Étale hyperdescent

1

§ Definition and basic properties

Conventions/notations: (1) Site: 1-category together w/ a Grothendieck topo. + assume ∃ final object.

for X qcqs: X_{et}, X_{Nis} étale / Nisnevich site

Et_X : cat of étale qcqs X-schemes

(2) for $\mathcal{C}$ a site: $Sh(\mathcal{C})$, $Sh(\mathcal{C}, Sp)$: ∞-cat of sheaves of anima / spectra on $\mathcal{C}$.

(3) $\mathcal{P}$ = set of prime numbers. A spectrum X is $\mathcal{P}$-local if mult. by $q \notin \mathcal{P}$ is invertible on X.
$Sp_{\mathcal{P}}$ = ∞-cat of $\mathcal{P}$-local spectra.

(4) k field. The $\mathcal{P}$-local Galois coho dim is $\sup_{p \in \mathcal{P}} \underbrace{cd_p(k)}_{\text{p-local Galois coho dim of } k}$

Same for $\mathcal{P}$-local virtual.

Definition: $\mathcal{C}$ site, $\mathcal{F}$ sheaf of anima (resp. spectra) on $\mathcal{C}$.
(1) $\mathcal{F}$ is acyclic if $\pi_n \mathcal{F} = 0$ $\forall n \in \mathbb{N}$ (resp. $n \in \mathbb{Z}$).
(2) $\mathcal{F}$ is hypercomplete (or hypersheaf) if $Hom_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = 0$ $\forall \mathcal{G}$ acyclic.
(3) $\mathcal{F}$ is Postnikov complete if $\mathcal{F} \to \varprojlim_n \tau_{\leq n} \mathcal{F}$ is an iso.
(4) $Sh(\mathcal{C})$ is hypercomplete is $\forall \mathcal{F} \in Sh(\mathcal{C})$ $\mathcal{F}$ hypercomplete.

Properties: (1) The category of hypercomplete sheaves of anima (resp spectra) is stable under limits.
(2) A morphism $\mathcal{F} \to \mathcal{G}$ of hypersheaves is an isomorphism iff $\pi_i \mathcal{F} \to \pi_i \mathcal{G}$ is an iso $\forall i \in \mathbb{N}$ (resp. $i \in \mathbb{Z}$)
(3) Assume $Sh(\mathcal{C}, Set)$ has enough points. ~~Let x ∈ Sh(C, Set) a point~~
A morphism $\mathcal{F} \to \mathcal{G}$ of hypersheaves is an iso iff $\mathcal{F}_x \to \mathcal{G}_x$ iso $\forall x$.
(4) If $\pi_i \mathcal{F} = 0$ for $i \gg 0$ then $\mathcal{F}$ is hypercomplete.
(5) $\mathcal{F}$ Postnikov complete $\Rightarrow$ $\mathcal{F}$ hypercomplete.
not an eq in general.

Cases where (5) is an eq:

Definition. $\mathcal{C}$ site, $\mathcal{F}$ sheaf of connective $\mathcal{P}$-local spectra on $\mathcal{C}$.
(1) A sheaf of abelian groups on $\mathcal{C}$, $H^i(\mathcal{F}, A) := \pi_0 Hom_{Sh(\mathcal{C}, Sp)}(\mathcal{F}, \Sigma^i A)$.
(2) $\mathcal{F}$ has $\underline{\mathcal{P}\text{-local coho dim} \leq d}$ if $H^i(\mathcal{F}, A) = 0$ $i > d$ $\forall A$ sh of $\mathcal{P}$-local abelian gps
(3) $\mathcal{C}$ finitary site. $\mathcal{C}$ has $\underline{\mathcal{P}\text{-local coho dim} \leq d}$ if the $\mathcal{P}$-localization of $\Sigma^\infty_+ h_x$ has, $\forall x \in \mathcal{C}$.

(4) We say that $\mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B},\geq 0})$ has <u>enough objects of $\mathcal{B}$-local coho dim $\leq d$</u> if $\forall\, \mathcal{G} \in \mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B},\geq 0})$ $\exists\, f : \mathcal{H} \to \mathcal{G}$ in $\mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B},\geq 0})$ such that $\pi_0(f)$ epimorphism and $\mathcal{H}$ has $\mathcal{B}$-local coho dim $\leq d$.

