K-Theory of Adic Spaces

Topology: X — compact Hausdorff space

$$K(X) = \text{the (complex) K-Theory spectrum of } X$$

an object of algebraic topology, has "homotopy groups"

One way to think about them:

E_∞-groups are spectra with no negative homotopy groups

difficult to define

$$K_i(X) := \pi_i(K(X))$$

$K_0(X)$ = the group completion of the monoid of isoclasses of $\mathbb{C}$-v.b. on X

Let's see what it is:

$\mathrm{Vect}^{\mathbb{C}}(X)$ = monoid wrt direct sum

If $X = *$, $\mathrm{Vect}^{\mathbb{C}}(X) \simeq \mathbb{N} \Rightarrow K_0(*) \simeq \mathbb{Z}$

Algebra / Algebraic Geometry:

X — an algebraic variety / scheme

(already the case $X = \operatorname{Spec} \mathbb{C}$ is interesting)

$\rightsquigarrow$

$K(X)$ — the K-theory spectrum of X

As in the topological situation, the definition is difficult.

$K_0(X)$ admits a similar description for nice schemes (in particular, affine)

One nice class is formed by so called qcqs schemes with resolution property

Question: X — a proper algebraic variety $/\mathbb{C}$

What is the relationship between $K(X)$ and $K(X(\mathbb{C}))$?

Interesting question even for $X = \operatorname{Spec} \mathbb{C}$.

Generalities on spectra: Let A be a spectrum. Then we have the following pullback diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes \mathbb{Q} \arrow[d] \\ \prod \hat{A}_p \arrow[r] & \prod \hat{A}_p \otimes \mathbb{Q} \end{tikzcd}$$

One construction of K-Theory:

$$\left(\mathrm{Vect}^{\simeq}(X)\right)^{\infty\text{-grp}} \longrightarrow \left(\mathrm{Vect}^{\simeq,\mathbb{C}}(X(\mathbb{C}))\right)^{\infty\text{-grp}}$$

$$\begin{array}{ccc} \Vert\wr & & \Vert\wr \\ K(X) & & K(X(\mathbb{C})) \end{array}$$

Example: $X = \mathrm{Spec}\,\mathbb{C}$

$$\left(\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{disc}}\right)^{\infty\text{-grp}} \longrightarrow \left(\mathrm{Vect}^{\simeq,\mathrm{top}}\right)^{\infty\text{-gr}}$$

$$\begin{array}{ccc} \Vert\wr & & \Vert\wr \\ K(\mathrm{Spec}\,\mathbb{C}) & & K(*) \end{array}$$

Remark: later we will change the perspective, which will be crucial to all our considerations

Theorem

$$K(\operatorname{Spec}\mathbb{C})^\wedge_p \longrightarrow K(*)^\wedge_p$$

is an isomorphism

Question Is there an analogue of the RHS in non-archimedean geometry? In particular, just for $\mathbb{Q}_p/\mathbb{Z}_p$.

$$\boxed{\text{Adic Geometry} = \text{a non-archimedean analog of complex analytic geometry}}$$

Setup: (R, R^+) — a complete $\underbrace{\text{Huber pair}}$

a certain class of pairs of topological rings

$R^+ \subseteq R$ open integrally closed

$\rightsquigarrow$ $\operatorname{Spa}(R, R^+)$ = the adic spectrum of (R, R^+)

Def. $\mathrm{Spa}(R, R^+) = \{ \nu : R \to \Gamma \sqcup \{0\} \mid \nu$ is a cont. valuation, $\forall x \in R^+ \ \nu(x) \leq 1 \} / \sim$

Γ — ordered ab group

Example:

- $(\mathbb{Z}_p, \mathbb{Z}_p)$ — its adic space has two points
- $(\mathbb{Q}_p, \mathbb{Z}_p)$ — its adic space has just one point
- $(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) \rightsquigarrow \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) =$ the closed unit disk over $\mathbb{Q}_p$
- $\mathbb{A}^1_{\mathbb{Q}_p}$, $\mathbb{P}^1_{\mathbb{Q}_p}$, ...

General situation: glue $\mathrm{Spa}(R, R^+)$ along open embeddings

Goal: construct K-theory for arbitrary* complete Huber pairs and maybe extend it to analytic adic spaces

$\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ is not analytic

$\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ is analytic

Reasons:

- usual algebraic K-theory does not take the topology of R into account
- furthermore, it does not satisfy descent

i.e. $\mathcal{U} \longmapsto K(\mathcal{U})$ is <u>not</u> a sheaf

$\mathcal{U} \subseteq X$ open

- Continuity fails:

$$K(R) \xrightarrow{\times} \varprojlim K(R/I^n)$$

R — an I-adically complete Noetherian ring

Our K-Theory will satisfy all of these properties. In the discrete case, it will just agree with algebraic K-theory. Furthermore, it will admit an abstract description which is convenient for many purposes.

