RELATIVE POINCARÉ DUALITY IN NONARCHIMEDEAN GEOMETRY

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Abstract. We prove a conjecture of Bhatt–Hansen that derived pushforwards along proper morphisms of rigid-analytic spaces commute with Verdier duality on Zariski-constructible complexes. In particular, this yields duality statements for the intersection cohomology of proper rigid-analytic spaces. In our argument, we construct cycle classes in analytic geometry as well as trace maps for morphisms that are either smooth or proper or finite flat, with appropriate coefficients. As an application of our methods, we obtain new, significantly simplified proofs of p-adic Poincaré duality and the preservation of $\mathbf{F}_p$-local systems under smooth proper higher direct images.

Contents

1. Introduction

1.1. **Main results.** Let K be a nonarchimedean field of characteristic 0 and residue characteristic $p \geq 0$. Set $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for some integer $n > 0$. When n is coprime to p, Berkovich and Huber independently developed in [Ber93] and [Hub96] (see also [dJvdP96]) a robust theory of étale cohomology of rigid-analytic spaces over K which shares most of the nice properties of the algebraic theory of étale cohomology developed in [AGV71, Del77, SGA77].

However, things become significantly more complicated when $n = p$. Many of the most basic properties fail: for example, finiteness of (compactly) supported cohomology (see Lemma 5.5.21 and Remark 5.1.14), proper base change, etc. On the other hand, Scholze recently proved in his seminal paper [Sch13a] that the $\mathbf{F}_p$-cohomology groups of smooth *proper* rigid-analytic spaces are finite dimensional.[1] This led to a significant interest in studying p-adic étale cohomology groups in p-adic analytic geometry. For instance, [Zav21b] and [Man22] showed that smooth and proper rigid-analytic spaces satisfy Poincaré duality for $\mathbf{F}_p$-local systems (see also [LLZ23] and [CGN23] for rational variants of duality), while [BH22] developed a robust theory of Zariski-constructible sheaves, dualizing complexes, Verdier duality, and perverse t-structures.

Despite all these advances, one question that has remained open is whether there is a relative version of Poincaré duality with coefficients (more general than local systems). In [BH22], Bhatt and Hansen put forward a conjecture that relates the behavior of derived proper pushforward and dualizing complexes (in the sense of [BH22, Th. 3.21]). Our main result is the proof of this conjecture:

Theorem 1.1.1 (Bhatt–Hansen's conjecture, Theorem 7.5.18)**.** *Let $f\colon X \to Y$ be a proper morphism of rigid-analytic spaces over K, and let ω_X and ω_Y be the dualizing complexes on X and Y respectively. Then there is a canonical trace morphism $\mathrm{Tr}_f\colon \mathrm{R}f_*\omega_X \to \omega_Y$ such that the induced duality morphism*

$$\mathrm{PD}_f\colon \mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{F}, \omega_X) \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{F}, \mathrm{R}f_*\omega_X) \xrightarrow{\mathrm{Tr}_f \circ -} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{F}, \omega_Y)$$

is an equivalence for any $\mathcal{F} \in D_{\mathrm{zc}}(X_{\acute{e}t}; \Lambda)$. In other words, derived pushforward along a proper morphism commutes with Verdier duality on Zariski-constructible complexes.

First, we want to note that, if n is coprime to p, then Theorem 1.1.1 follows from [BH22, Th. 3.21]. In fact, *loc. cit.* implies that Theorem 1.1.1 admits a compactly supported version for an arbitrary taut separated f and arbitrary coefficients. However, a version of Theorem 1.1.1 for non-proper f (or proper f and non-Zariski-constructible $\mathcal{F}$) fails miserably when $n = p$ (see Remark 6.4.11). This lack of a local version makes the proof of Theorem 1.1.1 quite difficult for two (somewhat related) reasons: one cannot run standard arguments to reduce to the case of a relative affine (or projective) line and, more importantly, the definition of the dualizing complex ω_X is *local* on X, so it is not well-adapted for proving global results like Theorem 1.1.1. For these reasons, our approach to Theorem 1.1.1 is completely different from [BH22, Th. 3.21] and from the classical approach in algebraic geometry. Moreover, it works uniformly for all n, divisible by p or not.

Another issue we want to point out is that, in order to formulate Theorem 1.1.1 precisely, one first needs to construct a trace morphism. In fact, constructing a trace map satisfying some sufficiently nice properties

[1]The smoothness assumption was later removed in [Sch13b, Th. 3.17].

is one of the key steps in the proof of Theorem 1.1.1. To do this, we first develop a robust theory of trace morphisms for smooth (but not necessarily proper) morphisms and revisit Poincaré duality for smooth proper morphisms by giving a new easy and essentially diagrammatic proof.

Before we discuss these results in more detail in the next subsection, we want to mention several immediate corollaries of Theorem 1.1.1 which look more similar to the classical Poincaré duality results. First, we note that Theorem 1.1.1 implies a version of Poincaré duality for some class of smooth non-proper rigid-analytic spaces:

Corollary 1.1.2 (Corollary 7.5.26)**.** *Let $\overline{X}$ be a proper rigid-analytic space over K and $X \subset \overline{X}$ be a smooth Zariski-open rigid-analytic subspace of equidimension d. Let $\mathbf{L}$ be a local system of finite free Λ-modules on $X_{\text{ét}}$ and let $\mathbf{L}^\vee$ be its Λ-linear dual. Set $C \coloneqq \widehat{\overline{K}}$. Then the groups $\mathrm{H}^i_c(X_C, \mathbf{L})$ and $\mathrm{H}^{2d-i}(X_C, \mathbf{L}^\vee)$ are finite and there is a Galois-equivariant isomorphism*

$$\mathrm{H}^i_c(X_C, \mathbf{L})^\vee \simeq \mathrm{H}^{2d-i}(X_C, \mathbf{L}^\vee)(d)$$

which is functorial in $\mathbf{L}$.

As a second application of Theorem 1.1.1, we prove another conjecture of Bhatt and Hansen predicting duality of intersection cohomology on proper rigid-analytic spaces (see [BH22, Paragraph after Th. 4.13]). In fact, we show a slightly stronger statement:

Corollary 1.1.3 (Bhatt–Hansen's conjecture, Theorem 7.5.22)**.** *Let $\overline{X}$ be a proper rigid-analytic space over K and $U \subset X \subset \overline{X}$ be two rigid-analytic subspaces which are both Zariski-open in $\overline{X}$. Assume that U is smooth of equidimension d. Let $\mathbf{L}$ be a local system of finite free Λ-modules on $U_{\text{ét}}$ and let $\mathbf{L}^\vee$ be its Λ-linear dual. Set $C \coloneqq \widehat{\overline{K}}$. Then the groups $\mathrm{IH}^i_c(X_C, \mathbf{L})$ and $\mathrm{IH}^{-i}(X_C, \mathbf{L}^\vee)$ are finite and there is a Galois-equivariant isomorphism*

$$\mathrm{IH}^i_c(X_C, \mathbf{L})^\vee \simeq \mathrm{IH}^{-i}(X_C, \mathbf{L}^\vee)(d)$$

which is functorial in $\mathbf{L}$.

1.2. **Trace and duality for smooth maps.** In this subsection, we discuss our main duality results for smooth morphisms. Unlike in the previous subsection, the results of this subsection hold for arbitrary locally noetherian[2] analytic adic spaces. For this subsection, we fix an integer $n > 0$ and put $\Lambda = \mathbf{Z}/n\mathbf{Z}$.

We start by discussing the construction of trace maps for separated taut smooth morphisms.[3]

Theorem 1.2.1 (Theorem 6.1.1, Proposition 6.2.4, Corollary 6.2.7, Lemma 6.2.8)**.** *There is a unique way to assign to any separated taut smooth of equidimension d morphism $f \colon X \to Y$ of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$ a trace map $\mathrm{tr}_f \colon \mathrm{R}f_!\underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$ in $D(Y_{\text{ét}}; \Lambda)$ such that:*

1. tr *is compatible with compositions;*
2. tr *is compatible with pullbacks;*
3. *if f is étale, then tr_f is given by the counit*
$$\mathrm{R}f_!\underline{\Lambda}_X \simeq \mathrm{R}f_!f^*\underline{\Lambda}_X \to \underline{\Lambda}_Y$$
of the adjunction between $\mathrm{R}f_!$ and f^*; and
4. *If f is the analytification of the structure morphism $\mathbf{P}^1_C \to \operatorname{Spec} C$ for some complete, algebraically closed nonarchimedean field C, then tr_f is the analytification of the algebraic trace.*

Furthermore, these trace maps satisfy the following compatibilities:

(1) whenever A is a strongly noetherian Tate affinoid algebra and f is the analytification of a finite type separated smooth of equidimension d morphism of locally finite type A-schemes, our tr_f is the analytification of the algebraic trace map;

[2] We have to impose the locally noetherian assumption only because [Hub96] works out general theories of smooth morphisms and étale cohomology of analytic adic spaces under the locally noetherian assumption. We never use this noetherianness assumption in any serious way.

[3] We also impose the taut separated assumption on f simply because the $\mathrm{R}f_!$-functor has been defined only for such morphisms in [Hub96]. Theorem 1.2.1 can be formally extended to all smooth morphism of equidimension d once one works out a robust theory of $\mathrm{R}f_!$ for a general morphism f of finite type (see [Zav23a, Th. 9.4] for the case when $n \in (\mathcal{O}_Y^+)^\times$).

(2) *whenever K is a nonarchimedean field and $f\colon X \to Y$ is a partially proper smooth morphism of equidimension d between rigid-analytic spaces, our tr_f coincides with the Berkovich trace t_f from* [Ber93, Th. 7.2.1] *(see also* [Zav21b, Th. 5.3.3] *for the translation into the language of adic spaces);*
(3) tr_f *is compatible with the trace of Lan–Liu–Zhu from* [LLZ23, Th. 1.3] *whenever the latter is defined.*

When n is invertible in $\mathcal{O}_Y^+$, the trace map was previously constructed by Huber in [Hub96, Th. 7.3.4]. When n is only invertible in $\mathcal{O}_Y$ and $f\colon X \to Y$ is a *partially proper* smooth morphism of rigid-analytic spaces over a non-archimedean field K, the trace map f was constructed by Berkovich in [Ber93, Th. 7.2.1]. Our construction of the trace map is independent of either of these constructions[4] and, as we explain after Remark 1.2.2, it crucially uses the techniques of universal compactifications and the existence of higher rank points, both of which are only available in Huber's formalism of adic spaces.

Remark 1.2.2. Our main motivation for developing a general theory of smooth trace maps comes from the needs of our proof of Theorem 1.1.1. Indeed, to prove Theorem 1.1.1, we need to construct proper trace maps with coefficients in dualizing complexes, for which the full strength of Theorem 1.2.1 is used. Namely, even though we define trace maps with coefficients in dualizing complexes only for proper maps, it is indispensable for the construction to have smooth trace available for non-partially proper morphisms. We elaborate on this more in Section 1.3.

We now explain the main ideas behind the construction of tr_f in Theorem 1.2.1. A dévissage similar to the one in algebraic geometry [AGV71, Exp. XVII] allows us to reduce the construction of smooth trace maps in general to the situation when $f\colon X \to Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ is a smooth connected affinoid curve over an algebraically closed nonarchimedean field C. In this case, we crucially use the geometry of adic spaces: The complement of X inside its universal compactification X^c is a pseudo-adic space that consists of finitely many points corresponding to valuations of rank 2. The second components of these valuations give rise to a map $\mathrm{H}^1(X^c \smallsetminus X, \mu_n) \to \mathbf{Z}/n$. We then show that this map descends to an analytic trace map $\mathrm{tr}_X\colon\ \mathrm{H}^2_c(X, \mu_n) \to \mathbf{Z}/n$ via the exact excision sequence

$$\mathrm{H}^1(X, \mu_n) \to \mathrm{H}^1(X^c \smallsetminus X, \mu_n) \to \mathrm{H}^2_c(X, \mu_n) \to 0$$

and that the thus constructed analytic trace maps are compatible with étale morphisms and (algebraic) trace maps for algebraic curves. The verification of these claims is extremely subtle and occupies most of Section 5.

We note that it is somewhat surprising that the trace map exists when n is not invertible in $\mathcal{O}_Y^+$ due to the observation that, for a smooth connected affinoid rigid-analytic curve over an algebraically closed nonarchimedean field C, the top degree compactly supported cohomology group $\mathrm{H}^2_c(X, \mu_p)$ behaves pretty wildly. In fact, the group $\mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p)$ is already quite pathological due to the following observations:

Remark 1.2.3. Even though the truncated smooth trace $\mathrm{tr}_{\mathbf{D}^1_C}\colon\ \mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p) \to \mathbf{F}_p$ is still an epimorphism (see Lemma 6.2.3), it is certainly not an isomorphism in contrast to the situation in algebraic geometry (or when $n \in (\mathcal{O}_Y^+)^\times$). Furthermore, the trace map does not induce any kind of "weak" Poincaré duality in general (see Remark 6.4.11) and the group $\mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p)$ is infinite (see Lemma 5.5.21) and depends on the choice of the algebraically closed ground field C (see Example 6.3.3). Moreover, different points $x, y \in \mathbf{D}^1_C$ might have different cycle classes in $\mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p)$ (see Lemma 5.5.21) and it is unclear if they generate the entire cohomology group. As a consequence, many of the usual tricks familiar from algebraic geometry cannot be applied anymore.

Once we have a trace morphism at hand, we can give an easy and essentially formal proof of Poincaré duality for smooth proper morphisms and locally constant coefficients; many cases were treated before in [Ber93, Th. 7.3.1], [Hub96, Cor. 7.5.5], [Zav21b, Th. 1.1.2], and [Man22, Cor. 3.10.22]:

Theorem 1.2.4 (Theorem 6.4.1, Theorem 6.4.10)**.** *Let $f\colon X \to Y$ be a smooth proper morphism of equidimension d between locally noetherian analytic adic spaces such that $n \in \mathcal{O}_Y^\times$. Let $\mathcal{E} \in D_{\mathrm{lis}}(X_{\text{ét}}; \Lambda)$ with dual $\mathcal{E}^\vee := \mathrm{R}\mathscr{H}om(\mathcal{E}, \Lambda)$. Then duality morphism*

$$\mathrm{PD}_f\colon \mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E}, \underline{\Lambda}_X(d)[2d]) \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E}, \mathrm{R}f_*\underline{\Lambda}_X(d)[2d]) \xrightarrow{\mathrm{tr}_f\,\circ -} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E}, \underline{\Lambda}_Y)$$

[4] In fact, the construction of Huber is very specific to the case of n being invertible in $(\mathcal{O}_Y^+)^\times$, while the construction of Berkovich is very specific to the partially proper case.

is an isomorphism.

When n is invertible in $\mathcal{O}_Y^+$, Theorem 1.2.4 was first proven by Berkovich in [Ber93, Th. 7.3.1] and by Huber in [Hub96, Cor. 7.5.5] independently (and it was later revisited in [Zav23b, Th. 1.3.2]). When X and Y are rigid-analytic spaces over a nonarchimedean field K of mixed characteristic $(0, p)$ and $n = p$, Theorem 1.2.4 was proven in [Zav21b] and [Man22] independently. Both proofs crucially rely on the theory of perfectoid spaces, $\mathcal{O}^+/p$-cohomology groups, and the Grothendieck–Serre duality in characteristic p. In particular, the previous proofs only apply either when $(n, p) = 1$ or when $n = p$ and the strategies in the two cases are completely different.

In contrast, our proof of Theorem 1.2.4 is different from any of the four proofs mentioned above (instead, it is somewhat motivated by the proof presented in [Zav23b, Th. 1.3.2]). It uses a bare minumum of the perfectoid theory, works uniformly for any integer $n \in \mathcal{O}_Y^\times$, and is essentially diagrammatic once the trace map is constructed.

These methods are quite formal and, thus, they can be used in different cohomological setups. For instance, in an ongoing joint project with Nizioł, we expect to generalize the techniques developed in this paper to prove a version of Poincaré duality for pro-étale $\mathbf{Q}_p$-local systems on smooth proper X.

We now explain the main ideas behind our proof. We first reduce the question to showing that, for a dualizable $\mathcal{E} \in D(X_{\text{ét}}; \Lambda)$, the complex $\mathrm{R}f_*\mathcal{E}$ is dualizable in $D(Y_{\text{ét}}; \Lambda)$ with the dual given by $\mathrm{R}f_*\mathcal{E}^\vee(d)[2d]$. Then we need to construct the evaluation and coevaluation morphisms and check that certain compositions are the identity. The evaluation map essentially comes from the trace map constructed in Theorem 1.2.1, while the coevaluation map essentially comes from the Künneth isomorphism established in Corollary 6.3.9[5] and the cycle class map of the diagonal (see Section 3 and Construction 6.4.2). The verification that certain compositions are equal to the identity boil down to the computation that $\mathrm{tr}_{\mathrm{pr}}(c\ell_\Delta) = \mathrm{id}$ for a projection $\mathrm{pr}\colon X \times_Y X \to X$ and the diagonal morphism $\Delta\colon X \hookrightarrow X \times_Y X$ and to the fact that the braiding morphism on $\left(\underline{\Lambda}_{X\times_Y X}(d)[2d]\right)^{\otimes 2}$ is the identity morphism.

As a formal consequence of our proof of Theorem 6.4.10, we also get that derived pushforwards along smooth and proper morphisms preserve locally constant sheaves:

Corollary 1.2.5 (Corollary 6.4.8)**.** *Let $f\colon X \to Y$ be a smooth proper morphism of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$. Let $\mathcal{E} \in D_{\mathrm{lis}}(X_{\text{ét}}; \Lambda)$ be a lisse complex. Then $\mathrm{R}f_*\mathcal{E}$ lies in $D_{\mathrm{lis}}(Y_{\text{ét}}; \Lambda)$. If $\mathcal{E}$ is locally bounded (resp. perfect), then so is $\mathrm{R}f_*\mathcal{E}$.*

If $n \in (\mathcal{O}_Y^+)^\times$, Corollary 1.2.5 was first shown in [Hub96, Cor. 6.2.3] under some extra assumptions and also recently revisited in [Zav23b, App. 1.3.4(4)] in full generality. If Y is a rigid-analytic space over $\mathrm{Spa}\,(K, \mathcal{O}_K)$ and $n = p$ is equal to the characteristic of the residue field of $\mathcal{O}_K$, this result was shown in [SW20, Th. 10.5.1], using the full strength of the perfectoid and diamond machinery. In contrast to these two proofs, our proof is uniform in n, is essentially formal, and remains largely in the world of locally noetherian analytic adic spaces.

1.3. **Trace and duality for proper maps.** In this section, we go back to the discussion of Theorem 1.1.1 and explain the main ideas behind its proof. We fix a nonarchimedean field K of characteristic 0 and residue characteristic $p \geq 0$. Set $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for some integer $n > 0$.

For general (not necessarily smooth) proper morphisms of rigid-analytic spaces over K, one cannot expect to have trace maps with constant coefficients as in Theorem 1.2.1. Instead, it is more reasonable to expect trace maps with coefficients in the dualizing complexes $\omega_X \in D^b_{zc}(X; \Lambda)$; see [BH22, Th. 3.21] for their definition. One first main result is the actual construction of such morphisms:

Theorem 1.3.1 (Digest of Section 7.4)**.** *For any proper morphism $f\colon X \to Y$ of rigid-analytic spaces over K, the complex $\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$. Furthermore, one can assign to every such proper morphism $f\colon X \to Y$ a trace map $\mathrm{Tr}_f\colon \mathrm{R}f_*\omega_X \to \omega_Y$ such that:*

(1) Tr *is compatible with compositions;*
(2) Tr_f *is étale local on Y;*
(3) if f is a closed immersion, then Tr_f is adjoint to the natural isomorphism $\omega_X \xrightarrow{\sim} \mathrm{R}f^!\omega_Y$ from [BH22, Th. 3.21(1)] *(see Construction 7.2.1);*

[5]The proof of Corollary 6.3.9 is the only place which needs, via [BH22, Lem. 3.25], the perfectoid machinery; see Remark 6.3.2.

(4) if f is smooth and proper, Tr_f *is compatible with the smooth trace map from Theorem 1.2.1 (via Construction 7.2.7);*

(5) Tr *is compatible with extensions of base fields.*

Just like in Theorem 1.2.1, we can modify the list of compatibility axioms in Theorem 1.3.1 to characterize our proper traces uniquely; see Theorem 7.4.1. We do not do so here in order to avoid complicated notations.

We point out that the proof of Theorem 1.1.1 is not so difficult given the construction of trace maps from Theorem 1.3.1. Indeed, Theorem 1.3.1 (3) eventually allows us to reduce to the case $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$. In this case, we use resolution of singularities, explicit generators in $D_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$, and the compatibilities of Tr_f from Theorem 1.3.1 again to eventually reduce the question to the (very special case of) Theorem 1.2.4.

Remark 1.3.2. Theorem 1.2.1 suggests that it is reasonable to expect that one can assign a reasonable trace morphism $\mathrm{Tr}_f \colon \mathrm{R}f_! \omega_X \to \omega_Y$ to any taut separated morphism $f \colon X \to Y$ of rigid-analytic spaces over K. The main obstacle to apply our method for such f is that we do not know whether $\mathrm{R}\mathscr{H}om(\mathrm{R}f_! \omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$ in such generality (see Remark 7.3.15 for more detail). It would certainly be interesting to have trace maps constructed in greater generality, even though the analog of Theorem 1.1.1 cannot hold for nonproper f. Therefore, we decided not to pursue this direction in this paper.

Now we mention the main ideas that go into the proof of Theorem 1.3.1. We first construct the trace map when X is smooth and Y is quasicompact and separated. In this case, we factor f through its graph Γ_f as

$$X \xhookrightarrow{\Gamma_f} X \times Y \xrightarrow{\mathrm{pr}_2} Y.$$

Since Γ_f is a closed embedding by our separatedness assumption, it has a closed trace map thanks to [BH22, Th. 3.21(1)] (cf. Construction 7.2.1). On the other hand, pr_2 is smooth taut separated, hence it has a trace map thanks to Theorem 1.2.4 (cf. Construction 7.2.7). The composition of these traces gives a trace map for f; we call it the *smooth-source trace*.

Remark 1.3.3. We want to emphasize that the morphism pr_2 is not (partially) proper unless X is (partially) proper. For this reason, Theorem 1.2.1 for partially proper morphisms is inadequate for the purpose of proving Theorem 1.3.1 (at least via our methods).

Then we use resolution of singularities, the constructed above smooth-source trace, and some dévissage to reduce the claim that $\mathrm{R}\mathscr{H}om(\mathrm{R}f_* \omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$ to the case when $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$ and X is smooth. In this case, this coconnectivity claim boils down to the classical estimates on the cohomological dimension of X (see [Hub96, Prop. 5.5.8]).

Finally, we use the coconnectivity established above to reduce the question to the case of quasicompact separated Y. Then we use resolution of singularities and the smooth-source trace again to reduce to the case when, in addition, X is smooth of equidimension d. In this case, the smooth-source trace does the job again. After that, we have to figure out all the desired compatilibities which requires quite delicate arguments and diagram chases; this occupies the most of the proof of Theorem 7.4.1.

Organization of the paper. In Section 2, Section 3, and Section 4, we provide some technical background on finite morphisms, trace maps for finite flat morphisms, cycle classes, and curves in the setting of analytic adic geometry, for which we often could not find suitable references in the literature. We recommend the reader to skip these sections on the first reading. Section 5 is one of the key sections of the paper, in which we construct analytic trace map for smooth affinoid rigid-analytic curves and verify its properties. The first half of Section 6 extends the construction of smooth traces to an arbitrary separated smooth taut morphism of equidimension d, while the second half of this section gives our "diagrammatic" proof of Theorem 1.2.4. This is another key novelty of this paper (an impatient reader may start reading there and take for granted the existence of smooth trace map). Section 7 discusses the construction of proper traces and general Poincaré duality for proper morphisms, leading up to the proofs of our main results Theorem 1.3.1 and Theorem 1.1.1.

Acknowledgments. We would like to thank Bhargav Bhatt, David Hansen, Sasha Petrov, and Peter Scholze for useful conversations.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1926686. In addition, S.L. was also supported by the National Natural Science Foundation of China

(No. 12288201) and during a stay at the Institute for Advanced Study by the Ky Fan and Yu-Fen Fan Endowed Fund. We are grateful to the following institutions for their financial support and for providing excellent working conditions while we completed parts of this project: the School of Mathematics of the Institute for Advanced Study (S.L., E.R., B.Z.), the University of Michigan (S.L.), the Morningside Center of Mathematics (S.L.), the Max Planck Institute for Mathematics in Bonn (E.R., B.Z.), the Institute des Hautes Études Scientifiques (E.R.), and Princeton University (B.Z.).

Notation and conventions. In this paper, a topological field K is called *nonarchimedean* if its topology is induced by a valuation of rank 1 on K and if K is complete with respect to this topology; beware that while the completeness assumption is somewhat standard, it is not imposed in [Hub96, Def. 1.1.3]. We usually denote the ring of integers of K by $\mathcal{O}_K$, its maximal ideal by $\mathfrak{m}_K$, and its residue field by $k := \mathcal{O}_K/\mathfrak{m}_K$.

A *rigid-analytic space* over a nonarchimedean field K is always understood to be an adic space which is locally of finite type over $\mathrm{Spa}\,(K, \mathcal{O}_K)$; by [Hub94, Prop. 4.5], the resulting category of quasiseparated rigid-analytic K-spaces is equivalent to the category of quasiseparated rigid-analytic K-varieties in the classical sense as in, say, [BGR84, Def. 9.3.1/4]. An analytic adic space is *locally noetherian* if every $x \in X$ is contained in an open affinoid subspace $U \subset X$ for which $\mathcal{O}(U)$ is a strongly noetherian Tate ring. An *affinoid field* means a Huber pair (k, k^+) such that k is a field and $k^+ \subset k$ is an open and bounded microbial valuation ring; it is not assumed to be complete. Note that this definition is slightly narrower than [Hub96, Def. 1.1.5], which also allows k to be discrete.

An *admissible* formal $\mathcal{O}_K$-scheme is a flat, locally finitely presented formal $\mathcal{O}_K$-scheme $\mathscr{X}$. To any formal scheme $\mathscr{X}$ which is locally of finite presentation over $\mathrm{Spf}(\mathcal{O}_K)$, we attach the *special fiber* $\mathscr{X}_s := \mathscr{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spec}\,(k)$, which is a locally finitely presented k-scheme, and the *rigid-analytic generic fiber* $\mathscr{X}_\eta$ in the sense of [Hub96, Prop. 1.9.1], which is a quasiseparated rigid-analytic space over $\mathrm{Spa}\,(K, \mathcal{O}_K)$. Given a rigid-analytic space X over $\mathrm{Spa}\,(K, \mathcal{O}_K)$, any admissible $\mathscr{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$ with $\mathscr{X}_\eta \simeq X$ is called a *formal model* of X.

We usually denote the points of an adic space X by x, y, etc. and valuations in their equivalence class by v_x, v_y, etc. Following Huber, we use multiplicative notation for valuations and value groups.

Given a locally noetherian analytic adic space X, the d-dimensional *(closed) unit disk* over X is defined as $\mathbf{D}^d_X := \mathrm{Spa}\,(\mathbf{Z}[T_1, \dots, T_d], \mathbf{Z}[T_1, \dots, T_d]) \times_{\mathrm{Spa}\,(\mathbf{Z},\mathbf{Z})} X$. Alternatively, one can set $\mathbf{D}^d_X := \mathrm{Spa}\,(A\langle T\rangle, A\langle T\rangle^+)$ when $X = \mathrm{Spa}\,(A, A^+)$ is affinoid; since this construction is functorial in X, this gives $\mathbf{D}^d_X$ for general X via gluing. Likewise, the d-dimensional *open unit disk* over X is $\mathring{\mathbf{D}}^d_X := \mathrm{Spa}\,(\mathbf{Z}[\![T_1, \dots, T_d]\!], \mathbf{Z}[\![T_1, \dots, T_d]\!]) \times_{\mathrm{Spa}\,(\mathbf{Z},\mathbf{Z})} X$ and the d-dimensional *affine space* over X is $\mathbf{A}^{d,\mathrm{an}}_X := \mathrm{Spa}\,(\mathbf{Z}[T_1, \dots, T_d], \mathbf{Z}) \times_{\mathrm{Spa}\,(\mathbf{Z},\mathbf{Z})} X$. When the base is understood from the context (mainly over an algebraically closed field), we drop it from the notation and simply write $\mathbf{D}^d$, $\mathring{\mathbf{D}}^d$, $\mathbf{A}^{d,\mathrm{an}}$, etc.

Given a locally noetherian analytic adic space X and a finite commutative ring Λ, we denote the (triangulated) derived category of étale sheaves of Λ-modules on X by $D(X_{\text{ét}}; \Lambda)$. We write $D^{(b)}(X_{\text{ét}}; \Lambda)$ for the full subcategory spanned by locally bounded complexes and $D_{\mathrm{lis}}(X_{\text{ét}}; \Lambda)$ for the full subcategory spanned by complexes with lisse cohomology sheaves. When X is a rigid-analytic space over K, we also consider the full subcategory $D_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$ spanned by complexes with Zariski-constructible cohomology sheaves. If $f\colon X \to Y$ is a morphism of locally noetherian analytic adic spaces and f_* (for f finite) or $f_!$ (for f étale) is exact, we often drop the "R" from the notation of the associated derived functors $\mathrm{R}f_*$ or $\mathrm{R}f_!$, respectively.

We denote the unit of an adjunction of functors $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ by $\eta\colon \mathrm{id} \to G \circ F$ and its counit by $\epsilon\colon F \circ G \to \mathrm{id}$. In the special case when $f\colon X \to Y$ is a morphism of locally noetherian analytic adic spaces and $(F, G) = (f^*, \mathrm{R}f_*)$ or (for f étale) $(F, G) = (\mathrm{R}f_!, f^*)$ or (for f finite) $(F, G) = (f_*, \mathrm{R}f^!)$, we also use η_f and ϵ_f. We denote *canonical* isomorphisms by $\simeq$ and *noncanonical* isomorphisms by $\cong$.