<u>Proposition</u> - $\mathcal{C}$ site, $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B}})$. Assume $\mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B}})$ has enough obj. of $\mathcal{B}$-local coho dim $\leq d$. Then:

$$\mathcal{F} \text{ hypercomplete} \Leftrightarrow \mathcal{F} \text{ Postnikov - complete.}$$

<u>Sketch of proof</u> - (1) Enough to prove $\alpha : \mathcal{F} \to \lim_{\leftarrow n} \tau_{\leq n} \mathcal{F}$ is a π_*-iso.

(2) Enough to prove $\pi_0 \alpha$ iso.

(3) Enough to prove $\pi_0(\beta : \lim_{\leftarrow n} \tau_{\leq n} \mathcal{F} \to \tau_{\leq d+1} \mathcal{F})$ iso

(4) Write $\mathcal{F}$ for the fiber of $\lim_{\leftarrow n} \tau_{\leq n} \mathcal{F} \to \tau_{\leq d+1} \mathcal{F}$, $\exists\, f : \mathcal{H} \to \mathcal{F}_{\geq 0}$ in $\mathrm{Sh}(\mathcal{C}, \mathrm{Sp}_{\mathcal{B},\geq 0})$ with $\pi_0(f)$ epi and $\mathcal{H}$ of $\mathcal{B}$-local coho dim $\leq d$.

$\rightsquigarrow [\mathcal{H}, \mathcal{F}] = 0$ and $[\mathcal{H}, \Sigma \mathcal{F}] = 0 \rightsquigarrow \pi_0 \mathcal{F} = 0,\ \pi_{-1} \mathcal{F} = 0 \Rightarrow \pi_0 \beta$ iso.

$\square$

<u>Proposition</u> (Local-global principle) - $\mathcal{C}$ site, $\{X_i\}_{i \in I}$ covering of the final obj of $\mathcal{C}$. $\mathcal{F}$ sheaf of anima or spectra

$\mathcal{F}$ is hypercomplete (resp Postnikov complete) $\Leftrightarrow$ its restriction to $\mathcal{C}_{/X_i}$ is, $\forall i$.

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§ <u>Smashing hypercompletion and nilpotence criterion</u>.

$\mathcal{E}$ = ∞-cat of sheaves of $\mathcal{B}$-local spectra on $\mathcal{C}$.

$\mathcal{E}^h$ = ∞-cat of hypercomplete sheaves of ______

$\mathcal{E}^h \hookrightarrow \mathcal{E}$ admits a left adjoint $\mathcal{E} \to \mathcal{E}^h$ "hypercompletion" $\mathbb{1} \mapsto \mathbb{1}^h$ where $\mathbb{1}$ monoidal unit.

<u>Proposition / definition</u>: Equivalent:

(1) $\mathcal{E}^h \hookrightarrow \mathcal{E}$ is closed under colim and $- \otimes X$ for $X \in \mathcal{E}$.

(2) $\forall X \in \mathcal{E}$ $\mathbb{1}^h \otimes X$ hypercomplete.

(3) $\forall X \in \mathcal{E}$ $\mathbb{1}^h \otimes X$ isomorphic to the hypercompletion of X.

(4) $\mathrm{Mod}_{\mathbb{1}^h}(\mathcal{E}) \to \mathcal{E}$ is fully faithful w/ essential image $\mathcal{E}^h$.
forgetful functor

(5) $\forall X \in \mathcal{E}$ with a module structure over $A \in \mathrm{Alg}(\mathcal{E}^h)$ then $X \in \mathcal{E}^h$

$\rightsquigarrow$ We say that hypercompletion is <u>smashing</u> for $\mathcal{E}$.

Nilpotence criterion.

Definition. $m \geq 0$. (1) A filtered object in Sp $\cdots \to X_{-1} \to X_0 \to X_1 \to \cdots$ is called m-nilpotent (resp. weakly m-nilpotent) when

$$\forall i \quad X_i \to X_{i+m+1} \text{ nullhomotopic (resp. induces } 0 \text{ on } \pi_*).$$

(2) The augmented cosimplicial diagram $X^\bullet \in Fun(\Delta^+, Sp)$ is m-rapidly converging (resp. weakly m-rapidly converging) if the tower $\{cofib(X^{-1} \to Tot_n(X^\bullet))\}_n$ is m-nilpotent (resp. weakly m-nilpotent).

