tensor products will follow from (2) since basic nuclears are clearly closed under tensor products.

Now, lets move on to proving 2). We remark that basic nuclears are $\aleph_1$-compact in $\mathcal{C}$, hence in $Nuc(\mathcal{C})$. (by definition)

Claim: Basic nuclear objects form a stable subcategory of Nuc, closed under countable colimits. Enough to verify closure under cones & countable direct sums.

For stability under cones, let

$f: X \to X'$ be map (of basic nuclear objects),

i.e. $\mathrm{colim}\, X_i \to \mathrm{colim}\, X_i'$.

Up to reindexing the X_i', can assume individual maps $X_i \to X_i'$.

Let $Q_i := \mathrm{Cone}(X_i \to X_i')$.

WTS $Q_i \to Q_{i+2}$ trace class $\forall i$. (not true generally that $Q_i \to Q_{i+1}$ is trace class.)

We need to define $1 \to Q_i^\vee \otimes Q_{i+2}$ witnessing this map.

(can view this as fiber of map

$(X_i')^\vee \otimes Q_{i+2} \to (X_i)^\vee \otimes Q_{i+2}$).

It is therefore enough to define (5)

$1 \to (X_i')^\vee \otimes X''_{i+2}$ whose images in $X_i^\vee \otimes X'_{i+2}$ agree.

& $1 \to X_i^\vee \otimes X_{i+2}$

To define this, pick sections

$\alpha: 1 \to X_i^\vee \otimes X_{i+1}$

$\beta: 1 \to (X'_{i+1})^\vee \otimes X'_{i+2}$

witnessing the $X_i \to X_{i+1}$, $X'_{i+1} \to X'_{i+2}$ are trace class.

Now take the induced maps

$\alpha: 1 \to X_i^\vee \otimes X_{i+2}$

$\beta: 1 \to (X_i')^\vee \otimes X'_{i+2}$

this shows thus that $Q_i \to Q_{i+2}$ is trace class, so basic nuclears are closed under cones. (finite colimits)

It remains to check for countable direct sums. Choose a representative of each term as a sequential colimit along trace class maps. Then rewrite countable direct sum as a sequential colimit of finite direct sums. Pass to diagonal in sequential colimit of sequential colimit.

Thus, we see that every basic nuclear is $\aleph_1$-compact, and that ~~they~~ are closed under countable colimits.

Just say this part...

It remains to show, $\forall x \in Nuc(\mathcal{C})$,

if $Hom(y, x) = 0$ $\forall$ basic nuclear y, then $x = 0$.

If not, then there is some compact c_0, with non-zero map $c_0 \to x$.

By nuclearity, $(c_0^\vee \otimes x)(*) = Hom(c_0, x)$.

Now, since both side of the above commute with filtered colimits &
since $x \simeq \varinjlim y_i$, with y_i compact, there is a lift to some class in $\pi_0(c_0^\vee \otimes y_i)(*)$ for some i, giving a <u>trace-class map</u>.
Let $c_1 = y_i$.

Iterating this, we find a nonzero map from some basic nuclear $colim(c_0 \to c_1 \to \cdots)$ to c, which is a contradiction

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§ <u>Nuclearity & dualizability</u>

<u>Proposition</u>: With the above assumptions, an object $x \in \mathcal{C}$ is dualizable iff it is compact & nuclear.

<u>Proof</u>: It is a general fact that if the tensor unit is compact, then all dualizable objects are compact as well.
Indeed

$$\begin{aligned} Hom(x, \varinjlim N_i) &\simeq Hom(1, x^\vee \otimes \varinjlim N_i) \\ &\simeq Hom(1, \varinjlim (x^\vee \otimes N_i)) \\ &\simeq \varinjlim Hom(1, x^\vee \otimes N_i) \\ &\simeq \varinjlim Hom(x, N_i) \end{aligned}$$

Out of this we also see that every dualizable object is basic nuclear, as it follows from the axioms that $id_M : M \to M$ is trace class. (this is an if & only if)

For the reverse implication, assume M is nuclear & compact. Then, for any compact obj P, we have

$$(P^\vee \otimes M)(*) = Hom(P, M),$$

but we also have $(P^\vee \otimes M)(*) \simeq Hom(1, P^\vee \otimes M)$ for any P.

Let $P = M$, to get a map $\eta : 1 \to M^\vee \otimes M$, coming from identity map $id_M : M \to M$.

One now checks that $\eta : 1 \to M^\vee \otimes M$ & $ev : M^\vee \otimes M \to 1$ give the duality.

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§ Nuclearity in $D(A, M)$.

Let $\mathcal{C} = D(A, M)$, for (A, M) an analytic (animated) ring.

Then nuclearity will be phrased as:

$\forall$ extremally disc. set S,

$$(M[S]^\vee \otimes^L X)(*) \to Hom(M[S], X) = X(S)$$

is an equivalence in $D(Ab)$.

Rmk: If M is compact, then it is a retract of a finite complex with terms of the form $\bigoplus_{i=1}^{n} M[S_i]$ for some non-negative n, & profinite sets S_i.

There is in fact, an obvious "inner nuclearity" variant which in this case will be satisfied. 8

Rem: Before stating the following proposition, we describe how to extend the functor of measures to all profinite sets.

for $S \in (*)_{\text{proét}}$,

$$\mathcal{M}[S] := A[S] \otimes^{L}_{A} (A, \mathcal{M})$$

(Compute by taking hypercover $S_\bullet \to S$ of S by ext. disc.'s stuff...)

