Dualizable stable ∞-categories

Talk in analytic K-theory seminar

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Last week, we defined algebraic K-theory of a stable ∞-category. For rings R,

$$K(R) = K(\mathrm{Perf}(R)).$$

In some sense, the more fundamental stable ∞-category attached to a ring R is $\mathcal{D}(R)$. However, it is "too big":

Lemma 1. *If $\mathcal{C}$ admits countable coproducts, $K(\mathcal{C}) = 0$.*

Proof. We have the functor $F : \mathcal{C} \to \mathcal{C}$, $X \mapsto \bigoplus_{\mathbb{N}} \mathcal{C}$. It satisfies

$$F \simeq F \oplus \mathrm{id},$$

and so $K(F) = K(F) + K(\mathrm{id})$, so the identity is zero on $K(\mathcal{C})$ and thus $K(\mathcal{C}) = 0$. $\square$

So: *K-theory is only interesting for "small enough" categories!* We now want to see how $\mathcal{D}(R)$ and $\mathrm{Perf}(R)$ determine each other.

Definition 1. *An ∞-category I is filtered if every finite subcategory extends over a (right) cone. Colimits of shape I are called filtered colimits.*

An object X of an ∞-category $\mathcal{C}$ is called compact *if $\mathrm{Map}_{\mathcal{C}}(X, -)$ preserves filtered colimits.*

We write $\mathcal{C}^\omega$ for the full subcategory of compact objects.

Example 1. $\mathcal{D}(R)^\omega = \mathrm{Perf}(R)$.

Definition 2. *We call an ∞-category compactly generated if it admits all small colimits, and every object can be written as a colimit of compact objects.*

There are other characterisations, for example $\mathcal{C}$ is also compactly generated if it admits all small colimits and $\mathrm{Map}_{\mathcal{C}}(K, -)$ jointly detect equivalences. One can also see that compact objects are already closed under finite colimits, so it suffices to ask for filtered colimits.

Compactly generated categories are not just generated by $\mathcal{C}^\omega$, in a sense they are *freely* generated by $\mathcal{C}^\omega$. Given a small ∞-category $\mathcal{C}_0$, we can build one with $\mathcal{C}_0$ as its compact objects, by "freely adjoining filtered colimits".

Definition 3. *For a small ∞-category $\mathcal{C}_0$, we let*

$$\mathrm{Ind}(\mathcal{C}_0) \subseteq \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$$

be the full subcategory generated by the Yoneda image under filtered colimits.

(Equivalently, this agrees with the full subcategory $\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{An})$ if $\mathcal{C}_0$ admits finite colimits.)

From this construction, one can relatively directly see a universal property: filtered-colimit preserving functors

$$\mathrm{Ind}(\mathcal{C}_0) \to \mathcal{D}$$

are the same as functors $\mathcal{C}_0 \to \mathcal{D}$.

Mapping spaces in $\mathrm{Ind}(\mathcal{C}_0)$ can be computed as

$$\mathrm{Map}_{\mathrm{Ind}(cC_0)}(\mathrm{colim}_{i\in I} X_i, \mathrm{colim}_{j\in J} Y_j) \simeq \lim_{i\in I} \mathrm{colim}_{j\in J} \mathrm{Map}_{\mathcal{C}_0}(X_i, Y_j),$$

and from that we see that the filtered-colimit preserving functor

$$\mathrm{Ind}(\mathcal{C}^\omega) \to \mathcal{C}$$

is fully faithful. It is essentially surjective if and only if $\mathcal{C}$ is compactly generated, so *compactly generated categories are always of the form* $\mathrm{Ind}(\mathcal{C}^\omega)$. Conversely, if we start with a small ∞-category $\mathcal{C}_0$, $\mathrm{Ind}(\mathcal{C}_0)^\omega$ almost agrees with $\mathcal{C}_0$, with one caveat: Compact objects in $\mathrm{Ind}(\mathcal{C}_0)$ are easily seen to be retracts of objects from $\mathcal{C}_0$, but don't have to be from $\mathcal{C}_0$ itself. In fact, $\mathrm{Ind}(\mathcal{C}_0)^\omega$ is always the idempotent completion of $\mathcal{C}_0$. (Think $\mathrm{Free}(R) \subseteq \mathrm{Proj}(R)$)

