Efimov K-theory

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IAS/Princeton learning seminar on K-theory

Q. (Poincaré '89) Is every compact smooth manifold homeomorphic to a finite simplicial complex?

A. ('40 Whitehead) Yes.

Variant (Kneser '26) Is every compact topological manifold homeomorphic to a finite simplicial complex?

A. (Casson '90) No.

Q. Is every compact topological manifold homotopy equivalent to a finite simplicial complex?

Q. (Borsuk '54) Is every compact absolute neighborhood retract homotopy equivalent to a finite simplicial complex?

A. Yes (Kirby–Siebenmann '69).

A. Yes (West, 1977).

Baby version: yes, if the manifold is simply connected.

Note: if M is a compact manifold, then $H_*(M;\mathbb{Z})$ is f.g.

$C_*(M,\mathbb{Z})$ is a perfect complex in $D(\mathbb{Z})$

has universal property: $R\mathrm{Hom}(C_*(M,\mathbb{Z}), K) \cong C^*(M;K)$,

which commutes with filtered colimits in K b/c it's $R\Gamma(M, \underline{K})$.

More generally: if M is any topo. space s.t.

1) M is simply connected
2) $C_*(M,\mathbb{Z})$ is a perfect complex
3) M has the homotopy type of a CW complex

then M has the homotopy type of a finite CW complex.

2) Says: ~~$C^{red}_*(M)$~~ there is a quasi-iso. $C^{red}_*(M) \xleftarrow{\alpha} K$ finite free in degrees ≥ 2.

Idea: realize K as the cellular reduced chain complex of a finite CW complex Y and α as coming from a map $Y \to M$.

Construction: work one cell at a time. To extend a map to one more cell, need to know

$K' \subseteq K$
$\cong$
$C^{cellular}_*(Y')$

$\alpha_{K'}$ induced by some map $Y' \to M$

$\pi_{n}(M, Y') \to H_*(M, Y')$ is surjective (Hurewicz theorem)

What if M is not simply connected? Take the universal cover $\underset{G}{\overset{\circlearrowleft}{\tilde M}} \twoheadrightarrow M$,

$G = \pi_1(M)$

$\underset{G}{\overset{\circlearrowleft}{}} C_*(\tilde M, \mathbb{Z})$ is usually not a perfect complex of $\mathbb{Z}$-modules in gen'l.
but is a perfect complex of $\mathbb{Z}[G]$-modules b/c

$$R\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(\tilde M, \mathbb{Z}), K) \simeq C^*(M; \tilde K) \qquad \text{for } K \in D(\mathbb{Z}[G])$$

which commutes with filtered colimits in K.
~~this~~ same argument would G-equivariantly if knew that $C_*(\tilde M, \mathbb{Z})$ was quasi-iso. to a finite complex of (finite) free $\mathbb{Z}[G]$-modules.

$$\begin{array}{ccccc} & & [C_*(\tilde M, \mathbb{Z})] & & \\ & & \cap & & \\ \mathbb{Z} & \xrightarrow{\gamma} & K_0(\mathbb{Z}[G]) & \longrightarrow & K_0^{red}(\mathbb{Z}[G]) := \operatorname{coker}\gamma \\ 1 & \longmapsto & [\mathbb{Z}(G)] & & \end{array}$$

Wall finiteness obstruction: $[C_*(\tilde M, \mathbb{Z})] \in K_0^{red}(\mathbb{Z}[G]) = \operatorname{coker}\gamma$.

__Warning__ If G is any finitely presented gp., $\eta \in K_0^{red}(\mathbb{Z}[G])$, then can find a __finitely dominated__ space M with $\pi_1(M) = G$ and $\eta = [C_*(\tilde M, \mathbb{Z})]$.
compact in the ∞-cat. of spaces
($\Leftrightarrow$ retract of a finite CW complex)

$\rightsquigarrow$ Should use something about manifolds.

How to understand $[C_*(\tilde M, \mathbb{Z})] \in K_0(\mathbb{Z}[G])$?

