0. Snaith theorem

KU (top K-theory)

group completion of top. groupoid $|\mathrm{Vec}_{\mathbb{C}}^{top}|^{\simeq}$ of fin. dimensional $\mathbb{C}$-vector spaces

$BS^1 \simeq B\mathbb{C}^\times \longrightarrow |\mathrm{Vec}_{\mathbb{C}}^{top}|^{\simeq}$ (eg. $\beta \in KU_2$ induces a map $B^2\mathbb{Z} \to KU$)

image of β is the Bott element.

$\mu_{p^\infty} \subseteq S^1$

$\mathbb{S}\otimes\Sigma^\infty B^2\mathbb{Z} \simeq \Sigma^\infty_+ BS^1 \longrightarrow KU$

has $\pi_2(\ ) \simeq \mathbb{Z}$ by Hurewicz. let β be a generator.

after p-completing

$(\Sigma^\infty_+ B\mu_{p^\infty})^\wedge_p \longrightarrow KU^\wedge_p$

is

$(\Sigma^\infty_+ B^2\mathbb{Z}_p(1))^\wedge_p$

Thm (Snaith)

$(\Sigma^\infty_+ BS^1)[\beta^{-1}] \xrightarrow{\sim} KU$ is an isomorphism.

Want: $L_{K(1)}K^{cont} \simeq KU^\wedge_p$

as a sheaf on suff. nice adic spaces over $\mathrm{Spa}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}[\frac{1}{p}])$

However: ~~the above~~ (last time)

$L_{K(1)}K^{cont}(\mathbb{C})^\wedge_p \simeq KU^\wedge_p$ ⟹ Need to modify r.h.s.

~~But~~ $\pi_2(K^{cont}(\mathbb{C})^\wedge_p) \otimes \mathbb{Z} \simeq \mathbb{Z}_p(1)$.

$\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ act non-trivially on

1. Let X be a (nice) adic space over $\mathbb{Z}[\frac{1}{p}]$.

$X_{et} \to \mathrm{Spa}(\mathbb{Z}[\frac{1}{p}])_{et} \xrightarrow{\alpha} (B\mathbb{Z}_p^\times)_{et}$

corresponding to cyclotomic extension $\mathbb{Z}[\frac{1}{p}, \zeta_{p^\infty}]$

sheaves on ~~topos~~ of finite sets with an action of $\mathbb{Z}_p^\times$

$\mu_{p^\infty} := \mathbb{Q}_p/\mathbb{Z}_p$ with a natural $\mathbb{Z}_p^\times$-action

$\alpha^*\mu_{p^\infty} = \mu_{p^\infty}$

$\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$

Lemma: $\mathbb{Z}_p^\times \simeq \mu_{p-1} \times g\mathbb{Z}$ (torsion)

$\pi^*: \mathrm{Sh}^{hyp}(B\mathbb{Z}_p^\times, D_p^\wedge) \to \mathrm{PSh}(B(\mu \times \mathbb{Z}), \mathrm{Sp}_{p\text{-}c})$ is f.f. + the image consists of those spectra whose the action of $\mu \times \mathbb{Z}$ on π_* extends to a cont. action.

Note that as $K(\mathrm{Perf}\,X) \to \mathbb{K}^{cont}(X)$

loc. is the group compl. of $|\mathrm{Vect}(X)^{\simeq}|$

$B\mu_{p^\infty} \to B\mathbb{G}_m \to |\mathrm{Vect}(X)^{\simeq}|$

- can view $L_{K(1)}K(\mathbb{Z}[\frac{1}{p}, \zeta_{p^\infty}])$ as a hypersheaf of p-complete spectra on $B\mathbb{Z}_p^\times$

- $\Sigma^\infty_+ B\mu_{p^\infty} \longrightarrow L_{K(1)}K(\mathbb{Z}[\frac{1}{p}, \zeta_{p^\infty}])$ ... $L_{K(1)}K(\ )$ (as sheaf on $\mathbb{Z}[\frac{1}{p}]$)

can view as a ... of hypersheaves on $B\mathbb{Z}_p^\times$

get a map $L_{K(1)}(\Sigma^\infty_+ B\mu_{p^\infty})^{*}_{\text{ét}} \longrightarrow L_{K(1)} K^{\text{cont}}$

Thm (Andreychev)

This map is an equivalence.

