IAS study group
on analytic K-thy
MORROW

Recollections on K-theory
(OR: 50 years of math in 2 hours)

Warning: almost no proofs in this talk. But we will be very careful where things live.
Note: all K-thy in this talk is connective.

§1 K_0 and K_1 of rings

Let R be a unital, associative ring.

Def^n: An R-module is finite projective iff it is a direct sum of R^n for some $n \geq 0$. Let $\mathrm{Proj}(R)$ be the cat. of them. Let $K_0(R)$ be the group completion of the monoid

$$\left(\begin{array}{c}\text{isom classes} \\ \text{of objs of } \mathrm{Proj}(R)\end{array}, 0, \oplus \right)$$

ie) $K_0(R) := \dfrac{\text{Free ab. group on isom classes of } \mathrm{Proj}(R)}{\left\langle [P] = [P_1] + [P_2] \text{ whenever there is an isom } P \simeq P_1 \oplus P_2 \right\rangle}$

Eg 1) k a field. Then $\dim : \mathrm{Proj}(R) \to \mathbb{N}$ induces
$$\dim : K_0(R) \xrightarrow{\cong} \mathbb{Z}$$

Eg 2) D Dedekind domain. Then any nonzero fractional ideal $I \subseteq \mathrm{Frac}\,D$ is a fin proj D-module, so get $[I] \in K_0(D)$, and in fact a s.e.s. :

$$0 \to Cl(\mathrm{Frac}\,D) \to K_0(D) \xrightarrow{\text{rank}} \mathbb{Z} \to 0$$

<u>Def^n</u>:
$$GL_1(R) \subseteq GL_2(R) \subseteq GL_3(R) \subseteq \cdots \subseteq \bigcup_{n\geq 1} GL_n(R) =: GL(R)$$
$$r \mapsto \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$K_1(R) :=$ max^l abelian quotient of $GL(R)$
$= GL(R)/E(R)$ where $E(R) := [GL(R), GL(R)]$

<u>Lemma</u> (Whitehead) (1) A matrix $\alpha \in GL(R)$ lies in $E(R)$ iff it is equiv to 1 via row and column operations

(2) $E(R)$ is the max^l $\underbrace{\text{perfect}}_{\text{ie) equal to own commutator}}$ subgroup of $GL(R)$

ie) $K_1(R)$ classifies normal forms of invertible matrices

Eg 1) k a field. Gaussian elimination implies
$$\det : K_1(k) \xrightarrow{\cong} k^\times$$

Eg 2) For the Dedekind domain $D = \mathbb{R}[X,Y]/X^2+Y^2-1$
there is a ses

$$0 \to \mathbb{Z}/2 \to K_1(D) \xrightarrow{\det} D^\times \to 0$$
$$1 \longmapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

detect using
real top K-thy
of sphere.

Thm (Bass: Fundamental theorem of K_0). There is a natural exact sequence

$$0 \to K_1(R) \to K_1(R[T]) \oplus K_1(R[T^{-1}]) \to K_1(R[T^{\pm 1}]) \to K_0(R) \to 0$$

non-obvious belying

In particular, $K_1 : \mathrm{Rngs} \to \mathrm{Ab}$ refines $K_0 : \mathrm{Rngs} \to \mathrm{Ab}$.

60s/70s : Does $\exists\ K_2 : \mathrm{Rngs} \to \mathrm{Ab}$ refining K_1 in same way?
" K_3 ——— K_2 ——— ?

1973 Quillen : Yes!

§2 K-thy space of a ring : Quillen's + -construction

∞-categorical preliminaries on topological spaces :

(2a) The ∞-cat of spaces is
$\mathrm{Spc} :=$ homotopy coherent nerve of
simplicial cat. of Kan complexes

$\underset{\text{Lurie}}{\simeq}$ ∞-cat freely generated under sifted colimits by finite sets $=:$ Anima

(2b) <u>Theorem</u> (Kervaire, Quillen : $+$-construction) : There is a unique functor

$$Spc \longrightarrow Spc \quad X \longmapsto X^+$$

with following properties:

(1) $\pi_1(X^+)$ is hypoabelian (ie) it has no non-zero perfect subgroup)
(2) there is a nat. map $X \to X^+$, universal wrt property (1), and $X \to X^+$ is an equiv iff $\pi_1(X)$ is hypoabelian.