(In part. $X^\bullet$ is a limit diagr.)

Proposition (Nilpotence criterion). $\mathcal{C}$ finitary site of $\mathcal{B}$-local coho dim $\leq d$, $\mathcal{F}$ sheaf of $\mathcal{B}$-local sp.

Eq: (1) $\mathcal{F}$ is hypercomplete ($\Leftrightarrow$ Postnikov complete)

(2) $\forall$ truncated hypercover $\mathcal{Y}_\bullet$ in $\mathcal{C}$ of an object x in $\mathcal{C}$, the augmented cosimplicial spectrum $\mathcal{F}(x) \to \mathcal{F}(\mathcal{Y}_\bullet)$ is d-rapidly converging.

Proposition. Let $(X_i^\bullet)_{i\in I}$ be a filtered system of augmented cosimplicial spectra with I partially ordered set.

Suppose $\exists m \geq 0$ s.t. each $X_i^\bullet$ is weakly m-rapidly converging.

Then $\underset{i\in I}{colim}\, X_i^\bullet$ is m-rapidly cv and $\underset{i\in I}{colim}\, Tot(X_i^\bullet) \xrightarrow{\sim} Tot(\underset{i}{colim}\, X_i^\bullet)$.

Rmk. We can extend Nisnevich sheaves to ind étale objects:

$\mathcal{F}$ Nisnevich sh on $X = Spec(A)$, $\mathcal{F}$ defines a functor on the cat of étale A-alg

$B = colim\, A_i$ ind-étale alg. $\mathcal{F}(B) = colim\, \mathcal{F}(A_i)$

$\hookrightarrow$ defines a Nisnevich sheaf on $Spec B$.

Same if $X = Spa(A, A^+)$ affinoid adic space, Huber pair

$$(B, B^+) \simeq (colim(A_i, A_i^+))^\wedge$$

(not true for étale sheaves). ~~[illegible]~~.

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§ Hypersheaves and homotopy dimension

Definition. $\mathcal{C}$ site, $n \geq -1$.

(1) $\mathcal{F}$ sheaf of anima is n-connective if $\tau_{\leq n-1}\mathcal{F}$ is a final obj. in $Sh(\mathcal{C})$.

(2) $Sh(\mathcal{C})$ has homotopy dimension $\leq n$ if $\forall$ n-connective $\mathcal{F}$ in $Sh(\mathcal{C})$, $\exists$ a morphism $* \to \mathcal{F}$ (ie $\mathcal{F}$ ~~[illegible]~~ has a section).

Thm - $\mathcal{C}$ site - If $Sh(\mathcal{C})$ has homotopy dim $\leq n$ then it is hypercomplete and of cohomological dimension $\leq n$.
(In particular, every sheaf on $\mathcal{C}$ is Postnikov-complete).

Thm (Clausen-Mathew) X qcqs scheme of finite Krull dimension d.
For $x \in X$ write $i_x : \mathrm{Spec}\, \mathcal{O}_{X,x} \to X$, $\iota_x : \mathrm{Spec}\, \mathcal{O}^h_{X,x} \to X$.

(1) Let $\mathcal{F}$ be a sheaf of anima on the Zariski site of X.
If $\Gamma(\mathrm{Spec}\, \mathcal{O}_{X,x}, (i_x)^*_{Zar} \mathcal{F})$ is $(\dim \overline{\{x\}} + n)$-connective $\forall x \in X$
then $\mathcal{F}(X)$ n-connective. (In fact this is true for any spectral space of finite Krull-dim)

(2) Let $\mathcal{F}$ be a sheaf of anima on the Nisnevich site of X.
If $\Gamma(\mathrm{Spec}\, \mathcal{O}^h_{X,x}, (\iota_x)^*_{Nis} \mathcal{F})$ is $(\dim \overline{\{x\}} + n)$-connective $\forall x \in X$
then $\mathcal{F}(X)$ n-connective.

In particular the homotopy dim. of the Zariski (resp. Nisnevich) site of X is $\leq \dim X$.