2. Preliminaries

2.1. **Valuation rings.** In this subsection, we collect some facts about valuation rings that we will use throughout the paper. We assume that most results of this subsection are well-known to the experts, but it seems difficult to extract them from the existing literature.

We start by recalling the following classical definition:

Definition 2.1.1. Let $x \in X$ be a point of an analytic adic space $(X, \mathcal{O}_X, \{v_x\})$. Then

(1) the *residue field* $k(x)$ is the residue field of the local ring $\mathcal{O}_{X,x}$ and $k(x)^+ \subset k(x)$ is a valuation ring associated to the valuation v_x of $\mathcal{O}_{X,x}$. Then $\big(k(x), k(x)^+\big)$ is a (non-complete) affinoid field;

(2) the *completed residue field* $\widehat{k(x)}$ is the topological completion of $k(x)$. This comes with a canonical valuation subring $\widehat{k(x)}^+ \subset \widehat{k(x)}$ such that pair $\big(\widehat{k(x)}, \widehat{k(x)}^+\big)$ is a (complete) affinoid field.

Remark 2.1.2. If $\varpi \in k(x)^+$ is a pseudo-uniformizer, then [Hub93a, Lem. 1.6] ensures that $\widehat{k(x)}^+$ is the usual ϖ-adic completion of $k(x)^+$ and $\widehat{k(x)} = \widehat{k(x)}^+\big[\frac{1}{\varpi}\big]$.

We first classify all connected adic spaces which are finite over the adic spectrum of a complete affinoid field:

Lemma 2.1.3. *Let $Y = \mathrm{Spa}\,(K, K^+)$ be the adic spectrum of a complete affinoid field (K, K^+). Let y be the closed point of Y corresponding to a valuation $v\colon K \to \Gamma_v \cup \{0\}$, $\mathfrak{m} \subset K^+$ the maximal ideal. Let X be a reduced and connected adic space and let $f\colon X \to Y$ be a finite morphism. Then*

(i) *$X = \mathrm{Spa}\,(A, A^+)$ is an affinoid space with $A = L$ being a finite field extension of K, and A^+ being the integral closure[6] of K^+ in L;*

(ii) *the pre-image $f^{-1}(y)$ is equal to the set of valuations of L that extend v;*

(iii) *for each $x \in f^{-1}(y)$, we have $\mathrm{rk}\, x = \mathrm{rk}\, y$;*

(iv) *there is an equality $A^+ = \cap_{x\in f^{-1}(y)} L^+_x$, where L^+_x is the valuation ring of (the valuation corresponding to) x, i.e., $L^+_x = \{a \in L \mid v_x(a) \le 1\}$.*

(v) *for each $x \in f^{-1}(y)$, we have $\widehat{k(x)}^+ = L^+_x$.*

(vi) *A^+ is semi-local, and all maximal ideals are given by $\mathfrak{m}_x := \mathfrak{m}^+_x \cap A^+$ for $x \in f^{-1}(y)$ and $\mathfrak{m}^+_x \subset L^+_x$ the corresponding maximal ideal. Furthermore, the natural morphism $A^+_{\mathfrak{m}_x} \to L^+_x$ is an isomorphism for each $x \in f^{-1}(y)$;*

(vii) *we have $\mathrm{rad}(\mathfrak{m}A^+) = \bigcap_{x\in f^{-1}(y)} \mathfrak{m}_x$.*

Proof. (i) First, we note that [Hub93b, Satz 3.6.20 and Korollar 3.12.12] imply that $X = \mathrm{Spa}\,(A, A^+)$ is an affinoid and $(K, K^+) \to (A, A^+)$ is a finite morphism of Huber pairs, i.e., A is a finite K-algebra and A^+ is the integral closure of K^+ in A. Furthermore, the assumptions on X imply that A is a reduced finite K-algebra without idempotents. This implies that A must be a field L such that $K \subset L$ is a finite extension.

(ii) Now we note that, by definition of an adic space, we can identify $f^{-1}(y)$ with the set of valuation subrings $R_w \subset L$ such that

(A) $K^+ \subset R_w$ and the morphism $K^+ \to R_w$ is local;

(B) the corresponding valuation $w\colon L \to L^\times/R_w^\times \cup \{0\} = \Gamma_w \cup \{0\}$ is continuous;

(C) $w(A^+) \le 1$.

Therefore, for the purpose of proving (2), it suffices to show that condition **(A)** implies **(B)** and **(C)**. In other words, we need to show that any valuation w of L that extends v is automatically continuous and satisfies $w(A^+) \le 1$.

Since $w|_K = v$, $v(K^+) \le 1$, and A^+ is integral over K^+, we conclude that $w(A^+) \le 1$ as well. Therefore, it suffices to show continuity of w. We choose some compatible rings of definition $K_0 \subset K$, $A_0 \subset A^+$, and a pseudo-uniformizer $\varpi \in K_0$. Therefore, [Sem15, Cor. 9.3.3] ensures that it suffices to show that $w(\varpi)$ is cofinal in Γ_w and $w(\varpi) < w(a)$ for any $a \in A_0$.

First, [Bou98, Ch. VI, § 8, n. 1, Prop. 1] gives that Γ_w/Γ_v is a torsion group. Therefore $w(\varpi) = v(\varpi)$ is cofinal in Γ_w since it is cofinal in Γ_v due to [Sem15, Cor. 9.3.3]. In particular, $w(\varpi) < 1$. Now we note that $w(A^+) \le 1$, $v|_{K^+} = w|_{K^+}$, and thus

$$w(a\varpi) = w(a)w(\varpi) < w(a) \le 1$$

for any $a \in A_0 \subset A^+$. Therefore, we conclude that w is continuous.

(iii) This follows directly from [Bou98, Ch. VI, § 8, n. 1, Cor. 1].

[6] We warn the reader that A^+ is not necessarily a valuation ring unless K^+ is henselian along its maximal ideal.

(iv) This follows from (2) and [Mat89, Exercise 10.3].

(v) We first note, for every $x \in f^{-1}(y)$, [Sem15, L.14, "Caveat on residue fields" on pp. 2–3] implies that $\widehat{k(x)} = \widehat{L/\mathrm{supp}(x)} = \widehat{L} = L$ since $\mathrm{supp}(x) = (0)$ and L is already a complete field. Therefore, L_x^+ and $\widehat{k(x)}^+$ are both equal to the valuation rings defined as $\{a \in L = \widehat{k(x)} \mid v_x(a) \le 1\}$.

(vi) This follows directly from (2) and [Bou98, Ch. VI, § 8, n. 6, Prop. 6].

(vii) Since the ideal $\bigcap_{x \in f^{-1}(y)} \mathfrak{m}_x$ is radical, the equality $\mathrm{rad}(\mathfrak{m}A^+) = \bigcap_{x \in f^{-1}(y)} \mathfrak{m}_x$ means that the fiber of $\mathrm{Spec}\, A^+ \to \mathrm{Spec}\, K^+$ over the closed point consists exactly of the closed points in $\mathrm{Spec}\, A^+$. This follows from [Mat89, Th. 9.3(ii) and Th. 9.4(i)] □

Now we discuss the definition and basic properties of henselian affinoid fields.

Definition 2.1.4. An affinoid field (K, K^+) is *henselian* if K^+ is henselian with respect to its maximal ideal $\mathfrak{m} \subset K^+$.

Remark 2.1.5. We note that [EP05, Th. 4.1.3] implies that (K, K^+) is henselian in the sense of Definition 2.1.4 if and only if it is henselian in the sense of [EP05, p. 86]. In other words, (K, K^+) is henselian if and only if its valuation v uniquely extends to any finite field extension $K \subset L$.

It turns out that we can always canonically make any affinoid field into a henselian one.

Definition 2.1.6. Let (K, K^+) be an affinoid field. Its *henselization* is an affinoid field $\left(K^{\mathrm{h}}, K^{+,\mathrm{h}}\right)$ where $K^{+,\mathrm{h}}$ is the henselization of K^+ with respect to its maximal ideal and $K^{\mathrm{h}} = K^{+,\mathrm{h}} \otimes_{K^+} K$.

We note that $K^{+,\mathrm{h}}$ is a valuation ring by [Sta22, Tag 0ASK], and it is microbial since K^+ is. Thus $(K^{\mathrm{h}}, K^{+,\mathrm{h}})$ is indeed an affinoid field.

Warning 2.1.7. The notation K^{h} may be a bit misleading because this object depends on the valuation subring $K^+ \subset K$ and not just on K as a topological field.

Definition 2.1.8. Let $x \in X$ be a point of an analytic adic space $(X, \mathcal{O}_X, \{v_x\})$. Then

(1) the *henselized residue field* $k(x)^{\mathrm{h}}$ is the henselization of $k(x)$. In particular, $\left(k(x)^{\mathrm{h}}, k(x)^{+,\mathrm{h}}\right)$ is a (henselian) affinoid field;

(2) the *henselized completed residue field* $\widehat{k(x)}^h$ is the henselization of $\widehat{k(x)}$ (see Definition 2.1.1). In particular, $\left(\widehat{k(x)}^{\mathrm{h}}, \widehat{k(x)}^{+,\mathrm{h}}\right)$ is a (henselian) affinoid field.

The following lemma is well-known in the rank-1 case. Even though it is probably also well-known in the higher rank case to the experts, we cannot find this explicitly stated in the literature and therefore include a proof here.

Lemma 2.1.9. *Let (K, K^+) be a henselian affinoid field with the valuation $v_K \colon K \to \Gamma_K \cup \{0\}$, and $i_{K/L} \colon K \hookrightarrow L$ a finite field extension with the (unique) compatible valuation $v_L \colon L \to \Gamma_L \cup \{0\}$. Then*[7]

$$\left(v_L(-)\right)^{[L:K]} = v_L\left(i_{K/L} \mathrm{Nm}_{L/K}(-)\right),$$

where $\mathrm{Nm}_{L/K} \colon L^\times \to K^\times$ is the norm map.

Proof. Step 1. *We assume that L/K is normal.* We pick an element $f \in L^\times$. Then [Bou03, Ch. 5, § 8, n. 3, Prop. 4] implies that we have

$$\mathrm{Nm}_{L/K}(f) = \Big(\prod_{\sigma \in \mathrm{Aut}(L/K)} \sigma(f)\Big)^{[L:K]_i},$$

[7] Recall that we use the multiplicative notation for the group structure on any value group Γ.

where $[L:K]_i$ is the inseparable degree extension of L/K. Since K is henselian, there is a unique extension of v_K to v_L, so we conclude that $v_L(\sigma(f)) = v_L(f)$ for any $\sigma \in \mathrm{Aut}(L/K)$. Therefore, we conclude that

$$\begin{aligned} v_L\left(i_{K/L}\mathrm{Nm}_{L/K}(-)\right) &= v_L\left(\Big(\prod_{\sigma\in \mathrm{Aut}(L/K)} \sigma(f)\Big)^{[L:K]_i}\right) \\ &= \left(v_L\Big(\prod_{\sigma\in \mathrm{Aut}(L/K)} \sigma(f)\Big)\right)^{[L:K]_i} \\ &= \left(v_L(f)\right)^{[L:K]_i[L:K]_s} = \left(v_L(f)\right)^{[L:K]}. \end{aligned}$$

Step 2. *General case.* We consider a normal closure $L \subset M$ of L, so M/K is normal. We denote by $i_{K/L}\colon K \to L$ the corresponding inclusion (and similarly for $i_{K/M}$ and $i_{L/M}$), and by $j_{K/L}\colon \Gamma_K \to \Gamma_L$ the induced morphism on the value groups (and similarly for $j_{K/M}$ and $j_{L/M}$).

Since M/K is normal, we already know that

$$\begin{aligned} \left(j_{L/M}\left(v_L\left(f\right)\right)\right)^{[L:K]} &= \left(v_M\left(i_{L/M}(f)\right)\right)^{[M:K]} \\ &= v_M\left(i_{K/M}\mathrm{Nm}_{M/K} i_{L/M}(f)\right) \\ &= v_M\left(i_{L/M} i_{K/L}\mathrm{Nm}_{L/K}\mathrm{Nm}_{M/L} i_{L/M}(f)\right) \\ &= v_M\left(i_{L/M} i_{K/L}\mathrm{Nm}_{L/K}\left(f^{[M:L]}\right)\right) \\ &= \left(j_{L/M} v_L\left(i_{K/L}\mathrm{Nm}_{L/K}(f)\right)\right)^{[M:L]}. \end{aligned} \tag{$\boxdot$}$$

Indeed, the first equality is formal. The second equality follows from Step 1 applied to M/K and $i_{L/M}f$. The third equality follows from the transitivity of Norm maps and the inclusion morphisms. The fourth equality follows from the formula $\mathrm{Nm}_{M/L} i_{L/M} f = f^{[M:L]}$. And the last equality is also formal.

Now we note that the morphism $j_{L/M}\colon \Gamma_L \to \Gamma_M$ is injective, and both Γ_L and Γ_M are torsion-free (since they are totally ordered abelian groups). Therefore, Equation ($\boxdot$) implies that $\left(v_L(f)\right)^{[L:K]} = v_L\left(i_{K/L}\mathrm{Nm}_{L/K}(f)\right)$ □

2.2. **Curve-like affinoid fields.** In this subsection, we define and study a particular class of curve-like affinoid fields. We will later show that "boundary" points on the universal compactification of a rigid-analytic curve are necessarily curve-like (see Lemma 4.2.5). Throughout the subsection, we fix a nonarchimedean field C with a rank-1 valuation $|.|\colon C \to \Gamma_C \cup \{0\}$. We denote by $\mathcal{O}_C \subset C$ the corresponding valuation ring and by $\mathfrak{m}_C \subset \mathcal{O}_C$ its maximal ideal.

For the following definition, we fix a henselian affinoid field (K, K^+) and a finite field extension $K \subset L$. Remark 2.1.5 and [Mat89, Exercise 10.3] ensure that the integral closure L^+ of K^+ in L is a henselian valuation ring. So we denote by $\mathfrak{m}_K \subset K^+$ and $\mathfrak{m}_L \subset L^+$ the unique maximal ideals of K^+ and L^+ respectively.

Definition 2.2.1 ([Bou98, Ch. 6, § 8, n. 1])**.** The *ramification index* $e(L/K)$ is the cardinality of $|\Gamma_L/\Gamma_K|$. The *residue class* $f(L/K)$ is the degree $[L^+/\mathfrak{m}_L : K^+/\mathfrak{m}_K]$.

Remark 2.2.2. Note that [Bou98, Ch. 6, § 8, n. 1, Lem. 2] implies that we have an inequality $e(L/K)f(L/K) \leq [L:K]$. In particular, both $e(L/K)$ and $f(L/K)$ are finite numbers.

Definition 2.2.3. [Hub01] A henselian affinoid ring (K, K^+) is *defectless in every finite extension* if, for every finite field extension $K \subset L$, we have $e(L/K)f(L/K) = [L:K]$.

Finally, we are essentially ready to define the notion of a curve-like affinoid field.

Definition 2.2.4. A $(C, \mathcal{O}_C)$-*affinoid field* is an affinoid field (K, K^+) with a continuous morphism $(C, \mathcal{O}_C) \to (K, K^+)$.

For any $(C, \mathcal{O}_C)$-affinoid field, we have the natural induced morphism $j_K\colon \Gamma_C \to \Gamma_K$ of valuation groups.

Definition 2.2.5. A $(C, \mathcal{O}_C)$-affinoid field (K, K^+) is called *curve-like* if

(1) (K, K^+) is henselian and defectless in every finite extension;
(2) the set[8] $\{\gamma \in \Gamma_K \mid \gamma < 1\}$ has a greatest element γ_0;
(3) Γ_K is generated (as an abelian group) by $j_K(\Gamma_C)$ and the element γ_0.

The following lemma (in conjunction with Lemma 4.2.5) will be at the heart of our construction of the analytic trace map (see Definition 5.1.10):

Lemma 2.2.6. *Let (K, K^+) be a curve-like $(C, \mathcal{O}_C)$-affinoid field, and let $\Gamma_C \times \mathbf{Z}$ be a totally ordered abelian group with the lexicographical order. Then the natural morphism*

$$\alpha \colon \Gamma_C \times \mathbf{Z} \to \Gamma_K$$

$$\alpha(\gamma, n) = j_K(\gamma) \cdot \gamma_0^{-n}$$

is an isomorphism of totally ordered abelian groups.

Proof. In this proof, we will freely use [BGR84, Obs. 3.6/10] which guarantees that Γ_C is divisible. We will also denote by $\langle \gamma_0 \rangle \subset \Gamma_K$ the subgroup generated by γ_0. Since Γ_K is torsion-free, we conclude that $\langle \gamma_0 \rangle$ is isomorphic to $\mathbf{Z}$ as abelian groups.

Step 0. *The natural morphism $j_K \colon \Gamma_C \to \Gamma_K$ is injective.* Explicitly, we need to show that the natural morphism $C^\times/\mathcal{O}_C^\times \to K^\times/(K^+)^\times$ is injective. Since every element in $C^\times/\mathcal{O}_C^\times$ is either equal to the class of $\overline{\pi}$ or $\overline{\pi}^{-1}$ for some pseudo-uniformizer $\pi \in \mathcal{O}_C$, it suffices to show that the image of $\overline{\pi}$ is non-zero in $K^\times/(K^+)^\times$. Equivalently, we need to show that any pseudo-uniformizer $\pi \in \mathcal{O}_C$ does not become invertible in K^+. Since the morphism $C \to K$ is continuous, we conclude that $\pi \in K^+$ is a topologically nilpotent. In particular, it lies in the maximal ideal $\mathfrak{m}_{K^+}$, so it is not invertible.

Step 1. *We have $j_K(\Gamma_C) \cap \langle \gamma_0 \rangle = \{1\}$.* Suppose that there is an element $1 \neq \gamma \in j_K(\Gamma_C) \cap \langle \gamma_0 \rangle$. Without loss of generality, we can assume that $\gamma = \gamma_0^n$ for some *positive* integer n. Since Γ_C is divisible, we conclude that there is $\gamma' \in j_K(\Gamma_C) \subset \Gamma_K$ such that $(\gamma')^{2n} = \gamma_0^n$. Since Γ_K is torsion-free, we conclude that $\gamma_0 = (\gamma')^2$. Therefore, we conclude that $\gamma_0 < \gamma' < 1$. This contradicts the assumption that γ_0 is the greatest among elements < 1.

Step 2. *The map α is an isomorphism of abelian groups.* Our assumption on K implies that α is surjective, while Step 1 ensures that it is injective.

Step 3. *For any $\gamma \in j_K(\Gamma_C)$ such that $\gamma > 1$, we have $\gamma > \gamma_0^N$ for any integer N.* We argue by contradiction. Suppose there are $\gamma \in j_K(\Gamma_{C,>1})$ and an integer N such that $\gamma_0^N \geq \gamma$. Since $\gamma_0 < 1$ and $\gamma > 1$, we see that $N < 0$. So we write it as $N = -n$ for some positive n. Then using that Γ_C is divisible, we can find $\gamma' \in j_K(\Gamma_C)$ such that $\gamma = (\gamma')^{-n}$. The inequality $\gamma_0^{-n} \geq \gamma = (\gamma')^{-n} > 1$ implies that $\gamma_0 \leq \gamma' < 1$. The choice of γ_0 implies that $\gamma_0 = \gamma' \in j_K(\Gamma_C)$, but this is impossible due to Step 1.

Step 4. *The map α is an isomorphism of ordered abelian groups.* It suffices to show that the subgroup $\langle \gamma_0 \rangle \subset \Gamma_K$ is convex. This means that if $\gamma \in \Gamma_K$ satisfies $\gamma_0^n \leq \gamma < \gamma_0^m$ for some integers n and m, then $\gamma \in \langle \gamma_0 \rangle$. Since we already know that $\Gamma_K = \Gamma_C \times \langle \gamma_0 \rangle$ as an abelian group, it suffices to show that the only element $\gamma \in j_K(\Gamma_C)$ satisfying

$$\gamma_0^n \leq \gamma \leq \gamma_0^m \tag{2.2.7}$$

for some integers n and m is the neutral object 1. By passing to inverses, we can assume that $\gamma \geq 1$. But then it follows directly from Step 3. ∎

Definition 2.2.8. For a curve-like $(C, \mathcal{O}_C)$-affinoid field (K, K^+), we define the *reduction* morphism

$$\# \colon \Gamma_K \to \mathbf{Z}$$

to be the unique homomorphism that sends γ_0 to 1 and Γ_C to 0.

Warning 2.2.9. Note that $\#$ coincides with the composition $-\mathrm{proj}_2 \circ \alpha^{-1}$, where $\mathrm{proj}_2 \colon \Gamma_C \times \mathbf{Z} \to \mathbf{Z}$ is the projection onto the second factor. In particular, $\#(\alpha(0, 1)) = -1$.

[8] The element 1 in the next formula means the neutral element of the group Γ_K.

Lemma 2.2.10. *Let C be an algebraically closed nonarchimedean field, and $(K, K^+) \subset (L, L^+)$ be a finite extension of curve-like $(C, \mathcal{O}_C)$-affinoid fields. If the residue field $K^+/\mathfrak{m}_K$ is algebraically closed, then the diagram*

$$\begin{tikzcd} L^\times \arrow[r, "\# \circ v_L"] \arrow[d, "\mathrm{Nm}_{L/K}"'] & \mathbf{Z} \\ K^\times \arrow[ur, "\# \circ v_K"'] & \end{tikzcd}$$

commutes.

Proof. Let us choose minimal elements $\gamma_{L,0} \in \{\gamma \in \Gamma_L \mid \gamma < 1\}$ and $\gamma_{K,0} \in \{\gamma \in \Gamma_K \mid \gamma < 1\}$ respectively. Then Lemma 2.2.6 implies that

$$\Gamma_L = \Gamma_C \times \langle \gamma_{L,0} \rangle, \, \Gamma_K = \Gamma_C \times \langle \gamma_{K,0} \rangle$$

with the lexicographic order. Thus, we have a commutative diagram

$$\begin{tikzcd} L^\times \arrow[r, "v_L"] & \Gamma_L = \Gamma_C \times \langle \gamma_{L,0} \rangle \arrow[r, "\sharp"] & \mathbf{Z} \\ K^\times \arrow[r, "v_K"] \arrow[u, "i_{K/L}"] & \Gamma_K = \Gamma_C \times \langle \gamma_{K,0} \rangle \arrow[r, "\sharp"] \arrow[u, "j_{K/L}"] & \mathbf{Z}, \arrow[u, "\cdot e_{L/K}"'] \end{tikzcd} \tag{2.2.11}$$

where $j_{K/L}$ is the morphism of value groups induced by the inclusion $K \subset L$ and $e_{L/K} = |\Gamma_L/\Gamma_K|$ is the ramification index of L/K. Since $\mathbf{Z}$ is torsion-free, it suffices to show that

$$e_{L/K} \cdot \# \circ v_L(f) = e_{L/K} \cdot \# \circ v_K(\mathrm{Nm}_{L/K}(f))$$

for any $f \in L^\times$. Therefore, (2.2.11) implies that it suffices to show that

$$e_{L/K} \cdot \# \circ v_L(f) = \# \circ v_L \left(i_{K/L}\mathrm{Nm}_{L/K}(f)\right).$$

Now we show an even stronger[9] claim that $v_L(f)^{e_{L/K}} = v_L \left(i_{K/L}\mathrm{Nm}_{L/K}(f)\right)$. Since (K, K^+) is defectless in every finite extension and the residue field $K^+/\mathfrak{m}_K$ is algebraically closed, we conclude that $f_{L/K} = 1$ and $[L : K] = e_{L/K}$. Therefore, Lemma 2.1.9 implies that

$$v_L(f)^{e_{L/K}} = v_L \left(i_{K/L}\mathrm{Nm}_{L/K}(f)\right)$$

for any $f \in L^\times$. □

2.3. **Finite morphisms and residue fields.** In this subsection, we record some results about the behaviour of various residue field (see Definition 2.1.1 and Definition 2.1.8) with respect to finite morphisms. We expect that some of these results are probably well-known to the experts, but they do seem to appear in the existing literature.

That being said, we first study the behavior of the completed residue fields with respect to finite morphisms. We establish nice properties when x is a point of rank-1. To deal with higher rank points, we will need to pass to the henselized completed residue fields later in this subsection.

Lemma 2.3.1. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, and $x \in X$. Then x and $y = f(x)$ have equal ranks.*

Proof. Without loss of generality, we can assume that $Y = \mathrm{Spa}\left(\widehat{k(y)}, \widehat{k(y)}^+\right)$. Then we can replace X by its reduction, and then pass to connected components to assume that X is reduced and connected. In this case, the result follows from Lemma 2.1.3 (iii). □

Lemma 2.3.2. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, $y \in Y$ a rank-1 point, and $V \subset X$ an open subset of X containing $f^{-1}(y)$. Then there is an open $U \subset Y$ containing y such that $f^{-1}(U) \subset V$.*

[9]We recall again that we use the multiplicative notation for the group structure on any value group Γ.

Proof. The question is local on Y, so we can assume that Y is an affinoid. Since f is a finite morphism, we conclude that X is also affinoid due to [Hub93b, Korollar 3.12.12]. In particular, both underlying topological spaces $|X|$ and $|Y|$ are spectral. Since $f^{-1}(y)$ is a finite set, we can refine V to assume that it is quasi-compact. In particular, the complement $Z \coloneqq X \smallsetminus V$ is a constructible subset of X.

Now we note that any rank-1 point in Y is maximal due to [Hub96, Lem. 1.1.10 (ii)], so $y = \cap_{i\in I} U_i$, where $\{U_i\}_{i\in I}$ is the filtered poset of quasi-compact opens containing y. Since $f^{-1}(y) = \cap_{i\in I} f^{-1}(U_i)$, we note that the condition that $f^{-1}(y) \subset V$ is equivalent to

$$Z \bigcap \cap_{i\in I} f^{-1}(U_i) = \bigcap_{i\in I} (Z \cap f^{-1}(U_i)) = \varnothing. \tag{2.3.3}$$

Now each $f^{-1}(U_i)$ is a quasi-compact open subset of X, and so $Z \cap f^{-1}(U_i)$ is a constructible subset of X (in particular, it is closed in the constructible topology on X). Therefore, (2.3.3) and [Sta22, Tag 0A2W] guarantee that there is $i \in I$ such that $f^{-1}(U_i) \cap Z = \varnothing$. In other words, $f^{-1}(U_i) \subset V$. So $U = U_i$ does the job. □

Corollary 2.3.4. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, $y \in Y$ a point of rank-1, and $\{x_i\}_{i=1}^n = f^{-1}(y)$. Then there is an open Tate affinoid $U \subset Y$ neighborhood of y such that $f^{-1}(U) = \sqcup_{i=1}^n U_i$ and $x_i \in U_j$ if and only if $i = j$.*

Proof. Lemma 2.3.1 ensures that all x_i are points of rank-1. In particular, they are are maximal points of X. Therefore, there are no common generalizations among x_i's. So [Sta22, Tag 0904] implies that there are open neighborhoods $V_i \ni x_i$ such that $V_i \cap V_j = \varnothing$ if $i \neq j$.

Now we apply Lemma 2.3.2 to $V \coloneqq \cup V_i = \sqcup V_i \subset X$ to find $y \in U \subset Y$ such that $f^{-1}(U) = \sqcup_{i=1}^n U_i$ and $x_i \in U_j$ if and only if $i = j$. We can replace U with any open Tate affinoid $y \in U' \subset U$ to find the desired *affinoid* open subset. □

We first show that stalks at rank-1 points behave nicely with respect to finite morphisms of affinoid rings.

Lemma 2.3.5. *Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be a finite morphism of stronly noetherian Tate affinoids, and let $y \in Y$ be a point of rank-1. Then the natural morphism*

$$\mathcal{O}_{Y,y} \otimes_A B \to \prod_{x_i \in f^{-1}(y)} \mathcal{O}_{X,x_i},$$

is an isomorphism.

Proof. We use Corollary 2.3.4 to replace Y with U to assume that $X = \sqcup_i X_i$ and each X_i contains *exactly one* point over y. Therefore, we can replace X with X_i to assume that $f^{-1}(y) = \{x\}$. In this case, we need to show that the natural morphism

$$\mathcal{O}_{Y,y} \otimes_A B \to \mathcal{O}_{X,x} \tag{2.3.6}$$

is an isomorhism. This comes from the following sequence of isomorphisms:

$$\begin{aligned}
\mathcal{O}_{Y,y} \otimes_A B &\simeq \big(\mathrm{colim}_{V\ni y} \mathcal{O}_Y(V)\big) \otimes_A B \\
&\simeq \mathrm{colim}_{V\ni y} \big(\mathcal{O}_Y(V) \otimes_A B\big) \\
&\simeq \mathrm{colim}_{V\ni y} \mathcal{O}_X\left(f^{-1}(V)\right) \\
&\simeq \mathrm{colim}_{W\ni x} \mathcal{O}_X(W) \\
&\simeq \mathcal{O}_{X,x},
\end{aligned}$$

where the third isomorphism comes from [Zav24, Lem. B.3.6], and the fifth isomorphism comes from Lemma 2.3.2. □

Our next goal is to get an analogue of Lemma 2.3.5 for completed residue fields at rank-1 points (under some further assumptions). We start with the following preliminary lemma:

Lemma 2.3.7. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, $y \in Y$ be a point with the unique rank-1 generalization y_{gen}. Let $f^{-1}(y) = \{x_i\}_{i\in I}$, and $x_{i,\mathrm{gen}}$ the unique rank-1 generalization of x_i for $i \in I$. Then $f^{-1}(y_{\mathrm{gen}}) = \{x_{i,\mathrm{gen}}\}_{i\in I}$ (some $x_{i,\mathrm{gen}}$ might coincide).*

Proof. First, [Hub96, Lem. 1.1.10(iv)] implies that $\{x_{i,\mathrm{gen}}\}_{i\in I} \subset f^{-1}(y_{\mathrm{gen}})$. Now [Hub96, Lem. 1.4.5(ii)] implies that f is closed. Therefore, [Sta22, Tag 0066] implies that any $x \in f^{-1}(y_{\mathrm{gen}})$ is a generalization of some x_i. Furthermore, Lemma 2.3.1 ensures that x is of rank-1, so it must be $x_{i,\mathrm{gen}}$. □

Lemma 2.3.8. *Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be a finite morphism of strongly noetherian Tate affinoid adic spaces, and let $y \in Y$ be a rank-1 point. If $\widehat{k(y)} \otimes_A B$ is a reduced ring, then the natural morphism*

$$\widehat{k(y)} \otimes_A B \to \prod_{x_i\in f^{-1}(y)} \widehat{k(x_i)}$$

is an isomorphism.