$\mathrm{Loc}(M, D(\mathbb{Z}[G]))$ = ∞-cat. of local systems on M with valued in $D(\mathbb{Z}[G])$

$$\begin{array}{ccc} \| & & \\ \mathrm{Fun}(\mathrm{Sing}\, M, D(\mathbb{Z}[G])) & \xleftarrow{\text{const}} & D(\mathbb{Z}[G]) \\ \underline{K} & \xleftarrow{\quad\quad} & K \end{array}$$

has a left adjoint ~~...~~ $C_*(M, \mathcal{E})$
which preserves compact objects
(b/c right adjoint commutes with $\varinjlim$ filtered)

Get $\mathrm{Loc}^{comp}(M, D(\mathbb{Z}[G])) \longrightarrow \mathrm{Perf}(\mathbb{Z}[G])$

$$\begin{array}{ccc} \mathcal{L} & \longmapsto & C_*(\tilde M, \mathbb{Z}) \end{array}$$

$\mathcal{L}$ = pushforward of $\mathbb{Z}$ on $\tilde M$

$$K_0\left(\mathrm{Loc}^{comp}\left(M, D(\mathbb{Z}[G])\right)\right) \longrightarrow K_0(\mathbb{Z}[G])$$

$$[\mathcal{L}] \longmapsto [C_*(\tilde{M}, \mathbb{Z})]$$

Recall: local systems on M with values in a category $\mathcal{A}$ are objects $V \in \mathcal{A}$ with action of G.

$\mathrm{Loc}^{comp}(M, \mathrm{Mod}(\mathbb{Z}[G])) = \mathrm{Mod}(G \times G) \ni \mathcal{L}$ is a $\mathbb{Z}[G]$ as a bimodule over itself.

Idea: what if we replace the ∞-category of local systems by $\underbrace{\mathrm{Shv}(M, D(\mathbb{Z}[G]))}_{\text{dualizable category}}$.

What do we take K-thy. of?

1) Rings $R \longmapsto K_0(R)$
2) Abelian/Exact categories $\mathcal{C} \longmapsto K_0(\mathcal{C})$
3) stable ∞-categories $\mathcal{C} \longmapsto K_0(\mathcal{C})$. (e.g., $K_n(\mathrm{Perf}(R)) = K_n(R)$)
4) compactly generated stable ∞-categories $\mathcal{C} \longmapsto K_0^{Ef}(\mathcal{C}) := K_0(\mathcal{C}^{comp})$

$$K_0(\mathcal{C}) \to K_0^{Ef}(\mathrm{Ind}\,\mathcal{C}) = K_0(\mathrm{Ind}(\mathcal{C})^{comp}) = K_0(\mathcal{C}^{\text{idempotent complete}})$$

is an iso. iff $\mathcal{C}$ is idempotent complete

Given $\mathcal{C}$ idempotent complete, have a bijection

$$\left\{ \begin{matrix} \text{subcategories (full) subcategories} \\ \mathcal{C}_0 \subset \mathcal{C} \text{ with idempotent completion } \mathcal{C} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{subgroups of} \\ K_0(\mathcal{C}) \end{matrix} \right\}.$$

(complete (closed under iso.'s))

$$\mathcal{C}_0 \longrightarrow K_0(\mathcal{C}_0)$$

$$\mathcal{C}_A := \left\{ \begin{matrix} x \in \mathcal{C} \\ [x] \in A \end{matrix} \right\} \longleftarrow A \subseteq K_0(\mathcal{C}_0)$$

Thm. In this case, $K_n(\mathcal{C}_0) \to K_n(\mathcal{C})$ is an $\begin{cases} \text{iso. for } n > 0 \\ \text{injection for } n = 0. \end{cases}$

Thm. (Barwick, Waldhausen) If $\mathcal{C}$ is a stable ∞-category, $\mathcal{C}_0 \subseteq \mathcal{C}$ thick subcategory, $\mathcal{C}/\mathcal{C}_0$. Then have a fiber sequence

$$K(\mathcal{C}_0) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C}/\mathcal{C}_0).$$

$\rightsquigarrow$ have a LES $\cdots \to K_1(\mathcal{C}/\mathcal{C}_0) \to K_0(\mathcal{C}_0) \to K_0(\mathcal{C}) \to K_0(\mathcal{C}/\mathcal{C}_0) \to 0$

Assume $\mathcal{C}$ is idempotent complete

$$\begin{array}{ccccc} K(\mathcal{C}_0) & \longrightarrow & K(\mathcal{C}) & \longrightarrow & K(\mathcal{C}/\mathcal{C}_0) \\ \| & & \| & & \downarrow \\ K(\mathrm{Ind}\,\mathcal{C}_0) & \longrightarrow & K^{\mathrm{Ef}}(\mathrm{Ind}\,\mathcal{C}) & \longrightarrow & K^{\mathrm{Ef}}\big(\underbrace{\mathrm{Ind}(\mathcal{C})/\mathrm{Ind}(\mathcal{C}_0)}_{=\mathrm{Ind}(\mathcal{C}/\mathcal{C}_0)\ \text{not idempotent complete}}\big) \end{array}$$

fiber sequence of spaces, not of spectra.