Proof - (1) We will show that if $\forall x$ $\mathcal{F}_x$ is $\dim \overline{\{x\}}$-conn then $\mathcal{F}(X) \neq \emptyset$.
$\rightsquigarrow$ induction on the Krull dimension of X.

When $X = \emptyset$, $\mathcal{F}(X) = * \neq \emptyset$ ✓.
Assume $X = \mathrm{Spec}\, A$ with A ring of finite Krull dim.
The hypothesis is: if $\mathfrak{p} \in \mathrm{Spec}\, A$ $\mathcal{F}(A_\mathfrak{p})$ is $\dim(A/\mathfrak{p})$-connective. $(\star)$
$\mathcal{L}$ = collection of localizations $A[S^{-1}]$ of A s.t. $\mathcal{F}(A[S^{-1}]) = \emptyset$.
We want to show that $\mathcal{L}$ is empty. Assume it is not.
$\mathcal{L}$ is partially ordered and admits filtered colim
$\rightsquigarrow$ $\exists$ maximal element A' $\rightsquigarrow$ $\begin{cases} \mathcal{F}(A'') \neq \emptyset \text{ for } A'' \text{ localizat}^\circ \text{ of } A' \ (1) \\ \mathcal{F}(A') = \emptyset \ (2) \end{cases}$

(2) $\Rightarrow$ A' not local (because $(\star)$) and since $A' \neq 0$, $\exists f \in A'$ s.t.
f is not a unit and f is not in the Jacobson rad of A'.
$A'' = A'[S^{-1}]$ with $S = \{1 + gf \mid g \in A'\}$.
Have a pullback square:

$$\begin{tikzcd} \mathcal{F}(A') \arrow[r] \arrow[d] & \mathcal{F}(A'[1/f]) \neq \emptyset \text{ (by (1))} \arrow[d] \\ \emptyset \neq \mathcal{F}(A'') \arrow[r] & \mathcal{F}(A''[1/f]) \end{tikzcd}$$

$\mathcal{F}(A') \neq \emptyset \Leftarrow$ (by (1))
contradiction

and $\underbrace{\dim A''[\tfrac{1}{f}] < \dim A}_{\text{use the induction to prove } \mathcal{F}(A''[\frac{1}{f}]) \neq \emptyset}$

$\mathcal{F}(A''[\frac{1}{f}]) \neq \emptyset$:

$x \in \operatorname{Spec} A''[\frac{1}{f}]$, $\tilde{x}$ the image in $\operatorname{Spec} A$, $\dim \overline{\{x\}} < \dim \overline{\{\tilde{x}\}}$.

$\mathcal{F}_{\tilde{x}}$ is $\dim \overline{\{\tilde{x}\}}$ - connective $\Rightarrow$ $(\mathcal{F}|_{A''[\frac{1}{f}]})_x$ is $\dim\overline{\{x\}}$ - connective

\+ use induction.

Induction on $\dim X$.

<u>Proof of (2)</u>: It suffices to show that $\forall x \in X$ the Zariski stalk $\mathcal{F}_x$ is $\dim \overline{\{x\}}$ - connective.

Assume $X = \operatorname{Spec} A$ for A local ring with closed point x.

Nisnevich pullback of $\mathcal{F}$ to x is 0-connective, i.e. locally non-empty

So $\exists$ A' étale neighborhood of x s.t. $\mathcal{F}(A') \neq \emptyset$.

~~Let I be such that~~ $I \subset A$ finitely generated ideal s.t. $A \to A'$ iso along I.

Pullback square:

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r] \arrow[d] & \mathcal{F}(A') \neq 0 \text{ by hyp.} \arrow[d] \\ \text{by ind. } 0 \neq \mathcal{F}(\operatorname{Spec} A \setminus V(I)) \arrow[r] & \mathcal{F}(\operatorname{Spec} A' \setminus V(I)) \neq 0 \text{ by ind.} \end{tikzcd}$$

<u>Want</u>: $\mathcal{F}(A) \neq \emptyset$.

$y \in \operatorname{Spec} A \setminus V(I)$

$\tilde{y} \in A$ $\quad \dim \overline{\{y\}} < \dim \overline{\{\tilde{y}\}}$

Same for A'. because $\operatorname{Spec} A \setminus V(I) \to \operatorname{Spec} A$

$x \notin \operatorname{im}(\cdot)$ $\square$.