Proof. Throughout this proof, we will freely use [Zav24, Lem. B.3.6] that ensures that certain completed tensor products coincide with the usual tensor products.

That being said, we can use Lemma 2.3.1 and Corollary 2.3.4 to reduce to the situation $f^{-1}(y) = x$ is a unique rank-1 point. Then we can replace Y with $\mathrm{Spa}\left(\widehat{k(y)}, \widehat{k(y)}^\circ\right)$ to assume that $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$ for a nonarchimedean field K. Then our assumption on $X = \mathrm{Spa}\,(B, B^+)$ implies that it is a reduced adic space which is finite over $\mathrm{Spa}\,(K, \mathcal{O}_K)$, and $|X|$ is a singleton (in particular, it is connected). Therefore, Lemma 2.1.3 (i), (iv), (v) imply that $B \simeq \widehat{k(x)}$. Thus the map

$$\widehat{k(y)} \otimes_A B \simeq K \otimes_K \widehat{k(x)} \to \widehat{k(x)}$$

is obviously an isomorphism. □

Now we want to get an analogue of Lemma 2.3.8 for higher rank points. For this, we will need to work with henselized completed residue fields.

We start by proving the following general lemma in commutative algebra:

Lemma 2.3.9. *Let $I_1, I_2, \dots, I_n \subset A$ be ideals in a ring A. Suppose that for each $1 \le i, j \le n$, $I_i + I_j = A$. Then the natural morphism*

$$A^{\mathrm{h}}_{I_1\cap I_2\cap\cdots\cap I_n} \to \prod_{i=1}^{n} A^{\mathrm{h}}_{I_i}$$

is an isomorphism.

Proof. First, we notice that $(\cap_{j\neq i} I_j) + I_i = A$ for every $i = 1, \dots, n$. So we can assume that $n = 2$. Now we note $I_1 A^{\mathrm{h}}_{I_1}$ is a radical ideal, so the assumption $I_1 + I_2 = A$ implies $I_2 A^{\mathrm{h}}_{I_1} = A^{\mathrm{h}}_{I_1}$. Thus, $(I_1 \cap I_2)A^{\mathrm{h}}_{I_1} = I_1 A^{\mathrm{h}}_{I_1}$ and, similarly, $(I_1 \cap I_2)A^{\mathrm{h}}_{I_2} = I_2 A^{\mathrm{h}}_{I_2}$. Therefore, $A^{\mathrm{h}}_{I_1} \times A^{\mathrm{h}}_{I_2}$ is henselian along $I_1 \cap I_2$ and ind-étale over A. Thus, in order to check that the natural morphism

$$A^{\mathrm{h}}_{I_1\cap I_2} \to A^{\mathrm{h}}_{I_1} \times A^{\mathrm{h}}_{I_2}$$

is an isomorphism, it suffices to check it modulo $I_1 \cap I_2$. Using [Sta22, Tag 0AGU] and the equalities $(I_1 \cap I_2)A^{\mathrm{h}}_{I_i} = I_i A^{\mathrm{h}}_{I_i}$, we conclude that it suffices to show that $A/(I_1 \cap I_2) \to A/I_1 \times A/I_2$ is an isomorphism. This follows from our assumption that $I_1 + I_2 = A$. □

Finally, we are ready to prove the main result of this subsection:

Theorem 2.3.10. *Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be a finite morphism of strongly noetherian Tate affinoids, and let $y \in Y$ be a point with the unique rank-1 generalization y_{gen}. If $\widehat{k(y_{\mathrm{gen}})} \otimes_A B$ is reduced, then the natural morphism*

$$\widehat{k(y)}^{\mathrm{h}} \otimes_A B \to \prod_{x_i\in f^{-1}(y)} \widehat{k(x_i)}^{\mathrm{h}}$$

is an isomorphism.

Proof. Let us denote by $x_{i,\mathrm{gen}}$ the unique rank-1 generalization of x_i for each $x_i \in f^{-1}(y)$. Then Lemma 2.3.7 implies that the sets $\{x_{i,\mathrm{gen}}\}_{i\in I}$ and $f^{-1}(y_{\mathrm{gen}})$ coincide. Then Corollary 2.3.4 allows us to reduce to the situation when $x_{i,\mathrm{gen}} = x_{j,\mathrm{gen}}$ for any i, j. We denote this common rank-1 generalization by x_{gen}.

Now Lemma 2.3.8 implies that the natural morphism

$$\widehat{k(y_{\mathrm{gen}})} \otimes_A B \to \widehat{k(x_{\mathrm{gen}})} \tag{2.3.11}$$

is an isomorphism. In particular, $\widehat{k(x_{\mathrm{gen}})}$ is a finite extension of $\widehat{k(y_{\mathrm{gen}})}$. Let us denote by R^+ the integral closure of $\widehat{k(y)}^+$ in $\widehat{k(x_{\mathrm{gen}})}$. Then Lemma 2.1.3 implies that

$$R^+ = \bigcap_{i=1}^{n} \widehat{k(x_i)}^+.$$

Let us denote by $\mathfrak{m} \subset \widehat{k(y)}^+$ the maximal ideal in $\widehat{k(y)}^+$, and by $\mathfrak{m}'_i \subset \widehat{k(x_i)}^+$ the maximal ideal in $\widehat{k(x_i)}^+$. Now Lemma 2.1.3 (vi) implies that R^+ is a semi-local ring with maximal ideals given by $\mathfrak{m}_i \coloneqq R^+ \cap \mathfrak{m}'_i$, and that the natural morphism $R^+_{\mathfrak{m}_i} \to \widehat{k(x_i)}^+$ is an isomorphism of local rings. Furthermore, Lemma 2.1.3 (vii) implies that $\mathrm{rad}(\mathfrak{m}R^+) = \cap_{i=1}^n \mathfrak{m}_i$. Therefore, [Sta22, Tag 0DYE] and Lemma 2.3.9 (applied to the maximal ideals $\mathfrak{m}_i$) imply that

$$\widehat{k(y)}^{+,\mathrm{h}} \otimes_{\widehat{k(y)}^+} R^+ \simeq R^{+,\mathrm{h}}_{\cap_{i=1}^n \mathfrak{m}_i} = \prod_{i=1}^{n} R^{+,\mathrm{h}}_{\mathfrak{m}_i} = \prod_{i=1}^{n} \widehat{k(x_i)}^{+,\mathrm{h}}.$$

Now we recall that the natural morphism $\widehat{k(y)} \to \widehat{k(y_{\mathrm{gen}})}$ is an isomorphism due to [Hub96, Lem. 1.1.10(iii)]. Thus, after inverting a pseudo-uniformizer, we get the isomorphism

$$\widehat{k(y)}^{\mathrm{h}} \otimes_{\widehat{k(y_{\mathrm{gen}})}} \widehat{k(x_{\mathrm{gen}})} \simeq \prod_{i=1}^{n} \widehat{k(x_i)}^{\mathrm{h}}.$$

Combining it with (2.3.11), we conclude that the desired isomorphism

$$\widehat{k(y)}^{\mathrm{h}} \otimes_A B \simeq \widehat{k(y)}^{\mathrm{h}} \otimes_{\widehat{k(y_{\mathrm{gen}})}} \widehat{k(y_{\mathrm{gen}})} \otimes_A B \simeq \widehat{k(y)}^{\mathrm{h}} \otimes_{\widehat{k(y_{\mathrm{gen}})}} \widehat{k(x_{\mathrm{gen}})} \simeq \prod_{i=1}^{n} \widehat{k(x_i)}^{\mathrm{h}}. \qquad \square$$

Even though the conclusion of Theorem 2.3.10 does not hold for an arbitrary finite morphism, it can still be used to study properties of arbitrary finite morphisms:

Corollary 2.3.12. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, and let $x \in X$ be an arbitrary point with $y = f(x)$. Then $\widehat{k(y)}^{\mathrm{h}} \to \widehat{k(x)}^{\mathrm{h}}$ is a finite field extension.*

Proof. Without loss of generality, we can assume that $Y = \mathrm{Spa}\left(\widehat{k(y)}, \widehat{k(y)}^+\right)$. Then we can replace X by its reduction, and then pass to connected components to assume that X is reduced and connected. In this case, the assumption of Theorem 2.3.10 is obviously satisfied since $\widehat{k(y)} \simeq \widehat{k(y_{\mathrm{gen}})}$. Therefore the result follows directly from Theorem 2.3.10. $\square$

We finish the subsection by establishing two examples when the assumptions of Theorem 2.3.10 is automatic.

Example 2.3.13. The assumption of Theorem 2.3.10 is always satisfied if $f\colon X \to Y$ is a finite *étale* morphism.

Now we give another, more elaborate example:

Lemma 2.3.14. *Let K be a nonarchimedean field, let $f\colon X = \mathrm{Spa}\,(B, B^\circ) \to Y = \mathrm{Spa}\,(A, A^\circ)$ be a finite morphism of smooth rigid-analytic affinoid spaces over K, and let $y \in Y$ be a weakly Shilov point of Y (in the sense of* [BH22, Def. 2.5]*). Then $\widehat{k(y)} \otimes_A B$ is reduced.*

Proof. First, there is an open affinoid $U \subset Y$ neighborhood of y such that $y \in U$ is a Shilov point. Then [Zav24, Lem. B.3.6] guarantees that

$$\widehat{k(y)} \otimes_A B \simeq \widehat{k(y)} \otimes_{\mathcal{O}_Y(U)} \mathcal{O}_Y(U) \otimes_A B \simeq \widehat{k(y)} \otimes_{\mathcal{O}_Y(U)} \mathcal{O}_X(X_U).$$

Therefore, we can replace Y with U and assume that $y \in Y$ is a Shilov point. Then [BH22, Th. 2.25] implies that $\widehat{k(y)} \otimes_A B$ is a regular algebra. In particular, it is reduced. □

2.4. **Finite flat morphisms.** In this subsection, we collect some facts about flat and finite flat morphisms of locally noetherian analytic adic spaces. In particular, we show that the universal compactification of a finite flat morphism is always finite and flat.

We start by recalling the following definition:

Definition 2.4.1. A morphism $f\colon X \to Y$ of locally noetherian analytic adic spaces is *flat* if, for every point $x \in X$, the morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat morphism of local rings.

Remark 2.4.2. Flat morphisms are closed under compositions. But it is not clear (and probably false) whether they are closed under base change.

Lemma 2.4.3. *Let $f\colon X \to Y$ be a morphism of locally noetherian analytic adic spaces. Suppose that, for any rank-1 point $x \in X$, the morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Then f is flat.*

Proof. Pick any $x_0 \in X$, and let x be its unique rank-1 generalization (it exists by [Hub96, Lem. 1.1.10]). Then *loc. cit.* implies that $y \coloneqq f(x)$ is the unique rank-1 generalization of $y_0 \coloneqq f(x_0)$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x_0} & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y,y_0} & \longrightarrow & \mathcal{O}_{Y,y}. \end{array}$$

Loc. cit. ensures that the horizontal maps are local and flat (in particular, they are faithfully flat), and the right vertical arrow is flat by the assumption. This implies that the left vertical is flat as well. Since x was arbitrary, we conclude that f is flat. □

In general, it seems very difficult to test whether a morphism of locally noetherian adic spaces is flat. However, it turns out that the situation is much better in the case of finite morphisms:

Lemma 2.4.4. *Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be a morphism of strongly noetherian Tate affinoids. Then f is finite flat if and only if $(A, A^+) \to (B, B^+)$ is a finite morphism of Huber pairs and $A \to B$ is a flat morphism of rings.*

Proof. If f is finite and flat, then $(A, A^+) \to (B, B^+)$ is a finite morphism of Huber pairs by [Hub93b, Satz 3.6.20 and Korollar 3.12.12] and $A \to B$ is flat by [Zav24, Lem. B.4.3].

Now suppose that $X = \mathrm{Spa}\,(B, B^+)$ is an affinoid, $(A, A^+) \to (B, B^+)$ is finite and $A \to B$ is flat. Then $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A, A^+)$ is clearly finite. We only need to show that it is also flat. Let $x \in X$ is a rank-1 point, and $y = f(x)$. Then Lemma 2.3.5 implies that the morphism

$$\mathcal{O}_{Y,y} \to \prod_{x_i \in f^{-1}(y)} \mathcal{O}_{X,x_i} \simeq B \otimes_A \mathcal{O}_{Y,y}$$

is flat. In particular, $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. Therefore, Lemma 2.4.3 ensures that $X \to Y$ is flat. □

Remark 2.4.5. Lemma 2.4.4 and [Zav24, Lem. B.3.6] imply that finite flat morphisms are closed under arbitrary base change.

Finally, we are ready to show that universal compactifications preserve finite flat morphisms. We refer to Appendix A for a brief recollection on universal compactifications.

Lemma 2.4.6. *Let K be a nonarchimedean field, $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ is a finite (resp. finite flat) morphism of rigid-analytic affinoid spaces over K, and $f^c\colon X^c = \mathrm{Spa}\,(B, B'^+) \to Y^c = \mathrm{Spa}\,(A, A'^+)$ the induced morphism on the universal compactifications. Then f^c is a finite (resp. finite flat) morphism.*

Proof. Since universal compactifications do not change rings of rational functions (see Lemma A.0.3), Lemma 2.4.4 implies that it suffices to show that $(A, A'^+) \to (B, B'^+)$ is a finite morphism of Huber pairs. Cleary, $A \to B$ is finite, so it suffices to show that $A'^+ \to B'^+$ is integral.

Now [Hub93b, Lem. 3.12.10] (see also [Zav23a, Lem. 3.5]) implies that the integral closures B''^+ of A'^+ in B defines a Huber pair (B, B''^+). By construction, it contains $\mathcal{O}_K$. Lemma A.0.3 ensures that B'^+ is the minimal $+$-ring containing $\mathcal{O}_K$. Therefore, $B'^+ \subset B''^+$, and thus B'^+ is integral over A'^+. □

2.5. **Trace for finite flat morphisms.** The main goal of this subsection is to define a trace morphism for any finite flat morphism of locally noetherian analytic adic spaces. This construction will be an important tool in studying properties of the analytic trace map in Section 5.

Before we start the construction, we mention that the algebraic counterpart of the finite flat trace map has been defined in [AGV71, Exp. XVII, Th. 6.2.3] and [Sta22, Tag 0GKI]. We carry over a similar strategy to the nonarchimedean situation. We begin with some preliminary lemmas.

Lemma 2.5.1. *Let $f\colon X \to Y$ be a finite morphism of locally noetherian analytic adic spaces, $\overline{y} \to Y$ a geometric point, and $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$ a sheaf of abelian groups. Then there is a natural isomorphism*

$$\bigoplus_{\overline{x} \in f^{-1}(\overline{y})} \mathcal{F}_{\overline{y}} \xrightarrow{\sim} (f_* f^* \mathcal{F})_{\overline{y}}\,.$$

Proof. This follows from the sequence of isomorphisms

$$\bigoplus_{\overline{x} \in f^{-1}(\overline{y})} \mathcal{F}_{\overline{y}} \simeq \bigoplus_{\overline{x} \in f^{-1}(\overline{y})} (f^* \mathcal{F})_{\overline{x}} \simeq (f_* f^* \mathcal{F})_{\overline{y}}\,,$$

where the second isomorphism comes from [Hub96, Prop. 2.6.3]. □

Remark 2.5.2. Lemma 2.5.1 implies that for a finite morphism $f\colon X \to Y$ and an abelian sheaf $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$, the natural morphism

$$f_* \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{F} \to f_* f^* \mathcal{F}$$

is an isomorphism.

Remark 2.5.3. We also note that [Hub96, Prop. 2.6.3] guarantees that for a finite morphism $f\colon X \to Y$, the functor $f_*\colon \mathcal{A}b(X_{\text{ét}}) \to \mathcal{A}b(Y_{\text{ét}})$ is exact and commute with arbitrary base change along $Y' \to Y$.

Recall that for every strongly noetherian Tate affinoid $S = \mathrm{Spa}\,(A, A^+)$, [Hub96, Cor. 1.7.3, (3.2.8)] constructs a functorial morphism of étale topoi $c_S\colon S_{\text{ét}} \to (\mathrm{Spec}\, A)_{\text{ét}}$. In particular, for every morphism $f\colon T = \mathrm{Spa}\,(B, B^+) \to S = \mathrm{Spa}\,(A, A^+)$ with induced morphism $f^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, the following diagram commutes (up to canonical equivalence):

$$\begin{tikzcd} T_{\text{ét}} \arrow[r, "c_T"] \arrow[d, "f_{\text{ét}}"] & (\mathrm{Spec}\, B)_{\text{ét}} \arrow[d, "f^{\mathrm{alg}}_{\text{ét}}"] \\ S_{\text{ét}} \arrow[r, "c_S"] & (\mathrm{Spec}\, A)_{\text{ét}} \end{tikzcd}$$

Construction 2.5.4 (Relative analytification)**.** Likewise, for every strongly noetherian Tate affinoid $S = \mathrm{Spa}\,(A, A^+)$ and a locally finite type A-scheme $g\colon X \to \mathrm{Spec}\, A$, [Hub94, Prop. 3.8] defines the relative analytification $X^{\mathrm{an}/S}$ as an adic space which is locally of finite type over S. By [Hub96, Cor. 1.7.3, (3.2.8)], it comes equipped with a canonical morphism of étale topoi $c_{X/S}\colon X^{\mathrm{an}/S}_{\text{ét}} \to X_{\text{ét}}$ such that the diagram[10]

$$\begin{tikzcd} X^{\mathrm{an}/S}_{\text{ét}} \arrow[r, "c_{X/S}"] \arrow[d, "g^{\mathrm{an}/S}_{\text{ét}}"] & X_{\text{ét}} \arrow[d, "g_{\text{ét}}"] \\ S_{\text{ét}} \arrow[r, "c_S"] & \operatorname{Spec} A_{\text{ét}} \end{tikzcd}$$

commutes (up to canonical equivalence).

Lemma 2.5.5. *Let $f\colon X = \operatorname{Spa}(B, B^+) \to Y = \operatorname{Spa}(A, A^+)$ be a finite morphism of strongly noetherian Tate affinoids and $\mathcal{F} \in \mathcal{A}b(\operatorname{Spec} B_{\text{ét}})$. Then the natural morphism*

$$\gamma\colon c_Y^* f_*^{\mathrm{alg}} \mathcal{F} \longrightarrow f_* c_X^* \mathcal{F}$$

is an isomorphism.

Proof. First, we note that $c_Y^* f_*^{\mathrm{alg}}\mathcal{F}$ and $c_X^*\mathcal{F}$ are both overconvergent (in the sense of [Hub96, Def. 8.2.1]) because they are analytifications of algebraic sheaves. Furthermore, $f_* c_X^* \mathcal{F}$ is also overconvergent due to [Hub96, Prop. 8.2.3]. Therefore, it suffices to show that γ induces an isomorphism on stalks at geometric points of rank 1. Since both f_* and f_*^{alg} commute with arbitrary base change (see Remark 2.5.3 and [Sta22, Tag 0959]), we may thus assume that $Y = \operatorname{Spa}(C, \mathcal{O}_C)$ for an algebraically closed nonarchimedean field C. Then we can further replace X by X_{red} to assume that X is reduced. In this case, X decomposes as a finite disjoint union $X = \bigsqcup_{i=1}^n \operatorname{Spa}(C, \mathcal{O}_C)$ and the result becomes trivial. $\square$

Finally, we are ready to prove the main result of this subsection:

Theorem 2.5.6. *There is a unique way to assign to any finite flat morphism $f\colon X \to Y$ of locally noetherian analytic adic spaces and any abelian sheaf $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$, a trace map $\mathrm{tr}_{f,\mathcal{F}}\colon f_* f^* \mathcal{F} \to \mathcal{F}$ satisfying the following properties:*

(1) *(functoriality in $\mathcal{F}$) For any morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ in $\mathcal{A}b(Y_{\text{ét}})$, the following diagram is commutative:*

$$\begin{tikzcd} f_* f^* \mathcal{F} \arrow[r, "\mathrm{tr}_{f,\mathcal{F}}"] \arrow[d, "f_* f^*(\varphi)"] & \mathcal{F} \arrow[d, "\varphi"] \\ f_* f^* \mathcal{G} \arrow[r, "\mathrm{tr}_{f,\mathcal{G}}"] & \mathcal{G} \end{tikzcd}$$

(2) *(compatibility with compositions) For any two finite flat morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ and any $\mathcal{F} \in \mathcal{A}b(Z_{\text{ét}})$, the following diagram is commutative:*

$$\begin{tikzcd} (g \circ f)_*(g \circ f)^* \mathcal{F} \arrow[r, "\mathrm{tr}_{g\circ f,\mathcal{F}}"] \arrow[d, "\wr"] & \mathcal{F} \\ g_*\big(f_* f^*(g^*\mathcal{F})\big) \arrow[r, "g_*(\mathrm{tr}_{f,g^*\mathcal{F}})"] & g_* g^* \mathcal{F} \arrow[u, "\mathrm{tr}_{g,\mathcal{F}}"'] \end{tikzcd}$$

(3) *(compatibility with pullbacks) For any pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in which f and f' are finite flat and any $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$, the following diagram is commutative:

$$\begin{tikzcd} g^* f_* f^* \mathcal{F} \arrow[r, "g^*(\mathrm{tr}_{f,\mathcal{F}})"] \arrow[d, "\wr"] & g^* \mathcal{F} \\ f'_* g'^{,*} f^* \mathcal{F} \arrow[r, "\sim"] & f'_* f'^{,*} g^* \mathcal{F} \arrow[u, "\mathrm{tr}_{f',g^*\mathcal{F}}"] \end{tikzcd}$$

[10] If $X = S$, we denote $c_{X/S}$ simply by c_S.

(4) *(normalization) If f is of constant rank d, the composition $\mathcal{F} \to f_* f^* \mathcal{F} \xrightarrow{\mathrm{tr}_{f,\mathcal{F}}} \mathcal{F}$ is equal to the multiplication by d.*

Furthermore, this assignment satisfies the following properties:

(5) *(compatibility with étale traces) If f is finite étale, then $\mathrm{tr}_{f,\mathcal{F}}$ is given by the counit*
$$f_* f^* \mathcal{F} \simeq f_! f^* \mathcal{F} \to \mathcal{F}$$
of the adjunction between $f_!$ and f^*.

(6) *(compatibility with algebraic traces I) For any finite flat morphism $f\colon T = \mathrm{Spa}\,(B, B^+) \to S = \mathrm{Spa}\,(A, A^+)$ with associated morphism[11] $f^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ and any sheaf $\mathcal{F} \in \mathcal{A}b\big((\mathrm{Spec}\, A)_{\text{ét}}\big)$, the diagram*

$$\begin{tikzcd} c_S^* f_*^{\mathrm{alg}} f^{\mathrm{alg},*}\mathcal{F} \arrow[r, "c_S^*(\mathrm{tr}_{f^{\mathrm{alg}},\mathcal{F}})"] \arrow[d, "\wr"] & c_S^*\mathcal{F} \\ f_* c_T^* f^{\mathrm{alg},*}\mathcal{F} \arrow[r, "\sim"] & f_* f^* c_S^* \mathcal{F} \arrow[u, "\mathrm{tr}_{f, c_S^*\mathcal{F}}"'] \end{tikzcd} \tag{2.5.7}$$

commutes, where c_A and c_B are the morphisms of topoi defined just before Construction 2.5.4.

(7) *(compatibility with algebraic traces II) For any strongly noetherian Tate affinoid $S = \mathrm{Spa}\,(A, A^+)$, any finite flat morphism $g\colon X \to Y$ of locally finite type A-schemes with relative analytification $g^{\mathrm{an}/S}\colon X^{\mathrm{an}/S} \to X^{\mathrm{an}/S}$, and any sheaf $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$, the diagram*

$$\begin{tikzcd} c_{Y/S}^* g_* g^* \mathcal{F} \arrow[r, "c_{Y/S}^*(\mathrm{tr}_{g,\mathcal{F}})"] \arrow[d, "\wr"] & c_{Y/S}^*\mathcal{F} \\ g_*^{\mathrm{an}/S} c_{X/S}^* g^* \mathcal{F} \arrow[r, "\sim"] & g_*^{\mathrm{an}/S} g^{\mathrm{an}/S,*} c_{Y/S}^* \mathcal{F} \arrow[u, "\mathrm{tr}_{g^{\mathrm{an}/S}, c_{Y/S}^*\mathcal{F}}"'] \end{tikzcd} \tag{2.5.8}$$

commutes, where $c_{X/A}$ and $c_{Y/A}$ are the morphisms of topoi defined in Construction 2.5.4.

Proof. *Step 1. Uniqueness.* First, we note that [Hub96, Prop. 2.5.5] and (3) ensure that $\mathrm{tr}_{f,\mathcal{F}}$ is determined by the case when $Y = \mathrm{Spa}\,(C, C^+)$ for a complete algebraically closed nonarchimedean field C and an open bounded valuation subring $C^+ \subset C$. In that setting, we can write $X = \bigsqcup_{i=1}^n X_i$ with each X_i connected (and finite flat over Y); we denote by $f_i\colon X_i \to Y$ the induced morphisms and set $\mathcal{F}_i := \mathcal{F}\big|_{X_i}$. For the natural clopen immersions $j_i\colon X_i \to X$, properties (4) and (3) imply that the trace morphisms $\mathrm{tr}_{j_i,\mathcal{F}}\colon j_{i,*}\mathcal{F}_i \to \mathcal{F}$ must be the adjunction morphisms $j_{i,*}\mathcal{F}_i = j_{i,!}\mathcal{F}_i \to \mathcal{F}$. Then $\mathrm{tr}_{f,\mathcal{F}} = \sum_{i=1}^n \mathrm{tr}_{f_i,\mathcal{F}_i}$ for any $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$ due to (2). Therefore, it suffices to show uniqueness under the additional assumptions that $Y = \mathrm{Spa}\,(C, C^+)$ and X is connected.

In this case, we clearly have that $f\colon X \to Y$ is of constant rank. Since C is algebraically closed, Remark 2.1.5 implies that C^+ is henselian. Therefore, Lemma 2.1.3 ensures that $X_{\mathrm{red}} \simeq Y$, so $X_{\text{ét}} \simeq Y_{\text{ét}}$. In particular, $f_* f^* \mathcal{F} = \mathcal{F}$ for any $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$. Thus, (4) implies that $\mathrm{tr}_{f,\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}$ must be equal to the multiplication by $\dim_C \mathcal{O}_X(X)$ for any $\mathcal{F} \in \mathcal{A}b(Y_{\text{ét}})$. This finishes the proof of uniqueness.

Step 2. Construction of $\mathrm{tr}_{f,\mathcal{F}}$. We first construct trace maps $\mathrm{tr}_{f,\underline{\mathbf{Z}}}$ that are compatible with base change. Thanks to the base change compatibility, it suffices to do so locally on Y. Hence, we may assume that both $X = \mathrm{Spa}\,(B, B^+)$ and $Y = \mathrm{Spa}\,(A, A^+)$ are affinoid. Then Lemma 2.4.4 implies that $A \to B$ is a finite flat ring map, so the induced morphism $f^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is also finite flat. Therefore, we use Lemma 2.5.5 to define
$$\mathrm{tr}_{f,\underline{\mathbf{Z}}} \coloneqq c_Y^* \left(\mathrm{tr}_{f^{\mathrm{alg}},\underline{\mathbf{Z}}}\right),$$
where $\mathrm{tr}_{f^{\mathrm{alg}},\underline{\mathbf{Z}}}$ is the algebraic trace for finite flat morphisms constructed in [AGV71, Exp. XVII, Th. 6.2.3]. We note that $\mathrm{tr}_{f,\underline{\mathbf{Z}}}$ commutes with arbitrary base change since the same holds for the algebraic trace map.

For a general sheaf $\mathcal{F}$, we define $\mathrm{tr}_{f,\mathcal{F}}$ as the composition
$$f_* f^* \mathcal{F} \xrightarrow{\sim} f_* f^* \underline{\mathbf{Z}} \otimes \mathcal{F} \xrightarrow{\mathrm{tr}_{f,\underline{\mathbf{Z}}} \otimes \mathrm{id}} \underline{\mathbf{Z}} \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{F}, \tag{2.5.9}$$

[11] We note that f^{alg} is finite flat due to Lemma 2.4.4.

where the first map is the inverse of the projection formula isomorphism (see Remark 2.5.2). This definition satisfies (1) by its very construction. Next, we check that it also satisfies (6). For this, we first note that Diagram (2.5.7) commutes for $\mathcal{F} = \underline{\mathbf{Z}}$ by the very definition of $\mathrm{tr}_{f,\underline{\mathbf{Z}}}$. Secondly, we note that the construction of $\mathrm{tr}_{f^{\mathrm{alg}},\mathcal{F}}$ is analogous to (2.5.9) except that one uses $\mathrm{tr}_{f^{\mathrm{alg}},\underline{\mathbf{Z}}}$ in place of $\mathrm{tr}_{f,\underline{\mathbf{Z}}}$; see the proof of [Sta22, Tag 0GKG] and [Sta22, Tag 0GKI]. Since the algebraic projection formula analytifies to the analytic one, we conclude that Diagram (2.5.7) commutes for any $\mathcal{F}$.