Fact: If $(A, \mathcal{M})$ is an analytic ring over $\mathbb{Z}_\square$, then $\mathcal{M}[S]$ is concentrated in degree zero, compact & projective. Moreover, $\mathcal{M}[S_1] \otimes^{L}_{(A,\mathcal{M})} \mathcal{M}[S_2] \simeq \mathcal{M}[S_1 \times S_2]$.

Prop: Let $X \in \mathcal{D}(A, \mathcal{M})$ be nuclear. Then, for any profinite set S, the natural map

$$\mathcal{M}[S]^{\vee} \otimes^{L} X \to \underline{\mathrm{hom}}(\mathcal{M}[S], X)$$

is itself an equivalence.

Remark: on proof:

Seemingly stronger characterization of nuclearity: $X \in \mathrm{Nuc}$ $\forall N \in \mathcal{D}(A, \mathcal{M})$, $\left(\underline{\mathrm{hom}}(\mathcal{M}[S], N) \otimes^{L} X\right)(*) \to N \otimes X(S)$

Set $N = 1$, to get previous version.

(just in case...)

One shows, then that $\underline{\mathrm{hom}}(\mathcal{M}[S], N) \otimes X \simeq \underline{\mathrm{hom}}(\mathcal{M}[S], X)$ again

Pf: Need to check $\forall Q \in$ Ext. dis

$$\mathrm{hom}(\mathcal{M}[Q], \underline{\mathrm{hom}}(\mathcal{M}[S], N) \otimes X) \to \mathrm{hom}(\mathcal{M}[Q], \underline{\mathrm{hom}}(\mathcal{M}[S], N \otimes X))$$

use $\mathcal{M}[Q \times S] = \mathcal{M}[Q] \otimes \mathcal{M}[S]$

§ Nuclearity over discrete Huber pairs

Recall (A, A^+), A discrete (but not necessarily static)
$A^+ \subseteq \pi_0(A)$ integrally closed.

Then there is a fully faithful functor (induced structure from last time)

$$\{\text{discrete Huber pairs}\} \longrightarrow \text{AnRing}$$
$$(A, A^+) \longmapsto (A, \mathcal{M}_{A^+}),$$

with $\mathcal{M}_{A^+} : \text{ExDisc} \to \mathcal{D}_{\geq 0}(A)$ $\quad \left(S \mapsto A[S] \otimes^L_{A^+} A^+_{\square}\right)$

We will see that for analytic rings coming from discrete Huber pairs, nuclearity is the same as "discreteness". (in the condensed sense)

Def: Let $(A, \mathcal{M})_{\circledast}$. Viewing A as a condensed ring $\underline{A}$, we can take $\underline{A}(*)$, giving a functor

"relative" condition

$$\mathcal{D}(A, \mathcal{M}) \to \text{Mod}_{A(*)},$$

admitting an exact fully faithful left adjoint $M \mapsto M^{\delta} := M \otimes_{\underline{A(*)}} (A, \mathcal{M})$.
We say M is discrete if it lies in the image of the above.

Remark: Dealing with discrete Huber pairs, (A, A^+) it will be, then will in particular arise as colimits of A.

Proposition: Let (A, A_+) be a discrete Huber pair. Then $X \in \mathcal{D}(A, A^+)_{\circledast}$ is nuclear iff it is discrete.

$\longrightarrow$

__Proof:__ Fix X nuclear,

Pick $S = \varprojlim S_i$ extremally disc. with S_i finite.

then
$$
\begin{aligned}
\mathcal{M}_{A^+}[S]^\vee &= \underline{\mathrm{Hom}}_{A}(\mathcal{M}_{A^+}[S], A) \\
&= \underline{\mathrm{Hom}}_A(A[S], A) \\
&= \underline{\mathrm{Hom}}(S, A) \\
&= \varinjlim \mathrm{Hom}(S_i, A) = \varinjlim A^{S_i}.
\end{aligned}
$$

by analyticity

Thus, by nuclearity, we obtain
$$
\begin{aligned}
X(S) &= \left(\mathcal{M}_{A_+}[S]^\vee \otimes_{(A,\mathcal{M}_{A^+})} X\right)(*) = \left(\varinjlim A^{S_i} \otimes X\right)(*) \\
&= \varinjlim X^{S_i}(*) \\
&= \varinjlim X(S_i).
\end{aligned}
$$

This immediately implies that X is discrete.

Conversely, if X is discrete, it is a colimit of copies of A. But A is nuclear and nuclear objects are stable under colimits.

__§ Rigidity:__

Given $(A,\mathcal{M})_\blacksquare$, $\mathrm{Nuc}(A,\mathcal{M})_\blacksquare$ will be a rigid category.

These are $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ which can be given the following characterization

① {trace class maps}

$= \{$compact maps$\}$

$X \xrightarrow{f} Y$ compact iff $\exists \{Z_i\}$ s.t. $\varprojlim Z_i = 0$, $\forall\, Y \to Z_i$, $\exists\, j > i$ s.t. $\underbrace{X \xrightarrow{f} Y \to Z_i \to Z_j}_{0}$

② dualizable.

As a consequence, this category is self dual!

For $(A,M)_{\circledast}$, taking nuclear objects is the rigidification of $\mathcal{D}(A,M)_{\circledast}$.