Theorem 1. *Let Pr^L_ω denote the category whose objects are compactly generated ∞-categories, and morphisms are colimit-preserving functors $F : \mathcal{C} \to \mathcal{D}$ with $F(\mathcal{C}^\omega) \subseteq \mathcal{D}^\omega$.*

Let $\mathrm{Cat}^{\mathrm{perf}}_\infty$ denote the category whose objects are small idempotent-complete ∞-categories with finite colimits, and morphisms are finite colimit preserving functors.

Then Ind *and* $(-)^\omega$ *give inverse equivalences*

$$\mathrm{Cat}^{\mathrm{perf}}_\infty \simeq \mathrm{Pr}^L_\omega.$$

Proof. We sketched this above, except for the colimits. This amounts to a different universal property of Ind, relating functors

$$\mathrm{Ind}(\mathcal{C}_0) \to \mathcal{D}$$

preserving *all* colimits to functors

$$\mathcal{C}_0 \to \mathcal{D}$$

preserving finite colimits. This follows the philosophy

$$\text{small colimits} = \text{filtered colimits} + \text{finite colimits}$$

□

For example, $\mathrm{Perf}(R)$ and $\mathcal{D}(R)$ correspond to each other here.

The philosophy is that this allows us to think of $K(R)$ as something attached to the big category $\mathcal{D}(R)$ (by passing to compact objects). More generally, this makes sense for all compactly generated categories.

One can generalize this to higher cardinals than ω, but already for the next one, ω_1, $\mathcal{C}^{\omega_1}$ is closed under countable colimits, so has trivial K-theory. However, it turns out that there is a class of categories between compactly generated and ω_1-compactly generated ones, which K-theory can be extended to. This contains many natural examples, such as almost module categories and categories of sheaves on locally compact Hausdorff spaces.

This arises most naturally in the following setting:

Definition 4. *We call an ∞-category* presentable *if it has all small colimits and is generated by a small collection of κ-compact objects for some κ.*

Pr^L *is the category with objects presentable ∞-categories and morphisms colimit-preserving functors.*

Remark 1. *Note that Pr^L_ω is not a full subcategory of Pr^L, since we also restricted the morphisms to be compact-object preserving.*

Theorem 2. Pr^L *carries a symmetric-monoidal structure characterized by the universal property that colimit-preserving functors*

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$$

agree with functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in both variables separately. The unit is An.

This restricts to a symmetric-monoidal structure on $\mathrm{Pr}^L_{\mathrm{st}}$ with unit Sp.

Example 2.

1. $\mathrm{Sp} \otimes \mathcal{C}$ *is the stabilisation of $\mathcal{C}$, and agrees with $\mathcal{C}$ if $\mathcal{C}$ is already stable (so $\mathrm{Pr}^L_{\mathrm{st}}$ is modules over* Sp *in* Pr^L*).*

2. $\mathrm{Shv}(X; \mathrm{An}) \otimes \mathcal{C} = \mathrm{Shv}(X; \mathcal{C})$, *and* $\mathrm{Shv}(X; \mathrm{An}) \otimes \mathrm{Shv}(Y; \mathrm{An}) = \mathrm{Shv}(X \times Y; \mathrm{An})$.

3. $\mathcal{D}(R) \otimes \mathcal{D}(S) = \mathcal{D}(R \otimes_{\mathbb{S}} S)$ *for ring spectra R and S.*

4. $\mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{D}_0) = \mathrm{Ind}(\mathcal{C}_0 \otimes^{\mathrm{Rex}} \mathcal{D}_0)$, *where the inside tensor product is analogously defined with respect to finite-colimit preserving functors between small ∞-categories with finite colimits.*

Recall that a symmetric-monoidal structure gives rise to a notion of dualizability:

Definition 5. *An object $X \in \mathcal{C}$ is dualizable if there exists $X^\vee$ with maps*

$$\mathbf{1} \to X \otimes X^\vee$$

and

$$X^\vee \otimes X \to \mathbf{1}$$

such that the two composites

$$X \to X \otimes X^\vee \otimes X \to X$$

and

$$X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee$$

are identities.