In order to check (2), (3), and (4), we may assume that X, Y, Z, Y' are affinoid, and $\mathcal{F} = \underline{\mathbf{Z}}$. Then the results follow from (6) and the analogous properties of algebraic finite flat trace (see [AGV71, Exp. XVII, Th. 6.2.3]). Thus, we are only left to check (5) and (7).

We start with (5). We first note that it suffices to treat the case $\mathcal{F} = \underline{\mathbf{Z}}$. Since $\underline{\mathbf{Z}}$ and $f_*\underline{\mathbf{Z}}$ are overconvergent (see [Hub96, Prop. 8.2.3.(ii)]), we can check equality over geometric points of rank 1. Both $\mathrm{tr}_{f,\underline{\mathbf{Z}}}$ and the $(f_!, f^*)$-adjunction commute with arbitrary base change, so we can assume that $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for an algebraically closed nonarchimedean field C. Then $X = \bigsqcup_{i=1}^n Y$ and the result follows from (4).

Finally, we show (7). First, we observe that both $c^*_{Y/S} g_* g^* \mathcal{F}$ and $c^*_{Y/S}\mathcal{F}$ are overconvergent as analytifications of algebraic sheaves. Therefore, we can check that Diagram (2.5.8) commutes over geometric rank-1 points of $Y^{\mathrm{an}/S}$. Since both $\mathrm{tr}_{g^{\mathrm{an}/S}}$ and tr_g commute with arbitrary base change, we can assume that $A = C$ is an algebraically closed nonarchimedean field and $Y = \mathrm{Spec}\, C$. In this case, the result follows from (6). □

For later reference, we also discuss a version of étale traces for general étale morphisms. Recall that [Hub96, Lem. 2.7.6] guarantees that in this case, f^* admits an exact right adjoint functor $f_!$, which furthermore commutes with arbitrary base change.

Definition 2.5.10. Let $f\colon X \to Y$ be an étale morphism of locally noetherian analytic adic spaces and let $\mathcal{F} \in \mathcal{A}b(Y_{\mathrm{ét}})$ be any abelian sheaf on Y. Then the natural counit

$$\mathrm{tr}^{\mathrm{ét}}_{f,\mathcal{F}}\colon f_! f^* \mathcal{F} \to \mathcal{F}$$

for the adjunction between $f_!$ and f^* is called the *étale trace map* for f.

Notation 2.5.11. We will often use the étale trace in the situation where we have fixed a ring of coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$. In this case, we denote the map $\mathrm{tr}^{\mathrm{ét}}_{f,\Lambda}$ simply by $\mathrm{tr}^{\mathrm{ét}}_f$. Likewise, we denote $\mathrm{tr}^{\mathrm{ét}}_{f,\mu_n^{\otimes m}}$ by $\mathrm{tr}^{\mathrm{ét}}_f(m)$.

Lemma 2.5.12. *Let $f\colon X \to Y$ be an étale morphism of locally noetherian analytic adic spaces and let $\mathcal{F} \in \mathcal{A}b(Y_{\mathrm{ét}})$ be an abelian sheaf on Y. Then the étale trace map is compatible with compositions, pullbacks, and relative analytifications (in the sense of Theorem 2.5.6 (2), (3), (7)).*

Proof. The compatibility under compositions follows from the compatibility of counits of adjunctions with compositions (see e.g. [Lur24, Tag 02DS]). Now we show the compatibility under pullbacks. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback diagram of locally noetherian adic spaces in which f and f' are étale. Thanks to [Hub96, Prop. 2.5.5] and the observation that the étale trace commutes with taking stalks, it suffices to check that $g^*\mathrm{tr}_f$ agrees with $\mathrm{tr}_{f'}$ after passing to stalks at each geometric point ξ of Y'. Therefore, we can assume that the morphism g is of the form $Y' = \mathrm{Spa}\,(C', C'^+) \to Y = \mathrm{Spa}\,(C, C^+)$ for some algebraically closed nonarchimedean field C and C' and a faithfully flat morphism $C^+ \to C'^+$ of open bounded valuation subrings. Furthermore, in this case, it suffices to show that $\Gamma(Y', g^*\,\mathrm{tr}^{\mathrm{ét}}_f) = \Gamma(Y', \mathrm{tr}^{\mathrm{ét}}_{f'})$. The claim is local on X, so we can assume that $X = \mathrm{Spa}\,(A, A^+)$ is affinoid. Then [Hub96, Lem. 2.2.8] implies that X is an open subspace inside a finite étale Y-space $\overline{X}$. Since C is algebraically closed, we conclude that $\overline{X}$ is a finite disjoint union of copies of Y, so we can assume that $f\colon X \to Y$ is an open immersion. If f is an isomorphism, the claim is obvious. If f is an open immersion that does not meet the closed point of Y, we see that $\Gamma(Y', g^*\,\mathrm{tr}^{\mathrm{ét}}_f) = 0 = \Gamma(Y', \mathrm{tr}^{\mathrm{ét}}_{f'})$. □

3. Cycle classes

In this section, we develop a theory of cycle classes in analytic adic geometry. The cycle class considerations will be an important technical tool for verifying properties of the analytic trace in Section 5. Furthermore, they will be absolutely crucial for our "diagrammatic" approach to Poincaré duality (see Section 6.4).

Our approach closely follows its algebraic counterpart developed in [Del77] in the case of divisors, and in [Fuj02, § 1] in the case of lci immersion of higher codimension.

3.1. Cycle classes of divisors.

The goal of this subsection is to define the cycle class in the case of divisors. In later subsections, we will generalize this construction to higher codimensions as well.

We refer the reader to [Zav23a, § 5] for the definition and basic properties of effective Cartier divisors in analytic adic geometry.

Throughout this subsection, we fix a locally noetherian analytic adic space X, an integer $n \in \mathcal{O}_X^\times$ (we do not assume that n is invertible in $\mathcal{O}_X^+$), and set $\Lambda \coloneqq \mathbf{Z}/n\mathbf{Z}$.

3.1.1. *Construction of cycle classes.* The goal of this subsection is to construct a class $c\ell_X(D) \in \mathrm{H}^2_D(X, \mu_n)$ for any effective Cartier divisor $D \subset X$. In order to start the construction, we will need the following explicit characterization of (local) étale cohomology of $\mathbf{G}_m$:

Lemma 3.1.1. *Let X be a locally noetherian analytic adic space, and D a Zariski-closed subspace with the open complement U. Then there are functorial identifications*

$$\mathrm{H}^1(X, \mathbf{G}_m) \simeq \mathrm{Pic}(X),$$

$$\mathrm{H}^1_D(X, \mathbf{G}_m) \simeq \big\{(\mathcal{L}, \varphi) \mid \mathcal{L} \text{ a line bundle on } X, \varphi : \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}|_U\big\}/\sim.$$

Proof. The first claim follows from [Hub96, (2.2.7)]. Then the second statement follows from the argument identical to that of [Ols15, 2.13] (and it is essentially formal). □

For the following discussion, we fix an effective Cartier divisor $i\colon D \hookrightarrow X$. Our current goal is to leverage Lemma 3.1.1 and the Kummer exact sequence to define the cycle class $c\ell_X(D) \in \mathrm{H}^2_D(X, \mu_n)$. We start with the following definition:

Definition 3.1.2. The line bundle *associated to* $D \subset X$ is $\mathcal{O}_X$-module $\mathcal{O}_X(D) \coloneqq \big(\ker(\mathcal{O}_X \to i_*\mathcal{O}_D)\big)^\vee$. We denote its dual by $\mathcal{O}_X(-D) \simeq \ker(\mathcal{O}_X \to i_*\mathcal{O}_D)$.

By definition, we have the following short exact sequence,

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

By passing to duals, we get a canonical morphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ which is an isomorphism over $U \coloneqq X \smallsetminus D$. We denote the restriction of this morphism on U by

$$\varphi_D\colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_X(D)|_U. \tag{3.1.3}$$

Lemma 3.1.1 implies that the pair $(\mathcal{O}_X(D), \varphi_D)$ defines a class $[D] \in \mathrm{H}^1_D(X, \mathbf{G}_m)$. To get the (localized) cycle class of D, we combine the above discussion with the boundary map

$$\delta_X\colon \mathrm{H}^1_D(X, \mathbf{G}_m) \to \mathrm{H}^2_D(X, \mu_n)$$

coming from the Kummer exact sequence

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{f \mapsto f^n} \mathbf{G}_m \to 0.$$

More precisely, we give the following definition:

Definition 3.1.4. The (localized) *cycle class* $c\ell_X(D) \in \mathrm{H}^2_D(X, \mu_n)$ of an effective divisor $D \overset{i}{\hookrightarrow} X$ is defined to be $\delta_X([D]) \in \mathrm{H}^2_D(X, \mu_n)$.

Variant 3.1.5. Sometimes, it will be more convenient to think of the cycle class as a map

$$\mathrm{cl}_X(D)\colon i_*\underline{\Lambda}_D \to \mu_{n,X}[2].$$

Lemma 3.1.6 (Tranversal Base Change)**.** *Let*

$$\begin{array}{ccc} D' & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

be a cartesian diagram of locally noetherian analytic adic spaces such that the vertical arrows are effective Cartier divisors. Then $f^* c\ell_X(D) = c\ell_{X'}(D') \in \mathrm{H}^2_{D'}(X', \mu_n)$.

Proof. The boundary map coming from the Kummer exact sequence commutes with an arbitrary base change. Therefore, it suffices to show that the pair $(\mathcal{O}_X(D), \varphi_D)$ pullbacks to the pair $(\mathcal{O}_{X'}(D'), \varphi_{D'})$. This follows from [Zav23a, Lem. 5.7]. □

Now we show that our construction of cycle classes is compatible with the algebraic construction in [Del77, Def. 2.1.2].

Lemma 3.1.7. *Let* $S = \mathrm{Spa}\,(A, A^+)$ *be a strongly noetherian Tate affinoid, let* X *be a locally finite type* A*-scheme with the relative analytification* $X^{\mathrm{an}/S}$ *(see Construction 2.5.4), and let* $D \subset X$ *be an effective Cartier divisor. Then the natural comparison morphism*[12]

$$c^*_{X/S} \colon \mathrm{H}^2_D(X, \mu_n) \longrightarrow \mathrm{H}^2_{D^{\mathrm{an}/S}}(X^{\mathrm{an}/S}, \mu_n)$$

sends $c\ell_X(D) \in \mathrm{H}^2_D(X, \mu_n)$ *to* $c\ell_{X^{\mathrm{an}/S}}(D^{\mathrm{an}/S}) \in \mathrm{H}^2_{D^{\mathrm{an}/S}}(X^{\mathrm{an}/S}, \mu_n)$.

Proof. Let $[D]$ be the class in $\mathrm{H}^1_D(X, \mathbf{G}_m)$ corresponding to the pair $(\mathcal{O}_X(D), \varphi_D)$, where $\varphi_D \colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_X(D)|_U$ is the algebraic counterpart of the isomorphism constructed in (3.1.3). Then we note that $c\ell_X(D)$ is defined as $\delta_X([D])$, where δ_X is the boundary map in the algebraic Kummer sequence. Since the relative analytification of the pair $(\mathcal{O}_X(D), \varphi_D)$ is isomorphism to the pair $(\mathcal{O}_{X^{\mathrm{an}/S}}(D^{\mathrm{an}/S}), \varphi_{D^{\mathrm{an}/S}})$, we conclude that the natural map $c_{X/S} \colon \mathrm{H}^1_D(X, \mathbf{G}_m) \to \mathrm{H}^1_{D^{\mathrm{an}/S}}(X^{\mathrm{an}/S}, \mathbf{G}_m)$ sends the class $[D]$ to $[D^{\mathrm{an}/S}]$. Using the compatibility between algebraic and analytic Kummer exact sequences, we conclude that

$$c\ell_{X^{\mathrm{an}/S}}(D^{\mathrm{an}/S}) = \delta_{X^{\mathrm{an}/S}}([D^{\mathrm{an}/S}]) = \delta_{X^{\mathrm{an}/S}}(c^*_{X/S}[D]) = c^*_{X/S}\big(\delta_X([D])\big) = c^*_{X/S}(c\ell_X(D)),$$

where we slightly abuse notation and denote by $c_{X/S}$ both natural comparison morphisms $\mathrm{H}^1_D(X, \mathbf{G}_m) \to \mathrm{H}^1_{D^{\mathrm{an}/S}}(X^{\mathrm{an}/S}, \mathbf{G}_m)$ and $\mathrm{H}^2_D(X, \mu_n) \to \mathrm{H}^2_{D^{\mathrm{an}/S}}(X^{\mathrm{an}/S}, \mu_n)$. □

3.1.2. *First Chern classes.* In this subsection, we define the first Chern classes of line bundles. Although our discussion is essentially just a variant of Definition 3.1.4, it is convenient to treat this construction separately since it applies to a more general setup.

More precisely, the goal is to define a class $c_1(\mathcal{L}) \in \mathrm{H}^2(X, \mu_n)$ for any line bundle $\mathcal{L}$ on X. For this, we recall that Lemma 3.1.1 ensures that $\mathrm{H}^1(X, \mathbf{G}_m)$ is canonically isomorphic to $\mathrm{Pic}(X)$, so the isomorphism class of a line bundle $\mathcal{L}$ defines a class $[\mathcal{L}] \in \mathrm{H}^1(X, \mathbf{G}_m)$. Now we combine it with the boundary map

$$\delta_X \colon \mathrm{H}^1(X, \mathbf{G}_m) \to \mathrm{H}^2(X, \mu_n)$$

to get the first Chern class of $\mathcal{L}$:

Variant 3.1.8. The *first Chern class* $c_1(\mathcal{L}) \in \mathrm{H}^2(X, \mu_n)$ of a line bundle $\mathcal{L}$ on X is defined to be $\delta_X([\mathcal{L}]) \in \mathrm{H}^2(X, \mu_n)$.

Remark 3.1.9. Let $D \subset X$ be an effective Cartier divisor, and $\iota \colon \mathrm{H}^2_D(X, \mu_n) \to \mathrm{H}^2(X, \mu_n)$ the natural morphism. Then $\iota(c\ell_X(D)) = c_1(\mathcal{O}_X(D))$ as can be directly seen from the construction.

Remark 3.1.10. The formation of the first Chern class commutes with base change along an *arbitrary* morphism of locally noetherian analytic adic spaces $f \colon X' \to X$. The proof is analogous to that of Lemma 3.1.6 and boils down to the equality $f^*[\mathcal{L}] = [f^*\mathcal{L}] \in \mathrm{H}^2(X', \mu_n)$.

Now we show that the analytic first Chern classes are compatible with the algebraic first Chern classes:

[12] Here, we implicitly use [Zav23a, Cor. 6.5] (see also [GL21, Prop. 5.5]) that guarantees that $D^{\mathrm{an}/S} \subset X^{\mathrm{an}/S}$ is an effective Cartier divisor

Lemma 3.1.11. *Let $S = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid, let X be a locally finite type A-scheme with the relative analytification $X^{\mathrm{an}/S}$, and let $\mathcal{L} \in \mathrm{Pic}(X)$. Then the natural comparison morphism*

$$c^*_{X/S} \colon \mathrm{H}^2(X, \mu_n) \longrightarrow \mathrm{H}^2(X^{\mathrm{an}/S}, \mu_n)$$

sends $c_1(\mathcal{L}) \in \mathrm{H}^2(X, \mu_n)$ to $c_1(c^*_{X/A}\mathcal{L}) \in \mathrm{H}^2(X^{\mathrm{an}/S}, \mu_n)$.

Proof. The proof is essentially identical (and, in fact, easier) to that of Lemma 3.1.7. We leave details to the interested reader. □

3.2. **Projective bundle and blow-up formulas.** In this subsection, we prove the projective bundle and blow-up formulas. This will be the crucial ingredient in the extension of (localized) cycle classes from the case of divisors to the case of general lci immersions $Y \subset X$.

Throughout this subsection, we fix a locally noetherian analytic adic space X, an integer $n \in \mathcal{O}_X^\times$, and set $\Lambda \coloneqq \mathbf{Z}/n\mathbf{Z}$.

3.2.1. *Projective bundle formula.* The main goal of this subsection is to prove the projective bundle formula. For this, we fix a vector bundle $\mathcal{E}$ on X of rank $d+1$ and consider the associated projective bundle

$$f \colon P = \mathbf{P}_X(\mathcal{E}) \to X$$

with the universal line bundle $\mathcal{O}_{P/X}(1)$ (see [Zav23a, Rmk. 6.13] for the precise definition of $\mathbf{P}_X(\mathcal{E})$ in the context of adic spaces).

Construction 3.2.1. The first Chern class $c_1\big(\mathcal{O}(1)\big) \in \mathrm{H}^2(P, \mu_n)$ defines a morphism $\underline{\Lambda}_P \to \mu_n[2]$. After twisting, this becomes a morphism

$$c_1 \colon \underline{\Lambda}_P(-1)[-2] \to \underline{\Lambda}_X.$$

By the $(f^*, \mathrm{R}f_*)$-adjunction, this defines a morphism

$$c_1 \colon \underline{\Lambda}_X(-1)[-2] \to \mathrm{R}f_*\underline{\Lambda}_P$$

Using the multiplicative structure on $\mathrm{R}f_*\underline{\Lambda}_P$ coming from the cup-product map, we get a morphism

$$c_1^k \colon \underline{\Lambda}_X(-k)[-2k] \to \mathrm{R}f_*\underline{\Lambda}_P$$

for any $k \geq 0$.

Proposition 3.2.2. *(Projective Bundle Formula) Let $\mathcal{E}$ be a vector bundle on X of rank $d+1$. Let $f \colon P = \mathbf{P}_X(\mathcal{E}) \to X$ be the associated projective bundle. Then the natural morphism*

$$\gamma = \bigoplus_{i=0}^{d} c_1^i \colon \bigoplus_{i=0}^{d} \underline{\Lambda}_X(-i)[-2i] \to \mathrm{R}f_*\underline{\Lambda}_P$$

is an isomorphism.

Proof. By [Hub96, Prop. 8.2.3(ii)], $\mathrm{R}^i f_*\underline{\Lambda}_P$ is overconvergent for all $i \geq 0$. Therefore, it suffices to show that γ is an isomorphism over geometric points of rank-1. Since first Chern classes commute with arbitrary base change (see Remark 3.1.10) and the formation of $\mathrm{R}f_*\underline{\Lambda}_P$ commutes with taking stalks (see [Hub96, Prop. 2.6.1]), it suffices to prove the claim under the additional assumption that $X = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for an algebraically closed nonarchimedean field C. Then P algebraizes to a projective space $P^{\mathrm{alg}} = \mathbf{P}^d_C$. Therefore, the result follows from its algebraic counterpart established in [SGA77, Exp. VII, Th. 2.2.1] as well as the comparison results [Hub96, Th. 3.7.2] and Lemma 3.1.11. □

3.2.2. *Blow-up formula.* In this subsection, we discuss the blow-up formula in the context of adic spaces. We refer to [Zav23a, Def. 5.4] for the discussion of lci immersions in the context of adic spaces (see also [GL21, Def. 5.4]) and to [Zav23a, Def. 6.14] for the definition of a blow-up.

We fix a locally noetherian analytic adic space X with an lci immersion $i\colon Z \hookrightarrow X$ of pure codimension c. Let $\mathcal{I}_Z$ be the coherent ideal sheaf of the Zariski-closed immersion i. Define the conormal bundle to be $\mathcal{N}_{Z|X} \coloneqq \mathcal{I}_Z/\mathcal{I}_Z^2$; this is a vector bundle[13] over Z of rank c. We consider the blow-up morphism

$$\pi\colon \widetilde{X} \coloneqq \mathrm{Bl}_Z(X) \to X$$

and define the exceptional divisor via the formula

$$E \coloneqq \underline{\mathrm{Proj}}_X^{\mathrm{an}} \bigoplus_{n\geq 0} \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1} \simeq \mathbf{P}_Z(\mathcal{N}_{Z|X}).$$

Similarly as in algebraic geometry, one checks that E is naturally an effective Cartier divisor in $\mathrm{Bl}_Z(X)$ and that $\mathcal{O}_{\widetilde{X}}(-E) \simeq \mathcal{O}_{\widetilde{X}/X}(1)$.

In order to compute cohomology of the blow-up, we will need to use the construction of cup products in local cohomology. For this, we recall that given a Zariski-closed immersion $i\colon Z \hookrightarrow X$ and $\mathcal{F}, \mathcal{G} \in D(X_{\text{ét}}; \Lambda)$, there is a canonical morphism

$$w_{\mathcal{F},\mathcal{G}}\colon i^*\mathcal{F} \otimes^L \mathrm{R}i^!\mathcal{G} \to \mathrm{R}i^!(\mathcal{F} \otimes^L \mathcal{G}),$$

which is adjoint to $i_*(i^*\mathcal{F} \otimes^L \mathrm{R}i^!\mathcal{G}) \xrightarrow[\sim]{\mathrm{PF}^{-1}} \mathcal{F} \otimes^L i_*\mathrm{R}i^!\mathcal{G} \xrightarrow{\mathrm{id}\otimes^L \epsilon_i} \mathcal{F} \otimes^L \mathcal{G}$.

Construction 3.2.3 (Cup product in local cohomology)**.** (1) Let $i\colon Z \hookrightarrow X$ be a Zariski-closed immersion and let $\mathcal{F}, \mathcal{G} \in D(X_{\text{ét}}; \Lambda)$. Then, for each integers i and j, there is a functorial map

$$- \cup -\colon \mathrm{H}^i(Z, i^*\mathcal{F}) \otimes \mathrm{H}^j_Z(X, \mathcal{G}) \to \mathrm{H}^{i+j}_Z(X, \mathcal{F} \otimes^L \mathcal{G})$$

defined as the composition

$$\begin{aligned} &\mathrm{H}^i(Z, i^*\mathcal{F}) \otimes \mathrm{H}^j_Z(X, \mathcal{G}) \simeq \mathrm{Hom}\big(\underline{\Lambda}_Z, i^*\mathcal{F}[i]\big) \otimes \mathrm{Hom}\big(\underline{\Lambda}_Z, \mathrm{R}i^!\mathcal{G}[j]\big) \xrightarrow{(f,g)\mapsto f\otimes^L g} \\ &\qquad \mathrm{Hom}\big(\underline{\Lambda}_Z, i^*\mathcal{F}[i] \otimes^L \mathrm{R}i^!\mathcal{G}[j]\big) \xrightarrow{w_{\mathcal{F},\mathcal{G}}\circ -} \mathrm{Hom}\big(\underline{\Lambda}_Z, \mathrm{R}i^!(\mathcal{F} \otimes^L \mathcal{G})[i+j]\big) \simeq \mathrm{H}^{i+j}_Z(X, \mathcal{F} \otimes^L \mathcal{G}). \end{aligned}$$

(2) Let $i_1\colon Z_1 \hookrightarrow X$, $i_2\colon Z_2 \hookrightarrow X$ be two Zariski-closed immersions, let $i\colon Z \coloneqq Z_1 \times_X Z_2 \hookrightarrow X$ be their intersection, and let $\mathcal{F}, \mathcal{G} \in D(X_{\text{ét}}; \Lambda)$. Then, for each integers i and j, there is a functorial map

$$- \cup -\colon \mathrm{H}^i_{Z_1}(X, \mathcal{F}) \otimes \mathrm{H}^j_{Z_2}(X, \mathcal{G}) \to \mathrm{H}^{i+j}_Z(X, \mathcal{F} \otimes^L \mathcal{G})$$

defined as the composition

$$\begin{aligned} &\mathrm{H}^i_{Z_1}(X, \mathcal{F}) \otimes \mathrm{H}^j_{Z_2}(X, \mathcal{G}) \simeq \mathrm{Hom}\big(i_{1,*}\underline{\Lambda}_{Z_1}, \mathcal{F}[i]\big) \otimes \mathrm{Hom}\big(i_{2,*}\underline{\Lambda}_{Z_2}, \mathcal{G}[j]\big) \xrightarrow{(f,g)\mapsto f\otimes^L g} \\ &\qquad \mathrm{Hom}\big(i_{1,*}\underline{\Lambda}_{Z_1} \otimes^L i_{2,*}\underline{\Lambda}_{Z_2}, \mathcal{F}[i] \otimes^L \mathcal{G}[j]\big) \xrightarrow{\sim} \mathrm{Hom}(i_*\underline{\Lambda}_Z, \mathcal{F} \otimes^L \mathcal{G}[i+j]) \simeq \mathrm{H}^{i+j}_Z(X, \mathcal{F} \otimes^L \mathcal{G}), \end{aligned}$$

where the third map is given by precomposing with the inverse of the Künneth formula isomorphism

$$i_{1,*}\underline{\Lambda}_{Z_1} \otimes^L i_{2,*}\underline{\Lambda}_{Z_2} \xrightarrow{\sim} i_*\underline{\Lambda}_Z.$$

Convention 3.2.4. We denote by $c\ell_{\widetilde{X}}(-E)$ the cohomology class $-c\ell_{\widetilde{X}}(E) \in \mathrm{H}^2_E\big(\widetilde{X}, \Lambda(1)\big)$.

Proposition 3.2.5. *(Blow-up Formula) Let $Z \hookrightarrow X$ be an lci immersion of pure codimension c and let $\pi\colon \widetilde{X} \to X$ be the blow-up of Z in X. Then there is a canonical isomorphism*

$$\alpha\colon \bigoplus_{i=1}^{c-1} \mathrm{H}^{2(c-i)}\big(Z, \Lambda(c-i)\big) \oplus \mathrm{H}^{2c}_Z\big(X, \Lambda(c)\big) \to \mathrm{H}^{2c}_E\big(\mathrm{Bl}_Z(X), \Lambda(c)\big)$$

given by the formula

$$\alpha\big((\gamma_i), \gamma\big) = \sum_{i=1}^{c-1} \gamma_i \cdot c\ell_{\widetilde{X}}(-E)^i + \pi^*\gamma,$$

[13]This uses the lci assumption.

where $E \subset \mathrm{Bl}_Z(X)$ *is the exceptional divisor of the blow-up and the product* $\gamma_i \cdot c\ell_{\widetilde{X}}(-E)^i$ *is from Construction 3.2.3.*

Proof. The proof is a formal consequence of the projective bundle formula and the excision sequence. For example, the proof of [Fuj02, Lem. 1.1.1] applies verbatim in this context. □

3.3. **Higher-dimensional cycle classes.** In this subsection, we extend the theory of cycle classes to arbitrary lci immersions of pure codimension by following the strategy taken in [Fuj02]. The case of effective Cartier divisors was already treated in Section Section 3.1; the general case is reduced to this via certain blow-ups. Throughout this subsection, we fix a locally noetherian analytic adic space X, an integer $n \in \mathcal{O}_X^\times$, and denote $\Lambda \coloneqq \mathbf{Z}/n\mathbf{Z}$.

Let $i\colon Z \hookrightarrow X$ be an lci immersion of pure codimension c; our current goal is to define the (localized) cycle class $c\ell_X(Z) \in \mathrm{H}^2_Z\big(X, \Lambda(c)\big)$. For this, we consider the blow-up

$$\pi\colon \widetilde{X} \coloneqq \mathrm{Bl}_Z(X) \to X$$

with exceptional divisor $E \subset \widetilde{X}$. Now Definition 3.1.4 provides us with a class

$$c\ell_{\widetilde{X}}(-E) \in \mathrm{H}^2_E\big(\widetilde{X}, \Lambda(1)\big),$$

while the blow-up formula from Proposition 3.2.5 implies that there is a unique monic relation of degree c

$$c\ell_{\widetilde{X}}(-E)^c + \sum_{i=1}^{c} c_i \cdot c\ell_{\widetilde{X}}(-E)^{c-i} = 0 \in \mathrm{H}^{2c}_E\big(\widetilde{X}, \Lambda(c)\big) \tag{3.3.1}$$

with $c_i \in \mathrm{H}^{2i}\big(Z, \Lambda(i)\big)$ and $c_c \in \mathrm{H}^{2c}_Z\big(X, \Lambda(c)\big)$.

Definition 3.3.2. The *(localized) cycle class* $c\ell_X(Z) \in \mathrm{H}^{2c}_Z\big(X, \Lambda(c)\big)$ is the class $c_c \in \mathrm{H}^{2c}_Z\big(X, \Lambda(c)\big)$ from (3.3.1).

Variant 3.3.3. Similar to Variant 3.1.5, it will sometimes be more convenient to think of the cycle class as a map

$$\mathrm{cl}_i = \mathrm{cl}_X(Z)\colon i_*\underline{\Lambda}_Z \to \underline{\Lambda}_X(c)[2c].$$

Lemma 3.3.4 (Tranversal Base Change)**.** *Let*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

be a cartesian diagram of locally noetherian analytic adic spaces (still with $n \in \mathcal{O}_X^\times$*) such that the vertical arrows are lci immersions of pure codimension* c*. Then* $f^*c\ell_X(Z) = c\ell_{X'}(Z') \in \mathrm{H}^{2c}_{Z'}\big(X', \Lambda(c)\big)$*.*

Proof. Let $\mathcal{I}_Z$ and $\mathcal{I}_{Z'}$ be the ideal sheaves of the Zariski-closed immersions $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$, respectively. Then [Zav23a, Lem. 5.8] implies that $f^*\mathcal{I}_Z \simeq \mathcal{I}_{Z'}$. Then [Zav23a, Rmk. 7.8] ensures that there is a natural isomorphism

$$\alpha\colon \mathrm{Bl}_{Z'}(X') \xrightarrow{\sim} \mathrm{Bl}_Z(X) \times_X X'.$$

Denote by $E \subset \mathrm{Bl}_Z(X)$ and $E' \subset \mathrm{Bl}_{Z'}(X')$ the corresponding exceptional divisors. It is easy to see that α restricts to an isomorphism $\alpha\big|_{E'}\colon E' \xrightarrow{\sim} E \times_X X'$. In particular, the cartesian diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{Bl}_{Z'}(X') \arrow[r] & \mathrm{Bl}_Z(X) \end{tikzcd}$$

is transversal. After unraveling Definition 3.3.2, the question then boils down to showing that the natural morphism $\mathrm{H}^2_E(\mathrm{Bl}_Z(X), \mu_n) \to \mathrm{H}^2_{E'}(\mathrm{Bl}_{Z'}(X), \mu_n)$ sends $c\ell_{\mathrm{Bl}_Z(X)}(E)$ to $c\ell_{\mathrm{Bl}_{Z'}(X')}(E')$. This follows from Lemma 3.1.6. □

Now we study the behavior of cycle classes with respect to intersections.

