We first recall the Nisnevich topology. All schemes and adic spaces here are qcqs.

**Definition 1** (Nisnevich topology on schemes)**.** A family of étale maps of schemes $\{f_i : U_i \to X\}_{i \in I}$ is called a *Nisnevich cover* if for every $x \in X$, there exist $i \in I$ and $u \in U_i$, such that $f_i(u) = x$ and $k(u) = k(x)$. For a scheme $X$, the *(small) Nisnevich site* of $X$, denoted $X_{\mathrm{Nis}}$, is the site with underlying category that of schemes étale over $X$ and with covers Nisnevich covers.

**Definition 2** (Nisnevich squares of schemes)**.** A *Nisnevich square* of schemes is a commutative square

$$\begin{CD} W @>>> V \\ @VVV @VV{f}V \\ U @>>{j}> X \end{CD}$$

where $j$ is an open immersion, $f$ is étale, and $f$ is an isomorphism over $X \setminus U$.

**Proposition 3.** *Let $X$ be a scheme and $\{f_i : U_i \to X\}_{i \in I}$ is a family of étale maps. The following are equivalent:*

*(1) $\{f_i : U_i \to X\}_{i \in I}$ is a Nisnevich cover.*
*(2) There is a chain of finitely presented closed subschemes*

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_n = X$$

*and a family of étale maps $\{g_j : V_j \to X\}_{j=1}^{n}$ refining $\{f_i : U_i \to X\}_{i \in I}$, such that $g_j$ is an isomorphism over $Z_j \setminus Z_{j-1}$.*

*Proof.* This is a standard Zorn argument. See [BH21, Lemma A.1]. □

**Corollary 4.** *The Nisnevich topology is generated by the empty cover of the empty scheme and the families $\{j, f\}$ for all Nisnevich squares as in Definition 2.*

**Definition 5** (Étale maps of analytic adic spaces)**.**

(1) A map of analytic adic spaces is *finite étale* if it is locally on the target the Spa of a map $(R, R^+) \to (A, A^+)$ of Huber pairs, where $R \to A$ is finite étale and $A^+$ is the integral closure of $R^+$ in $A$.
(2) A map of analytic adic spaces is *étale* if, locally on both the source and the target, it can be written as an open immersion into a finite étale map.

**Definition 6** (Nisnevich topology on adic spaces)**.** A family of étale maps of schemes $\{f_i : U_i \to X\}_{i \in I}$ is called a *Nisnevich cover* if for every $x \in X$, there exist $i \in I$ and $u \in U_i$, such that $f_i(u) = x$ and $k(u) = k(x)$. For a scheme $X$, the *(small) Nisnevich site* of $X$ has underlying category the category of schemes étale over $X$, with Nisnevich covers as covers.

**Definition 7** (Nisnevich squares of analytic adic spaces)**.** A *Nisnevich square* of analytic adic spaces is a commutative square

$$\begin{CD} W @>>> V \\ @VVV @VV{f}V \\ U @>>{j}> X \end{CD}$$

where $j$ is an open immersion, $f$ is étale, and each $x \in X \setminus U$ has only one preimage in $V$ and it has the same residue field as $x$.

**Proposition 8.** *Let $X$ be an analytic adic space and $\{f_i \colon U_i \to X\}_{i \in I}$ is a family of étale maps. The following are equivalent:*

(1) *$\{f_i \colon U_i \to X\}_{i \in I}$ is a Nisnevich cover.*

(2) *There is a chain of finitely presented closed subspaces*

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_n = X$$

*and a family of étale maps $\{g_j \colon V_j \to X\}_{j=1}^{n}$ refining $\{f_i \colon U_i \to X\}_{i \in I}$, such that each $z_j \in Z_j \setminus Z_{j-1}$ has only one preimage under $g_j$ and it has the same residue field as $z_j$.*

*Proof.* This is more complicated than the scheme case. See [And23, Satz A.24]. □

**Corollary 9.** *The Nisnevich topology is generated by the empty cover of the empty scheme and the families $\{j, f\}$ for all Nisnevich squares as in Definition 7.*

We also need the basic theory of cd-structures.