<u>Thm</u>. X qcqs adic space w/ finite Krull dim, $\mathcal{F}$ sheaf of anima on the Nisnevich site of X, $\iota_x : (K_h(x), K_h^+(x)) \to X$, $n \geq 0$.

Then: (henselianization of the residue field of x.)

If $\Gamma(\operatorname{Spa}(K_h(x), K_h^+(x)), (\iota_x)^*_{Nis} \mathcal{F})$ $\dim \overline{\{x\}} + n$ - connective $\forall x \in X$

then $\mathcal{F}(X)$ is n - connective.

In particular the homotopy dim of $X \leq \dim X$.

<u>Proof</u>. Similar to the algebraic case. ~~(reduction to Zariski~~

(reduce to ~~prove~~ $X = \operatorname{Spa}(K, K^+)$ affinoid w/ finite Krull dim

closed point $x \in$ prove $\mathcal{F}(\operatorname{Spa}(K, K^+)) \neq \emptyset$

$\exists$ $\operatorname{Spa}(\tilde{K}, \tilde{K}^+)$ aff. / $\operatorname{Spa}(K, K^+)$ s.t. $\mathcal{F}(\operatorname{Spa}(\tilde{K}, \tilde{K}^+)) \neq \emptyset$.)

$\square$

§ ~~An~~ Hypercompleteness criterion.

Classifying topos of a profinite group:

Definition – G profinite group. $\mathcal{T}_G$ is the Grothendieck site such that

(1) The underlying category is the category of finite continuous G-sets.

(2) $\{S_i \to S\}_{i\in I}$ is a covering sieve if it is jointly surjective.

Definition – K finite group acting on a spectrum X. We say that the K-action is weakly m-nilpotent if the augmented cosimplicial sp:

$$X^{hK} \to X \rightrightarrows \prod_K X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$ is weakly m-rapidly converging.

Proposition – G profinite group. $\mathcal{F}$ sheaf of spectra on $\mathcal{T}_G$.
Suppose $\exists\, d \geq 0$ such that $\forall\, \underset{\text{normal}}{N \subset H}$, $\underset{\text{open}}{N, H \subset G}$, the action of H/N on $\mathcal{F}(G/N)$ is weakly d-nilpotent.
Then $\mathcal{F}$ is Postnikov complete.

Proof – Let $\hat{\mathcal{F}}$ be the Postnikov completion of $\mathcal{F}$. Want: $\mathcal{F} \to \hat{\mathcal{F}}$ eq.

(1) Reduce to prove the equivalence for the G-set $*$.

(By repeating the argument for $H \subset G$ open, $u : \mathcal{T}_H \to \mathcal{T}_G$, $S \mapsto G \times_H S$)

(2) In that case, we have:

$$\hat{\mathcal{F}}(*) \simeq \mathrm{Tot}\Big(\underbrace{\hat{\mathcal{F}}(G) \rightrightarrows \hat{\mathcal{F}}(G\times G) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots}_{\operatorname{colim}\big(\underbrace{\mathcal{F}(G/N) \rightrightarrows \mathcal{F}(G/N \times G/N) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots}_{\text{d-rapidly converging (ind of N)}}\big)}\Big)$$

$\Rightarrow$ can invert Tot and colim.

But $\mathrm{Tot}\big(\mathcal{F}(G/N) \rightrightarrows \mathcal{F}(G/N \times G/N) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots\big) \simeq \mathcal{F}(*)$ since $\mathcal{F}$ sheaf. $\square$

(Nilpotence criterion for $\mathcal{T}_G$).

Corollary – G profinite group of finite p-local coho dim $\leq d$, $\mathcal{F}$ sheaf of p-local sp.

Equivalent:

(1) $\mathcal{F}$ hypercomplete (or Postnikov complete)

(2) $\exists\, M$ such that $\forall\, \underset{\text{normal}}{N \subset H}$, $\underset{\text{open}}{N, H \subset G}$ the H/N-action on $\mathcal{F}(G/N)$ is weakly M-nilpotent.

(3) $\forall\, \underset{\text{normal}}{N \subset H}$, $\underset{\text{open}}{N, H \subset G}$ the H/N-action on $\mathcal{F}(G/N)$ is weakly d-nilpotent.

