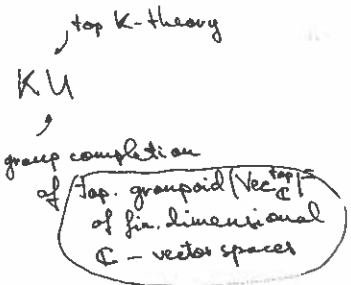


0. Swiath theorem



$BS^1 \simeq B\mathbb{C}^{\times} \rightarrow |\text{Vect}_{\mathbb{C}}^{\text{top}}|$ (eq. $\beta \in K\mathbb{Z}$ induces a map $B^2\mathbb{Z} \rightarrow KU$)

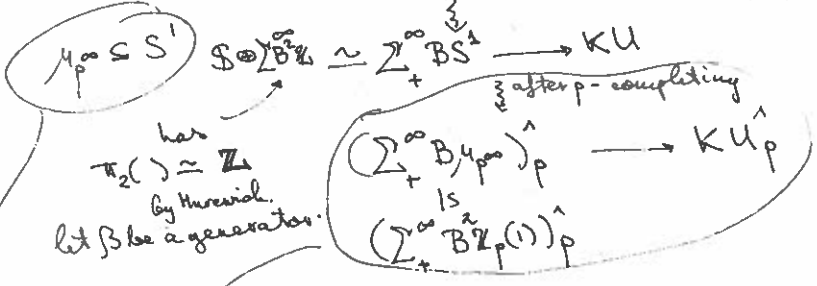


image of β is the Bott element.

Then (Swiath)

$(\sum_{+}^{\infty} BS^1)[\beta^{-1}] \xrightarrow{\sim} KU$ is an isomorphism.

Want: $\mathcal{L}_{K(1)} K^{\text{cont}} \simeq KU_{\hat{p}}$

as a sheaf on suff. nice adic spaces over $\text{Spa}(\mathbb{Z}_p^{\wedge}, \mathbb{Z}_p^{\wedge})$

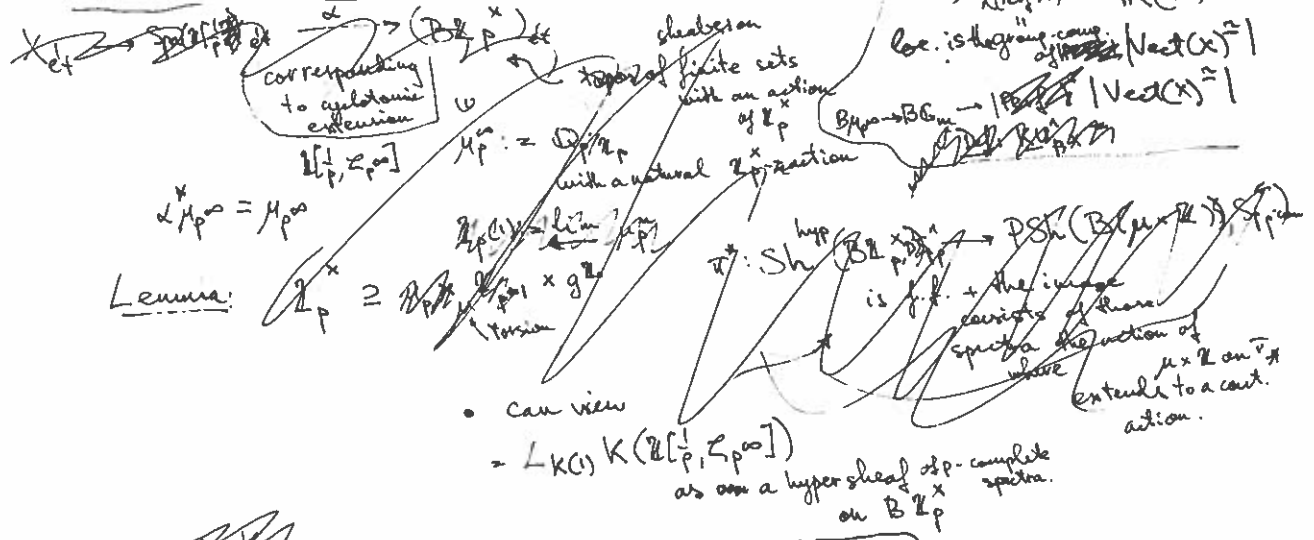
However: the above (last time)