Pf: will check on p-completed stalks.

$c_x : x = \mathrm{Spa}(\overline{k(x)}, \overline{k(x)^+}) \longrightarrow X$ be a geometric point.

$\operatorname{colim}_{x \in U \text{ affinoid nbhd}} L_{K(1)} K^{\dagger}_{\text{cont}}(\mathcal{O}_X(U))^\wedge_p \simeq L_{K(1)}(\operatorname{colim}_{x\in U} K^{\dagger}_{\text{cont}}(\mathcal{O}_X(U))) \overset{*}{\simeq}$

$L_{K(1)}$ doesn't commute with colimits but commutes mod p.

want to replace with usual K-theory.

~~Let ϖ be a uniformizer~~

Fix $U = \mathrm{Spa}(A, A^+)$ and take some sub ring of def'n $A_0 \subset A$.

Then $K(A_0)^\wedge_p \underset{\text{Gabber's rigidity}}{\simeq} K(A_0/\varpi)^\wedge_p \xleftarrow[\sim]{\text{Gabber's rig}} \lim_n K(A_0/\varpi^n)^\wedge_p \xleftarrow{\text{Efimov}} K^{\text{cont}}(A_0)^\wedge_p$

localizations.

$$\begin{array}{ccccc}
\mathrm{Tor}(\varpi^\infty) & \longrightarrow & D(A_0) & \longrightarrow & D(A_0[\tfrac{1}{\varpi}]) \\
\downarrow \wr & & \downarrow & & \downarrow \\
\mathrm{Tor}^{\mathrm{Nuc}}(\varpi^\infty) & \longrightarrow & \mathrm{Nuc}(A_0) & \longrightarrow & \mathrm{Nuc}(A_0[\tfrac{1}{\varpi}])
\end{array}$$

all discrete

$\Rightarrow$

$$\begin{array}{ccc}
K(A_0) & \longrightarrow & K(A_0[\tfrac{1}{\varpi}]) \\
\downarrow & \ulcorner & \downarrow \\
\mathrm{Nuc}(A_0) & \longrightarrow & \mathrm{Nuc}(A_0[\tfrac{1}{\varpi}])
\end{array}$$

$\Rightarrow K^{\text{cont}}(A)^\wedge_p \simeq K(A)^\wedge_p$.

$\underset{(*)}{\Rightarrow} \simeq L_{K(1)}(\operatorname{colim}_{x\in U} K(\mathcal{O}_X(U))) \simeq L_{K(1)} K(\operatorname{colim}_{x\in U} \mathcal{O}_X(U)) \simeq L_{K(1)} K(\mathcal{O}^h_{X,x})$

$\wr$ Gabber's rigidity

there the map is an isomorphism by Snaith's thm + last talk.

$L_{K(1)}(\overline{k(x)})$

here, $\{\zeta_{p^\infty}\} \subseteq \overline{k(x)}$ and

$L_{K(1)}(\Sigma^\infty_+ B\mu_{p^\infty}) \simeq (\Sigma^\infty_+ B\mu_{p^\infty})^\wedge_p[\beta^{-1}]$.

2. $L_{K(1)} K^{cont}(-)[\frac{1}{p}]$.

<u>Prop:</u> There is a natural equivalence of étale sheaves on X.

$$L_{K(1)} K^{cont}(-)[\tfrac{1}{p}] \overset{\widetilde{ch}}{\simeq} \bigoplus_n \underbrace{\mathbb{Q}_p(n)}_{(\varprojlim \mathbb{Z}/p^n)[\frac{1}{p}]}[2n]$$

<u>Proof (sketch):</u>

$$\begin{tikzcd} X_{proét} \arrow[r] \arrow[d, "\nu"] \arrow[rr, bend left, "\pi"] & \mathrm{Spa}(\mathbb{Z}[\frac{1}{p}])_{proét} \arrow[r] & (B\mathbb{Z}_p^\times)_{proét} \\ X_{ét} & & \end{tikzcd}$$

$\mu_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$ with the nat. $\mathbb{Z}_p^\times$-action

Define:

- $\hat{KU}_{p,proét} := (\Sigma^+_\infty B\mu_{p^\infty}[\beta^{-1}])^\wedge_p \in \mathrm{Shv}(B\mathbb{Z}_p^\times)_{proét}$

$\hat{KU}_p$ with an action by Adams oper.