**Definition 10** (cd-structure)**.** A *cd-structure* on a small category $\mathcal{C}$ is a family of square diagrams in it. More formally, a cd-structure on $\mathcal{C}$ is a full subcategory $\chi \subseteq \mathsf{Fun}([1]^2, \mathcal{C})$. For a cd-structure $\chi$ on $\mathcal{C}$, the *topology associated to* $\chi$, denoted $\tau_\chi$, is the topology on $\mathcal{C}$ generated by families $\{U \to X, V \to X\}$ for all

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \in \chi.$$

**Definition 11.** Let $\mathcal{C}$ be a small category with a cd-structure $\chi$. Let $\mathcal{D}$ be a category and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be a functor. We say that $F$ satisfies:

- $\chi$-*excision*, if $F$ takes diagrams in $\chi$ to pullbacks.
- $\chi$-*descent*, if $F$ is a $\tau_\chi$-sheaf.

Our statement here is slightly more general than [AHW17, Theorem 3.2.5].

**Theorem 12** (Voevodsky)**.** *Let $\mathcal{C}$ be a small category and $\chi$ be a cd-structure on $\mathcal{C}$. Consider the following conditions:*

(1) *$\mathcal{C}$ has pullbacks and $\chi$ is closed under them, namely for any*

$$Q = \begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \in \chi$$

*and any map $Y \to X$, we have*

$$Q \times_X Y = \begin{array}{ccc} W \times_X Y & \longrightarrow & V \times_X Y \\ \downarrow & & \downarrow \\ U \times_X Y & \longrightarrow & Y \end{array} \in \chi.$$

(2) *All diagrams in $\chi$ are pullbacks. Moreover, for any*

$$Q = \begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \in \chi,$$

*there exists $n \in \mathbb{N}$ such that both $U \to X$ and $V \to X$ are $n$-truncated, and*

$$\begin{array}{ccc} W & \longrightarrow & V \\ {\scriptstyle \Delta_{W/U}}\downarrow & & \downarrow{\scriptstyle \Delta_{V/X}} \\ W \times_U W & \longrightarrow & V \times_X V \end{array}, \quad \begin{array}{ccc} W & \xrightarrow{\Delta_{W/V}} & W \times_V W \\ \downarrow & & \downarrow \\ U & \xrightarrow[\Delta_{U/X}]{} & U \times_X U \end{array} \in \chi.$$

*If (1) holds, then $\chi$-excision implies $\chi$-descent. If furthermore (2) holds, then $\chi$-excision and $\chi$-descent are equivalent.*

*Proof.* For every

$$Q = \begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \in \chi,$$

consider the two natural maps

$$e_Q \colon h_U \sqcup_{h_W} h_V \to h_X,\ c_Q \colon \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} (h_U \sqcup h_V)^{\times_{h_X}(n+1)} \to h_X$$

in the presheaf category $\mathsf{P}(\mathcal{C})$, where $h_X$ denotes the Yoneda presheaf. Let

$$E_\chi = \{e_Q \mid Q \in \chi\},\ C_\chi = \{c_Q \mid Q \in \chi\},$$

and we want to compare the Bousfield localizations of $\mathsf{P}(\mathcal{C})$ with these two families of maps. Denote by $\overline{E_\chi}$ and $\overline{C_\chi}$ the families of maps that become invertible after the corresponding localizations. Then they contain $E_\chi$ and $C_\chi$ respectively, and they are closed under colimits. Consider the commutative diagram

$$\begin{array}{ccc} (h_U \sqcup_{h_W} h_V) \times_{h_X} \operatorname{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} (h_U \sqcup h_V)^{\times_{h_X}(n+1)} & \xrightarrow{\sim} & h_U \sqcup_{h_W} h_V \\ {\scriptstyle f_Q}\downarrow & & \downarrow{\scriptstyle e_Q} \\ \operatorname{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} (h_U \sqcup h_V)^{\times_{h_X}(n+1)} & \xrightarrow[c_Q]{} & h_X \end{array}$$

where the upper arrow is an isomorphism, because it is the pushout of the Čech nerves of the base changes of the map $h_U \sqcup h_V \to h_X$ to $h_U$, $h_V$, and $h_W$, which are all isomorphisms as they are Čech nerves of maps admitting sections. Therefore, $e_Q \in \overline{C_\chi}$ if and only if $f_Q \in \overline{C_\chi}$, and $c_Q \in \overline{E_\chi}$ if and only if $f_Q \in \overline{E_\chi}$.