Definition 3.3.5. A collection of effective Cartier divisors $\{D_i\}_{i=1,\dots,c}$ on X *crosses normally* if, for every subset $I \subset \{1, \dots, c\}$, the ("scheme-theoretic") intersection

$$D_I := \bigcap_{i \in I} D_i$$

is an lci Zariski-closed subspace of X of pure codimension $|I|$.

Lemma 3.3.6. *Let $\{D_i\}_{i=1,\dots,c}$ be a collection of effective Cartier divisors on X which crosses normally. Let $Z = \cap_{i \in I} D_i$ be their ("scheme-theoretic") intersection. Then $c\ell_X(Z)$ is given by the cup product*

$$c\ell_X(Z) = \bigcup_{i=1}^{c} c\ell_X(D_i) \in \mathrm{H}^{2c}_Z\big(X, \Lambda(c)\big).$$

Proof. We denote by $\widetilde{D}_i := \mathrm{Bl}_Z(D_i) \subset \widetilde{X} = \mathrm{Bl}_Z(X)$ the strict transform of D_i in the blow-up $\pi \colon \widetilde{X} \to X$. Since $Z \subset X$ is an lci Zariski-closed immersion, we see that $\widetilde{D}_i$ and $\pi^{-1}(D_i)$ are effective Cartier divisors and $\mathcal{O}_{\widetilde{X}}\big(\pi^{-1}(D_i)\big) \simeq \mathcal{O}_{\widetilde{X}}(\widetilde{D}_i) \otimes \mathcal{O}_{\widetilde{X}}(E)$. Therefore, we have an equality of cycle classes

$$c\ell_{\widetilde{X}}(\pi^{-1}(D_i)) = c\ell_{\widetilde{X}}(\widetilde{D}_i) + c\ell_{\widetilde{X}}(E) \in \mathrm{H}^2_{\pi^{-1}(D_i)}\big(\widetilde{X}, \Lambda(1)\big).$$

Now we consider the cup-product of all these classes to get

$$\bigcup_{i=1}^{c} \Big(c\ell_{\widetilde{X}}\big(\pi^{-1}(D_i)\big) + c\ell_{\widetilde{X}}(-E) \Big) = \bigcup_{i=1}^{c} c\ell_{\widetilde{X}}(\widetilde{D}_i), \tag{3.3.7}$$

where the equality takes place in $\mathrm{H}^{2c}_{\cap_{i=1}^{c} \pi^{-1}(D_i)}\big(\widetilde{X}, \Lambda(c)\big) = \mathrm{H}^{2c}_{E}\big(\widetilde{X}, \Lambda(c)\big)$. Since the intersection of strict transforms $\cap_{i=1}^{c} \widetilde{D}_i = \varnothing$ is empty, the product $\bigcup_{i=1}^{c} c\ell_{\widetilde{X}}(\widetilde{D}_i)$ factors through the natural morphism $\mathrm{H}^{2c}_{\varnothing}\big(\widetilde{X}, \Lambda(c)\big) = 0 \to \mathrm{H}^{2c}_{\cap_{i=1}^{c} \pi^{-1}(D_i)}\big(\widetilde{X}, \Lambda(c)\big)$. In particular, we see that $\bigcup_{i=1}^{c} c\ell_{\widetilde{X}}(\widetilde{D}_i) = 0$. Hence, (3.3.7) simplifies to the equation

$$\sum_{j=0}^{c} \sigma_j \Big(c\ell_{\widetilde{X}}\left(\pi^{-1}\left(D_1\right)\right), \dots, c\ell_{\widetilde{X}}\left(\pi^{-1}\left(D_c\right)\right) \Big) \cdot c\ell_{\widetilde{X}}(-E)^{c-j} = 0 \in \mathrm{H}^{2c}_{E}(\widetilde{X}, \Lambda(c)),$$

where σ_j denotes the j-th elementary symmetric polynomial and $\sigma_0 = 1$. Therefore, Definition 3.3.2 directly implies that

$$\pi^* c\ell_X(Z) = \sigma_c\Big(c\ell_{\widetilde{X}}\big(\pi^{-1}(D_1)\big), \dots, c\ell_{\widetilde{X}}\big(\pi^{-1}(D_c)\big) \Big) = \bigcup_{i=1}^{c} c\ell_{\widetilde{X}}\big(\pi^{-1}(D_i)\big).$$

Thus, Lemma 3.1.6 ensures that

$$\pi^* c\ell_X(Z) = \bigcup_{i=1}^{c} c\ell_{\widetilde{X}}(\pi^{-1}(D_i)) = \bigcup_{i=1}^{c} \pi^* c\ell_X(D_i) = \pi^* \left(\bigcup_{i=1}^{c} c\ell_X(D_i) \right).$$

Finally, Proposition 3.2.5 implies that $c\ell_X(Z) = \bigcup_{i=1}^{c} c\ell_X(D_i)$. □

Corollary 3.3.8. *Let $\{D_1, \dots, D_c\}$ be a collection of normally crossing effective Cartier divisors on X, and let c_1, c_2 be two nonnegative integers with $c_1 + c_2 \le c$. Set $Y := \cap_{i=1}^{c_1} D_i$ and $Z := \cap_{i=1}^{c_1+c_2} D_i$, giving rise to natural Zariski-closed immersions $Z \xhookrightarrow{i_1} Y \xhookrightarrow{i_Y} X$. Then the following composition commutes:*

$$\begin{tikzcd}
i_{Y,*}\big(i_{1,*}\underline{\Lambda}_Z\big) \arrow[r, "i_{Y,*}(\mathrm{cl}_{i_1})"] \arrow[d, "\wr"] & i_{Y,*}\underline{\Lambda}_Y(c_2)[2c_2] \arrow[d, "\mathrm{cl}_{i_Y}(c_2)[2c_2]"] \\
(i_Y \circ i_1)_*\underline{\Lambda}_Z \arrow[r, "\mathrm{cl}_{i_Y \circ i_1}"] & \underline{\Lambda}_X(c_1 + c_2)[2(c_1 + c_2)]
\end{tikzcd}$$

Proof. Let us put $W := \cap_{i=c_1+1}^{c_1+c_2} D_i$, resulting in a Cartesian diagram

$$\begin{tikzcd}
Z \arrow[r, "i_1"] \arrow[d, "i_2"] \arrow[dr, "i_Z"] & Y \arrow[d, "i_Y"] \\
W \arrow[r, "i_W"] & X.
\end{tikzcd}$$

Our assertion follows directly once we show that the following diagram commutes:

$$\begin{tikzcd}
i_{Y,*}\big(i_{1,*}\underline{\Lambda}_Z\big) \arrow[rrr, bend left, "\sim"] \arrow[dr, "i_{Y,*}(\mathrm{cl}_{i_1})"'] & i_{Y,*}\big(i_Y^* i_{W,*}\underline{\Lambda}_W\big) \arrow[l, "i_{Y,*}(\mathrm{BC})"'] \arrow[d, "i_{Y,*}(i_Y^*\mathrm{cl}_{i_W})"] & i_{Y,*}\underline{\Lambda}_Y \otimes i_{W,*}\underline{\Lambda}_W \arrow[l, "\mathrm{PF}_{i_Y}"'] \arrow[r, "\mathrm{KM}"] \arrow[d, "\mathrm{id}\otimes \mathrm{cl}_{i_W}"] \arrow[dr, "\mathrm{cl}_{i_Y}\otimes \mathrm{cl}_{i_W}"] \arrow[r, phantom, bend left, "(3.3.9)"'] & i_{Z,*}\underline{\Lambda}_Z \arrow[d, "\mathrm{cl}_{i_Z}"] \\
 & i_{Y,*}\underline{\Lambda}_Y(c_2)[2c_2] \arrow[rr, bend right, "\mathrm{cl}_{i_Y}(c_2)[2c_2]"'] & i_{Y,*}\underline{\Lambda}_Y \otimes \underline{\Lambda}_X(c_2)[2c_2] \arrow[l, "\mathrm{PF}_{i_Y}"] \arrow[r, "\mathrm{cl}_{i_Y}\otimes \mathrm{id}"] & \underline{\Lambda}_X(c_1+c_2)[2(c_1+c_2)]
\end{tikzcd}$$

The commutativity of (3.3.9) (as well as the meaning of KM, PF, and BC) expressing the factorization of Künneth map in terms of the projection formula and the base change map is explained in Lemma 6.3.8, whose proof is independent from the rest of our paper. The left triangle commutes by virtue of Lemma 3.3.4, whereas the upper right triangle commutes thanks to Lemma 3.3.6. The rest part is easily seen to be commutative. $\square$

Remark 3.3.10. One can adapt the proof of [ILO14, Exp. XVI, Th. 2.3.3] to show that, more generally, for an lci immersion $Z \overset{i}{\hookrightarrow} X$ of pure codimension c and an lci immersion $Y \overset{j}{\hookrightarrow} Z$ of pure codimension c', the diagram

$$\begin{tikzcd}
i_*\big(j_*\underline{\Lambda}_Y\big) \arrow[r, "i_*(\mathrm{cl}_j)"] \arrow[d, "\wr"] & i_*\big(\underline{\Lambda}_Z(c')[2c']\big) \arrow[d, "\mathrm{cl}_i(c')[2c']"] \\
(i\circ j)_*\underline{\Lambda}_Y \arrow[r, "\mathrm{cl}_{i\circ j}"] & \underline{\Lambda}_X(c+c')[2(c+c')].
\end{tikzcd}$$

commutes. We do not prove this claim as we never use it in this paper.

3.4. **Comparison with algebraic cycle classes.** Throughout this subsection, we fix a strongly noetherian Tate affinoid $S = \mathrm{Spa}\,(A, A^+)$, a locally finite type A-scheme X, an integer $n \in A^\times$, and denote $\Lambda := \mathbf{Z}/n\mathbf{Z}$.

Recall that [Del77, Cycle, Def. 2.1.2] defines a cycle class

$$c\ell_X(D) \in \mathrm{H}^2_D(X, \mu_n)$$

for any effective Cartier divisor $D \subset X$. This definition has been extended to general lci closed subschemes $Y \subset X$ of pure codimension c by Gabber in [Fuj02, Def. 1.1.2]. So *loc. cit.* defines the cycle class

$$c\ell_X(Y) \in \mathrm{H}^{2c}_Y\big(X, \Lambda(c)\big).$$

The main goal of this subsection is to show that this construction is compatible with Definition 3.3.2 via the relative analytification functor from Construction 2.5.4.

For this, we recall that [Zav23a, Cor. 6.5] (see also [GL21, Prop. 5.5]) ensures that, for an lci immersion $Y \subset X$ of pure codimension c, the relative analytification $Y^{\mathrm{an}/S} \subset X^{\mathrm{an}/S}$ is also an lci immersion of pure codimension c. Therefore, Definition 3.3.2 applies to this situation, providing us with the cycle class $c\ell_{X^{\mathrm{an}/S}}(Y^{\mathrm{an}/S}) \subset \mathrm{H}^{2c}_{Y^{\mathrm{an}/S}}\big(X^{\mathrm{an}/S}, \Lambda(c)\big)$.

Lemma 3.4.1. *Let X and Y be locally finite type A-schemes and let $Y \subset X$ be an lci immersion of pure codimension c. Then the natural morphism*

$$c^*_{X/S} \colon \mathrm{H}^{2c}_Y\big(X, \Lambda(c)\big) \longrightarrow \mathrm{H}^{2c}_{Y^{\mathrm{an}/S}}\big(X^{\mathrm{an}/S}, \Lambda(c)\big)$$

sends the algebraic cycle class $c\ell_X(Y)$ to the analytic cycle class $c\ell_{X^{\mathrm{an}/S}}(Y^{\mathrm{an}/S})$.

Proof. We recall that [Zav23a, Lem. 7.14] provides an isomorphism between the analytification of the algebraic blow-up and the analytic blow-up

$$\alpha\colon \mathrm{Bl}_Y(X)^{\mathrm{an}/S} \simeq \mathrm{Bl}_{Y^{\mathrm{an}}/S}(X^{\mathrm{an}}/S)$$

that restricts to an isomorphism of exceptional divisors. Thus, after unraveling the definitions and using the compatibility of the first Chern classes from Lemma 3.1.11, the question boils down to showing that the natural morphism

$$c_{\mathrm{Bl}_Y(X)/S}\colon \mathrm{H}^2_E(\mathrm{Bl}_Y(X), \mu_n) \longrightarrow \mathrm{H}^2_{E^{\mathrm{an}/S}}(\mathrm{Bl}_Y(X)^{\mathrm{an}/S}, \mu_n)$$

sends $c\ell_{\mathrm{Bl}_Y(X)}(E)$ to $c\ell_{\mathrm{Bl}_Y(X)^{\mathrm{an}/S}}(E^{\mathrm{an}/S})$, where E is the exceptional divisor of $\mathrm{Bl}_Y(X)$. Therefore, we may and do assume that $Y = D \subset X$ is an effective Cartier divisor. Then the result follows from Lemma 3.1.7. $\square$

3.5. **Cycle class of point.** In this subsection, we discuss a variant of Definition 3.3.2 that takes values in compactly supported cohomology groups. Throughout this subsection, we fix an algebraically closed nonarchimedean field C, an integer $n \in C^\times$, and set $\Lambda \coloneqq \mathbf{Z}/n\mathbf{Z}$.

Recall that, for every taut, separated, finite type adic C-space X and every complex $\mathcal{F} \in D^+(X_{\text{ét}}; \Lambda)$, [Hub96, Def. 5.4.4] defines the compactly supported étale cohomology complex $\mathrm{R}\Gamma_c(X, \mathcal{F})$. Now suppose that X is a rigid-analytic space over C of equidimension d. Let $x \in X(C)$ be a classical point of X. Then [Zav23a, Cor. 5.11] implies that $i\colon \{x\} \hookrightarrow X$ is an lci immersion of pure codimension d. Thus, Definition 3.3.2 provides us with the cycle class $c\ell_X(x) \in \mathrm{H}^{2d}_x\big(X, \Lambda(d)\big)$. Now since x is proper over $\mathrm{Spa}\,(C, \mathcal{O}_C)$, the $\mathrm{R}\Gamma_c(X, -)$-functor applied to the counit morphism

$$i_*\mathrm{R}i^!\underline{\Lambda}_X(c) \longrightarrow \underline{\Lambda}_X(c)$$

yields a morphism

$$\mathrm{R}\Gamma_x\big(X, \Lambda(c)\big) \longrightarrow \mathrm{R}\Gamma_c\big(X, \Lambda(c)\big).$$

In particular, this induces a canonical morphism

$$\mathrm{H}^{2c}_x\big(X, \Lambda(c)\big) \longrightarrow \mathrm{H}^{2c}_c\big(X, \Lambda(c)\big). \tag{3.5.1}$$

Definition 3.5.2. In the situation above, the *(compactly supported) cycle class* $c\ell_X(x) \in \mathrm{H}^{2c}_c\big(X, \Lambda(c)\big)$ is the image of $c\ell_X(x) \in \mathrm{H}^{2c}_x\big(X, \Lambda(c)\big)$ under the morphism

$$\mathrm{H}^{2c}_x\big(X, \Lambda(c)\big) \longrightarrow \mathrm{H}^{2c}_c\big(X, \Lambda(c)\big)$$

from (3.5.1).

The main result of this subsection is that the compactly supported version of the cycle class of a point behaves well with respect to étale morphisms and Zariski-closed immersions. We begin with the case of étale morphisms:

Lemma 3.5.3. *Let $f\colon X \to Y$ be an étale morphism between smooth separated taut rigid-analytic spaces over C of equidimension d. Let $x \in X(C)$ be a classical point. Then the natural morphism*

$$\mathrm{H}^{2d}_c\big(X, \Lambda(d)\big) \longrightarrow \mathrm{H}^{2d}_c\big(Y, \Lambda(d)\big)$$

sends $c\ell_X(x)$ to $c\ell_Y\big(f(x)\big)$.

Proof. We first consider the case of an open immersion $f\colon X \hookrightarrow Y$. In this case, the claim follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^{2d}_x\big(X, \Lambda(d)\big) \arrow[d] & \mathrm{H}^{2d}_{f(x)}\big(Y, \Lambda(d)\big) \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{H}^{2d}_c\big(X, \Lambda(d)\big) \arrow[r] & \mathrm{H}^{2d}_c\big(Y, \Lambda(d)\big)
\end{tikzcd}$$

Now we treat the general case. The case of open immersions implies that, for the purposes of proving the statement, we may replace X with any open neighborhood of x and Y with any open neighborhood of $f(x)$. Therefore, we can first reduce to the case when X and Y are affinoids. Then [Hub96, Cor. 1.7.4] implies that

f is quasi-finite. Thus, [Hub96, Lem. 1.4.7, Lem. 1.5.2.(f)] imply that we can further localize to the case of a finite étale morphism f.

Choose a pseudo-uniformizer $\varpi \in \mathcal{O}_C$. We have

$$\widehat{\mathcal{O}}^+_{Y,f(x)}\left[\frac{1}{\varpi}\right] \simeq \widehat{k\big(f(x)\big)}^+\left[\frac{1}{\varpi}\right] \simeq C,$$

where $\widehat{(-)}$ stands for the ϖ-adic completion (see e.g. [Sch12, Prop. 2.25] or [Bha, Prop. 7.5.5.5]). Since $\mathcal{O}^+_{Y,f(x)}$ is ϖ-henselian (as a colimit of ϖ-complete, hence ϖ-henselian rings), [GR03, Prop. 5.4.53] implies that

$$\mathcal{O}_{Y,f(x)_{\text{fét}}} \simeq C_{\text{fét}}.$$

In particular, any finite étale $\mathcal{O}_{Y,f(x)}$-algebra is split. Therefore, a standard approximation argument implies that, after passing to an open neighborhood of $f(x)$, we can assume that $X = \bigsqcup_{i=1}^n Y$. Then, by passing to a neighborhood of x, we can assume that $X = Y$. In this case, the claim is obvious. $\square$

Lemma 3.5.4. *Let X and Y be smooth separated taut rigid-analytic spaces over C of equidimension d_X and d_Y respectively. Let $i\colon X \hookrightarrow Y$ be a Zariski-closed immersion, and let $x \in X(C)$ be a classical point. Then the morphism*

$$\mathrm{H}^{2d_X}_c\big(Y, \mathrm{cl}_i(d_X)\big)\colon \mathrm{H}^{2d_X}_c\big(X, \Lambda(d_X)\big) \to \mathrm{H}^{2d_Y}_c\big(Y, \Lambda(d_Y)\big)$$

sends $c\ell_X(x)$ to $c\ell_Y(x)$.

Proof. Lemma 3.5.3 implies that we can replace Y with any open subspace $x \in U \subset Y$ (and X with $X \cap U$). Therefore, we can assume that there is a collection of normally crossing (in the sense of Definition 3.3.5) effective Cartier divisors $\{D_1, \dots, D_{d_Y}\}$ on Y such that $X = \cap_{i=1}^{d_Y-d_X} D_i$ and $\{x\} = \cap_{i=1}^{d_Y} D_i$. Then the result follows automatically from Corollary 3.3.8. $\square$

4. Rigid-analytic curves

In this section, we study some properties of smooth rigid-analytic curves over an algebraically closed nonarchimedean field. These results will be crucial in the construction of the trace map in étale cohomology.

Throughout this section, we fix an algebraically closed nonarchimedean field C. We denote its ring of integers by $\mathcal{O}_C$, its maximal ideal by $\mathfrak{m}_C \subset \mathcal{O}_C$, and its residue field by $k_C := \mathcal{O}_C/\mathfrak{m}_C$. We also choose a pseudo-uniformizer $\varpi \in \mathcal{O}_C$.

4.1. **Geometry of curves.** In this subsection, we collect some results about the geometry of rigid-analytic curves. In particular, we recall that smooth (quasicompact and separated) rigid-analytic curves behave similarly to algebraic curves. We also discuss that rigid-analytic curves often admit particularly nice formal models.

Definition 4.1.1. A rigid-analytic C-space X is a *rigid-analytic C-curve* if X is of pure dimension 1 (in the sense of [Hub96, Def. 1.8.1]).

We start with a technical lemma that will be handy in a number of situations later in this paper.

Lemma 4.1.2. *Let $f\colon X \to Y$ be a finite morphism of smooth rigid-analytic curves over C. Then f is flat.*

Proof. Without loss of generality, we can assume that both X and Y are affinoid. In this situation, $f\colon X = \mathrm{Spa}\,(B, B^\circ) \to Y = \mathrm{Spa}\,(A, A^\circ)$ is induced by a finite morphism $f^\#\colon A \to B$ of K-affinoid domains.

Then it suffices to show that, for every maximal ideal $\mathfrak{m} \subset B$ with the pre-image $\mathfrak{n} := f^{\#,-1}(\mathfrak{m})$ (this prime ideal is automatically maximal), the natural morphism $A_\mathfrak{n} \to B_\mathfrak{m}$ is flat. Now [Hub96, Lem. 1.8.6.(ii)] and [FK18, Th. II.10.1.8] implies that $\dim A_\mathfrak{n} = \dim B_\mathfrak{m} = 1$. Furthermore, [FvdP04, Th. 3.6.3] ensures that $A_\mathfrak{n}$ are $B_\mathfrak{m}$ are regular local rings. Therefore, flatness of $A_\mathfrak{n} \to B_\mathfrak{m}$ follows directly from Miracle Flatness (see [Sta22, Tag 00R4]). $\square$

Now we discuss formal models of rigid-analytic curves over C. We start with the following definition of semi-stable formal $\mathcal{O}_C$-curves:

Definition 4.1.3. We say that an admissible formal $\mathcal{O}_C$-scheme $\mathscr{X}$ is a *semi-stable formal $\mathcal{O}_C$-curve* if the special fiber has pure dimension 1 and for any point $x \in \mathscr{X}$, either the structure map $f\colon \mathscr{X} \to \mathrm{Spf}(\mathcal{O}_C)$ is smooth at x, or there is a pointed affine formal $\mathcal{O}_C$-scheme $(\mathrm{Spf}\, A, y)$ such that there exists a diagram of pointed formal schemes

$$\begin{tikzcd} & (\mathrm{Spf}\, A, y) \arrow[dl, "g"'] \arrow[dr, "h"] & \\ (\mathscr{X}, x) \arrow[dr] & & \left(\mathrm{Spf}\, \dfrac{\mathcal{O}_C\langle S, T\rangle}{(ST-\alpha_x)}, \{u, 0, 0\}\right) \arrow[dl] \\ & (\mathrm{Spf}\, \mathcal{O}_C, u), & \end{tikzcd}$$

where u is the only point of $|\mathrm{Spf}\, \mathcal{O}_C|$, g and h are étale morphisms, and $\alpha_x \in \mathfrak{m}_C$.

Remark 4.1.4. A semi-stable formal $\mathcal{O}_C$-curve $\mathscr{X}$ is rig-smooth if and only if all α_x in Definition 4.1.3 are non-zero.

Our next goal is to discuss formal models of (smooth) rigid-analytic curves in more detail. We start with the following algebraization result that seems difficult to find in the existing literature:

Lemma 4.1.5. *Let $\mathscr{X}$ be a finitely presented proper formal $\mathcal{O}_C$-scheme such that the special fiber $\mathscr{X}_s$ is of pure dimension 1. Then there is a finitely presented projective $\mathcal{O}_C$-scheme $\mathfrak{X}$ and an $\mathcal{O}_C$-isomorphism $\widehat{\mathfrak{X}} \simeq \mathscr{X}$, where $\widehat{\mathfrak{X}}$ is the ϖ-adic completion of $\mathfrak{X}$. Furthermore, if $\mathscr{X}$ is admissible (resp. rig-smooth), then $\mathfrak{X}$ is $\mathcal{O}_C$-flat (resp. $\mathfrak{X}_\eta$ is C-smooth).*

Proof. First, we note that [Sta22, Tag 09NZ] implies that the special fiber $\mathscr{X}_s$ admits an ample line bundle $\mathcal{L}_s$. Then a standard approximation argument implies that we can find a pseudo-uniformizer $\pi \in \mathcal{O}_C$ such that $\mathscr{X}_0 \coloneqq \mathscr{X} \times_{\mathrm{Spec}\, \mathcal{O}_C} \mathrm{Spec}\, \mathcal{O}_C/(\pi)$ admits a line bundle $\mathcal{L}_0$ such that its restriction $\mathcal{L}_0|_{\mathscr{X}_s}$ is isomorphic to $\mathcal{L}_s$. Furthermore, [Sta22, Tag 0D2S] implies that $\mathcal{L}_0$ is automatically ample.

Now, for each integer $n \geq 0$, we denote by $\mathscr{X}_n$ the $\mathcal{O}_C/\pi^{n+1}$-scheme $\mathscr{X} \times_{\mathrm{Spf}\, \mathcal{O}_C} \mathrm{Spec}\, \mathcal{O}_C/(\pi^{n+1})$. Then [Sta22, Tag 0C6R] and vanishing of $\mathrm{H}^2(\mathscr{X}_n, \pi^n \mathcal{O}_{\mathscr{X}_{n+1}})$ imply that $\mathrm{Pic}(\mathscr{X}_n) \to \mathrm{Pic}(\mathscr{X}_{n-1})$ is surjective for any $n \geq 1$. Therefore, we can find a compatible sequence of line bundles $\mathcal{L}_n$ on $\mathscr{X}_n$ such that $\mathcal{L}_n|_{\mathscr{X}_{n-1}} \simeq \mathcal{L}_{n-1}$. By passing to the limit, we get a line bundle $\mathcal{L}$ on $\mathscr{X}$ such that $\mathcal{L}|_{\mathscr{X}_0} \simeq \mathcal{L}_0$ is ample. Therefore, [FK18, Prop. I.10.3.2] implies that there is a finitely presented proper $\mathcal{O}_C$-scheme $\mathfrak{X}$ with an isomorphism $\widehat{\mathfrak{X}} \simeq \mathscr{X}$.

Now suppose that $\mathscr{X}$ is admissible. We wish to show that $\mathfrak{X}$ is then $\mathcal{O}_C$-flat. Choose an open subscheme $\mathrm{Spec}\, A \subset \mathfrak{X}$. Then [FGK11, Prop. 4.3.4 and Th. 7.3.2] imply that the map $A \to \widehat{A}$ is flat. Therefore, the map $A \to \widehat{A} \times A[\frac{1}{\varpi}]$ is faithfully flat. In particular, it is injective. Our assumption that $\mathscr{X}$ is admissible implies that $\widehat{A}$ is $\mathcal{O}_C$-flat (i.e. has no ϖ-torsion). Therefore, $A \hookrightarrow \widehat{A} \times A[\frac{1}{\varpi}]$ has no ϖ-torsion as well. Thus, it is $\mathcal{O}_C$-flat.

Finally, we assume that $\mathscr{X}$ is rig-smooth. Then [Con99, Th. A.3.1] implies that there is an isomorphism $\mathfrak{X}_\eta^{\mathrm{an}} \simeq \mathscr{X}_\eta$. Our assumption implies that $\mathfrak{X}_\eta^{\mathrm{an}}$ is C-smooth. Therefore, $\mathfrak{X}_\eta$ is C-smooth due to [Con99, Th. A.2.1]. $\square$

Finally, we recall the following version of the semi-stable reduction for rigid-analytic curves. This result will be crucial in our proof of Theorem 5.4.2:

Proposition 4.1.6.

(i) ([FM86, Th. 2]) *Every irreducible quasi-compact separated rigid-analytic curve over C is either affinoid or proper;*
(ii) ([Lüt16, Section 1.8]) *The category of smooth proper rigid-analytic curves over C and the category of smooth proper algebraic curves over C are equivalent;*
(iii) ([vdP80, Th. 1.1]) *Every smooth affinoid rigid-analytic curve X over C is an open subdomain of a smooth proper rigid-analytic C-curve $\overline{X}$.*

(iv) *Every pair $X \subset \overline{X}$ over C in (iii) arises as the rigid generic fiber of open immersions of admissible formal $\mathcal{O}_C$-schemes $\mathscr{X} \subset \overline{\mathscr{X}}$ where $\overline{\mathscr{X}}$ is a semi-stable formal $\mathcal{O}_C$-curve;*

(v) *In particular, every smooth quasi-compact separated rigid C-curve admits a semi-stable formal model over $\mathcal{O}_C$;*

(vi) *For a proper and smooth C-scheme X of equidimension 1 and an integer $n \in C^\times$, the natural morphism $\mathrm{R}\Gamma(X, \mu_n) \to \mathrm{R}\Gamma(X^{\mathrm{an}}, \mu_n)$ is an isomorphism.*

Of course, [Hub96, Th. 3.7.2] guarantees that the conclusion of (vi) holds for any proper C-scheme X. However, we prefer to give a different elementary argument below.

Proof. (i)- (iii) are stated and proven in the said references. (v) follows from combining (i)- (iv). Since we did not find the exact statement (iv) in literature, we give a proof below.

By [Bos14, Lem. 2], we can find an admissible formal $\mathcal{O}_C$-scheme $\overline{\mathscr{X}}$ and an open formal subscheme $\mathscr{X} \subset \overline{\mathscr{X}}$ such that the generic fiber of this immersion is equal to $X \subset \overline{X}$. Now [Tem00, Cor. 4.4 and 4.5] imply that $\overline{\mathscr{X}}$ is automatically a proper admissible formal $\mathcal{O}_C$-scheme, and [Zav21b, Cor. B.4] implies that the special fiber $\overline{\mathscr{X}}_s$ is of pure dimension 1. Therefore, Lemma 4.1.5 ensures that there is a projective, finitely presented, flat $\mathcal{O}_C$-scheme $\overline{\mathfrak{X}}$ such that $\widehat{\overline{\mathfrak{X}}} \simeq \overline{\mathscr{X}}$. Furthermore, the generic fiber $\overline{\mathfrak{X}}_\eta$ is C-smooth.