~~$K^{\text{cont}}(\mathbb{C})_{\hat{p}} \simeq KU_{\hat{p}}$~~

But $\pi_2(K^{\text{cont}}(\mathbb{C})_{\hat{p}}) \simeq \mathbb{Z} \simeq \mathbb{Z}_p(1)$ \Rightarrow Need to modify r.h.s.

$\text{Aut}_{\mathbb{C}}(\mathbb{C})$ act non-trivially on

1. Let X be a (nice) adic space over \mathbb{Z}_p^{\wedge} .



$\sum_{+}^{\infty} B M_p \rightarrow \mathcal{L}_{K(1)} K(\mathbb{Z}_p^{\wedge}[\zeta_p^{\infty}]) \rightarrow \mathcal{L}_{K(1)} K(\mathbb{Z}_p^{\wedge})$

can view as a hyper sheaf on $B\mathbb{Z}_p^{\times}$

get a map $L_{K(i)}(\sum_+^{\infty} B\mu_p^{\infty}) \rightarrow L_{K(i)} K^{\text{cont}}$

Thm (Andreychev)

This map is an equivalence.

PP: will check on p -completed stalks.

$$i_x: x = \text{Spa}(K(x), \overline{K(x)^+}) \rightarrow X \text{ be a geometric point.}$$

$$\text{colim}_{x \in U} L_{K(i)} K_{\text{cont}}^{\wedge}(\mathcal{O}_x(\mathcal{U}))_p^{\wedge} \cong L_{K(i)} (\text{colim}_{x \in U} K_{\text{cont}}^{\wedge}(\mathcal{O}_x(\mathcal{U})))_p^{\wedge} \cong *$$

$L_{K(i)}$ doesn't commute with colimits but commutes mod p .

want to replace with usual K -theory.

let ω be a uniformizer

Then $\text{Fib } \mathcal{U} = \text{Spa}(A, A^+)$ and take some ring of def'n $A_0 \subset A$.

$$K(A_0)_p^{\wedge} \xrightarrow{\sim} K(A_0/\omega)_p^{\wedge} \xleftarrow{\sim} \varprojlim_n K(A_0/\omega^n)_p^{\wedge} \xleftarrow{\sim} K^{\text{cont}}(A_0)_p^{\wedge}$$

localizations.

$$\begin{array}{ccccc} \text{Tor}(\omega^{\infty}) & \longrightarrow & D(A_0) & \longrightarrow & D(A_0[\frac{1}{\omega}]) \\ \downarrow S & & \downarrow & & \downarrow \\ \text{Tor}^{\text{Nuc}}(\omega^{\infty}) & \longrightarrow & \text{Nuc}(A_0) & \longrightarrow & \text{Nuc}(A_0[\frac{1}{\omega}]) \\ \uparrow \text{all discrete} & & \uparrow & & \downarrow \\ & & K(A_0) & \longrightarrow & K(A_0[\frac{1}{\omega}]) \\ & & \downarrow \Gamma & & \downarrow \\ & & \text{Nuc}(A_0) & \longrightarrow & \text{Nuc}(A_0[\frac{1}{\omega}]) \end{array}$$

$$\Rightarrow K^{\text{cont}}(A)_p^{\wedge} \cong K^{\text{cont}}(A)_p^{\wedge}$$

$$\Rightarrow \cong L_{K(i)} (\text{colim}_{x \in U} K^{\text{cont}}(\mathcal{O}_x(\mathcal{U})))_p^{\wedge} \cong L_{K(i)} (\text{colim}_{x \in U} K(\mathcal{O}_x(\mathcal{U})))_p^{\wedge} \cong L_{K(i)} K(\mathcal{O}_{x,x}^h)$$