Proposition. G profinite group of finite ~~…~~ S-local virtual coho dimension. Then the hypercompletion functor is smashing for S-local sheaves of spectra on $\mathcal{T}_G$.

Hypercompleteness criterion

Rmk. X qcqs scheme, $\iota_x : x \to X$ or a qcqs adic space.

Then $\iota_x^* : Sh(X_{Nis}) \to Sh(x_{Nis})$ can be seen as a functor from sheaves on X_{Nis} to product preserving presheaves on $\mathcal{T}_x$

$\mathcal{T}_x := \mathcal{T}_{Gal(k(x)^{sep}/k(x))}$ or $Gal(k_h(x)^{sep}/k_h(x))$

$$(x^*\mathcal{F})(y \to x) = \lim_{\substack{\text{etale nghb. of } y \text{ in } X \\ y \to U \to X}} \mathcal{F}(U \to X).$$

ie $y \to X$ factors through U

To pass from qcqs schemes (or qcqs adic space) to affine schemes (resp affinoids) we will need the following:

Proposition. X qcqs scheme (resp ~~affinoid~~ adic space).
$\mathcal{F}$ Nisnevich sheaf of spectra on X. If any of the following is true for $f^*\mathcal{F}$, $f : U \to X$ etale, U affine (resp affinoid), then the analogous statement holds for $\mathcal{F}$.

(1) $f^*\mathcal{F}$ is a sheaf on the affine etale site of U (resp affinoid etale)

(2) $f^*\mathcal{F}$ is hypercomplete ———

(3) $f^*\mathcal{F}$ is Postnikov complete ———

(4) $f^*\mathcal{F}$ is zero as a presheaf ———
(resp. $f^*\mathcal{F}$ iso to the trivial sheaf on the affinoid etale site)

Thm. X qcqs scheme, $\mathcal{F}$ presheaf of S-local spectra on Et_X.
Suppose $\exists$ uniform bound N on the S-local etale coho dim of each $U \to X \in Et_X$ (or cofinal collection, eg $U \to X$ affine). Then

$\mathcal{F}$ hypercomplete etale sheaf $\iff$ $\mathcal{F}$ hypercomplete Nisnevich sheaf + the presheaf of spectra $x^*\mathcal{F}$ on $\mathcal{T}_x$ is an hypercomplete etale sheaf $\forall x \in X$.

Proof - Using the local-global principle + previous proposition, we can reduce the proof to $X = \mathrm{Spec}\, A$ affine.

$\boxed{\Rightarrow}$ Assume $\mathcal{F}$ étale hypercomplete, $x \in X$.
Want $x^*\mathcal{F}$ hypercomplete étale on $\mathrm{Spec}\, k(x)$.
$B_1 \to B_2$ faithfully flat map between finite étale A_B^h-alg.
"colim" $B_{1,i}$ "colim" $B_{2,i}$

. $\mathcal{F}(B_1) = \mathrm{colim}\, \mathcal{F}(B_{1,i})$
$\simeq \mathrm{colim}\, \mathrm{Tot}(\mathcal{F}(B_{2,i}) \rightrightarrows \mathcal{F}(B_{2,i} \otimes B_{2,i}) \overset{\to}{\rightrightarrows} \dots)$ } $\mathcal{F}$ hypercomplete
and $\mathcal{F}(B_{1,i}) \to (\quad)$ $B_{1,i}$ is N-nilpotent.

. Can invert colim and Tot by the nilpotence criterion.
Get $\mathcal{F}(B_1) \simeq \mathrm{Tot}(\mathcal{F}(B_2) \rightrightarrows \mathcal{F}(B_2 \underset{B_1}{\otimes} B_2) \overset{\to}{\rightrightarrows} \dots)$
and the diagr. is N-nilpotent
$\rightsquigarrow$ use nilpotence criterion for $\mathcal{C}_x$.

$\boxed{\Leftarrow}$ Suppose that ~~the Nisnevich pullbacks~~ $x^*\mathcal{F}$ are hypercomplete $\forall x \in X$.
$\mathcal{F}$ Nisnevich hypercomplete
Write $\mathcal{F}^h$ for the étale hypercompletion of $\mathcal{F}$.