Before we proceed, let's consider $\mathbb{Z}_p$.

$$\begin{tikzcd}
K(\mathbb{Z}_p) \arrow[r] \arrow[d] & K(\mathbb{Z}_p)\otimes\mathbb{Q} \arrow[d] \\
\prod_\ell K(\mathbb{Z}_p)^\wedge_\ell \arrow[r] & \prod_\ell K(\mathbb{Z}_p)^\wedge_\ell \otimes \mathbb{Q}
\end{tikzcd}$$

$K(\mathbb{Z}_p)^\wedge_\ell \to \varprojlim K(\mathbb{Z}/p^k\mathbb{Z})^\wedge_\ell$ is an iso $\forall \ell$

$\Downarrow$

we will only change the rational part

Modern approach to K-theory of schemes

Let X be a qcqs scheme. Consider its ∞-derived category of quasi-coherent sheaves

$$D_{qc}(X) \simeq \{ A \in D(\mathcal{O}_X\text{-Mod}) \mid H^i(A) \in QCoh(X) \}$$

Fact $D_{qc}(X)$ is compactly generated (by its full subcategory of perfect complexes)

Example $X = \mathrm{Spec}\, R$. In this case

$$D_{qc}(\mathrm{Spec}\, R) = D(R)$$

$$\mathrm{Perf}(\mathrm{Spec}\, R) = \left\{ \begin{array}{l} \text{bounded complexes} \\ \text{with projective terms} \end{array} \right\}$$

Definition:

scheme
geometric object (i.e. adic space)
$\rightsquigarrow$
dualizable
compactly generated stable ∞-category
$\xrightarrow{\text{K-theory functor}}$
spectrum

For schemes:

$$K(X) = K(D_{qc}(X))$$

Reasons: • ∞-group completion is very complicated

• we can see K-theoretic behavior at the level of categories

What does condensed math do?

It makes the first step possible.

Before the invention of condensed math, we had no category of "quasi-coherent sheaves" even on nice adic spaces

Construction: (R, R^+) — complete Huber pair

Consider R^+_{disc} with its discrete topology

$R^+_{disc, \blacksquare}$ — an analytic ring in the sense of condensed math

Now consider the following map of condensed rings:

$$R^+_{disc} \longrightarrow \underline{R}$$

We can induce an analytic structure on $\underline{R}$ via this map:

concentrated in degree 0

$$(R, R^+)_{\blacksquare} = \underline{R} \overset{L}{\underset{R^+_{disc,\blacksquare}}{\otimes}} R^+_{disc,\blacksquare}$$

<u>Example</u>: • $\mathbb{Z}_{p,\blacksquare}[S] \simeq \prod_I \mathbb{Z}_p$

• $\mathbb{Q}_{p,\blacksquare}[S] = \prod_I \mathbb{Z}_p[1/p]$

To every analytic ring we can associate the category $D((R, R^+)_{\blacksquare})$. However, this category is too big from the point of view of K-theory.

It's compactly generated but:

$$0 \to \mathbb{Z}_p \to \prod \mathbb{Z}_p \xrightarrow{\text{id}-\text{shift}} \prod \mathbb{Z}_p \to 0$$

so the class of 1 is trivial in K_0.

We need to pass to the full subcategory of nuclear objects:

— $(\mathcal{A}, \mathcal{M})$ — an analytic ring

$N / (\mathcal{A}, \mathcal{M})$ is called nuclear if $\forall S$ extr. disc

$$\mathcal{M}[S]^\vee \otimes N \quad (*) \xrightarrow{\sim} N(S)$$
$$\underline{\mathrm{Hom}}(\mathcal{M}[S], \mathcal{A})$$

— $\underline{\mathrm{Hom}}(K, M) \otimes N \simeq \underline{\mathrm{Hom}}(K, M \otimes N)$

$\forall K$ compact $\forall M$ $/(\mathcal{A}, \mathcal{M})$

Examples: • $(\mathbb{R}, \mathbb{R}^+)$ — discrete Huber pair

Then $\mathrm{Nuk}\left((\mathbb{R}, \mathbb{R}^+)_{\blacksquare}\right) = D(\mathbb{R})$

In particular, $\mathrm{Nuk}(\mathbb{Z}_{\blacksquare}) = D(\mathbb{Z})$

• $\mathbb{Z}_p$ or $\mathbb{Q}_p$:

everything is generated by

$C(S, \mathbb{Z}_p)$ S profinite

$C(S, \mathbb{Q}_p)$

In other words:

$$\mathbb{Z}_p\langle T\rangle \quad \text{or} \quad \mathbb{Q}_p\langle T\rangle$$

Theorem: Let (R, R^+) be a nice complete Huber pair. Then

— $\mathrm{Nuc}((R, R^+)_{\blacksquare})$ is independent of R^+

$$\mathrm{Nuc}((R,R^+)_{\blacksquare}) \subseteq D(\underline{R})$$

— $\forall S$ profinite: $C(S, R)$ is nuclear

— $C(S, R) \overset{L}{\otimes} C(S', R) \simeq C(S \times S', R)$

— the category of nuclear modules is dualizable

— it satisfies (étale) descent on analytic adic spaces

Def. $K^{nuc}(\mathbb{Z}_p) = K(\mathrm{Nuc}\,\mathbb{Z}_p)$

Theorem

$$K^{alg}(\mathbb{Z}_p) \longrightarrow K^{nec}(\mathbb{Z}_p)$$

is an iso modulo every m