Now [Tem10, Th. 1.5] implies that there is an η-modification[14] $f \colon \overline{\mathfrak{X}}' \to \overline{\mathfrak{X}}$ such that $\overline{\mathfrak{X}}'$ is a semi-stable $\mathcal{O}_C$-curve. We note that the completion $\widehat{\overline{\mathfrak{X}}}'$ is a semi-stable formal $\mathcal{O}_C$-curve in the sense of Definition 4.1.3 (see, for example, [Zav21a, Lem. B.11]) and the morphism

$$\widehat{f} \colon \overline{\mathscr{X}}' := \widehat{\overline{\mathfrak{X}}}' \to \widehat{\overline{\mathfrak{X}}} \simeq \overline{\mathscr{X}}$$

is a rig-isomorphism (see, for example, [Zav21a, Lem. B.8]). Thus, the inclusion $X \subset \overline{X}$ arises as the rigid generic fiber of an open immersion $\widehat{f}^{-1}(\mathscr{X}) \subset \overline{\mathscr{X}}'$ where $\overline{\mathscr{X}}'$ is a semi-stable formal $\mathcal{O}_C$-curve.

Lastly we give a proof of (vi). Using [Zav23b, Lem. 6.1.4.(2)] and [Sta22, Tag 03RT], it suffices to show that the natural morphism $\mathrm{H}^i(X, \mu_n) \to \mathrm{H}^i(X^{\mathrm{an}}, \mu_n)$ is an isomorphism for $i \le 2$. We use [Zav23b, Lem. 6.1.4.(3)], [Sta22, Tag 03RM], and the schematic and analytic Kummer exact sequence to reduce the question to proving that the natural morphisms

$$\mathrm{H}^0(X, \mathbf{G}_m) = \mathcal{O}_X(X)^\times \to \mathrm{H}^0(X^{\mathrm{an}}, \mathbf{G}_m) = \mathcal{O}_{X^{\mathrm{an}}}(X^{\mathrm{an}})^\times \text{ and}$$
$$\mathrm{H}^1(X, \mathbf{G}_m) = \mathrm{Pic}(X) \to \mathrm{H}^1(X^{\mathrm{an}}, \mathbf{G}_m) = \mathrm{Pic}(X^{\mathrm{an}})$$

are isomorphisms. Finally, we note that these maps are isomorphisms due to the rigid-analytic GAGA and properness of X (see [FK18, Th. II.9.4.1 and Cor. II.9.4.4]) $\square$

Now we end the subsection with a version of the Noether normalization result for semi-stable curves over $\mathcal{O}_C$. This result, in conjunction with Lemma 4.1.2 and Proposition 4.1.6, will be a very useful tool to reduce questions about general smooth rigid-analytic curves to the case of the closed unit disc.

Lemma 4.1.7. *Let $\mathscr{X}$ be a semi-stable proper formal $\mathcal{O}_C$-curve, and let $\{x_1, \dots, x_n\}$ be a finite set of closed points in $|\mathscr{X}| = |\mathscr{X}_s|$ that meets each irreducible component of $|\mathscr{X}|$. Then there is a finite morphism $f \colon \mathscr{X} \to \widehat{\mathbf{P}}^1_{\mathcal{O}_C}$ such that $f_s^{-1}(\{\infty\})$ is set-theoretically equal to $\{x_1, \dots, x_n\}$.*

Proof. We first construct the map over the residue field k_C. Since $\mathscr{X}_s$ is nodal, we know that there is an effective Cartier divisor on $\mathscr{X}_s$ whose set-theoretic support is the set of nodes $\{x_1, \dots, x_n\}$. Its associated line bundle $\mathcal{L}_s$ is ample on $\mathscr{X}_s$ because $\{x_1, \dots, x_n\}$ hits every irreducible component of $\mathscr{X}_s$ (see [Sta22, Tag 0B5Y]). Furthermore, $\mathcal{L}_s$ comes with a canonical section $\delta_s \in \mathcal{L}_s(\mathscr{X}_s)$ such that the vanishing locus $\mathrm{V}(\delta_s)$ is set-theoretically equal to $\{x_1, \dots, x_n\}$. Therefore, [Ked05, Lem. 6] ensures that there is an integer d and a section $\alpha \in \mathcal{L}_s^{\otimes d}(\mathscr{X}_s)$ such that the natural morphism $\mathcal{O}_{\mathscr{X}_s}^{\oplus 2} \xrightarrow{\alpha_s + \delta_s^{\otimes d}} \mathcal{L}_s^{\otimes d}$ is surjective and defines a finite map $f_s \colon \mathscr{X}_s \to \mathbf{P}^1_k$ such that the pre-image $f_s^{-1}(\{\infty\})$ is set-theoretically equal to $\{x_1, \dots, x_n\}$. By replacing $\mathcal{L}_s$ with $\mathcal{L}_s^{\otimes d}$ (and δ with $\delta^{\otimes d}$), we may and do assume that $d = 1$.

[14] An η-modification is a proper morphism $f \colon \overline{\mathfrak{X}}' \to \overline{\mathfrak{X}}$ such that the generic fiber $\overline{\mathfrak{X}}'_\eta$ is schematically dense in $\overline{\mathfrak{X}}'$ and f_η is an isomorphism. This automatically implies that $\overline{\mathfrak{X}}'$ is $\mathcal{O}_C$-flat and f is finitely presented.

Now a standard approximation argument implies that we can find a pseudo-uniformizer $\pi \in \mathcal{O}_C$ and a line bundle $\mathcal{L}_0$ on $\mathscr{X}_0 := \mathscr{X} \times_{\mathrm{Spf}\, \mathcal{O}_C} \mathrm{Spec}\, \mathcal{O}_C/(\pi)$ with two global sections s_0 and α_0 such that $\mathcal{L}_0|_{\mathscr{X}_s} \simeq \mathcal{L}_s$, $\delta_0|_{\mathscr{X}_s} = \delta_s$, $\alpha_0|_{\mathscr{X}_s} = \alpha_s$, and the natural morphism $\mathcal{O}_{\mathscr{X}_0}^{\oplus 2} \xrightarrow{\alpha_0+\delta_0} \mathcal{L}_0$ is surjective and defines a finite morphism $f_0 \colon \mathscr{X}_0 \to \mathbf{P}^1_{\mathcal{O}_C/(\pi)}$. By a standard deformation theory argument (see the proof of Lemma 4.1.5), we can lift $\mathcal{L}_0$ to an ample line bundle $\mathcal{L}$ on $\mathscr{X}$. Therefore, after replacing $\mathcal{L}$ with its high enough power (and replacing sections α_0 and δ_0 with their powers as well), we can assume that $\mathrm{H}^1(\mathscr{X}, \mathcal{L}) = 0$. With this cohomology vanishing, we can lift the sections δ_0 and α_0 to some sections $\delta \in \mathcal{L}(\mathscr{X})$ and $\alpha \in \mathcal{L}(\mathscr{X})$ such that $\delta|_{\mathscr{X}_0} = \delta_0$ and $\alpha|_{\mathscr{X}_0} = \alpha_0$. Lastly, Nakayama's lemma ensures these sections define a surjection $\mathcal{O}_{\mathscr{X}}^{\oplus 2} \xrightarrow{\alpha+\delta} \mathcal{L}$ which, in turn, defines a finite morphism $f \colon \mathscr{X} \to \widehat{\mathbf{P}}^1_{\mathcal{O}_C}$ such that $f|_s = f_s$. This implies that $f_s^{-1}(\{\infty\})$ is set-theoretically equal to $\{x_1, \dots, x_n\}$. □

4.2. **Universal compactifications of curves.** In this subsection, we study universal compactifications of curves; cf. Appendix A. The description of universal compactifications obtained in this subsection will be an important input in our construction of analytic trace maps in Section 5.

Lemma 4.2.1. *Let $f \colon X = \mathrm{Spa}\,(B, B^\circ) \to Y = \mathrm{Spa}\,(A, A^\circ)$ be a finite morphism of rigid-analytic affinoid C-curves, inducing a finite morphism $f^c \colon X^c \to Y^c$ of universal compactifications. Then $f^{c,-1}(Y^c \setminus Y) = X^c \setminus X$.*

Proof. Clearly, $f^{c,-1}(Y^c \setminus Y) \subset X^c \setminus X$. Therefore, it suffices to show that $X^c \setminus X \subset f^{c,-1}(Y^c \setminus Y)$. Equivalently, it suffices to show that the natural morphism

$$j \colon X \to X' := X^c \times_{Y^c} Y$$

is an isomorphism. Since $X \to X^c$ is an open immersion, we conclude that j is an open immersion as well. Since $\mathcal{O}_{Y^c}(Y^c) \simeq \mathcal{O}_Y(Y)$, $\mathcal{O}_{X^c}(X^c) \simeq \mathcal{O}_X(X)$, and X, X^c, Y, and Y^c are all affinoids (see Lemma A.0.3), we conclude that the natural morphism $\mathcal{O}_{X'}(X') \simeq \mathcal{O}_Y(Y) \widehat{\otimes}_{\mathcal{O}_{Y^c}(Y^c)} \mathcal{O}_{X^c}(X^c) \to \mathcal{O}_X(X)$ is a topological isomorphism.

Now Lemma 2.4.6 implies that f^c is a finite morphism, and so j is a morphism of finite adic Y-spaces. Therefore, j is itself a finite morphism. Thus, [Hub96, Lem. 1.4.5.(ii)] implies that topologically j is a closed morphism, and so [Zav24, Lem. B.6.14] ensures that j is a closed immersion. Therefore, [Zav24, Cor. B.6.9] and the established above isomorphism $\mathcal{O}_{X'}(X') \xrightarrow{\sim} \mathcal{O}_X(X)$ implies that j is an isomorphism. □

Lemma 4.2.2. *Let $X = \mathbf{D}^1$ be a one-dimensional closed unit disc. Then $X^c \smallsetminus X$ consists of a unique rank-2 point x_+. Furthermore, under the isomorphism $X^c = \mathrm{Spa}\,(C\langle T\rangle, \mathcal{O}_C + T\mathfrak{m}_C\langle T\rangle)$ of Lemma A.0.3, the point x_+ comes from the valuation*

$$v_{x_+} \colon C\langle T\rangle \to \big(\Gamma_C \bigoplus \mathbf{Z}\big) \cup \{0\}$$
$$v_{x_+}\Big(\sum_n a_n T^n\Big) = \sup_n \big\{(|a_n|, n)\big\}.$$

Proof. First, Lemma A.0.3 implies that $X^c = \mathrm{Spa}\,(C\langle T\rangle, \mathcal{O}_C + T\mathfrak{m}_C\langle T\rangle)$. Therefore, [Hub93a, Prop. 3.9] implies that

$$|\mathrm{Spa}\,(C\langle T\rangle, \mathcal{O}_C + T\mathfrak{m}_C\langle T\rangle)| \smallsetminus |\mathrm{Spa}\,(C\langle T\rangle, \mathcal{O}_C\langle T\rangle)| = |\mathrm{Spa}\,(C[T], \mathcal{O}_C + T\mathfrak{m}_C[T])| \smallsetminus |\mathrm{Spa}\,(C[T], \mathcal{O}_C[T])|,$$

where we endow both $\mathcal{O}_C[T]$ and $\mathcal{O}_C + T\mathfrak{m}_C[T]$ with the ϖ-adic topology. Therefore, the result follows directly from [Sem15, Lecture 11, Example 11.3.14] or [Hub01, Ex. 5.2]. □

In order to get some intuition of how the valuation v_{x_+} works, let us do the following easy computation:

Example 4.2.3. By how it is defined, we see that $v_{x_+}(a_n T^n) \le 1$ is equivalent to either $|a_n| < 1$ or $|a_n| = 1$ and $n = 0$. We also see that the subset of $C\langle T\rangle$ defined by the condition $v_{x_+} \le 1$ is precisely $\mathcal{O}_C + T\mathfrak{m}_C\langle T\rangle$.

Lemma 4.2.4 ([Hub01, Lem. 5.12])**.** *Let X be a separated, quasi-compact rigid-analytic C-curve with universal compactification $X \hookrightarrow X^c$. Then*

(i) *$|X^c| \smallsetminus |X|$ is finite and discrete;*
(ii) *each point $x \in |X^c| \smallsetminus |X|$ is a rank-2 point;*
(iii) *if X is affinoid, then $|X^c| \smallsetminus |X|$ is non-empty.*

Proof. (i) follows directly from [Hub01, Lem. 5.12]. For (ii), we first note that Lemma A.0.4 implies that $\mathrm{rk}\,\Gamma_x > 1$. Now [Hub96, Cor. 1.8.8, Cor. 5.1.14] imply that $\mathrm{tr.c}(\widehat{k(x)}/C) \leq 1$. Therefore, [Bou98, Ch. VI.10.3, Cor. 1] ensures that $\dim_{\mathbf{Q}}(\Gamma_x/\Gamma_C \otimes_{\mathbf{Z}} \mathbf{Q}) \leq 1$. Therefore, [Bou98, Ch.$\tilde{\text{V}}$I.10.2, Prop. 3] implies that $\mathrm{rk}\,\Gamma_x \leq \mathrm{rk}\,\Gamma_C + 1 \leq 2$. This implies that $\mathrm{rk}\,\Gamma_x = 2$.

To see (iii), we note that [Hub96, Prop. 1.4.6] implies that X is not proper. Then [Hub96, Cor. 5.1.6] ensures that $X \neq X^c$. □

Given $x \in X$ with residue field $k(x)$, we denote the associated valuation by $v_x \colon k(x) \to \Gamma_x \cup \{0\}$. We slightly abuse the notation and also denote by $v_x \colon \widehat{k(x)}^{\mathrm{h}} \to \Gamma_x \cup \{0\}$ the induced valuation of the henselized completed residue field $\widehat{k(x)}^{\mathrm{h}}$ (see [Sta22, Tag 0ASK] for the fact that henselization does not change the value group).

Our next goal is to study the henselized completed residue field $\widehat{k(x)}^{\mathrm{h}}$ for $x \in |X^c| \smallsetminus |X|$. It turns out that all these affinoid fields are curve-like in the sense of Definition 2.2.5.

Lemma 4.2.5 ([Hub01, Prop. 5.1])**.** *Let X be a separated rigid-analytic C-curve and x a rank-2 point on X^c. Then $\widehat{k(x)}^{\mathrm{h}}$ is a curve-like affinoid field and the secondary residue field $\widehat{k(x)}^{+,\mathrm{h}}/\mathfrak{m}_x\widehat{k(x)}^{+,\mathrm{h}} \simeq k(x)^+/\mathfrak{m}_x$ is isomorphic to k_C.*

Proof. By construction, $\widehat{k(x)}^{\mathrm{h}}$ is henselian. Therefore, it suffices to show that $\widehat{k(x)}^{\mathrm{h}}$ is defectless in every finite extension, $(\Gamma_x)_{<1}$ has a greatest element γ_x, and Γ_x is generated by Γ_C and γ_x.

Now we note that the point x is of Type III in the sense of [Hub01, § 5, p. 184]. Therefore, [Hub01, Lem. 5.3.(i, ii, iii)] implies that $\widehat{k(x)}^{\mathrm{h}}$ is defectless in every finite extension, while [Hub01, Lem. 5.1.(iii)] implies that $(\Gamma_x)_{<1}$ admits a greatest element γ_x and that it is generated by Γ_C and γ_x. Furthermore, *loc. cit.* implies that $\widehat{k(x)}^{+,\mathrm{h}}/\mathfrak{m}_x\widehat{k(x)}^{+,\mathrm{h}} \simeq k(x)^+/\mathfrak{m}_x$ is isomorphic to k_C. □

Lemma 4.2.6. *Let X and $x \in X$ be as in Lemma 4.2.5, and let x_{gen} be the unique rank-1 generalization of x. Then x_{gen} is weakly Shilov in the sense of* [BH22, Def. 2.5].

Proof. Since x_{gen} admits a proper specialization, it is of type II in the sense of [Hub01, § 5, p. 184]. Therefore, [Hub01, Prop. 5.1(ii)] implies that the secondary residue field of x_{gen} has transcendence degree 1 over C. Therefore, [BH22, Prop. 2.9] implies that x_{gen} is weakly Shilov. □

Definition 4.2.7. Let X be a rigid-analytic C-curve and x a rank-2 point on X^c with the corresponding valuation $v_x \colon \widehat{k(x)}^{\mathrm{h}} \to \Gamma_x \cup \{0\}$. We define the *reduction* morphism $\# \colon \Gamma_x \to \mathbf{Z}$ to be the morphism from Definition 2.2.8.

4.3. **Relation to formal models.** The main goal of this subsection is to give a geometric interpretation of the reduction morphism from Definition 4.2.7 in terms of formal models, which follows essentially from the discussion in [Hub01, pp. 199–200]; see also [Kob23] for a discussion in the setting of more general quasi-compact, separated rigid spaces. We expand the argument here for the convenience of the reader. This interpretation will play the key role in showing the compatibility of the analytic and algebraic trace maps (see Theorem 5.4.2).

Before we discuss this interpretation, we need to recall the construction of the specialization morhpism:

Construction 4.3.1 ([Hub96, Prop. 1.9.1])**.** Let $\mathscr{X}$ be an admissible formal $\mathcal{O}_C$-scheme with generic fiber $X = \mathscr{X}_\eta$. Then we recall that X is equipped with a *specialization morphism* $\mathrm{sp}_{\mathscr{X}} \colon (X, \mathcal{O}_X^+) \to (\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ that is universal among such maps from rigid spaces. This construction is affine local on $\mathscr{X}$; when $\mathscr{X} = \mathrm{Spf}(A_0)$, then $\mathscr{X}_\eta = \mathrm{Spa}\left(A_0\left[\frac{1}{\varpi}\right], A_0\left[\frac{1}{\varpi}\right]^\circ\right)$ and $\mathrm{sp}_{\mathscr{X}}$ sends a valuation $v \colon A_0\left[\frac{1}{\varpi}\right] \to \Gamma_v$ to the prime ideal $v^{-1}(\Gamma_{v,<1}) \cap A_0 \subset A_0$, which is open due to continuity of v.

When there is no risk for confusion, we also often write sp instead of $\mathrm{sp}_{\mathscr{X}}$.

We now discuss the behaviour of the specialization map at some specific class of points of X:

Lemma 4.3.2. *Let $\mathscr{X}$ be an admissible formal $\mathcal{O}_C$-scheme with reduced special fiber, and let $\zeta \in |\mathscr{X}_s|$ be a generic point in the special fiber. Then*

(i) $\mathrm{sp}^{-1}(\zeta)$ *consists of a unique point z;*
(ii) *the local ring $\mathcal{O}_{\mathscr{X},\zeta}$ is a rank-1 valuation ring which is ϖ-adically separated and ϖ-adically henselian;*
(iii) *the natural morphism $\mathcal{O}_{\mathscr{X},\zeta} \to k(z)^+$ is an isomorphism;*
(iv) *the ideal $\mathfrak{m}_C\mathcal{O}_{\mathscr{X},\zeta}$ is the maximal ideal of $\mathcal{O}_{\mathscr{X},\zeta}$;*
(v) *the natural morphism of value groups $\Gamma_C \to \Gamma_z$ is an isomorphism.*

Proof. We first show (i). For this, we recall that the underlying topological space of $\mathscr{X}_\eta$ is given by $|\mathscr{X}_\eta| \simeq \lim_{\mathscr{X}'\to\mathscr{X}} |\mathscr{X}'|$, where the limit is taken over all admissible blow-ups $\mathscr{X}' \to \mathscr{X}$ (see [ALY22, § 2.2] or [FK18, Th. II.A.4.7]). Furthermore, the specialization morphism is given simply by the projection $\mathrm{sp}\colon |\mathscr{X}_\eta| \simeq \lim_{\mathscr{X}'\to\mathscr{X}} |\mathscr{X}'| \to |\mathscr{X}|$. Therefore, it suffices to show that, for any admissible blow-up $f\colon \mathscr{X}' \to \mathscr{X}$, there is a dense open subset $\mathscr{U} \subset \mathscr{X}$ such that f is an isomorphism over $\mathscr{U}$. This follows directly from [Zav21b, Cor. B.14].

Now we note that [ALY22, Lem. A.2.(a)] implies that $\mathscr{X}$ is η-normal in the sense of [ALY22, Def. A.1]. Therefore, (ii)- (iii) follow from the combination of [ALY22, Def. A.11, Lem. A.12, and Prop. A.15]. We note that (iv) follows from the observation that $\mathcal{O}_{\mathscr{X},\zeta}/\mathfrak{m}_C\mathcal{O}_{\mathscr{X},\zeta} \simeq \mathcal{O}_{\mathscr{X}_s,\zeta}$ is a field because ζ is a generic point of $\mathscr{X}_s$.

Finally, we show (v). The question is Zariski-local on $\mathscr{X}$, so we can assume that $\mathscr{X} = \mathrm{Spf}\, A_0$ is affine, smooth, and connected (thus, irreducible). Put $A \coloneqq A_0\left[\frac{1}{\varpi}\right]$, then [Lüt16, Prop. 3.4.1] and the assumption that C is algebraically closed imply that $A^\circ = A_0$ and $A^{\circ\circ} = \mathfrak{m}_C A^\circ = \mathfrak{m}_C A_0$. Therefore, $A^\circ/A^{\circ\circ} = A_0/\mathfrak{m}_C A_0$ is an integral domain, and so [BGR84, Prop. 6.2/5] implies that the supremum semi-norm $|.|_{\mathrm{sup}}\colon A \to \Gamma_C \cup \{0\}$ is a valuation[15] of A. The supremum norm is bounded on A° due to [Bos14, Th. 3.1/17] and is continuous due to [Sem15, L. 9, Cor. 9.3.3(2)], thus it defines a point $z' \in \mathrm{Spa}\,(A, A^\circ) = \mathscr{X}_\eta$. Now [Bos14, Cor. 3.1/18] implies that $|.|_{\mathrm{sup}}^{-1}(\Gamma_{C,<1}) = A^{\circ\circ} = \mathfrak{m}_C A^\circ$. Therefore, we conclude that $\mathrm{sp}(z') = \zeta$. So (i) ensures that $z = z'$. Then $\Gamma_z = \Gamma_{z'} = \Gamma_C$ by the very construction. □

Lemma 4.3.3. *Let $\mathscr{X}$ be a quasi-compact admissible separated formal $\mathcal{O}_C$-scheme, then its rigid generic fiber X is separated and taut over* $\mathrm{Spa}\,(C, \mathcal{O}_C)$. *Let $x \in |X^c| \smallsetminus |X|$ with its unique rank-1 generalization $x_{\mathrm{gen}} \in X$. Then* $\mathrm{sp}(x_{\mathrm{gen}}) \in \mathscr{X}_s$ *is* not *a closed point.*

Proof. By [BL93, Prop. 4.7], we know that X is separated. Since $\mathscr{X}$ is quasi-compact, we conclude that X is quasi-compact. Since it is also separated, [Hub96, Lem. 5.1.3.(ii)] implies that it is taut. All rank-1 points on $|X^c|$ already lie on $|X|$ thanks to Lemma A.0.4.

Let us show the last statement, suppose to the contrary that $\mathrm{sp}(x_{\mathrm{gen}})$ is a closed point. Let $\mathscr{U}$ be an affine open neighborhood of $\mathrm{sp}(x_{\mathrm{gen}})$, let $U = \mathscr{U}_\eta$ be its generic fiber, and let $j^c\colon U^c \to X^c$ be the morphism induced by the natural open immersion $j\colon U \to X$. Then [Hub96, Cor. 1.3.9] implies that $\overline{\{x_{\mathrm{gen}}\}} \subset U^c$, where the closure $\overline{\{x_{\mathrm{gen}}\}}$ is taken inside X^c. This implies that x lies in U^c, so we can replace $\mathscr{X}$ with $\mathscr{U}$ to assume that $\mathscr{X} = \mathrm{Spf}\, A_0$ is affine.

In this situation, we put $A \coloneqq A_0[1/\varpi]$. We see that $X = \mathrm{Spa}\,(A, A^\circ)$ and Lemma A.0.3 implies that $X^c = \mathrm{Spa}\,(A, \mathcal{O}_C[A^{\circ\circ}]^+)$. Then the point x (resp. x_{gen}) defines a continuous morphism $r_x\colon \mathcal{O}_C[A^{\circ\circ}]^+ \to \widehat{k(x)}^+$ (resp. $r_{\mathrm{gen}}\colon A^\circ \to \widehat{k(x_{\mathrm{gen}})}^+$). We note that [Hub96, Th. 1.1.10] implies that $\widehat{k(x_{\mathrm{gen}})}^+$ is a rank-1 valuation ring and that $\mathrm{Frac}\Big(\widehat{k(x_{\mathrm{gen}})}^+\Big) = \mathrm{Frac}\Big(\widehat{k(x)}^+\Big)$. Thus, the specialization relation $x_{\mathrm{gen}} \rightsquigarrow x$ can be realized as the commutative diagram:

$$\begin{tikzcd} \mathcal{O}_C[A^{\circ\circ}]^+ \arrow[r, "r_x"] \arrow[d, hook] & \widehat{k(x)}^+ \arrow[d, hook] \\ A^\circ \arrow[r, "r_{\mathrm{gen}}"] & \widehat{k(x_{\mathrm{gen}})}^+. \end{tikzcd}$$

[15]Here, we implicitly use that C is algebraically closed, [BGR84, Obs. 3.6/10], and [Bos14, Prop. 3.1/16] to ensure that the value group of the supremum semi-norm is equal to Γ_C.

Since r_{gen} is continuous, we conclude that $r_{\mathrm{gen}}(\varpi)$ is a pseudo-uniformizer in $\widehat{k(x_{\mathrm{gen}})}^+$. Since $\widehat{k(x_{\mathrm{gen}})}^+$ is a rank-1 valuation ring, we conclude that $\mathfrak{m}_C\widehat{k(x_{\mathrm{gen}})}^+ = \mathfrak{m}_{\mathrm{gen}}$ is the maximal ideal of $\widehat{k(x_{\mathrm{gen}})}^+$. Therefore, we see that our assumption on $\mathrm{sp}(x)$ implies that the map of k_C-algebras

$$r_{\mathrm{gen}}|_{A_0} \bmod \mathfrak{m}_C A_0 \colon A_0/(\mathfrak{m}_C \cdot A_0) \to \widehat{k(x_{\mathrm{gen}})}^+/\mathfrak{m}_{\mathrm{gen}} = \widehat{k(x_{\mathrm{gen}})}^+/(\mathfrak{m}_C\widehat{k(x_{\mathrm{gen}})}^+)$$

factors through the natural morphism $k_C \to \widehat{k(x_{\mathrm{gen}})}^+/\mathfrak{m}_{\mathrm{gen}}$. Since $\mathfrak{m}_{\mathrm{gen}}$ and $\mathcal{O}_C$ lie in $\widehat{k(x)}^+$ (see [Mat89, Th. 10.1] for the former claim), we conclude that the image of $A_0 \to \widehat{k(x_{\mathrm{gen}})}^+$ lies inside $\widehat{k(x)}^+$. Finally, we use [BGR84, Prop. 6.3.4/1] and the fact that $\widehat{k(x)}^+$ is integrally closed in $\mathrm{Frac}\big(\widehat{k(x)}^+\big) = \mathrm{Frac}\big(\widehat{k(x_{\mathrm{gen}})}^+\big)$ to conclude that the morphism $r_{\mathrm{gen}} \colon A^\circ \to \widehat{k(x_{\mathrm{gen}})}^+$ factors through $\widehat{k(x)}^+$. This contradicts the assumption that $x \in |X^c| \smallsetminus |X|$. $\square$

Finally, we are almost ready to discussed the promised above relation between Definition 4.2.7 and formal models. But before we do this, we need to recall the following two lemmas:

Lemma 4.3.4. *Let k° be a rank-1 valuation ring with fraction field k, and let $\widehat{k}$ be the completion of k (with respect to the valuation topology). Let $\mathcal{V}(k, k^\circ)$ (resp. $\mathcal{V}(\widehat{k}, \widehat{k}^\circ)$) be the set of valuation rings A on k (resp. valuation rings B on $\widehat{k}$) such that $A \subset k^\circ$ (resp. $B \subset \widehat{k}^\circ$). Then the map*

$$\mathcal{V}(k, k^\circ) \to \mathcal{V}(\widehat{k}, \widehat{k}^\circ)$$
$$(A \subset k^\circ) \mapsto (\widehat{A} \subset \widehat{k}^\circ),$$

is a bijection with the inverse given by

$$(B \subset \widehat{k}^\circ) \mapsto (B \cap k \subset k^\circ).$$

Proof. In this proof, we denote by $\mathfrak{m}$ the maximal ideal of k° and choose a pseudo-uniformizer $\pi \in \mathfrak{m}$. Then we note that [BV18, Lem. 1(iii)] implies that $\widehat{k}^\circ = \widehat{k^\circ}$. Since $\widehat{k}^\circ/(\pi) \simeq k^\circ/(\pi)$ (see [Sta22, Tag 05GG]), we conclude that $\widehat{\mathfrak{m}}$ is equal to the maximal ideal of $\widehat{k}^\circ$. Now [Mat89, Th. 10.1] implies that $\mathcal{V}(k, k^\circ)$ and $\mathcal{V}(\widehat{k}, \widehat{k}^\circ)$ are both in bijection with the set of valuation rings $R \subset \kappa \coloneqq k^\circ/\mathfrak{m} \simeq \widehat{k}^\circ/\widehat{\mathfrak{m}}$. Furthermore, both bijections are realized by taking the pre-image of R along the reduction morphism $k^\circ \to \kappa$ or $\widehat{k}^\circ \to \kappa$.