Note that $f_Q$ is a colimit of base changes of $e_Q$ along some representable maps, so if (1) holds, these base changes remain in $E_\chi$, so $f_Q \in \overline{E_\chi}$ and thus $e_Q \in \overline{E_\chi}$. Hence $\chi$-excision implies $\chi$-descent.

Assume furthermore that (2) holds. We do induction on integers $n, m \geq -2$ to prove that, if $U \to X$ is $m$-truncated and $V \to X$ is $n$-truncated, then $e_Q \in \overline{C_\chi}$. By the above, this is equivalent to $f_Q \in \overline{C_\chi}$. If $m$ or $n$ is $-2$, this is obvious, as then $e_Q$ is an isomorphism. If $m, n > -2$, consider the commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{\Delta_{W/V}} & W \times_V W & \xrightarrow{\mathrm{pr}_1} & W \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow[\Delta_{U/X}]{} & U \times_X U & \xrightarrow[\mathrm{pr}_1]{} & U \end{array}$$

where the right square is the base change $Q_U$ of $Q$ along $U \to X$. Since the left square belongs to $\chi$ by assumption and is more truncated than $Q$, the induction hypothesis implies that its $e$ map is in $\overline{C_\chi}$. In other words, the left square becomes a pushout square after $\tau_\chi$-sheafification. Obviously the large square does, too. Hence so does the right square, namely $e_{Q_U} \in \overline{C_\chi}$. Similarly, $e_{Q_V} \in \overline{C_\chi}$. Since $\chi$ is closed under base change, we can base change the above diagram along any map $Y \to U$ before doing the same reasoning, so whenever a map $Y \to X$ factors through either $U$ or $V$, we have $e_{Q_Y} \in \overline{C_\chi}$. Now note that $f_Q$ is a colimit of such $e_{Q_Y}$'s, so $f_Q \in \overline{C_\chi}$, and thus $e_Q \in \overline{C_\chi}$. Hence $\chi$-descent implies $\chi$-excision. □

**Corollary 13.** *A Nisnevich presheaf is a Nisnevich sheaf if and only if it satisfies Nisnevich excision and takes the empty scheme to the initial object.*

**Corollary 14.** *Let $X$ be a scheme, $\mathcal{C}$ a presentable stable category, and $E\colon \mathsf{Pr}^{\mathrm{dual}}_{\mathsf{D}(X)} \to \mathcal{C}$ a localizing invariant. Then $E(\mathsf{D}(-))$ is a sheaf on $X_{\mathrm{Nis}}$.*

*Proof.* By faithfully flat descent of quasicoherent complexes, $\mathsf{D}(-)$ is even an fpqc sheaf, so in particular it is a Nisnevich sheaf. Thus Corollary 13 implies that $\mathsf{D}(\emptyset) = 0$, and that for any Nisnevich square

$$\begin{array}{ccc} W & \xrightarrow{j'} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & Y \end{array}$$

the resulting square

$$\begin{array}{ccc} \mathsf{D}(W) & \xleftarrow{j'^*} & \mathsf{D}(V) \\ {\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathsf{D}(U) & \xleftarrow[j^*]{} & \mathsf{D}(Y) \end{array}$$

is a pullback. Denote $Z = Y \setminus U = V \setminus W$ and $\mathsf{D}_Z(Y) = \ker(j^*) = 0 \times_{\mathsf{D}(U)} \mathsf{D}(Y)$. Then by the three pullback lemma, this is also $\mathsf{D}_Z(V) = \ker(j'^*)$. Since both $j^*$ and $j'^*$ has fully faithful right adjoints, namely the lower stars, we see that the horizontal sequences of the diagram

$$\begin{array}{ccccc} \mathsf{D}(W) & \xleftarrow{j'^*} & \mathsf{D}(V) & \longleftarrow & \mathsf{D}_Z(V) \\ {\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f^*} & & \| \\ \mathsf{D}(U) & \xleftarrow[j^*]{} & \mathsf{D}(Y) & \longleftarrow & \mathsf{D}_Z(Y) \end{array}$$

are exact sequences of dualizable categories and hence are mapped by $E$ to fiber sequences in $\mathcal{C}$. Therefore, $E(\mathsf{D}(-))$ maps the empty scheme to 0 and Nisnevich squares to fiber squares, so again by Corollary 13 it is a Nisnevich sheaf. □

To adapt the argument above to nuclear modules on analytic adic spaces, it suffices to verify Nisnevich descent for $\mathsf{Nuc}(-)$, as we also have the adjoint pair $(j^*, j_*)$ with $j_*$ fully faithful in the nuclear setting. In fact there is étale descent:

**Proposition 15.** $\mathsf{Nuc}(-)$ *is an étale sheaf on analytic adic spaces.*

*Proof.* By definition it is a sheaf for the analytic topology, so by the definition of étale covers of analytic adic spaces, it suffices to verify finite étale descent. In other words, it suffices to verify for any finite étale cover of analytic Huber pairs $(A, A^+) \to (B, B^+)$ that

$$\mathsf{Nuc}(A) = \lim_{n\in\mathbf{\Delta}^{\mathrm{op}}} \mathsf{Nuc}(B^{\otimes_A(n+1)}).$$

By the definition of finite étale maps, the analytic ring structure of $(B, B^+)_{\blacksquare}$ is induced from $(A, A^+)_{\blacksquare}$. Also, $B$ is nuclear as an $(A, A^+)_{\blacksquare}$-module. So by previous lectures, $\mathsf{Nuc}(B)$ is the category of $B$-modules in $\mathsf{Nuc}(A)$. Since $B \in \mathsf{Nuc}(A)$ is the image of $B \in \mathsf{D}(A)$ under the canonical functor $\mathsf{D}(A) \to \mathsf{Nuc}(A)$ which is a symmetric monoidal left adjoint, we see that $\mathsf{Nuc}(B) = \mathsf{D}(B) \otimes_{\mathsf{D}(A)} \mathsf{Nuc}(A)$. Similarly, $\mathsf{Nuc}(B^{\otimes_A(n+1)}) = \mathsf{D}(B^{\otimes_A(n+1)}) \otimes_{\mathsf{D}(A)} \mathsf{Nuc}(A)$. Now the proposition follows from the lemma below, noting that $B$, as a finite projective $A$-module supported on the whole $\mathrm{Spec}(A)$, is obviously descendable. Alternatively, one can also deduce this from that $\mathsf{D}(A) = \lim_{n\in\mathbf{\Delta}^{\mathrm{op}}} \mathsf{D}(B^{\otimes_A(n+1)})$ and that $- \otimes_{\mathsf{D}(A)} \mathsf{Nuc}(A)$ commutes with limits as $\mathsf{Nuc}(A)$ is dualizable over $\mathsf{D}(A)$. $\square$

**Lemma 16** ([Mat16, Corollary 3.42])**.** *Let $A \to B$ be a descendable ring map. Then for any $\mathsf{D}(A)$-module $\mathcal{C} \in \mathsf{Pr}^{\mathrm{L}}$,*

$$\mathcal{C} = \lim_{n\in\mathbf{\Delta}^{\mathrm{op}}} (\mathsf{D}(B^{\otimes_A(n+1)}) \otimes_{\mathsf{D}(A)} \mathcal{C}).$$

*Proof.* We first check comonoidality of the left adjoint $\mathcal{C} \to \mathsf{D}(B) \otimes_{\mathsf{D}(A)} \mathcal{C}$. By Barr–Beck–Lurie, this amounts to checking that every augmented cosimplicial diagram $F\colon \mathbf{\Delta}_+ \to \mathcal{C}$ that splits after $\mathsf{D}(B) \otimes_{\mathsf{D}(A)} -$ is a limit diagram. For this, consider

$$\left\{ M \in \mathsf{D}(A) \;\middle|\; (M \otimes_A F(-1))_{n\in\mathbb{N}} \to \left(M \otimes_A \lim_{\mathbf{\Delta}_{\le n}} F\right)_{n\in\mathbb{N}} \text{ is a pro-equivalence} \right\}.$$

It is obviously a thick $\otimes$-ideal containing $B$, so by descendability it contains $A$, giving the desired comonoidality. Therefore, $\mathcal{C}$ is the category of comodules of the comonad $B \otimes_A -$ on $\mathsf{D}(B) \otimes_{\mathsf{D}(A)} \mathcal{C}$, which unwinds to the limit in the statement. $\square$

**Corollary 17.** *Let $X$ be an analytic adic space, $\mathcal{C}$ a presentable stable category, and $E\colon \mathsf{Pr}^{\mathrm{dual}}_{\mathsf{Nuc}(X)} \to \mathcal{C}$ a localizing invariant. Then $E(\mathsf{Nuc}(-))$ is a sheaf on $X_{\mathrm{Nis}}$.*

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