~~$\hat{KU}_{p,proét}$~~

Then • $\underbrace{L_{K(1)}(\Sigma^+ B\mu_{p^\infty})}_{\text{étale sheaf}} \simeq \nu_* \pi^* \hat{KU}_{p,proét}$

• $L_{K(1)}(\Sigma^+ B\mu_{p^\infty})[\frac{1}{p}] \simeq \nu_*(\pi^* \hat{KU}_{p,proét}[\frac{1}{p}])$

enough to split.

Via solid formalism reduces to the claim that

$$KU \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}(n)[2n]$$

eq. w.r.t. to the Adams oper. (Maybe give more details...)

$$KU \simeq \mathbb{S}[B^2\mathbb{Z}][\beta^{-1}]$$

Smith $\otimes \mathbb{Q}$

<u>Def:</u> this is the chern character!

<u>Rem:</u> ~~the map of sheaves~~ Let us normalize the isom. so that

$(\Sigma^\infty_+ B^{\mathbb{Z}_p(1)}_{\mu_{p^\infty}}[\frac{1}{p}]) \longrightarrow (\mathbb{Z}_p^\times)_{proét}$ ~~$L_{K(1)}K^{cont}$~~ $\bigoplus_n \mathbb{Q}_p(n)[2n]$

π_2 here is isom. to ~~Bott elem~~

~~Def: Chern character map $ch: K^{cont}(X) \to \bigoplus_n \mathbb{Q}_p(n)(X)[2n]$ is defined as composition $K^{cont}(X) \to L_{K(1)}K^{cont}(X)[\frac{1}{p}] \simeq \bigoplus_n \mathbb{Q}_p(n)$~~

3. <u>Pushforwards</u> ~~Riemann-Roch~~. Let $f: X \to Y$ be a ~~(smooth proper)~~ map of smooth proper analytic spaces over S

Notation: 1) $\mathcal{H}_{ét}(X, \mathbb{Q}_p) := \bigoplus_n (p_X)_* \mathbb{Q}_p(n)[2n]$

2) $\mathbb{K}_{X/S} = p_{X*} L_{K(1)} K(-)$

<u>Remark:</u> f induces a map $f_*: \mathbb{K}_{X/S} \longrightarrow \mathbb{K}_{Y/S}$

f proper $\Rightarrow f_* = f_!$ $\Rightarrow$ f_* has right adjoint: f is lci $\Rightarrow$ $f^! = f^* \otimes \omega_{X/S}$ perfect complex.

$$\begin{array}{ccc} \mathbb{K}_{X/S} & \xrightarrow{f_*} & \mathbb{K}_{Y/S} \\ \downarrow{\scriptstyle \widetilde{ch}} & & \downarrow{\scriptstyle ch} \\ H^*_{et}(X) & \xrightarrow{f_*^{et}} & H^*_{et}(Y) \end{array}$$

doesn't commute!

Needs to be conjugated!

defined using Poincare duality

$p_X^! \, \mathbb{Z}/p^k \simeq \mathbb{Z}/p^k(d_X)[2d_X]$

$f_* p_X^! \, \mathbb{Z}/p^k(n) \to p_Y^! \, \mathbb{Z}/p^k(n)$

$\} (p_Y)_*$

$p_{X*} p_X^! \, \mathbb{Z}/p^k \mathbb{Z}(n)[2n] \longrightarrow p_{Y*} p_Y^! (\mathbb{Z}/p^k(n)[2n])$

$\downarrow{\scriptstyle S} \qquad \downarrow{\scriptstyle S}$

$p_{X*} (\mathbb{Z}/p^k(n+d_X)[2n+2d_X] \to p_{Y*} (\mathbb{Z}/p^k(n)[2\ldots$

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4. ~~Digression~~, characteristic classes.