<u>Rmks</u> (1) $\pi_1(X^+) = \pi_1(X)/\text{max}^{\ell}$ perfect subgroup
(2) $H_n(X,\mathbb{Z}) \xrightarrow{\cong} H_n(X^+,\mathbb{Z}) \quad \forall\, n \geq 0$
(3) But $\pi_n(X) \longrightarrow \pi_n(X^+)$ <u>not</u> usually isom for $n \geq 2$

Now we can define K-thy space of R "by deriving K_1":

$$\begin{array}{ccccc} & & GL(R) & & \\ & & \wr\downarrow & & \\ GL(R) & \xleftarrow{\pi_1} & BGL(R) & \xrightarrow{\pi_n,\, n\geq 2} & 0 \\ & & \wr\downarrow\,{+\text{-const.}} & & \\ K_1(R) & \xleftarrow{\pi_1} & BGL(R)^+ & \xrightarrow{\pi_n,\, n\geq 2} & ? \end{array}$$

$K_1(R)$ by Rmk(1) and Whiteheads Lemma(2)

? $\Leftarrow$ Quillen realized these are interesting groups.

<u>Def^{n}</u> (Quillen) The <u>K-thy space</u> of R is
$$\mathrm{Spc} \ni K_0(R) \times BGL(R)^+,$$
and $K_n(R) := \pi_n$ of this space, for $n \geq 0$
$$= \begin{cases} K_0(R) \text{ from §1} & \text{if } n=0 \\ K_1(R) \text{ from §1} & \text{if } n=1 \\ \pi_n(BGL(R)^+) & \text{if } n\geq 1. \end{cases}$$

<u>Rmk</u>: This is a useful def. for certain computations (eg) K-groups of finite field, of number field, ..) but it is not very structured/categorical.

<u>§3 K-thy spectrum of a ring: ∞-group completion</u>

∞-cat preliminaries on monoids in spaces:

(3.1) $\mathrm{Mon}_{E_1}(\mathrm{Spc}) := E_1$-monoids in $\mathrm{Spc}, \times$

...an object here is a simplicial space $M_\bullet$ s.t., for each $n \geq 0$, the n maps $[1] \longrightarrow [n] = \{0, \ldots, n\}$
$$\{0,1\} = 0,1 \longmapsto i, i+1$$
induce
$$M_n \xrightarrow{\simeq} \prod_{i=1}^{n} M_1$$

· informally $M_\bullet$ is data of pointed space
$$* \simeq M_0 \longrightarrow M_1$$
together with "multiplication" $M_1 \times M_1 \xrightarrow{\mu} M_1$
which is coherently associative. $\quad \wr\!\wr$ $M_2 \xrightarrow{[1]\to[2]}$

(3ii) adjoint pair:

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Spc}) \arrow[r, bend left, "{\text{Geom. realisation } |\cdot|, \text{ usually denoted } B \text{ in this context}}"] & \mathrm{Spc}_* \arrow[l, bend left, "\Omega = \text{loop space}"] \end{tikzcd} \quad := \text{pointed spaces}$$

(3iii) Ω actually lands in grouplike $\mathbb{E}_1$-monoids:

$$\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Spc}) \supseteq \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{Spc}) := \left\{ M \text{ s.t. the monoid } \pi_0(M) \text{ is a group} \right\}$$

(as $\pi_0(\Omega X) \cong \pi_1(X)$), and diagram restricts to inverse equivalences

$$\begin{tikzcd} \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{Spc}) \arrow[r, shift left, "B"] \arrow[r, phantom, "\simeq"] & \mathrm{Spc}_{*,\mathrm{conn}} \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

"<u>Cor</u>": The inclusion $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Spc}) \supseteq \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{Spc})$ has a left adjoint

$$M \longmapsto M^{\infty\text{-grp}} := \Omega B(M)$$

called the <u>∞-group completion</u> of M.