/s Gabber's rigidity

there the map is an isomorphism by Snaith's thm + last talk.

here, $\{x_p\} \subseteq \overline{K(x)}$ and

$$L_{K(i)}(\sum_+^{\infty} B\mu_p^{\infty}) \cong (\sum_+^{\infty} B\mu_p^{\infty})_p^{\wedge} [\beta^{-1}]$$

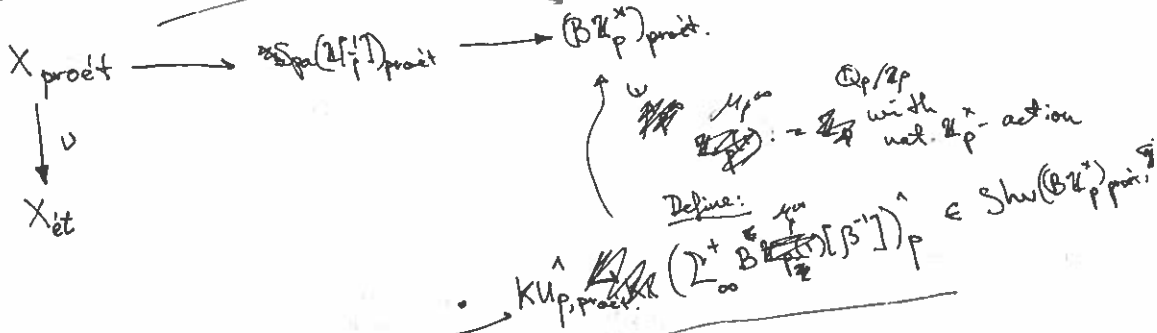
2. $L_{K(1)} K^{cont}(-) [\frac{1}{p}]$.

Prop: there is a natural equivalence of sheaves on X .

$$L_{K(1)} K^{cont}(-) [\frac{1}{p}] \xrightarrow{\sim} \bigoplus_n \mathbb{Q}_p(n) [2n]$$

($\lim_{\leftarrow} \frac{\mathbb{Z}}{p^n} [\frac{1}{p}]$)

Proof (sketch):



KU_p^{\wedge} with an action by Adams oper.

Then $L_{K(1)}(\mathbb{Z}_p^x/B\mathbb{Z}_p^x)_{prodt}^{\wedge} \simeq v_* \pi^* KU_{p,prodt}^{\wedge}$

circle sheaf

$L_{K(1)}(\mathbb{Z}_p^x/B\mathbb{Z}_p^x)_{prodt}^{\wedge} [\frac{1}{p}] \simeq v_*(\pi^* KU_{p,prodt}^{\wedge} [\frac{1}{p}])$

enough to split.

Via solid formalism reduces to the claim that.

$$KU \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{i \in \mathbb{Z}} \mathbb{Q}(i) [2i]$$

eq. w.r.t. to the Adams oper.

(Maybe give more details...)

$$KU \cong \mathbb{S}[B\mathbb{Z}_2][\beta^{-1}]$$

Smith $\circ \mathbb{Q}^{\wedge}$

Def: this is the Chern character!

Rem: the map of sheaves $(\mathbb{Z}_p^x/B\mathbb{Z}_p^x)_{prodt} \rightarrow (\mathbb{Z}_p^x/B\mathbb{Z}_p^x)_{prodt}^{\wedge}$ here is isom. to \mathbb{Z}_2 stem. Let us normalize the isom. so that.

$$L_{K(1)} K^{cont}(-) [\frac{1}{p}] \xrightarrow{\sim} \bigoplus_n \mathbb{Q}_p(n) [2n]$$

Def: Chern character map $ch: K^{cont}(X) \rightarrow \bigoplus_n \mathbb{Q}_p(n) [2n]$ is defined as composition $K^{cont}(X) \xrightarrow{L_{K(1)}} L_{K(1)} K^{cont}(X) [\frac{1}{p}] \xrightarrow{\sim} \bigoplus_n \mathbb{Q}_p(n) [2n]$