X = Noeth. analytic adic space. sat. cond...

Construction:

I. $c_1 : \mathrm{Pic}(X) \simeq H^1_{et}(X, \mathbb{G}_m) \longrightarrow H^2_{et}(X, \mathbb{Z}_p(1))$

first Chern class.

II.

$V \longmapsto H^*(X, \mathbb{Q}_p)$, $V \in \mathrm{Vect}(X)$

(2) for $0 \to V' \to V \to V'' \to 0$

Can restrict $\widetilde{ch}$ to $|\mathrm{Vect}(X)|^{\simeq}$.

This gives a Chern character

$$ch : \mathrm{Vect}(X) \longrightarrow H^*(X, \mathbb{Q}_p) \simeq \bigoplus_n H^{2n}(X, \mathbb{Q}_p(n))$$

1) functorial w.r.t. pull-backs

2) is additive: $0 \to V' \to V \to V'' \to 0$

$ch(V') + ch(V'') = ch(V)$

3) $\otimes \longmapsto$ mult. $ch(V \otimes W) = ch(V) \cdot ch(W)$

4) $ch(\mathcal{L}) = e^{c_1(\mathcal{L})}$ (check in the universal case of $\mathcal{O}(1)$ on $B\mathbb{G}_m$)

For $\mathcal{L} \in \mathrm{Pic}(X)$;

III. There exists a unique association $V \longmapsto Td(V) \in H^*(X, \mathbb{Q}_p)$, $V \in \mathrm{Vect}(X)$

1) as above

2) for $0 \to V' \to V \to V'' \to 0$ we have $Td(V) = Td(V') \cdot Td(V'')$

3) for $\mathcal{L} \in \mathrm{Pic}(X)$ we have $Td(\mathcal{L}) = \dfrac{c_1(\mathcal{L})}{1 - e^{-c_1(\mathcal{L})}}$

5. Riemann-Roch thm.

Let X, Y be smooth & proper over S.
Then there exists a commutative square in $D_{ét}(S, \mathbb{Q}_p)$

$$\begin{tikzcd} \mathbb{K}_{X/S} \arrow[r, "f_*"] \arrow[d, "Td(T_X)ch(-)"'] & \mathbb{K}_{Y/S} \arrow[d, "Td(T_Y)ch(-)"] \\ \mathcal{H}^*_{et}(X, \mathbb{Q}_p) \arrow[r, "f_*"] & \mathcal{H}^*_{et}(Y, \mathbb{Q}_p) \end{tikzcd}$$

~~Proposition~~: Instance of a general situation:

Let Man be the cat. of sm. proper ~~and~~ adic spaces over S.

~~Def: Let (C, ⊗) be a monoidal 1-category~~

~~An abstract C-valued coh. theory on Man~~ $H^*: \mathrm{Man} \xrightarrow{mon.} \mathcal{C}$

~~there is~~
1. pull-backs are functorial ~~self duality on both sides~~ and $H^*(X)$ splits off $H^*(\mathbb{P}_X(\mathcal{E}))$ for any $\mathcal{E}$
2. ~~there is~~
3. there is base change for transversal fiber products.
4. easy version of the projection formula
5. $H^*(\mathbb{P}_X(L \oplus 1)) \xrightarrow{(\pi_*, \infty^*)} H^*(X) \otimes H^*(X)$ is an iso.

Then Given a transformation
$\alpha : H^* \longrightarrow (H^*)'$
commuting with pull-backs
it commutes with pushforwards if and only if it commutes with Euler classes.

$e(L) = 0^* 0_*(1)$ for $X \hookrightarrow \mathbb{P}_X(L^\vee \oplus \mathcal{O})$

Ex: • for $\mathcal{H}^*_{et}(X)$, $e(L) = c_1(\mathcal{L})$ (viewed as a map $\mathbb{Q}_p \to p_{X*}\mathcal{H}^*_{et}(1, \mathbb{Q}_p)$)

• for $\mathbb{K}_{X/S}$, $e(L) = ch(\mathcal{O}_X) - ch(\mathcal{L}^{-1}) = 1 - e^{-c_1}$.