<u>Rmk</u>: The group $\pi_0(M^{\infty\text{-grp}})$ is the usual group completion of the monoid $\pi_0(M)$

(3iv) In fact we only require the commutative variant of above:

analgue of fact that B(abelian group) is not just a simplicial set, but a simplicial abelian group.

$$\begin{tikzcd} \mathrm{Mod}_{E_\infty}(\mathrm{Spc}) \arrow[r, shift left, "B"] \arrow[d, "\text{Forget comm. conditions}"'] & \mathrm{Mod}_{E_\infty}(\mathrm{Spc}) \arrow[l, shift left, "\Omega"] \arrow[d, "\text{Underlying space}"] \\ \mathrm{Mod}_{\mathbb{E}_1}(\mathrm{Spc}) \arrow[r, shift left, "B"] & \mathrm{Spc}_* \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

So again $\mathrm{Mod}_{\mathbb{E}_\infty}(\mathrm{Spc}) \supseteq \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$ has left adjoint $M \longmapsto M^{\infty\text{-grp}} := \Omega B(M)$

__Rmk__: Functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc}) \longrightarrow \mathrm{Sp} :=$ ∞-cat of spectra
$:= \lim\left(\mathrm{Spc}_* \xleftarrow{\Omega} \mathrm{Spc}_* \xleftarrow{\Omega} \cdots\right)$

$$M \longmapsto (M, BM, B^2M, \ldots)$$

This induces $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc}) \xrightarrow{\sim} \mathrm{Sp}_{\mathrm{conn}}$.

So we tend to call objects here spectra.

(3v) from ∞-cats to spaces:

__Prop__ (Joyal): For an ∞-cat $\mathcal{C}$, TFAE:
- all arrows in $\mathcal{C}$ are equivs
- $\mathrm{Ho}(\mathcal{C})$ is a 1-groupoid (ie) all arrows isoms)
- $\mathcal{C}$ is a Kan complex

In this case we call $\mathcal{C}$ an __∞-groupoid__, and so have equiv. of ∞-cats

$$\infty\text{-Grpds} \simeq \mathrm{Spc}.$$

Moreover, inclusion $\mathrm{Cat}_\infty \supseteq \infty\text{-Grps}$ has right adjt

$$\mathcal{C} \longmapsto \mathcal{C}^{\simeq} := \text{max}^{l} \text{ Kan sub simp. set of } \mathcal{C}$$

called <u>core</u> of $\mathcal{C}$

<u>Conclusion</u>: Have core functor $\mathrm{Cat}_\infty \longrightarrow \mathrm{Spc}$

$$\mathcal{C} \longmapsto \mathcal{C}^{\simeq}$$

[End of ∞-cat prelims]

We finally reach the goal of this section, by "deriving K_0":

<u>Def^n</u>: The <u>K-theory spectrum</u> of a ring R is

$$K(R) := \left(N(\mathrm{Proj}\,R)^{\simeq}\right)^{\infty\text{-grp}} \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$$

Lets parse this:

$R \rightsquigarrow$ sym. monoidal cat $\mathrm{Proj}\,R$ $\overset{\text{nerve}}{\rightsquigarrow}$ sym monoidal ∞-cat $N(\mathrm{Proj}\,R)$ ie) $\mathbb{E}_\infty$-monoid in Cat_∞ $\overset{\text{core}}{\rightsquigarrow}$ $N(\mathrm{Proj}\,R)^{\simeq} \in \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})$

$\Big\downarrow$ ∞-group complete as in (3iv)

$K(R) \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$

<u>Rel^n to K_0 of §1:</u>

As noted in (3iii), have

$$\pi_0 K(R) = \text{grp compl. of } \underbrace{\pi_0\left(N(\mathrm{Proj}\,R)^{\simeq}\right)}_{\text{isom classes of objs of Proj } R}$$

$$= K_0(R)$$

<u>Rel^n to K-thy space of §2 (sketch)</u>

Let Free R be the subcat $\{0, R, R^2, R^3, \ldots\} \subseteq \mathrm{Proj}\,R$.