Example 3. *1. Dualizable objects in* Vect_K *are exactly the finite-dimensional vector spaces.*

2. Dualizable objects in $\mathcal{D}(R)$ *are exactly* $\mathrm{Perf}(R)$*.*

What are the dualizable objects in $\mathrm{Pr}^L_{\mathrm{st}}$? (This question does not make much sense for Pr^L itself). Note

$$\mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}_0), \mathrm{Sp}) = \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}_0, \mathrm{Sp}) = \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0, \mathrm{Sp}) = \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}_0, \mathrm{An}) = \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$$

So it seems that $\mathrm{Ind}(\mathcal{C}_0)$ and $\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$ are dual! Indeed:

$$\mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Sp}$$

can be defined as Ind-extension of $\mathrm{Map}_{\mathcal{C}_0}$, and

$$\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}}) = \mathrm{Ind}(\mathcal{C}_0 \otimes \mathcal{C}_0^{\mathrm{op}})$$

corresponds to an object of $\mathrm{Ind}(\mathcal{C}_0 \otimes \mathcal{C}_0^{\mathrm{op}})$ which agrees with

$$\mathrm{Fun}^{\mathrm{Lex}}((\mathcal{C}_0 \otimes \mathcal{C}_0^{\mathrm{op}})^{\mathrm{op}}, \mathrm{An}),$$

where we can *again* take $\mathrm{Map}_{\mathcal{C}_0}$.

So compactly generated categories are dualizable! But there are more dualizable categories.

Definition 6. *We call a morphism* $X \to Y$ *in an* ∞*-category* $\mathcal{C}$ compact *if for each map* $Y \to \mathrm{colim}_{i \in I} Z_i$ *into a filtered colimit there exists a dashed lift*

$$\begin{array}{ccc} X & \dashrightarrow & Z_i \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{colim}_{i \in I} Z_i. \end{array}$$

We call an object $X \in \mathcal{C}$ compactly exhausted *if it can be written as*

$$X = \mathrm{colim}(X_0 \to X_1 \to \dots),$$

where all morphisms are compact.

We call $\mathcal{C}$ compactly assembled *if filtered colimits are exact and* $\mathcal{C}$ *is generated by compactly exhausted objects under colimits.*

If $\mathcal{C}$ is compactly generated, the compact morphisms are exactly those which factor through compact objects. But this notion makes sense even if we do not have enough compact objects, and turns out to be the correct "intrinsic" characterisation of dualizable stable ∞-categories. More precisely, we have the following:

Theorem 3. *For $\mathcal{C} \in \mathrm{Pr}^L_{\mathrm{st}}$, the following are equivalent.*

1. *$\mathcal{C}$ is dualizable.*

2. *$\mathcal{C}$ is compactly assembled.*

3. *The "colimit" functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a* left *adjoint $\widehat{j} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.*

4. *$\mathcal{C}$ is ω_1-compactly generated and $k : \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ admits a left adjoint $\widehat{j} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\omega_1})$.*

5. *$\mathcal{C}$ is a retract of a compactly generated category in Pr^L (so along colimit-preserving functors).*

6. *Filtered colimits distribute over small limits, i.e.*

$$\lim_K \operatorname{colim}_I = \operatorname{colim}_{I^K} \lim_K$$

in $\mathcal{C}$. (Analogue of AB5 + AB6)

Proof. (1) $\Rightarrow$ (5) works by tensoring $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$ with $\mathcal{C}^\vee$ and lifting the coevaluation through $\mathcal{C}^\vee \otimes \mathrm{Ind}(\mathcal{C}^\kappa)$, i.e. the identity of $\mathcal{C}$ through $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^\kappa)$. Conversely, (5) implies (1) since retracts of dualizables are dualizable. (5) implies (3) by observing that we have such an adjoint for $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ (Yoneda "inside"), and observing that adjointability here is stable under retracts. (3) is also equivalent to (6) by an explicit analysis of what k preserving limits means.