. $\mathcal{F}^h$ and $\mathcal{F}$ are hypercomplete for the Nisnevich topo $\rightsquigarrow$ it suffices to show that ~~they are equal on their Nisnev~~ their Nisnevich stalks are equal.
ie $\mathcal{F}^h(B) = \mathcal{F}(B)$ for B ind-étale A-alg, B henselian local.

. On the finite étale site of $\mathrm{Spec}\, B$: $\mathcal{F}$ and $\mathcal{F}^h$ are (étale hypercomplete
+ they have the same stalks
$\Rightarrow \mathcal{F}(B) = \mathcal{F}^h(B)$.

$\hookrightarrow$ by assumption for $\mathcal{F}$
by previous § ($\Rightarrow$) for $\mathcal{F}^h$.

$\square$

Thm - X qcqs adic space, suppose $\exists$ uniform bound of the $\mathbb{S}$-local étale coho dimensions of $U \to X$ étale. Then
$\mathcal{F}$ étale hypercomplete sheaf $\Leftrightarrow$ $\begin{cases} \mathcal{F} \text{ Nisnevich hypercomplete} \\ x^*\mathcal{F} \text{ is étale hypercomplete on } \mathcal{C}_x \quad \forall x \in X. \end{cases}$

Proof - Same as in the algebraic case
$K_R(x) \to k \to k'$ finite separable field ext
$k^+ =$ closure of $K_R(x)^+$ in k
$(k, k^+) \to (k', k'^+)$ can be written $(\underset{x \in U}{\mathrm{colim}}(B_U, B_U^+) \to (B'_U, B'^+_U))^*$
Nisnevich nghb

§ Etale hyperdescent for localizing invariant.

$\mathrm{Cat}_\infty^{\mathrm{perf}}$: ∞-category of small, stable, closed under retract ∞-categories.

Definition. $\mathcal{D}$ stable ∞-category. A localizing invariant with values in $\mathcal{D}$ is a functor $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}$ sending final object to final object
Verdier seq to fiber seq.

1) The algebraic case. Fix p a prime number.

Definition. X qcqs scheme, $\mathcal{D}$ as above. A localizing invariant for X with values in $\mathcal{D}$ is a presheaf $\mathcal{F}$: $\begin{cases} \{\text{qc open} \subset X\} \longrightarrow \mathcal{D} \\ U \longmapsto F(\mathrm{Perf}(U)) \end{cases}$

where $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}$ is a localizing invariant.

$\mathrm{Perf}(U)$: ∞-category of ~~complexes~~ perfect modules on U

Proposition. X qcqs scheme, $\mathcal{F}$ localizing invariant with values in $\mathcal{D}$. Then $\mathcal{F}$ is a sheaf with ~~[illegible]~~ values in $\mathcal{D}$ for the Nisnevich topology.

Theorem. X qcqs scheme, F localizing invariant with values in L_n^f-local spectra. Then $\mathcal{F} : U \mapsto F(\mathrm{Perf}(U))$ defines an etale sheaf.

Theorem (Clausen-Mathew) X qcqs scheme with finite Krull dimension. Assume $\exists$ uniform bound of the p-local virtual Galois coho dimensions of the residue fields of X. Then $\mathcal{F} : U \mapsto F(\mathrm{Perf}(U))$ is an etale hypersheaf, where F localizing invariant with values in L_n^f-local spectra.

Sketch of proof. (1) Presheaf $K_{\geq 0}$ acts on the presheaf $U \mapsto F(\mathrm{Perf}\, U)$
(2) Previous thm: $(K_{\geq 0})^{et}$ acts on the etale sheaf $U \mapsto F(\mathrm{Perf}\, U)$.
(3) $(K_{\geq 0})^{et}$ is hypercomplete. (compare to Selmer K-theory)
(4) Use that the hypercompletion functor is smashing for p-local sheaves of spectra on T_n.

$\square$.

2) The analytic case. stable dualizable cats.

Recall: (Hoyois) $\mathcal{D}$ stable ∞-category, the functor $\begin{cases} \mathrm{Fun}(\mathcal{C}^{\mathrm{dual}}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{D}) \\ F \longmapsto F \circ \mathrm{Ind} \end{cases}$ induces an eq between the subcategories of localizing invariants.