So

$$N(\mathrm{Proj}\,R)^{\simeq} \supseteq N(\mathrm{Free}\,R)^{\simeq}$$

$$\| $$

$$\coprod_{n \geq 0} BGL_n(R) =: G \in \mathrm{Mod}_{\mathbb{E}_\infty}(\mathrm{Spc})$$

Have $\pi_0(G) = \mathbb{N}$ as monoids, let $s \in \pi_0(G)$ be generator. The action $s \curvearrowright G$ is induced by

$$\underset{\substack{\text{as in}\\ \S 1}}{GL_n(R) \hookrightarrow GL_{n+1}(R)}$$

($s \leftrightarrow BGL_1(R)$)

Since the image of s in $\pi_0 K(R)$ is invertible we see that the map of $\mathbb{E}_\infty$-monoids $G \to K(R)$ factors through a map

$$\underbrace{\mathrm{colim}\left(G \xrightarrow{s} G \xrightarrow{s} G \to \cdots\right)}_{\| \atop BGL(R)} \longrightarrow K(R)$$

Next, since $\pi_1 K(R)$ is abelian (true for π_1 of any $\mathbb{E}$ monoid), this induces a map of spaces

$$BGL(R)^+ \longrightarrow K(R)$$

This fits into an equiv of spaces

Thm (Quillen, McDuff–Segal): $K_0(R) \times BGL(R)^+ \xrightarrow{\sim} K(R)$.

Not natural in R. What is natural is $BGL(R)^+ \xrightarrow{\sim}$ connected comp. of $1 \in K(R)$

§4 K_0 of stable ∞-categories

∞-cat preliminaries:

(4i) An ∞-cat $\mathcal{C}$ is stable if it has all finite limits + colimits, and pushout and pullback squares coincide

$\Rightarrow$ $\mathcal{C}$ has zero object $0 \in \mathcal{C}$ and there are inverse equivs

$$X \longmapsto \mathrm{cofib}(X \to 0) = \mathrm{pushout}\begin{pmatrix} X \to 0 \\ \downarrow \\ 0 \end{pmatrix}$$

$$\mathcal{C} \underset{\Omega}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathcal{C}$$

$$\mathrm{pullback}\begin{pmatrix} & 0 \\ & \downarrow \\ 0 \to & Y \end{pmatrix} = \mathrm{fib}(0 \to Y) \longleftarrow\!\shortmid\ Y$$

Eg) R ring $\rightsquigarrow$ Perf R

A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-cats is exact if it preserves finite limits and colimits.

Let $\mathrm{Cat}_\infty^{ex} :=$ ∞-cat of stable ∞-cats and exact functors between them

$\underline{\text{Def}^n}$: $K_0(\mathcal{C}) := \underbrace{\pi_0(\mathcal{C}^{\simeq})}$ / equiv rel^n gen by $Y = X \oplus Z$ whenever $\exists$ fib seq $X \to Y \to Z$.

- objs of $\mathcal{C}$ up to equiv.
- monoid under $\oplus$

Note: $K_0(\mathcal{C})$ is an abelian group: inverse of $X \in \mathcal{C}$ is ΣX since $X \to 0 \to \Sigma X$ implies $0 = X + \Sigma X$ in $K_0(\mathcal{C})$

Rel^n to K_0 of §1:
For a ring R, it is easy to check that

$$\left\{ \begin{matrix} \text{fin proj} \\ R\text{-mods} \end{matrix} \right\} \longrightarrow K_0(\mathrm{Perf}\, R)$$
$$P \longmapsto P$$

induces
$$K_0(R) \xrightarrow{\cong} K_0(\mathrm{Perf}\, R)$$
(inverse is induced by:
$$0 \to P_a \to \cdots \to P_b \to 0 \longmapsto \sum_{i=a}^{b} (-1)^i P_i \quad)$$

§5 K-thy of stable ∞-cats: Waldhausen's $S_\bullet$ construction

Two cat. prelims:

- $[n] := $ 1-cat $\{0 < 1 < \cdots < n\}$
- $Ar[n] := $ arrow cat. of $[n]$
  $= \{$ objs (j, i) for $0 \le j \le i \le n$

$\left\{ \text{unique morphism } (i,j) \to (i',j') \text{ iff } i \le i' \text{ and } j \le j' \right.$

Waldhausen's construction:

$\mathcal{C}$ a stable ∞-cat. Let

$$S_n(\mathcal{C}) \subseteq \mathrm{Fun}(Ar[n], \mathcal{C})$$

be full subcat of functors F s.t.