<u>Construction</u> Let $\mathcal{C}$ be a idempotent complete stable ∞-category. We will construct a spectrum $\mathbb{K}(\mathcal{C})$ s.t. $K(\mathcal{C}) = \Omega^\infty \mathbb{K}(\mathcal{C})$.

Consider $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}) \to (\mathrm{Ind}(\mathcal{C})/\mathcal{C})^{\text{idempotent completion}}$ to get

$$K(\mathcal{C}) \to * \to K\big(\underbrace{(\mathrm{Ind}(\mathcal{C})/\mathcal{C})^{\text{idempotent completion}}}_{=:\mathcal{C}_1}\big).$$

$$K(\mathcal{C}) = \Omega K(\mathcal{C}_1)$$

$$K(\mathcal{C}_1) = \Omega K(\mathcal{C}_2) \quad \cdots$$

$\rightsquigarrow$ get a spectrum $\mathbb{K}(\mathcal{C})$ with $\Omega^{\infty-n}(\mathbb{K}(\mathcal{C})) = K(\mathcal{C}_n)$.

If $\mathcal{C}$ is compactly generated,

$$\mathbb{K}^{\mathrm{Ef}}(\mathcal{C}) = \mathbb{K}(\mathcal{C}^{\text{compact objects}})$$

If $\mathcal{C}_0 \subseteq \mathcal{C}$, get

$$\begin{array}{ccccc} \mathbb{K}^{\mathrm{Ef}}(\mathrm{Ind}\,\mathcal{C}_0) & \to & \mathbb{K}(\mathrm{Ind}\,\mathcal{C}) & \to & \mathbb{K}(\mathrm{Ind}(\mathcal{C}/\mathcal{C}_0)) \\ \| & & \cong & & \cong \\ \mathbb{K}(\mathcal{C}_0) & \longrightarrow & \mathbb{K}(\mathcal{C}) & \longrightarrow & \mathbb{K}\big((\mathcal{C}/\mathcal{C}_0)^{\text{idempotent completion}}\big) \end{array}$$

Now have $K_{-n}(\mathcal{C}) := \pi_{-n}\mathbb{K}(\mathcal{C}) = K_0(\mathcal{C}_n)$.

<u>Thm.</u> (Bass) If R is a commutative regular ring, then

$$K_{-n}(R) := K_{-n}(\mathrm{Perf}(R)) = 0 \quad \text{for } n > 0.$$

<u>Pf. idea</u>

$$\mathrm{Perf}_{t\text{-nil}}(R[t]) \subseteq \mathrm{Perf}(R[t]) \to \mathrm{Perf}(R[t^{\pm 1}]) \qquad K_{n+1}(R[t^{\pm 1}])$$

set $t=0$ $\uparrow\downarrow$ forget t $\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$\mathrm{Perf}(R)$ $\quad \rightsquigarrow K_n(R)$ is a summand of $K_n(\mathrm{Perf}_{t\text{-nil}}(R[t]))$

$$\xrightarrow{\text{composite vanishes}} K_n(R[t])$$

b/c $M \mapsto 0 \to M[t] \xrightarrow{t} M[t] \to M \to 0$

<u>Conclusion</u> have a surjection $K_{n+1}(R[t^{\pm 1}]) \to K_n(R)$.

$\rightsquigarrow$ suffices to treat $n=-1$. and need to show

$$K_0(R[t]) \to K_0(R[t^{\pm 1}]) \quad \text{is surjective.}$$

surjective when R is regular (b/c can use G-theory). $\square$

<u>Recall from above</u>: $\mathbb{K}^{Ef}(\mathcal{C}) = \mathbb{K}(\mathcal{C}^{comp})$

5) "Observation (Efimov): let $\mathcal{C}$ be a dualizable stable ∞-category.
then there is a functor $\mathcal{C} \xrightarrow[f]{} \mathrm{Ind}(\mathcal{C})$ that preserves colimits
(when $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, then $f = \mathrm{Ind}(\mathcal{C}_0 \to \mathrm{Ind}(\mathcal{C}_0))$)
<u>Claim</u> $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is left adjoint to $\mathrm{colim} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.
Then $\mathrm{Ind}(\mathcal{C})/F(\mathcal{C})$ is compactly generated again.

<u>Better</u>: $\mathrm{Ind}\,\mathcal{C} \overset{Q}{\twoheadrightarrow} \mathrm{Ind}\,\mathcal{C}/\mathcal{C}$ preserves compact objects.