Once one has $\widehat{j}$, one can characterize compact morphisms $X \to Y$ as those where $jX \to jY$ factors as $jX \to \widehat{j}Y$. This allows one to construct enough compactly exhausted objects to prove (2). Conversely, given (2), $\widehat{j}$ is characterized as filtered-colimit preserving functor which takes a compactly exhausted

$$X = \operatorname{colim} X_n$$

to the Ind-object $\operatorname{colim} jX_n$. This also shows that $\widehat{j}$ takes those to ω_1-compact objects and therefore in total takes values in $\mathrm{Ind}(\mathcal{C}^{\omega_1})$. This shows (4). Finally, (4) clearly implies (5). □

Note that all versions except (1) makes sense also unstably!

Example 4. $\mathrm{Shv}(X; \mathcal{C})$ *for $\mathcal{C}$ compactly assembled and X a locally compact Hausdorff space is compactly assembled (=dualizable if $\mathcal{C}$ is stable)*

(It is typically not compactly generated, $\mathrm{Shv}(X)$ *only is compactly generated if X is locally profinite.)*

Proof. It suffices to do this for $\mathcal{C} = \mathrm{An}$. If $U \subseteq V$ is an inclusion of opens with a compact K in between, then restriction

$$\Gamma(V, \operatorname{colim} \mathcal{F}_i) \to \Gamma(U, \operatorname{colim} \mathcal{F}_i)$$

factors through $\Gamma(K, \operatorname{colim} \mathcal{F}_i) = \operatorname{colim} \Gamma(K, \mathcal{F}_i)$, so we have dashed lifts in

$$\begin{array}{ccc} \underline{U} & \dashrightarrow & \mathcal{F}_i \\ \downarrow & & \downarrow \\ \underline{V} & \longrightarrow & \operatorname{colim} \mathcal{F}_i. \end{array}$$

so these are compact morphisms. If we have a "compactly exhaustible open" in X, this shows that $\underline{U}$ is compactly exhaustible. In a locally compact Hausdorff space, "compactly exhaustible opens" form a basis. So $\mathrm{Shv}(X)$ is generated compactly exhaustible objects, so it is compactly assembled. □

Example 5. *Let R be some ring, $I \subseteq R$ an ideal, and assume the canonical map $I \otimes_R I \to I$ is an equivalence (all tensor products derived!) Then*

$$\mathcal{D}(R, I) = \ker(\mathcal{D}(R) \to \mathcal{D}(R/I))$$

is dualizable.

Proof. The inclusion $\mathcal{D}(R, I) \to \mathcal{D}(R)$ admits a right adjoint which takes M to

$$\mathrm{fib}(M \to M \otimes_R (R/I)).$$

That $\mathrm{map}(N, \mathrm{fib}(M \to M \otimes_R (R/I))) = \mathrm{map}(N, M)$ if $N \in \mathcal{D}(R, I)$ is clear, but for

$$\mathrm{fib}(M \to M \otimes_R (R/I)) \in \mathcal{D}(R, I)$$

we need $(R/I) \otimes_R (R/I) = R/I$. As the left hand side is the total cofiber of the square

$$\begin{array}{ccc} I \otimes_R I & \longrightarrow & I \\ \downarrow & & \downarrow \\ I & \longrightarrow & R, \end{array}$$

this is equivalent to $I \otimes_R I = I$, which we assumed.

This adjoint provides a retraction $\mathcal{D}(R) \to \mathcal{D}(R, I)$, and evidently preserves colimits, so $\mathcal{D}(R, I)$ is a retract of $\mathcal{D}(R)$ in Pr^L and thus compactly assembled. It is typically not compactly generated, since compact objects in $\mathcal{D}(R, I)$ would be compact in $\mathcal{D}(R)$ as well, but for example if I is local, compact objects in $\mathcal{D}(R)$ in the kernel of $\mathcal{D}(R) \to \mathcal{D}(R/I)$ are zero by Nakayama. □