Therefore, we are only left to show that the composite bijection $\mathcal{V}(k, k^\circ) \to \mathcal{V}(\widehat{k}, \widehat{k}^\circ)$ is given by taking the π-adic completion, and the other composite bijection $\mathcal{V}(\widehat{k}, \widehat{k}^\circ) \to \mathcal{V}(k, k^\circ)$ is given by taking the intersection with k. The first claim follows from the fact that, for any $A \in \mathcal{V}(k, k^\circ)$, we have[16] $A/\mathfrak{m} \simeq \widehat{A}/\widehat{\mathfrak{m}}$. The second claim can be easily seen by unravelling the definitions. $\square$

Lemma 4.3.5. [Hub96, Lem. 1.3.6.i)] *Let X be an analytic adic space, $y \in X$ a point, $x \in X$ a generalization of y inducing the morphism $\iota \colon k(y) \to k(x)$ of residue fields. Then there is a unique valuation ring $A_{y\to x} \subset k(x)^+$ such that $\iota^{-1}(A_{y\to x}) = k(y)^+$. Furthermore, the natural morphism*

$$\widehat{k(y)}^+ \to \widehat{A}_{y\to x}$$

is an isomorphism.

Proof. This follows directly from Lemma 4.3.4 and the observation that $\widehat{k(x)} \simeq \widehat{k(y)}$ (see [Hub96, Lem. 1.1.10.iii)]). $\square$

Now we get to the promised geometric interpretation of the reduction morphism from Definition 4.2.7 in terms of formal models.

Lemma 4.3.6 ([Hub01])**.** *Let $\mathscr{X}$ be a quasi-compact admissible formal $\mathcal{O}_C$-scheme such that the special fiber $\mathscr{X}_s$ is a reduced separated scheme of pure dimension 1. Let $\mathscr{X}_s^c$ be a schematically dense compactification of $\mathscr{X}_s$, and let $\nu \colon \mathscr{X}_s^{c,n} \to \mathscr{X}_s^c$ be its normalization. Let $X := \mathscr{X}_\eta$ be the rigid generic fiber of $\mathscr{X}$ with its universal compactification X^c. Then:*

[16]We note that [Mat89, Th. 10.1] implies that $\mathfrak{m} \subset A$.

(i) *For any generic point* $\zeta \in \mathscr{X}$ *with* $z = \mathrm{sp}^{-1}(\zeta)$*, there is a bijection*

$$\mu_\zeta \colon \overline{\{z\}} \xrightarrow{\sim} |Y_\zeta|$$

between points of the closure $\overline{\{z\}}$ *of* z *in* X^c*, and points of the corresponding connected component* $Y_\zeta \subseteq \mathscr{X}_s^{c,n}$*; when* $y \in \overline{\{z\}} \cap X$*, then* $\nu(\mu_\zeta(y)) = \mathrm{sp}(y)$.

(ii) *Let* $y \in \overline{\{z\}}$ *be a specialization of* z*. Then the ring* $A_{y\to z}$ *from Lemma 4.3.5 is the preimage of* $\mathcal{O}_{\mathscr{X}_s^{c,n}, \mu_\zeta(y)}$ *under*

$$k(z)^+ \xleftarrow{\sim} \mathcal{O}_{\mathscr{X},\zeta} \to \mathcal{O}_{\mathscr{X}_s,\zeta} \simeq \mathcal{O}_{\mathscr{X}_s^{c,n},\zeta} \simeq \mathrm{Frac}\big(\mathcal{O}_{\mathscr{X}_s^{c,n},\mu_\zeta(y)}\big)$$

where the first isomorphism comes from Lemma 4.3.2. (iii). In particular, $k(y)^+ = A_{y\to z} \cap k(y)$*;*

(iii) *Under the identification of (ii), the map* $\# \circ v_y$ *is the composition*

$$k(y)^+ \to \mathcal{O}_{\mathscr{X}_s^{c,n},\mu_\zeta(y)} \xrightarrow{\mathrm{ord}_{\mu_\zeta(y)}} \mathbf{Z},$$

where $\mathrm{ord}_{\mu_\zeta(y)}$ *is given by order of vanishing at* $\mu_\zeta(y) \in \mathscr{X}_s^{c,n}$*;*

(iv) *The bijections from (i) induce a bijection*

$$\mu \colon |X^c| \smallsetminus |X| \xrightarrow{\sim} |\mathscr{X}_s^{c,n}| \smallsetminus |\mathscr{X}_s^{n}|.$$

Proof. We note that Lemma 4.3.3 implies that X is separated and taut, so the universal compactification X^c exists due to Theorem A.0.1.

(i). Fix a generic point $\zeta \in \mathscr{X}_s$. Let $Y_\zeta \subseteq \mathscr{X}_s^{c,n}$ be the corresponding connected component of $\mathscr{X}_s^{c,n}$ and let $z := \mathrm{sp}_{\mathscr{X}}^{-1}(\zeta) \in X$ be the corresponding rank-1 point from Lemma 4.3.2 (i) with closure $\overline{\{z\}}$ in X^c. By Lemma 4.3.2 (iii), the natural map $\mathcal{O}_{\mathscr{X},\zeta} \to k(z)^+$ is an isomorphism.

By sending $y \in \overline{\{z\}}$ to $A_{y\to z} \subset k(z)^+$ (see Lemma 4.3.5), the valuative criterion for properness [Hub96, Lem. 1.3.6, Cor. 1.3.9] gives a correspondence between the points of $\overline{\{z\}}$ and valuations rings $V \subseteq k(z)^+$ such that $V \cap C = \mathcal{O}_C$. Now we note that [Mat89, Th. 10.1] and Lemma 4.3.2 (iii), (iv) imply that such valuation rings are in bijection[17] with valuation rings $\widetilde{V}$ on $k(z)^+/\mathfrak{m}_C k(z)^+ \simeq k(Y_\zeta)$ that contain $k_C = \mathcal{O}_C/\mathfrak{m}_C$. Now we apply [Bou98, Ch. VI, § 10.3, Cor. 2 and 3] to $K = k_C$ with the trivial valuation and $K' = k(Y_\zeta)$ to conclude that any such $\widetilde{V}$ is either trivial or a discrete valuation. By the valuative criterion for properness for the smooth proper curve Y_ζ, the natural map $\mathrm{Spec}\, k(Y_\zeta) \to Y_\zeta$ extends uniquely to $j_{\widetilde{V}} \colon \mathrm{Spec}\, \widetilde{V} \to Y_\zeta$; the image of the closed point of $\mathrm{Spec}\, \widetilde{V}$ under $j_{\widetilde{V}}$ is a closed point $u_{\widetilde{V}} \in Y_\zeta(k_C)$. Since the resulting map

$$\{\text{discrete valuation rings } k_C \subset \widetilde{V} \subset k(Y_\zeta)\} \longrightarrow \{u \in Y_\zeta(k_C) \text{ closed}\}, \quad \widetilde{V} \mapsto u_{\widetilde{V}} \tag{4.3.7}$$

is a bijection (the inverse sends a closed point $u \in Y_\zeta(k_C)$ to $\mathcal{O}_{Y_\zeta,u}$), the result follows directly.

(ii) and (iii) follow from chasing the construction in the proof of (i).

(iv). We fix a generic point $\zeta \in \mathscr{X}_s$ and the corresponding rank-1 point $z = \mathrm{sp}^{-1}(\zeta) \in X$. We start the proof by showing the following claim:

Claim 4.3.8. *A point* $y \in \overline{\{z\}} \subset X^c$ *lies in* X *if and only if* $\mu_\zeta(y) \in \mathscr{X}_s^n \subseteq \mathscr{X}_s^{c,n}$.

Proof. If $y \in \overline{\{z\}} \cap X$, then (i) implies that $\nu\big(\mu_\zeta(y)\big) = \mathrm{sp}(y) \in \mathscr{X}_s$. Therefore, $\mu_\zeta(y) \in \mathscr{X}_s^n$.

Now we pick a point $y \in \overline{\{z\}} \subset X^c$ such that $\mu_\zeta(y) \in \mathscr{X}_s$. We wish to show that y lies in X.

We first treat the case when $\mathscr{X} = \mathrm{Spf}\, A$ is an affine admissible formal $\mathcal{O}_C$-scheme. In this situation, $X = \mathrm{Spa}\,(A[\frac{1}{\varpi}], A^+)$, where A^+ is the integral closure of A in $A[\frac{1}{\varpi}]$. Then $y \in X$ if and only if $v_y(A^+) \le 1$, or equivalently $v_y(A) \le 1$. Now Lemma 4.2.5 and Lemma 2.2.6 imply that the valuation $v_y \colon A[\frac{1}{\varpi}] \to \Gamma_y$ has value group $\Gamma_y \simeq \Gamma_C \times \mathbf{Z}$. Now Lemma 4.3.2 (v) (applied to z) and the fact that $z \in X$ imply that, for every $a \in A$, the first coordinate $v_y(a)$ is less or equal to 1. Thus, we only need to show that $\# \circ v_y(a) \le 0$ for every $a \in A$. By (iii), this is equivalent to showing that

$$\mathrm{ord}_{\mu_\zeta(y)}(A/\mathfrak{m}_C A) \ge 0. \tag{4.3.9}$$

[17] Explicitly, this bijection sends a valuation ring $V \subset k(z)^+$ to $\widetilde{V} := V/\mathfrak{m}_C k(z)^+ \subset k(z)^+/\mathfrak{m}_C k(z)^+ \simeq k(Y_\eta)$. Its inverse is given by the map sending $\widetilde{V} \subset k(z)^+/\mathfrak{m}_C k(z)^+$ to $\pi^{-1}(\widetilde{V}) \subset k(z)^+$, where $\pi \colon k(z)^+ \to k(z)^+/\mathfrak{m}_C k(z)^+$ is the natural projection.

Under the bijection (4.3.7), (4.3.9) is equivalent to the condition that the image of $A/\mathfrak{m}_C A$ in $k_C(Y_\zeta)$ is contained in the valuation ring $\mathcal{O}_{Y_\zeta,\mu_\zeta(y)}$, i.e., $\mu_\zeta(y) \in |\mathscr{X}_s^n|$.

Now we explain how to reduce the case of a general $\mathscr{X}$ to the case of an affine $\mathscr{X}$. For this, we choose some open affine formal subscheme $\mathscr{U} := \operatorname{Spf} A \subset \mathscr{X}$ that contains the point $\nu\big(\mu_\zeta(y)\big)$. By construction, $\mathscr{U}$ also contains the point $\zeta \in |\mathscr{X}_s| = |\mathscr{X}|$, so the generic fiber $U := \mathscr{U}_\eta = \operatorname{sp}^{-1}(\mathscr{U}_s)$ contains the point $z \in X$. To clarify the notation later on, we denote the point z considered as a point of U by z_U. Arguing as in the proof of Lemma 4.3.3, we conclude that $y \in |U^c| \smallsetminus |U|$. The construction of μ_ζ in (i) is compatible with the open immersion $\mathscr{U} \to \mathscr{X}$, so it suffices to prove the claim for $\mathscr{U}$ that was treated above. □

As a consequence, μ_ζ restricts to a natural bijection

$$\mu_\zeta\colon \overline{\{z\}} \cap (|X^c| \smallsetminus |X|) \xrightarrow{\sim} |Y_\zeta| \cap (|\mathscr{X}_s^{c,n}| \smallsetminus |\mathscr{X}_s^n|).$$

By the last sentence of Lemma 4.3.3, we see that the disjoint union of the left hand side (where z runs through all preimages of generic points of $\mathscr{X}_s$ under specialization map) is exactly $|X^c| \smallsetminus |X|$. Since $|X^c| \smallsetminus |X|$ is finite and discrete, we can combine the various μ_ζ for all generic points of $\mathscr{X}_s$ to a bijection $\mu\colon |X^c| \smallsetminus |X| \xrightarrow{\sim} |\mathscr{X}_s^{c,n}| \smallsetminus |\mathscr{X}_s^n|$. □

Remark 4.3.10. We note that Lemma 4.3.6 (iv) implies that a smooth point $x \in \mathscr{X}_s^c \smallsetminus \mathscr{X}_s$ defines a unique rank-2 point $u_x \in X^c \smallsetminus X$ such that $\operatorname{sp}(u_x) = x$. Likewise, a nodal point $x \in \mathscr{X}_s^c \smallsetminus \mathscr{X}_s$ defines two rank-2 points $v_x, w_x \in |X^c| \smallsetminus |X|$ such that $\operatorname{sp}(v_x) = \operatorname{sp}(w_x) = x$.

4.4. **Relation to formal models: nodes.** Given a quasi-compact rigid curve X with a (quasi-compact) admissible formal model $\mathscr{X}$, Lemma 4.3.6 describes additional rank-2 points in $|X^c| \smallsetminus |X|$ in terms of "points at infinity" of the normalized special fiber $\mathscr{X}_s^n$. In this subsection, we explain the role played by the normalization, at least when $\mathscr{X} = \operatorname{Spf} \mathcal{O}_C\langle S, T\rangle/(ST-\pi)$ is a model rig-smooth semistable curve in the sense of Definition 4.1.3.

We recall that $\varpi \in \mathcal{O}_C$ is a fixed pseudo-uniformizer in $\mathcal{O}_C$. For the rest of this subsection, we choose another pseudo-uniformizer $\pi \in \mathfrak{m}_C \smallsetminus \{0\}$.

Notation 4.4.1. We set $R := \Big(\big(\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}\big)^{\mathrm{h}}_{(S,T)}\Big)^\wedge_\varpi$ to be the ϖ-completion of the (S,T)-adic henselization of the standard rig-smooth semi-stable nodal curve. In particular $(S, T, \mathfrak{m}_C \cdot R)$ is a maximal ideal in R. We also set $\widetilde{R} := R^\wedge_{(S,T,\varpi)}$ to be the (S,T,ϖ)-adic completion of R.

Lemma 4.4.2. *The natural map* $\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)} \to R$ *induces an isomorphism* $\frac{\mathcal{O}_C[\![S,T]\!]}{(ST-\pi)} \xrightarrow{\sim} \widetilde{R}$.

Proof. We put $\widetilde{R}_n := \frac{\left(\frac{\mathcal{O}_C}{(\varpi^n)}\right)[S,T]}{(ST-\pi,S^n,T^n)} \simeq \frac{\mathcal{O}_C[S,T]}{(ST-\pi,S^n,T^n,\varpi^n)}$. Then $\frac{R}{(S^n,T^n,\varpi^n)} \simeq \big(\widetilde{R}_n\big)^{\mathrm{h}}_{(S,T)}$ due to [Sta22, Tag 0DYE]. The ring $\widetilde{R}_n$ is already (S,T)-adically henselian due to [Sta22, Tag 0F0L], so we conclude that $\widetilde{R}_n \simeq \frac{R}{(S^n,T^n,\varpi^n)}$. Therefore, we conclude that

$$\frac{\mathcal{O}_C[\![S,T]\!]}{(ST-\pi)} \simeq \Big(\frac{\mathcal{O}_C[S,T]}{(ST-\pi)}\Big)^\wedge_{(S,T,\varpi)} \simeq \lim_n \widetilde{R}_n \simeq \lim_n \frac{R}{(S^n,T^n,\varpi^n)} \simeq \widetilde{R}. \qquad \square$$

Lemma 4.4.2 implies that any element $f \in \widetilde{R}[\frac{1}{\varpi}]$ can be written uniquely as

$$f = a_0 + \sum_{i\geq 1} b_i S^i + \sum_{j\geq 1} c_j T^j$$

where a_0, b_i, c_j are elements in C such that $\{a_0, b_i, c_i;\ i \in \mathbf{Z}_{\geq 1}\} \subset C$ is a bounded subset. The two Gauss norms on the annulus $\big(\operatorname{Spf}(\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)})\big)_\eta$ extend to norms on $\widetilde{R}[\frac{1}{\varpi}]$ in the following fashion:

Definition 4.4.3. Let $f \in \widetilde{R}[\frac{1}{\varpi}]$.

(i) The *S-Gauss norm* and *T-Gauss norm of f are given by the following valuations:*

$$|f|_S := \sup\{|a_0|, |b_i|, |c_j \cdot \pi^j| \mid i \geq 1, j \geq 1\} \quad \textit{and} \quad |f|_T := \sup\{|a_0|, |b_i \cdot \pi^i|, |c_j| \mid i \geq 1, j \geq 1\}$$

(ii) *We say that f is S-regular (resp. T-regular) if the supremum $|f|_S$ (resp. $|f|_T$) is attained by an element of the set. We say f is* regular *if it is both S-regular and T-regular.*

(iii) *We denote the sets of S-regular (resp. T-regular, resp. regular) elements of $\widetilde{R}[\frac{1}{\varpi}]$ by $\widetilde{R}[\frac{1}{\varpi}]_{S\text{-reg}}$ (resp. $\widetilde{R}[\frac{1}{\varpi}]_{T\text{-reg}}$, resp. $\widetilde{R}[\frac{1}{\varpi}]_{\text{reg}}$).*

To relate these norms to the classical Gauss norm, we need to introduce some further notation:

Notation 4.4.4. We set $\widetilde{R}_S \coloneqq (\mathcal{O}_C[\![S]\!][\frac{1}{S}])^\wedge_\varpi$ and $\widetilde{R}_T \coloneqq (\mathcal{O}_C[\![T]\!][\frac{1}{T}])^\wedge_\varpi$. Then we see that $\widetilde{R}$ admits two ring-homomorphisms

$$\beta_S\colon \widetilde{R} \to \widetilde{R}_S \quad \text{and} \quad \beta_T\colon \widetilde{R} \to \widetilde{R}_T$$

defined by the rule

$$\beta_S\Big(a_0 + \sum_{i\geq 1} b_i S^i + \sum_{j\geq 1} c_j T^j\Big) = a_0 + \sum_{i=1}^{\infty} b_i S^i + \sum_{i=1}^{\infty} c_i \cdot \pi^i S^{-i},$$

$$\beta_T\Big(a_0 + \sum_{i\geq 1} b_i S^i + \sum_{j\geq 1} c_j T^j\Big) = a_0 + \sum_{i=1}^{\infty} b_i \cdot \pi^i T^{-i} + \sum_{i=1}^{\infty} c_i T^i.$$

By abuse of notation, we denote by $\beta_S\colon \widetilde{R}[\frac{1}{\varpi}] \to \widetilde{R}_S[\frac{1}{\varpi}]$ and by $\beta_T\colon \widetilde{R}[\frac{1}{\varpi}] \to \widetilde{R}_T[\frac{1}{\varpi}]$ the natural morphisms induced by β_S and β_T from above.

Remark 4.4.5. Let $|-|\colon \widetilde{R}_S \to \Gamma_C \cup \{0\}$ be the classical Gauss norm $|\sum_{i\in\mathbf{Z}} a_i S^i| = \sup(|a_i|)$. Then we have the following equality

$$|-|_S = |-| \circ \beta_S\colon \widetilde{R} \to \Gamma_C \cup \{0\}.$$

We say that an element $f = \sum_{i\in\mathbf{Z}} a_i S^i \in \widetilde{R}_S[\frac{1}{\varpi}]$ is *regular* if $|f| = |a_i|$ for some $i \in \mathbf{Z}$.

To justify the name of the S- and T-Gauss norms, we make the following observations. First, both $|-|_S$ and $|-|_T$ are injective, submultiplicative, and satisfy the nonarchimedean triangle inequality. Therefore, they do define norms on $\widetilde{R}$ (in the sense of [BGR84, Def. 1.2.1/1]). One can also check that S- and T-Gauss norms are multiplicative by reducing the question to the Gauss norm on $\widetilde{R}_S$ and then approximating this norm with the Gauss norms on the closed disks of radius $r < 1$. In this paper, we never use this multiplicativity, so we leave the details to the interested reader.

Now we are ready to formulate one of the key results of this subsection:

Lemma 4.4.6. *The image of the natural map $R[\frac{1}{\varpi}] \to \widetilde{R}[\frac{1}{\varpi}]$ is contained in $\widetilde{R}[\frac{1}{\varpi}]_{\text{reg}}$.*

In order to present the proof, we will first need to study the map $R \to \widetilde{R}$ in more detail.

Lemma 4.4.7. *Let B be a ring. Then the natural maps $\big(B[S]\big)^{\text{h}}_S \to B[\![S]\!]$ and $\big(\frac{B[S,T]}{(ST)}\big)^h_{(S,T)} \to \frac{B[\![S,T]\!]}{(ST)}$ are injective.*

Proof. We show it for the map $\big(\frac{B[S,T]}{(ST)}\big)^{\text{h}}_{(S,T)} \to \frac{B[\![S,T]\!]}{(ST)}$, a similar proof applies to $\big(B[S]\big)^{\text{h}}_S \to B[\![S]\!]$. We start by writing $B \simeq \text{colim}_{i\in I} B_i$ as a filtered colimit of its finitely generated $\mathbf{Z}$-subalgebras. Then the natural morphism $\text{colim}_I\big(\frac{B_i[\![S,T]\!]}{(ST)}\big) \to \frac{B[\![S,T]\!]}{(ST)}$ is injective and the natural morphism $\text{colim}_I\big(\frac{B_i[S,T]}{(ST)}\big)^{\text{h}}_{(S,T)} \to \big(\frac{B[S,T]}{(ST)}\big)^{\text{h}}_{(S,T)}$ is an isomorphism (see [Sta22, Tag 0A04]). Therefore, it suffices to show the lemma under the additional assumption that B is a finitely generated $\mathbf{Z}$-algebra. In this case, the result follows directly from [Sta22, Tag 0AGV]. ☐

Already we are getting some interesting statements concerning the map $R \to \widetilde{R}$.

Lemma 4.4.8. *The maps $R/\pi \to \widetilde{R}/\pi$, $R \to \widetilde{R}$, and $R[\frac{1}{\varpi}] \to \widetilde{R}[\frac{1}{\varpi}]$ are injective. Moreover, we have a pullback diagram of rings*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \widetilde{R} \arrow[d] \\ R[\frac{1}{\varpi}] \arrow[r] & \widetilde{R}[\frac{1}{\varpi}]. \end{tikzcd}$$

Proof. By [Sta22, Tag 0DYE] and Lemma 4.4.2, the map $R/\pi \to \widetilde{R}/\pi$ is identified with the map

$$\Big(\frac{\frac{\mathcal{O}_C}{\pi}[S,T]}{(S,T)}\Big)^{\mathrm{h}}_{(S,T)} \to \Big(\frac{\frac{\mathcal{O}_C}{\pi}[S,T]}{(S,T)}\Big)^{\wedge}_{(S,T)}.$$

Thus, Lemma 4.4.7 implies that $R/\pi \to \widetilde{R}/\pi$ is injective. Since R is π-adically separated (it is even π-adically complete) and both R and $\widetilde{R}$ are π-torsionfree, we conclude that $R \to \widetilde{R}$ is injective as well. This formally implies that $R[\frac{1}{\varpi}] \to \widetilde{R}[\frac{1}{\varpi}]$ is injective. For the last statement, the vertical and horizontal maps are injective, so an easy diagram chase implies that it suffices to show that the natural map

$$\frac{R[\frac{1}{\varpi}]}{R} \simeq \frac{R[\frac{1}{\pi}]}{R} \to \frac{\widetilde{R}[\frac{1}{\pi}]}{\widetilde{R}} \simeq \frac{\widetilde{R}[\frac{1}{\varpi}]}{\widetilde{R}}$$

is injective. Since both R and $\widetilde{R}$ are π-torsionfree, we can identify this map with the map $\mathrm{colim}_n R/\pi^n \to \mathrm{colim}_n \widetilde{R}/\pi^n$. This follows formally from injectivity of $R/\pi \to \widetilde{R}/\pi$, injectivity of $R \to \widetilde{R}$, and the observations that R is π-adically separated and both R and $\widetilde{R}$ are π-torsionfree. □

With these technical preliminaries out of the way, we can prove Lemma 4.4.6.

Proof of Lemma 4.4.6. We fix an element $f \in R[\frac{1}{\varpi}]$ and show that its image $\widetilde{f} \in \widetilde{R}[\frac{1}{\varpi}]$ is S-regular (T-regularity follows as the role of S and T is symmetric). By inspection, we see that scaling by $C^\times$ or powers of S does not change S-regularity. So we may and do assume that $f \in R$. Then we choose a non-negative integer n such that

$$|\pi^{n+1}| < |\widetilde{f}|_S \le |\pi^n|.$$

The definition of the "S-Gauss norm" and the assumption that $f \in R$ imply that the image $\frac{S^n \cdot \widetilde{f}}{\pi^n} \in \widetilde{R}$. Therefore, the last statement in Lemma 4.4.8 ensures that $\frac{S^n \cdot f}{\pi^n} \in R$. In particular, we can further replace f with $\frac{S^n \cdot f}{\pi^n}$ to assume that $f \in R$ and $|\pi| < |\widetilde{f}|_S \le |1|$.

We put $\mathfrak{n} := \{c \in \mathcal{O}_C \mid |c| < |\widetilde{f}|_S\}$. Our assumptions on f imply that $(\pi) \subset \mathfrak{n}$. Suppose that $\widetilde{f}$ is not S-regular, then f lies in the kernel of the natural map $R \to \frac{R}{(\mathfrak{n},T)} \simeq \big(\frac{\mathcal{O}_C}{\mathfrak{n}}[S]\big)^{\mathrm{h}}_{(S)} \hookrightarrow \frac{\mathcal{O}_C}{\mathfrak{n}}[\![S]\!]$. The last map is injective due to Lemma 4.4.7, so f lies in the kernel of the natural morphism $R \to \frac{R}{(\mathfrak{n},T)}$. This means that $f = \pi' \cdot f_1 + T \cdot f_2$ for some $\pi' \in \mathfrak{n}$ and $f_1, f_2 \in R$. This leads to the contradiction $|\widetilde{f}|_S \le \max\{|\pi'|, |\pi|\} < |\widetilde{f}|_S$ and finishes the proof. □

Definition 4.4.9. (i) (Secondary Gauss valuation) Let $f \in (\widetilde{R}_S[\frac{1}{\varpi}])_{\mathrm{reg}} \subset (\widetilde{R}_S[\frac{1}{\varpi}]) = \big(\mathcal{O}_C[\![S]\!][S^{-1}]\big)^{\wedge}_{\varpi}[\frac{1}{\varpi}]$. We define the map $v\colon (\widetilde{R}_S[\frac{1}{\varpi}])_{\mathrm{reg}} \to \mathbf{Z}$ by the following rule

$$v\Big(\sum_{i\in\mathbf{Z}} a_i S^i\Big) := \min\{r \in \mathbf{Z} \mid |a_r| = |f|\};$$

(ii) (Secondary S-Gauss valuation) Let $f \in \widetilde{R}[\frac{1}{\varpi}]_{S\text{-reg}}$. We define the map $v_S\colon \widetilde{R}[\frac{1}{\varpi}]_{S\text{-reg}} \to \mathbf{Z}$ by the following rule

$$v_S(f) = v\big(\beta_S(f)\big),$$

where $\beta_S\colon \widetilde{R}[\frac{1}{\varpi}] \to \widetilde{R}_S[\frac{1}{\varpi}]$ is the morphism from Notation 4.4.4. We define the map $v_T\colon \widetilde{R}[\frac{1}{\varpi}]_{T\text{-reg}} \to \mathbf{Z}$ in a similar way.

Lemma 4.4.10. *The set $\widetilde{R}[\frac{1}{\varpi}]_{S\text{-reg}}$ is stable under multiplication. Furthermore, we have $|f \cdot g|_S = |f|_S \cdot |g|_S$ and $v_S(f \cdot g) = v_S(f) + v_S(g)$ for any $f, g \in \widetilde{R}[\frac{1}{\varpi}]_{S\text{-reg}}$. The same results hold for $\widetilde{R}[\frac{1}{\varpi}]_{T\text{-reg}}$ and v_T.*

Proof. By construction, it suffices to show that $(\widetilde{R}_S[\frac{1}{\varpi}])_{\mathrm{reg}} \subset (\widetilde{R}_S[\frac{1}{\varpi}])$ is closed under multiplication, and we have $|f \cdot g| = |f| \cdot |g|$ and $v(f \cdot g) = v(f) + v(g)$ for any $f, g \in (\widetilde{R}_S[\frac{1}{\varpi}])_{\mathrm{reg}}$. This is a standard exercise on the usual Gauss norms; we leave details to the interested reader. □

We come to the main statement of this subsection, which provides a concrete description of the valuations which correspond to nodes in the special fiber under the bijection μ_ζ from Lemma 4.3.6. (i). For the rest of the subsection, we fix an affine admissible formal $\mathcal{O}_C$-scheme $\mathscr{X} = \operatorname{Spf} A$ with a nodal point $q \in \mathscr{X}_s$ and a ϖ-completely étale morphism $g\colon \big(\operatorname{Spf} A, q\big) \to \left(\operatorname{Spf}\left(\frac{\mathcal{O}_C\langle S,R\rangle}{(ST-\pi)}\right), \{0,0\}\right)$ for $\pi \in \mathfrak{m}_C \smallsetminus \{0\}$.

Then the natural morphism

$$R = \Big(\big(\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}\big)^{\mathrm{h}}_{(S,T)}\Big)^{\wedge}_{\varpi} \xrightarrow{g^*} (\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q})^{\wedge}_{(\varpi)}$$

is an isomorphism, where $\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q}$ denotes the localization of the local ring with respect to the maximal ideal. By slight abuse of notation, we define the map $|-|_1\colon A\big[\frac{1}{\varpi}\big] \to \Gamma_C \cup \{0\}$ as the composition

$$A\big[\frac{1}{\varpi}\big] \to (\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q})^{\wedge}_{(\varpi)}\big[\frac{1}{\varpi}\big] \xrightarrow[\sim]{(g^*)^{-1}} R\big[\frac{1}{\varpi}\big] \to \widetilde{R}\big[\frac{1}{\varpi}\big] \xrightarrow{|-|_S} \Gamma_C \cup \{0\},$$

similarly we define $|-|_2\colon A \to \Gamma_C \cup \{0\}$ using $|-|_T$ in place of $|-|_S$. Likewise, we define the map $v_1\colon A\big[\frac{1}{\varpi}\big] \to \mathbf{Z}$ as the composition

$$A\big[\frac{1}{\varpi}\big] \to (\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q})^{\wedge}_{(\varpi)}\big[\frac{1}{\varpi}\big] \xrightarrow[\sim]{(g^*)^{-1}} R\big[\frac{1}{\varpi}\big] \to \widetilde{R}\big[\frac{1}{\varpi}\big] \xrightarrow{v_S} \mathbf{Z},$$

and similarly we define $v_2\colon A\big[\frac{1}{\varpi}\big] \to \mathbf{Z}$ using v_T in place of v_S.