<u>Need</u>: $Q(h(C)) \in \mathrm{Ind}(\mathcal{C})/F(\mathcal{C})$ is compact.
"constant objects"

Have a right adjoint $\mathrm{Ind}(\mathcal{C})/F(\mathcal{C}) \hookrightarrow \mathrm{Ind}(\mathcal{C})$ whose image are objects $Y \in \mathrm{Ind}(\mathcal{C})$ s.t. $\mathrm{Hom}(F(C), Y) = 0$ for all C

$$\mathrm{Hom}(F(C),Y) \cong \mathrm{Hom}(C, \mathrm{colim}\, Y).$$

$$\mathbb{K}^{Ef}(\mathcal{C}) \qquad K^{Ef}(\mathcal{C}) \qquad K^{Ef}(\mathrm{Ind}(\mathcal{C})/\mathcal{C})$$

$$K^{Ef}(\mathcal{C}) = K(\mathcal{C}_0) \qquad K^{Ef}(\mathrm{Ind}(\mathcal{C})/\mathcal{C}) = K((\mathrm{Ind}\,\mathcal{C}_0/\mathcal{C}_0)^{\text{idempotent completion}})$$

This defines a spectrum $\mathbb{K}^{Ef}(\mathcal{C})$ for $\mathcal{C}$ dualizable.

thm (Efimov) Let X be a compact Hausdorff space, $\mathcal{C}$ dualizable stable ∞-cat
then $\mathrm{Shv}(X, \mathcal{C})$ is dualizable (version of Verdier duality).

$$\mathbb{K}^{Ef}(\mathrm{Shv}(X,\mathcal{C})) = C^*(X, \mathbb{K}^{Ef}(\mathcal{C}))$$

$$K(\mathrm{Loc}^{comp}(M, \mathcal{D}(\mathbb{Z}[G]))) = \mathbb{K}^{Ef}(\mathrm{Loc}(M, \mathcal{D}(\mathbb{Z}[G]))).$$

Let now again M is a compact manifold (or compact absolute neighborhood retract) $\rightsquigarrow$ $\mathcal{C} \leftarrow \mathrm{Shv}(M, \mathcal{C})$ ← $\mathbb{Z}$-linear stable ∞-cat., e.g., $\mathcal{C} = \mathcal{D}(\mathbb{Z})$, $\mathcal{C} = \mathcal{D}(\mathbb{Z}[G])$

$C \in \mathcal{C}$; $\exists$ left adjoint (since M is a manifold) can think in terms of 6-functors

The left adjoint (on local systems) is $\mathcal{E} \mapsto C_*(M;\mathcal{E})$, it is a compact map, so get

$$\begin{array}{ccc} \mathbb{K}^{Ef}(Shv(M,\mathcal{C})) & \longrightarrow & \mathbb{K}^{Ef}(\mathcal{C}) \\ \| & \nearrow & \\ C^*(M;\mathbb{K}^{Ef}(\mathcal{C})) & & \\ \| & & \\ C^*(M;\mathbb{K}(\mathbb{Z})) \otimes_{\mathbb{K}(\mathbb{Z})} \mathbb{K}^{Ef}(\mathcal{C}) & & \end{array}$$

In the case $\mathcal{C} = \mathcal{D}(\mathbb{Z})$, get $C^*(M,K(\mathbb{Z})) \longrightarrow K(\mathbb{Z})$ given by a class in $H_0(M,K(\mathbb{Z}))$. There is an Atiyah–Hirzebruch SS:

$$H_m(X;K_n(\mathbb{Z})) \Rightarrow H_{m+n}(M,\mathbb{K}(\mathbb{Z}))$$

Get: $\underset{\overset{\|}{\mathbb{Z}} \text{ if } M \text{ is connected}}{H_0(M,K_0(\mathbb{Z}))} \simeq H_0(M,\mathbb{K}(\mathbb{Z})) \ni \chi$

Take $\mathcal{C} = \mathcal{D}(\mathbb{Z}[G])$. ~~Have~~ Get (here $G = \pi_1(M)$)

$$\begin{array}{ccc} K_0^{Ef}(Shv(M;\mathcal{C})) & \xrightarrow[\text{with } \chi]{\text{"cap product"}} & K_0(\mathbb{Z}[G]) \\ \Psi & & \Psi \\ [\mathcal{L}] & \longmapsto & [C_*(\tilde{M},\mathbb{Z})] = \underset{\text{Euler characteristic } \chi(M).}{\chi} \cdot [\mathbb{Z}[G]] \end{array}$$

Get the desired vanishing in reduced K-theory.