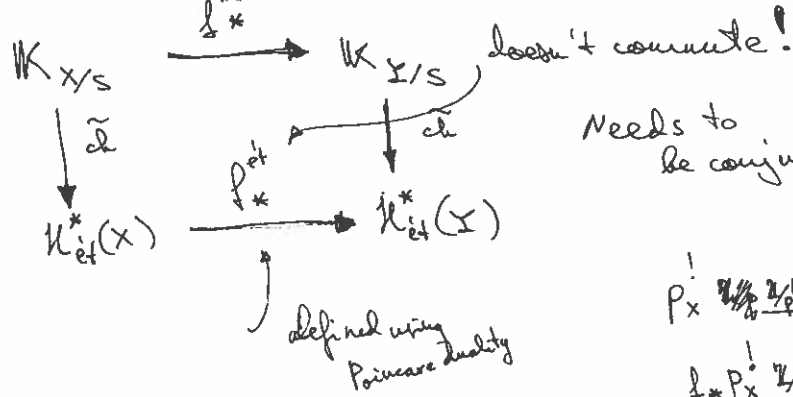
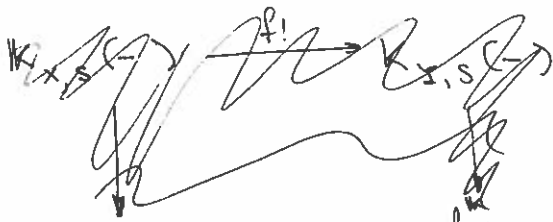
3. Pushforward Riemann-Roch. Let $f: X \rightarrow Y$ be a (smooth proper) map of smooth proper analytic spaces over S .

Notation: 1) $\mathcal{H}_{et}(X, \mathbb{Q}_p) := \bigoplus_n (p_X)_* \mathbb{Q}_p(n) [2n]$

2) $\mathcal{K}_{X/S} = p_{X*} L_{K(1)} K(-)$

Remark: f induced a map $f_*: \mathcal{K}_{X/S} \rightarrow \mathcal{K}_{Y/S}$

f proper $\Rightarrow f_* = f_!$ $\Rightarrow f_*$ has right adjoint $f^!: p$ is l.c.i. $\Rightarrow f^! = f^* \otimes \mathbb{L}_{X/Y}$ perfect complex.



Needs to be conjugated!

$$P_X^! \mathbb{Z}/p \mathbb{Z} \cong \mathbb{Z}/p^k(d_X)[2d_X]$$

$$f_* P_X^! \mathbb{Z}/p \mathbb{Z}(n) \rightarrow P_Y^! \mathbb{Z}/p \mathbb{Z}(n)$$

$$f_* \mathbb{Z}/p \mathbb{Z}(n+d_X)[2n+2d_X] \rightarrow \mathbb{Z}/p \mathbb{Z}(n)$$

$$P_X^! P_X^! \mathbb{Z}/p \mathbb{Z}(n)[2n] \rightarrow P_X^! P_X^! (\mathbb{Z}/p \mathbb{Z}(n)[2n])$$

$$P_X^! (\mathbb{Z}/p \mathbb{Z}(n+d_X)[2n+2d_X]) \rightarrow P_X^! (\mathbb{Z}/p \mathbb{Z}(n)[2n])$$

4. Digression, characteristic classes.

Construction: $X = \text{North. analytic adic space. set. cond. ...}$

$$\mathbb{Z} \subset C_2: \text{Pic}(X) \cong H_{et}^1(X, \mathbb{G}_m) \rightarrow H_{et}^2(X, \mathbb{Z}/p \mathbb{Z}(1))$$

first Chern class.