(i) $F(i,i) \simeq 0$ for $i = 0, \dots, n$

(ii) For all $0 \le i \le j < n$, the square

$$\begin{tikzcd}
F(i,j) \arrow[r] \arrow[d] & F(i,j+1) \arrow[d] \\
F(i+1,j) \arrow[r] & F(i+1,j+1)
\end{tikzcd}$$

is a pushout

ie) F is following data:

$$\begin{tikzcd}
\underset{\simeq 0}{F(0,0)} \arrow[r] & F(0,1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(0,2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdots \arrow[r] & F(0,n) \arrow[d] \\
 & F(1,1) \arrow[r] & F(1,2) \arrow[r] & \cdots & F(1,n) \arrow[d] \\
 & & & \arrow[d] & \vdots \arrow[d] \\
 & & & \arrow[r] & \underset{=0}{F(n,n)}
\end{tikzcd}$$

: Easy to see that $n \mapsto S_n(\mathcal{C})$ defines a simplicial ∞-cat, and so get simplicial space by taking cores:

$$S_\bullet(\mathcal{C})^{\simeq} : n \mapsto S_n(\mathcal{C})^{\simeq} \in \mathrm{Spc}$$

<u>**Def^n**</u>: The <u>K-thy</u> of $\mathcal{C}$ is

$$K(\mathcal{C}) := \Omega \, | S_\bullet(\mathcal{C})^{\simeq} |$$

We unpack where $K(\mathcal{C})$ lives:

– The functors

$$\text{"}\mathrm{Cat}_\infty^{ex} \longrightarrow \mathrm{Spc}$$
$$\mathcal{C} \longmapsto S_n(\mathcal{C})^{\simeq}, \; |S_\bullet(\mathcal{C})^{\simeq}|$$

naturally upgrade, using $\oplus$ in $\mathcal{C}$, to functors

$$\mathrm{Cat}_\infty^{ex} \longrightarrow \mathrm{Mod}_{\mathbb{E}_\infty}(\mathrm{Spc})$$

which we denote in same way.

To apply this argument need to know that $|S_\bullet(-)^{\simeq}|$ and $S_n(-)$ preserve products; use $S_n(\mathcal{C}) \simeq \mathrm{Fun}([n-1], \mathcal{C})$ to prove this.

(Formal proof: semiadditivity of Cat_∞^{ex} implies $\mathrm{Mod}_{\mathbb{E}_\infty}(\mathrm{Cat}_\infty^{ex}) \xrightarrow[\text{underlying}]{\sim} \mathrm{Cat}_\infty^{ex}$.)

Now apply Ω as in (3iv) to get that

$$K(\mathcal{C}) = \Omega \, | S_\bullet(\mathcal{C})^{\simeq} | \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$$

ie) we have defined the K-thy spectrum $K(\mathcal{C})$!

<u>Comparison with K_0 of §4</u> : We first build a natural map of spectra $\mathcal{C}^{\simeq} \to K(\mathcal{C})$.

For any simplicial space $X_\bullet$ there is a pushout in Spc:

$$\begin{array}{ccc} X_1 \sqcup X_1 & \longrightarrow & |sk_0(X_\bullet)| \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & |sk_1(X_\bullet)| \end{array}$$

In case of $X_\bullet = S_\bullet(\mathcal{C}^{\simeq})$ we have $X_0 = \{\text{zero objs of } \mathcal{C}\} \simeq *$ and $X_1 \simeq \mathcal{C}^{\simeq}$, so get pushout

$$\begin{array}{ccc} \mathcal{C}^{\simeq} \sqcup \mathcal{C}^{\simeq} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{C}^{\simeq} & \longrightarrow & |sk_1(S_\bullet(\mathcal{C})^{\simeq})| \end{array}$$

Take fibs of vertical arrows to get maps of $\mathbb{E}_\infty$-monoids

$$\mathcal{C}^{\simeq} \longrightarrow \Omega|sk_1| \longrightarrow K(\mathcal{C}).$$

<u>Lemma</u> (by direct calc.): For any stable ∞-cat $\mathcal{C}$, the induced map $\underbrace{\pi_0(\mathcal{C}^{\simeq})}_{=\text{ Isom classes of objs of } \mathcal{C}} \longrightarrow \pi_0 K(\mathcal{C})$ descends to an isom $K_0(\mathcal{C}) \xrightarrow{\simeq} \pi_0 K(\mathcal{C})$