$\mathcal{O}(\mathbb{P}_X(L^\vee \oplus \mathcal{O}))(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_X(L^\vee \oplus \mathcal{O})} \longrightarrow \mathcal{O}_*(\mathcal{O}_X)$

$1_S \longleftarrow H^*(X) \otimes H^*(X)$

$H^*(X)^\vee \xrightarrow{\sim} H^*(X)$

Proof (sketch) 1. $H^*(X)$ is self dual (by cup-product with $(\Delta_X)_*$) and f_* is dual to f^*,
So it is enough to show commutation with pushforwards in the case of $\Delta_X : X \longrightarrow X \times X$

Can show more generally for closed immersion $Z \hookrightarrow X$.

I. Assume for simplicity codim $= 1$.

Let $L = \mathcal{O}_X(Z)$

$s: X \to \mathrm{Tot}(L)$ = non-zero section with zeros given by Z

$o: X \to \mathrm{Tot}(L)$

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow i & \ulcorner & \downarrow s \\ X & \xrightarrow{o} & \mathbb{P}_X(L^\vee \oplus \mathcal{O}_X) \end{array}$$

We have $s_* = o_*$ (enough to check for ~~proj~~ π_* or o^*) obvious

$i_*^*(1) = o^* s_*(1) = o^* o_*(1) = e(L)$

$\Downarrow$

$\alpha(i_*(1)) = i_*^!(1)$

True more generally for classes that are pull-backs from X

$$\begin{array}{ccc} \mathbb{P}_Z(L^\vee|_Z \oplus \mathcal{O}) & \to & \mathbb{P}_X(L^\vee \oplus \mathcal{O}) \\ \uparrow i_Z & & \uparrow i_X \\ Z & \to & X \end{array}$$

$j_*((i_2)^*(i_2)_*(1))$ is in the image of j^*

$j^* i_X^* (i_X)_*(1)$

II. $o: X \hookrightarrow \mathbb{P}_X(E^\vee \oplus \mathcal{O})$

$\alpha(o_*(1)) = o^!(1)$

by splitting principle

$0 \subset E_0 \subset E_1 \subset \dots \subset E_n = E$

full flag

$X = \mathbb{P}_X(\mathcal{O}_X) \subset \mathbb{P}_X(E_1^\vee \oplus \mathcal{O}_X) \subset \mathbb{P}_X(E_2^\vee \oplus \mathcal{O}_X) \subset \dots$
$= \mathbb{P}_X(E^\vee \oplus \mathcal{O}_X)$

inductively on dim.

For simplicity just take 2.

$\mathbb{P}_X(\mathcal{O}_X) \overset{\pi_1}{\underset{i_1}{\subset}} \mathbb{P}_X(E_1^\vee \oplus \mathcal{O}_X) \underset{i_2}{\subset} \mathbb{P}_X(E_2^\vee \oplus \mathcal{O}_X)$

enough to show $(i_1)_*(1)$ comes as ~~pull-back~~ image of i_2^*.

zero of a section of $\mathcal{O}(1) \otimes (\pi_1)^* E_1$

$X \times \mathbb{P}^1 \leftarrow Bl_{Z \times \{0\}}(X \times \mathbb{P}^1)$

$Z \times \{0\}$

$N_Z(\ast \oplus \mathcal{O}) \cup X$ $\quad Bl_{Z \times \{0\}}(X \times \mathbb{P}^1) \quad X \leftarrow Z$

$i_{2,1}: Z \times \{1\} \hookrightarrow Z \times \mathbb{P}^1$

$i_{2,0}: Z \times \{0\} \hookrightarrow Z \times \mathbb{P}^1$

$\{0\} \to \mathbb{P}^1 \leftarrow \{0\}$

$(i_{2,0})_*(1) = (i_{2,1})_*(1) = e(\mathcal{O}(1))$

Some version of deformation to normal cone.