II. Extends to $\mathcal{V} \in \text{Vect}(X)$ s.t. \mathcal{H} it commutes with pull b

there is unique $\mathcal{H}: \text{Vect}(X) \rightarrow H^*(X, \mathbb{Q}_p)$

Can restrict \mathcal{H} to $|\text{Vect}(X)|^{\text{cl}}$. This gives a Chern character

(1) computing with base change
 (2) for $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$

$$\text{ch}: \text{Vect}(X) \rightarrow H^*(X, \mathbb{Q}_p) \cong \bigoplus_n H^{2n}(X, \mathbb{Q}_p(n))$$

1) functorial wr.t. pull-backs

2) is additive: $\text{ch}(V') + \text{ch}(V'') = \text{ch}(V)$

$$\text{ch}(V \otimes W) = \text{ch}(V) \cdot \text{ch}(W)$$

3)

4) $\text{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})}$ (check in the universal case of $\mathcal{O}(1)$ on \mathbb{P}^1)

For $\mathcal{L} \in \text{Pic}(X)$,

III. There exists a unique association $\mathcal{V} \mapsto \text{Td}(\mathcal{V}) \in H^*(X, \mathbb{Q}_p)$

1) we have =

2) for $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ we have $\text{Td}(V) = \text{Td}(V') \cdot \text{Td}(V'')$

3) for $\mathcal{L} \in \text{Pic}(X)$ we have $\text{Td}(\mathcal{L}) = \frac{c_1(\mathcal{L})}{1 - e^{-c_1(\mathcal{L})}}$

5. Riemann - Roch Thm:

Let X, Y be smooth & proper over S .
 then there exists a commutative square in $\text{Det}(S, \mathbb{Q}_p)$

$$\begin{array}{ccc}
 K_{X/S} & \xrightarrow{f_*} & K_{Y/S} \\
 \downarrow \text{Td}(X) \text{ch}(-) & & \downarrow \text{Td}(Y) \text{ch}(-) \\
 \mathbb{R}K_{\text{ét}}^*(X, \mathbb{Q}_p) & \xrightarrow{f_*} & \mathbb{R}K_{\text{ét}}^*(Y, \mathbb{Q}_p)
 \end{array}$$

Properties: Instance of a general situation:

Def: Let Man be the cat. of sm. proper adic spaces over S .
 Let (\mathcal{E}, \otimes) be a monoidal \mathbb{Z} -category.
 (1) An abstract \mathcal{E} -valued col. theory on Man $H^*: \text{Man} \rightarrow \mathcal{E}$

- there is
- (1) pull-backs are functorial.
 - (2) there is self-duality on both sides and $H^*(X)$ splits off $H^*(\mathbb{P}_X^1(\mathcal{E}))$ for any \mathcal{E} .
 - (3) there is base change for transversal fiber products.
 - (3) easy version of the projection formula
 - (4) $H^*(\mathbb{P}_X(L \otimes \mathbb{1})) \xrightarrow{(\pi_X, \text{co})^*} H^*(X) \otimes H^*(X)$ is an iso.

Then Given a transformation
 $\alpha: H^* \rightarrow (H^*)'$
 commuting with pull-backs

it commutes with pushforwards if and only if it commutes with Euler classes.

Ex: • for $\mathbb{R}K_{\text{ét}}^*(X)$, $e(L) = \mathcal{O}_X^*(\mathbb{1})$ for $X \hookrightarrow \mathbb{P}_X(L^{\vee} \otimes \mathbb{1})$
 • for $\mathbb{R}K_{\text{ét}}^*(X/S)$, $e(L) = \text{ch}(\mathcal{O}_X) - \text{ch}(\mathcal{L}^{-1}) = 1 - e^{-c_1}$

$$\mathcal{O}(\mathbb{P}_X^1(L^{\vee} \otimes \mathbb{1})) \rightarrow \mathcal{O}_{\mathbb{P}_X^1} \rightarrow \mathcal{O}_X^*(\mathbb{1}) \xrightarrow{(\Delta_X)_*} H^*(X) \otimes H^*(X)$$

(by cup-product with $(\Delta_X)_*$)

$$\mathbb{1}_S \xrightarrow{\quad} H^*(X) \otimes H^*(X) \xrightarrow{H^*(X)^{\vee}} H^*(X)$$

Proof (sketch) 1. $H^*(X)$ is self dual and f_* is dual to f^* .
 So it is enough to show commutation with pushforwards in the case of $\Delta_X: X \rightarrow X \times X$

Can show more generally for closed immersion $Z \hookrightarrow X$.