<u>Comparison to K(R) from §3</u>

R rings ⇝ stable ∞-cat $\mathrm{Perf}\,R$, and map of ∞-cats $N(\mathrm{Proj}\,R) \to \mathrm{Perf}\,R$

$$\rightsquigarrow N(\mathrm{Proj}\,R)^{\simeq} \to (\mathrm{Perf}\,R)^{\simeq} \xrightarrow[\text{above}]{\text{as}} K(\mathrm{Perf}\,R)$$

maps of $\mathbb{E}_\infty$-monoids in Spc

$\overset{\infty\text{-grp}}{\underset{\text{compl.}}{\rightsquigarrow}}$ maps of spectra

$$\begin{array}{c} \left(N(\mathrm{Proj}\,R)^{\simeq}\right)^{\infty\text{-grp}} \to (\mathrm{Perf}\,R)^{\simeq,\infty\text{-grp}} \to K(\mathrm{Perf}\,R) \\ \| \\ K(R) \text{ in } §3 \end{array}$$

<u>Thm</u> (Gillet–Waldhausen resolution): The composition $K(R) \to K(\mathrm{Perf}\,R)$ is an equiv. of spectra.

§6 A universal characterisation of $\mathcal{C} \mapsto K(\mathcal{C})$

Def^{n}s

- A stable ∞-cat $\mathcal{C}$ is idempotent complete if $\mathrm{Ho}(\mathcal{C})$ is idem. complete in 1-categorical sense.
- Let $\mathrm{Cat}_\infty^{perf}$:= idem. complete stable ∞-cats:

$$\mathrm{Cat}_\infty^{ex} \supseteq \mathrm{Cat}_\infty^{perf}$$

this is sym monoidal under Lurie's $\otimes$. Unit is Sp^{ω} := compact objs in Sp

left adjoint called idempotent completion $\mathcal{C} \longmapsto \mathrm{Idem}(\mathcal{C})$ in $\mathrm{Cat}_\infty^{perf}$

- A sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ is split exact if
  - it is both a fibre & cofibre seq in $\mathrm{Cat}_\infty^{perf}$
  - and f, g admit right adjoints
    $$\mathcal{C} \xleftarrow{i} \mathcal{D} \xleftarrow{j} \mathcal{E}$$
  - s.t $i \circ f \simeq id_{\mathcal{C}}$ and $g \circ j \simeq id_{\mathcal{E}}$

- A functor $F : \mathrm{Cat}_\infty^{perf} \to Sp$ is additive if
  - for every split exact seq, the map
    $$f \oplus j : F(\mathcal{C}) \oplus F(\mathcal{E}) \to F(\mathcal{D})$$
    is an equiv of spectra.
  - F commutes with filtered colimits
  - for $\mathcal{C}$ any stable ∞-cat, $F(\mathcal{C}) \xrightarrow{\sim} F(\mathrm{Idem}\,\mathcal{C})$

Non-trivial eg) $K : \mathrm{Cat}_\infty^{ex} \to Sp$, as defined in §5, is additive.

<u>Theorem</u> (Blumberg–Gepner–Tabuada): The functor

$$\mathrm{Fun}^{add}(\mathrm{Cat}_\infty^{ex}, Sp) \longrightarrow Sp$$
$$E \longmapsto E(Sp^\omega)$$

is corepresented by $K \in \mathrm{Fun}^{add}(\mathrm{Cat}_\infty^{ex}, Sp)$

ie) Have natural equiv of spectra

$$E(Sp^\omega) \simeq \mathrm{Map}_{\mathrm{Fun}^{add}}(K, E)$$

In fact, BGT prove more. There is an equiv in Fun^{add}:

$$K(-) \simeq \mathrm{Maps}_{\mathrm{Fun}^{add}}(\mathcal{U}(Sp^\omega), \mathcal{U}(-))$$

as functors $\mathrm{Cat}_\infty^{ex} \to Sp$ (where $\mathcal{U}: \mathrm{Cat}_\infty^{ex,op} \to \mathrm{Fun}^{add}(\mathrm{Cat}_\infty^{ex}, Sp)$ is Yoneda.)