Write $\mathcal{F}_{cont} \in \mathrm{Fun}(\mathcal{A}^{dual}, \mathcal{D})$ corresponding to $\mathcal{F} \in \mathrm{Fun}(\mathrm{Cat}^{perf}_{\infty}, \mathcal{D})$ localizing invariant.

Def. X qcqs adic space, $\mathcal{D}$ stable ∞-category.
A localizing invariant of X with values in $\mathcal{D}$ is a presheaf

$$f_{cont} : \begin{cases} \{qc \text{ open} \subset X\} \longrightarrow \mathcal{D} \\ \quad U \longmapsto \mathcal{F}_{cont}(\mathrm{Nuc}\,\mathcal{D}) \end{cases}$$

for $\mathcal{F}$ localizing invariant with values in $\mathcal{D}$.

Proposition. f_{cont} is a Nisnevich sheaf ($\Rightarrow$ Nisnevich hypersheaf).

Theorem. X qcqs analytic adic space with finite Krull dimension. Assume $\exists$ uniform bound of the p-local virtual coho dim of the residue fields of X. Then any f_{cont} localizing invariant with values in L_n^f-local spectra is an étale hypersheaf.

Proof. (1) f_{cont} defines an étale sheaf on X:

(Étale descent = Nisnevich + Galois descent $\forall x$.)

$x \in X$, $\iota_x : \mathrm{Spa}(k_x(x), k_x(x)^+) \longrightarrow X$, $x^* f_{cont}$ presheaf on $\mathcal{C}_x$.

$A := \mathcal{O}_x$ stalk of the Nisnevich str sheaf at x.

$A \simeq \mathrm{colim}\, A_i \qquad A_i = \mathcal{O}_X(U_i) \qquad x \in U_i$

Morphisms of $\mathcal{C}_x$: $A' \longrightarrow A''$ $\qquad A', A''$ finite étale $/A$.

$\mathrm{colim}\, A'_i \qquad$ finite étale $\qquad \cancel{A'_i = A'_0 \otimes_{A'_0} A}$

$\cancel{A''_i}$

$x^* f_{cont}(A') \simeq \mathrm{colim}\, \mathcal{F}_{cont}(\mathrm{Nuc}(A'_i))$

$\simeq \mathrm{colim}\, \cancel{\mathrm{Cat}}\, \mathcal{F}_{cont}(\mathrm{Perf}(A'_0) \otimes_{\mathrm{Perf}(A'_0)} \mathrm{Nuc}(A'_i))$

$\simeq \mathrm{Tot}\,\mathrm{colim}\, \mathcal{F}_{cont}(\mathrm{Perf}(A''_0) \otimes_{\mathrm{Perf}(A'_0)} \mathrm{Nuc}(A'_i) \rightrightarrows \mathrm{Perf}(A''_0 \otimes_{A'_0} A''_0) \otimes_{\mathrm{Perf}(A'_0)} \mathrm{Nuc}(A'_i) \rightrightarrows \dots)$

$\qquad \simeq \mathrm{Tot}\,\mathcal{F}_{cont}(\mathrm{Nuc}(A''_i) \rightrightarrows \mathrm{Nuc}(A''_i \otimes_{A'_i} A''_i) \rightrightarrows \dots)$

$\qquad\qquad \Vert\ \mathrm{Perf}(A''_0) \otimes_{\mathrm{Perf}(A'_0)} \mathrm{Nuc}(A'_i)$

$\qquad\qquad$ because $A''_i = A''_0 \otimes_{A'_0} A'_i$

$\Rightarrow x^* f_{cont}$ is a sheaf on $\mathcal{C}_x$.

Consider the presheaf $\mathcal{G} : A' \longmapsto \mathrm{colim}\, \mathcal{F}(\mathrm{Perf}(A'_i))$ on $\mathcal{C}_x$

(2) $x^* f_{cont}$ étale hypersheaf: $\quad$ finite étale$/A$ $\quad$ hypersheaf by alg. case

And $x^* f_{cont}$ is a module over $\mathcal{G}$ $\rightsquigarrow$ use that hypercompletion functor is smashing for $\mathcal{C}_x$ $\Rightarrow$ $x^* f_{cont}$ étale hypersheaf.