I. Assume for simplicity $\text{codim} = 1$.

Let $L = \mathcal{O}_X(Z)$

$S: X \rightarrow \mathbb{A}^1 \text{Tot}(L) = \text{non-zero section with zeros given by } Z$

$o: X \rightarrow \mathbb{A}^1 \text{Tot}(L)$

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow i & \nearrow & \downarrow s \\ X & \xrightarrow{o} & \mathbb{P}_X(L^\vee \otimes \mathcal{O}_X) \end{array}$$

We have $S_* = o_*$ (enough to check for $\text{Proj } \pi_*$ or o_*)
obvious

$i_*^{\#}(1) = o_*^{\#}(S) = o_*^{\#}(o) = e(L)$

$d(i_*(1)) = i_*^{\#}(1)$

True more generally for classes that are pull-backs from X

$\mathbb{P}_Z(L^\vee \otimes \mathcal{O}_Z) \rightarrow \mathbb{P}_X(L^\vee \otimes \mathcal{O}_X)$
 $\downarrow i_2 \downarrow i_1 \downarrow i$
 $Z \rightarrow X$

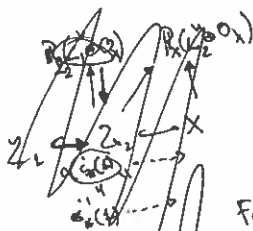
$i_2^*(i_1^*(1))$
is image of i_2^*

II. $o: X \rightarrow \mathbb{P}(E^\vee \otimes \mathcal{O}_X)$

$L(\mathcal{O}_X(1)) = \mathcal{O}_X(1)$

by splitting principle
 $o \subset E_0 \subset E_1 \subset \dots \subset E_n = E$
full flag

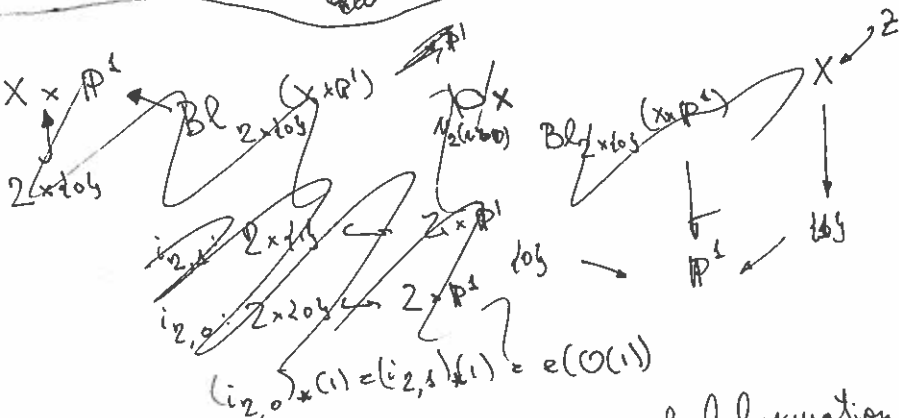
$X = \mathbb{P}_X(\mathcal{O}_X) \subset \mathbb{P}_X(E_1^\vee \otimes \mathcal{O}_X) \subset \mathbb{P}_X(E_2^\vee \otimes \mathcal{O}_X) \subset \dots = \mathbb{P}_X(E^\vee \otimes \mathcal{O}_X)$



For simplicity just take 2.

inductively on dim.

$\mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(E_1^\vee \otimes \mathcal{O}_X) \subset \mathbb{P}(E_2^\vee \otimes \mathcal{O}_X)$
 $\downarrow i_1 \downarrow i_2$
enough to show $(i_2)_*(1)$ comes as pull back image of i_2^* .
zero of a section of $\mathcal{O}(1) \otimes \pi_1^* E_1$



Some version of deformation to normal cone.