Proposition 4.4.11. *Let $g\colon \big(\mathscr{X}, q\big) = \big(\operatorname{Spf} A, q\big) \to \left(\operatorname{Spf}\left(\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}\right), \{0,0\}\right)$ be an étale map as above. Let ζ_1 and ζ_2 be the two generic points of $\mathscr{X}_s$ corresponding to the two irreducible components containing q, whose images are the open prime ideals $(\mathfrak{m}_C, T)$ and $(\mathfrak{m}_C, S)$, respectively. For $i = 1, 2$, let $z_i = \operatorname{sp}^{-1}_{\mathscr{X}}(\zeta_i) \in \mathscr{X}_\eta$ (see Lemma 4.3.2. (i)), let $q_i \in \mathscr{X}^{\mathrm{n}}_s$ be the points of the normalization lying over the node q that are on the component corresponding to ζ_i, and let $y_i \in \overline{\{z_i\}} \subset \mathscr{X}_\eta$ be the points corresponding to q_i under the bijection from Lemma 4.3.6. (i). Then the rank-1 points z_1 and z_2 correspond to the valuations $|-|_1, |-|_2\colon A\big[\frac{1}{\varpi}\big] \to \Gamma_C \cup \{0\}$ and the points y_1 and y_2 correspond to the valuations $\big(|-|_1, -v_1(-)\big)$, $\big(|-|_2, -v_2(-)\big)\colon A\big[\frac{1}{\varpi}\big] \to \Gamma_C \times \mathbf{Z} \cup \{0\}$, respectively.*

Proof. First, we note that $|-|_i$ and $\big(|-|_i, -v_i(-)\big)$ are multiplicative thanks to Lemma 4.4.6 and Lemma 4.4.10. Thus, one easily deduces that they define valuations on A. Furthermore, it is evident from the definition that they are continuous and bounded above by 1 on $A = \big(A\big[\frac{1}{\varpi}\big]\big)^{\circ}$. Therefore, all these valuations define elements of $\mathscr{X}_\eta = \operatorname{Spa}\big(A\big[\frac{1}{\varpi}\big], A\big)$.

By the étaleness of g, we may replace $\mathscr{X}$ by an open neighborhood of q to ensure that ζ_1 is the only preimage of the open prime ideal $(\mathfrak{m}_C, T) \subset \frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}$, and that q is the only preimage of the open maximal ideal $(\mathfrak{m}_C, S, T) \subset \frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}$. Then Lemma 4.3.2. (i) and Construction 4.3.1 imply that, in order to show that the point z_1 corresponds to $|-|_1$, it suffices to show that

$$\mathfrak{p}_1 \coloneqq |-|_1^{-1}(\Gamma_{C,<1}) \cap A = (\mathfrak{m}_C, T) \cdot A.$$

Since g is étale and $g^{-1}(g(\zeta_1)) = \{\zeta_1\}$, it suffices to show $g^{\#,-1}(\mathfrak{p}_1) = (\mathfrak{m}_C, T) \subset \frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}$, which can be seen by direct inspection.

Now we show that the point y_1 corresponds to the valuation $\big(|-|_1, -v_1(-)\big)$. The previous step directly implies that $\big(|-|_1, -v_1(-)\big)$ lies in $\overline{\{z_1\}}$. Therefore, Lemma 4.3.6. (i) and Construction 4.3.1 imply that it suffices to show that

$$\mathfrak{p}_1' \coloneqq \Big(\big(|-|_1, -v_1(-)\big)^{-1}\big((\Gamma_C \times \mathbf{Z})_{<1}\big)\Big) \cap A = (\mathfrak{m}_C, S, T) \cdot A.$$

The same argument as above reduces the question to the case $A = \frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}$, where it can be seen directly from the definition. $\square$

4.5. **Line bundles on semi-stable formal curves.** In this subsection, we collect some results about line bundles on rig-smooth semi-stable formal $\mathcal{O}_C$-curves. Some of these results might be well-known to the experts. However, since these results play an important role in Section 5 and do not seem to appear in the literature, we decide to provide full proofs. We recall that $\varpi \in \mathcal{O}_C$ is a fixed pseudo-uniformizer.

We start with the case of more general smooth formal $\mathcal{O}_C$-schemes. In this case, we show that any line bundle on generic fiber can be extended to a line bundle integrally.

Lemma 4.5.1. *Let $\mathscr{X}$ be a smooth formal $\mathcal{O}_C$-scheme (resp. smooth $\mathcal{O}_C$-scheme), and let $\mathcal{F}$ be an adically quasi-coherent (resp. quasi-coherent) π-torsionfree $\mathcal{O}_{\mathscr{X}}$-module of finite type. Then $\mathcal{F}$ is a perfect complex.*

Proof. The claim is local on $\mathscr{X}$. So we can assume that $\mathscr{X} = \operatorname{Spf} R$ (resp. $\mathscr{X} = \operatorname{Spec} R$) and $\mathcal{F} = M^\Delta$ (resp. $\mathcal{F} = \widetilde{M}$) for some finitely generated ϖ-torsionfree R-module M. We wish to show that M is a perfect R-complex. We prove this by verifying all conditions of [Sta22, Tag 068X], with $A \to B$ in *loc. cit.* being $\mathcal{O}_C \to R$. Conditions (1) and (2) follows from the assumption on R, and condition (4) follows from the fact that M is ϖ-torsionfree and thus $\mathcal{O}_C$-flat (see [Sta22, Tag 0539]). In order to check condition (3), we first note that R is coherent, see [BL93, Prop. 1.3] (resp. [FK18, Cor. 0.9.2.8]). Hence, it suffices to show M is finitely presented over R. Since M is finitely generated over R and flat over $\mathcal{O}_C$, the result follows from [Bos14, Th. 7.3/4] (resp. [Sta22, Tag 053E]). □

Lemma 4.5.2. *Let $\mathscr{X}$ be a quasi-compact smooth formal $\mathcal{O}_C$-scheme (resp. a quasi-compact smooth $\mathcal{O}_C$-scheme). Then the natural map $\operatorname{Pic}(\mathscr{X}) \to \operatorname{Pic}(\mathscr{X}_\eta)$ is surjective.*

Proof. Let $\mathcal{L}$ be a line bundle on $\mathscr{X}_\eta$. Pick any adically quasi-coherent (resp. quasi-coherent) $\mathcal{O}_{\mathscr{X}}$-module of finite type $\mathcal{L}_0$ such that $\mathcal{L}_{0,\eta} \simeq \mathcal{L}$ (such $\mathcal{L}_0$ exists by virtue of [FK18, Cor. II.5.3.3] and its algebraic counterpart). Then we replace $\mathcal{L}_0$ with its ϖ-free torsionfree quotient $\mathcal{L}_0/\mathcal{L}_0[\varpi^\infty]$ to assume that $\mathcal{L}_0$ is π-torsionfree adically quasi-coherent (resp. quasi-coherent) $\mathcal{O}_{\mathscr{X}}$-module of finite type. Now Lemma 4.5.1 ensures that $\mathcal{L}_0$ is a perfect complex on $\mathscr{X}$. We use [KM76, Th. 2] to construct $\det(\mathcal{L}_0)$ which is the desired line bundle over $\mathscr{X}$ whose generic fibre is $\det(\mathcal{L}) \simeq \mathcal{L}$ as the determinant construction commutes with base change. □

Now we discuss line bundles on certain rig-smooth semi-stable formal curves over $\mathcal{O}_C$. The exact analogue of Lemma 4.5.2 is probably false in this case. Instead, we prove a weaker substitute showing that we can always trivialize a line bundle $\mathcal{L} \in \operatorname{Pic}(\mathscr{X}_\eta)$ étale localy on $\mathscr{X}$; this result is good enough for all applications in this paper. We start with the following basic lemma:

Lemma 4.5.3. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be an étale morphism of admissible formal $\mathcal{O}_C$-schemes, let $f_\eta\colon X \to Y$ be its generic fiber, and let $y \in Y(C)$ be a classical point. Then, for any two points $x_1, x_2 \in f_\eta^{-1}(y) \subset X(C)$, we have $\operatorname{sp}_{\mathscr{X}}(x_1) \neq \operatorname{sp}_{\mathscr{X}}(x_2)$.*

Proof. The point $y \in Y(C)$ uniquely extends to a morphism $i_y\colon \operatorname{Spf} \mathcal{O}_C \to \mathscr{Y}$. Since the formation of the specialization map is functorial with respect to morphisms of formal models, we can freely replace $\mathscr{Y}$ with $\operatorname{Spec} \mathcal{O}_C$ and $\mathscr{X}$ with $\mathscr{X} \times_{\mathscr{Y}} \operatorname{Spf} \mathcal{O}_C$. In this case, $\mathscr{X}$ is of the form $\mathscr{X} = \bigsqcup_{i \in I} \mathscr{Y}$, so the claim becomes trivial. □

Lemma 4.5.4. *Let $(\mathscr{X}, q)$ be a pointed rig-smooth semi-stable admissible formal $\mathcal{O}_C$-scheme of pure relative dimension 1, and let L be a line bundle on $X \coloneqq \mathscr{X}_\eta$. Then there is an étale morphism $f\colon (\mathscr{U}, u) \to (\mathscr{X}, q)$ of pointed admissible formal $\mathcal{O}_C$-schemes such that $f_\eta^* L \simeq \mathcal{O}_{\mathscr{U}_\eta}$.*

Proof. If $q \in \mathscr{X}$ is a smooth point, the result immediately follows from Lemma 4.5.2 (in fact, one can choose f to be an open immersion). Therefore, we can assume that $q \in \mathscr{X}$ is a nodal point. In this case, there is a pointed étale morphism $(\mathscr{U}, u) \to (\mathscr{X}, q)$ together with a pointed étale morphism

$$g\colon (\mathscr{U}, u) \to (\mathscr{Y}_\pi, 0) \coloneqq \Big(\operatorname{Spf} \frac{\mathcal{O}_C\langle S, T\rangle}{(ST - \pi)}, (0,0)\Big)$$

for some pseudo-uniformizer[18] $\pi \in \mathfrak{m}_C \smallsetminus \{0\}$. Therefore, we can replace $(\mathscr{X}, q)$ with $(\mathscr{U}, u)$ to assume that there is an étale morphism $g\colon (\mathscr{X}, q) \to (\mathscr{Y}_\pi, 0)$. Also, we can shrink $\mathscr{X}$ around q to assume that $g^{-1}(\{0\}) = \{q\}$.

Furthermore, we can assume that $\mathscr{X} = \mathrm{Spf}\, A_0$ is affine and connected, so $X = \mathrm{Spa}\,(A, A^\circ)$ with $A = A_0[\frac{1}{\varpi}]$. Our assumptions on $\mathscr{X}$ imply that X is a smooth affinoid rigid-analytic curve. Furthermore, [Zav21b, Lem. B.12] ensures that X is connected. In particular, $A = \mathcal{O}_X(X)$ is a Dedekind domain (see [FvdP04, Th. 3.6.3]). Thus, there is a Cartier divisor D on X such that $L = \mathcal{O}_X(D)$.

We put $Y_\pi \coloneqq \mathscr{Y}_{\pi,\eta}$. Then we consider the Cartier divisor $D' \coloneqq f_\eta(D)$ on Y_π and the associated line bundle $L' \coloneqq \mathcal{O}_{Y_\pi}(D')$. Then [vdP80, Th. 2.1] or [FvdP04, Th. 2.2.9.(3)] imply that $\mathrm{Pic}(Y_\pi) = 0$. So $L' \simeq \mathcal{O}_{Y_\pi}$ and there is a meromorphic function $h \in \mathrm{Frac}\big(\mathcal{O}_{Y_\pi}(Y_\pi)\big)$ such that $D' = \mathrm{V}_{Y_\pi}(h)$. Therefore, we conclude that

$$L \simeq L \otimes f^*(L')^\vee \simeq \mathcal{O}_X(D - f^{-1}(D')).$$

Now Lemma 4.5.3 and our assumption that $\{q\} = g^{-1}(\{0\})$ imply that $D - f^{-1}(D') = \sum_{x \in X(C)} a_x[x]$ such that this sum is finite and $\mathrm{sp}_{\mathscr{X}}(x) \neq q$ if $a_x \neq 0$. In other words, we can choose D such that $q \notin \mathrm{sp}_{\mathscr{X}}\big(\mathrm{Supp}(D)\big)$ and $\mathcal{O}_X(D) \simeq L$. Now we can choose an open subspace $q \subset \mathscr{U} \subset \mathscr{X}$ such that $\mathscr{U} \cap \mathrm{sp}_{\mathscr{X}}\big(\mathrm{Supp}(D)\big) = \varnothing$, so $L|_{\mathscr{U}_\eta} \simeq \mathcal{O}_{\mathscr{U}_\eta}$. This finishes the proof. □

5. The trace map for smooth affinoid curves

In this section, we build on the material developed in Section 4 and construct a trace morphism $\mathrm{H}^2_c(X, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$ for any smooth affinoid curve.

Throughout this section, we fix an algebraically closed nonarchimedean field C. We denote its ring of integers by $\mathcal{O}_C$, its maximal ideal by $\mathfrak{m}_C \subset \mathcal{O}_C$, and its residue field by $k_C \coloneqq \mathcal{O}_C/\mathfrak{m}_C$. We also choose a pseudo-uniformizer $\varpi \in \mathcal{O}_C$ and an integer $n \in C^\times$.

5.1. Construction of the analytic trace map.

The main goal of this subsection is to construct the analytic trace morphism for any smooth affinoid curve over C.

Setup 5.1.1. We work with a smooth affinoid curve $X = \mathrm{Spa}\,(A, A^\circ)$ over C. We recall that Lemma A.0.3, Lemma 4.2.4, Lemma 4.2.5, and Lemma 2.2.6 imply that X admits a universal compactification $j\colon X \hookrightarrow X^c \coloneqq \mathrm{Spa}\,(A, \mathcal{O}_C[A^{\circ\circ}]^+)$ such that the complement $|X^c| \smallsetminus |X|$ consists of finitely many points $x_1, \dots, x_r$ corresponding to rank-2 valuations $v_{x_i}\colon k(x_i) \to \Gamma_C \times \mathbf{Z} \cup \{0\}$ and meets every connected component of $|X^c|$. By slight abuse of notation, we denote by the same letter the induced valuation on the henselized completed residue field $v_{x_i}\colon \widehat{k(x_i)}^{\mathrm{h}} \to \big(\Gamma_C \times \mathbf{Z}\big) \cup \{0\}$.

We start by studying étale cohomology of X and $\{x_i\}$ (considered as a pseudo-adic space) with coefficients in μ_n.

Proposition 5.1.2. *We have*

$$\mathrm{H}^i_c(X^c, \mu_n) \simeq \mathrm{H}^i(X^c, \mu_n) \simeq \mathrm{H}^i(X, \mu_n) \simeq \begin{cases} \mu_n(C)^{\#\pi_0(|X|)} \cong (\mathbf{Z}/n\mathbf{Z})^{\#\pi_0(|X|)} & i = 0 \\ 0 & i \geq 2 \end{cases}$$

and a short exact sequence

$$0 \to A^\times/A^{\times,n} \to \mathrm{H}^1(X, \mu_n) \to \mathrm{Pic}(X)[n] \to 0. \tag{5.1.3}$$

Proof. The first isomorphism follows from the fact that X^c is proper over $\mathrm{Spa}\,(C, \mathcal{O}_C)$. The second isomorphism follows from [Hub96, Cor. 2.6.7.(ii)]. Since C is algebraically closed, μ_n is non-canonically isomophic to the constant sheaf $\underline{\Lambda}$. Therefore, we conclude that $\mathrm{H}^0(X, \mu_n) \simeq (\mu_n(C))^{\#\pi_0(|X|)} \cong (\mathbf{Z}/n\mathbf{Z})^{\#\pi_0(|X|)}$.

Now we show the vanishing of $\mathrm{H}^i(X, \mu_n)$ for $i \geq 2$. Since X is smooth of pure dimension 1, Elkik's approximation theorem [Elk73, Th. 7, Rmk. 2, p. 587] lets us pick an $\mathcal{O}_C$-algebra B of finite type such that the ϖ-adic completion of B is isomorphic to A° and $B\big[\frac{1}{\varpi}\big]$ is smooth over C of pure dimension 1 (see [Zav21b,

[18] The assumptions that $q \in \mathscr{X}$ is a nodal point and that $\mathscr{X}$ is rig-smooth imply that π must be a pseudo-uniformizer.

Lem. B.5] for the dimension claim). Applying [Hub96, Th. 3.2.1, Ex. 3.1.13.iii)] to the decompleted Huber pair $(B[\frac{1}{\varpi}], B^+)$, where B^+ denotes the integral closure of B in $B[\frac{1}{\varpi}]$, we get

$$\mathrm{H}^i(X, \mu_n) \simeq \mathrm{H}^i\big(\operatorname{Spec} B^{\mathrm{h}}_{(\varpi)}\big[\tfrac{1}{\varpi}\big], \mu_n\big);$$

the $B^{\mathrm{h}}_{(\varpi)}$ stands for the henselization of B along the principal ideal (ϖ). In particular, $\operatorname{Spec} B^{\mathrm{h}}_{(\varpi)}\big[\frac{1}{\varpi}\big]$ is ind-étale over $\operatorname{Spec} B\big[\frac{1}{\varpi}\big]$. Thus, a standard approximation argument (see [Sta22, Tag 09YQ]) and the Artin–Grothendieck vanishing theorem (see [Sta22, Tag 0F0V]) imply that $\mathrm{H}^i\big(\operatorname{Spec} B^{\mathrm{h}}_{(\varpi)}\big[\frac{1}{\varpi}\big], \mu_n\big) = 0$ for $i \geq 2$.

To get the short exact sequence describing $\mathrm{H}^1(X, \mu_n)$, we use the Kummer exact sequence

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{\cdot n} \mathbf{G}_m \to 0$$

on X and the fact that $\operatorname{Pic}(X)$ can be identified with $\mathrm{H}^1(X, \mathbf{G}_m)$ (see [Hub96, (2.2.7)]). □

Lemma 5.1.4. *Let $x_i \in |X^c| \smallsetminus |X|$, $i = 1, \dots, r$, and let $\{x_i\}$ be the pseudo-adic space (X^c, x_i) for each $i = 1, \dots, r$. Then*

$$\mathrm{H}^i(\{x_i\}, \mu_n) \simeq \begin{cases} \mu_n(C) \cong \mathbf{Z}/n\mathbf{Z} & i = 0 \\ \widehat{k(x_i)}^{\mathrm{h},\times} / \Big(\widehat{k(x_i)}^{\mathrm{h},\times}\Big)^n & i = 1 \\ 0 & i \geq 2 \end{cases}.$$

Proof. Theorem B.2.1 ensures that $\mathrm{H}^i(\{x_i\}, \mu_n) \simeq \mathrm{H}^i(\operatorname{Spec} \widehat{k(x_i)}^{\mathrm{h}}, \mu_n)$. To show the vanishing part $i \geq 2$ of the lemma, it suffices to prove that the p-cohomological dimension of $\widehat{k(x_i)}^{\mathrm{h}}$ is ≤ 1 for every prime $p \mid n$. We note that [Hub96, Lem. 2.8.4] and [Hub96, Lem. 1.8.6] imply that

$$\operatorname{cd}_p\big(\widehat{k(x_i)}^{\mathrm{h}}\big) \leq \operatorname{tr.c}\big(\widehat{k(x_i)}/C\big) \leq 1.$$

The computation of $\mathrm{H}^i(\{x_i\}, \mu_n)$ for $i = 0, 1$ is similar to the analogous computation in the proof of Proposition 5.1.2, noting that $\mathrm{H}^1(\operatorname{Spec} \widehat{k(x_i)}^{\mathrm{h}}, \mathbf{G}_m) = 0$ as $\widehat{k(x_i)}^{\mathrm{h}}$ is a field. □

Proposition 5.1.5. *Keep notation as above, let us further denote $s := \#\big(\pi_0(X)\big)$. Then we have*

$$\mathrm{H}^0_c(X, \mu_n) = \mathrm{H}^{\geq 3}_c(X, \mu_n) = 0$$

and a natural exact sequence

$$0 \to \mu_n(C)^{\oplus(r-s)} \to \mathrm{H}^1_c(X, \mu_n) \to \mathrm{H}^1(X^c, \mu_n) \to \bigoplus_{i=1}^r \widehat{k(x_i)}^{\mathrm{h},\times} / \Big(\widehat{k(x_i)}^{\mathrm{h},\times}\Big)^n \xrightarrow{\partial_X} \mathrm{H}^2_c(X, \mu_n) \to 0.$$

Proof. Everything except for the vanishing of $\mathrm{H}^0_c(X, \mu_n)$ follows from the long exact sequence in cohomology for the exact triangle

$$\mathrm{R}\Gamma_c(X, \mu_n) \to \mathrm{R}\Gamma(X^c, \mu_n) \to \bigoplus_{i=1}^r \mathrm{R}\Gamma(\{x_i\}, \mu_n) \tag{5.1.6}$$

coming out of the decomposition of pseudo-adic spaces $X \hookrightarrow X^c \hookleftarrow \{X, \coprod_{i=1}^r x_i\}$ (see Remark B.1.9) together with Lemma B.1.12, Proposition 5.1.2 and Lemma 5.1.4. Now we address vanishing of $\mathrm{H}^0_c(X, \mu_n)$. For this, we can assume that X is connected, then Lemma 4.2.4 (iii) ensures $X^c \smallsetminus X$ is non-empty. We pick some point $x \in X^c \smallsetminus X$. Then (5.1.6) implies that the vanishing of $\mathrm{H}^0_c(X, \mu_n)$ follows from the injectivity of the map $\mathbf{Z}/n\mathbf{Z} \cong \mathrm{H}^0(X^c, \mu_n) \to \mathrm{H}^0(\{x\}, \mu_n) \cong \mathbf{Z}/n\mathbf{Z}$. □

As a nice application of Proposition 5.1.5, we prove the following result:

Corollary 5.1.7. *Let X be a smooth affinoid curve over C. Then $\mathrm{H}^1_c(X, \mu_n)$ is finite. Furthermore, $\mathrm{H}^1_c(\mathbf{D}^1, \mu_n) = 0$.*

Proof. Proposition 5.1.5 shows that it suffices to show that the image of the map $\alpha_X\colon \mathrm{H}^1_c(X,\mu_n)\to \mathrm{H}^1(X,\mu_n)=\mathrm{H}^1(X^c,\mu_n)$ is finite. For this, we use Proposition 4.1.6 (iii) to get an algebraic proper compactifcation $j\colon X\hookrightarrow \overline{X}$. Then the natural morphism $j_!\mu_{n,X}\to \mathrm{R}j_*\mu_{n,X}$ canonically factors through $\mu_{n,\overline{X}}$. Therefore, α_X factors as the composition

$$\mathrm{H}^1_c(X,\mu_n)\to \mathrm{H}^1(\overline{X},\mu_n)\to \mathrm{H}^1(X,\mu_n).$$

Thus, it suffices to show that $\mathrm{H}^1(\overline{X},\mu_n)$ is finite. This follows from Proposition 4.1.6 (vi) and finiteness of algebraic étale cohomology. To see that $\mathrm{H}^1_c(\mathbf{D}^1,\mu_n)=0$, we use the embedding $j\colon \mathbf{D}^1\to\mathbf{P}^{1,\mathrm{an}}$ and a similar argument to reduce to showing that $\mathrm{H}^1(\mathbf{P}^{1,\mathrm{an}},\mu_n)$. This again follows from Proposition 4.1.6 (vi) and the analogous claim in algebraic geometry. □

Now we are ready to start constructing the analytic trace map:

Definition 5.1.8. For a smooth affinoid X, we define an *analytic pre-trace*

$$\widetilde{t}_X := \sum_{i=1}^r \#\circ v_{x_i}\colon \bigoplus_{i=1}^r \widehat{k(x_i)}^{\mathrm{h},\times}/\left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \to \mathbf{Z}/n\mathbf{Z},$$

where # is defined as in Definition 4.2.7.

In order to justify the name "pre-trace" morphism, we recall the exact sequence

$$\mathrm{H}^1(X^c,\mu_n)\xrightarrow{\mathrm{res}} \bigoplus_{i=1}^r \widehat{k(x_i)}^{\mathrm{h},\times}/\left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \xrightarrow{\partial_X} \mathrm{H}^2_c(X,\mu_n)\to 0$$

from Proposition 5.1.5. This ensures that the analytic pre-trace morphism $\widetilde{t}_X$ descends to a morphism $t_X\colon \mathrm{H}^2_c(X,\mu_n)\to\mathbf{Z}/n\mathbf{Z}$ if and only if $\widetilde{t}_X$ vanishes on the image of res. The following vanishing is one of the main results of this subsection:

Theorem 5.1.9. *In Setup 5.1.1, the composition*

$$\mathrm{H}^1(X^c,\mu_n)\to \bigoplus_{i=1}^r \widehat{k(x_i)}^{\mathrm{h},\times}/\left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \xrightarrow{\widetilde{t}_X} \mathbf{Z}/n\mathbf{Z}$$

is zero.

In Section 5.2 and Section 5.3 below, we give two different proofs of Theorem 5.1.9. But before we start discussing the proofs, we give the official definition of the analytic trace map assuming validity of Theorem 5.1.9:

Definition 5.1.10. The *analytic trace morphism* $t_X\colon \mathrm{H}^2_c(X,\mu_n)\to\mathbf{Z}/n\mathbf{Z}$ is the unique group homomorphism such that the composition

$$\bigoplus_{i=1}^r \widehat{k(x_i)}^{\mathrm{h},\times}/\left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \xrightarrow{\partial_X} \mathrm{H}^2_c(X,\mu_n)\xrightarrow{t_X}\mathbf{Z}/n\mathbf{Z}$$

is equal to $\widetilde{t}_X$.

Remark 5.1.11. We note that each morphism $\#\circ v_{x_i}\colon \widehat{k(x_i)}^{\mathrm{h},\times}/\left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \xrightarrow{\widetilde{t}_X}\mathbf{Z}/n\mathbf{Z}$ is surjective. Then Lemma 4.2.4 (iii) formally implies that $t_X\colon \mathrm{H}^2_c(X,\mu_n)\to\mathbf{Z}/n\mathbf{Z}$ is surjective for any smooth affinoid curve X.

Before embarking on the proof of Theorem 5.1.9, let us explicate its statement and the resulting Definition 5.1.10 in the simple case of the 1-dimensional closed unit disk. This will require the following sequence of lemmas:

Lemma 5.1.12. *We have* $\mathrm{Pic}(\mathbf{D}^1)=0$.

Proof. First, [Ked19, Th. 1.4.2] ensures that $\mathrm{Pic}(\mathbf{D}^1)\simeq\mathrm{Pic}(C\langle T\rangle)$. Now [Bos14, Cor. 2.2/10] implies that $C\langle T\rangle$ is a UFD, thus $\mathrm{Pic}(C\langle T\rangle)=0$ by [Sta22, Tag 0BCH]. □

Lemma 5.1.13. *As an abelian group, we have a decomposition*

$$C\langle T\rangle^{\times} = C^{\times} \times (1 + \mathfrak{m}_C T\langle T\rangle, \times).$$

Here $1 + \mathfrak{m}_C T\langle T\rangle := \{f = \sum_i a_i T^i \in C\langle T\rangle \mid a_0 = 1, a_{\geq 1} \in \mathfrak{m}_C\}$, viewed as a group via multiplication.

Proof. We first note that if $f = \sum a_i T^i$ is a unit, then a_0 is a unit in C. Moreover, by [Bos14, Cor. 2.2/4], $f \in C\langle T\rangle$ is a unit if and only if $|a_0| > |a_i|$ for $i > 1$. Then one simply has $f = f(0) \cdot \frac{f}{f(0)}$ proving the lemma. □

Remark 5.1.14. Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$. Then Lemma 5.1.13 implies that there is a surjection

$$\mathrm{H}^1(\mathbf{D}^1, \mu_p) = (1 + \mathfrak{m}_C T\langle T\rangle, \times)/p \to \mathfrak{m}_C/p\mathfrak{m}_C$$

defined by the rule $1 + \sum_{i\geq 1} a_i T^i \mapsto a_1$. In particular, $\mathrm{H}^1(\mathbf{D}^1, \mu_p)$ is infinte and its cardinality is at least cardinality of $\mathfrak{m}_C/p\mathfrak{m}_C$.

Example 5.1.15. We explain Theorem 5.1.9 in case $X = \mathbf{D}^1$. By Lemma 4.2.2, the universal compactification of the 1-dimensional closed unit disk $\mathbf{D}^1 \subset \mathbf{D}^{1,c}$ consists of one additional point x_+ of rank-2 "pointing toward ∞" which corresponds to the valuation $v_{x_+} \colon k(x_+) \to \big(\Gamma_C \times \mathbf{Z}\big) \cup \{0\}$. The explicit description of the corresponding valuation v_{x_+} in Lemma 4.2.2 shows that $\# \circ v_{x_+} \colon \widehat{k(x_+)}^{\mathrm{h},\times} \to \mathbf{Z}$ vanishes on the image of the morphism $C\langle T\rangle^{\times} \to \widehat{k(x_+)}^{\mathrm{h},\times}$: by Lemma 5.1.13, this boils down to the fact that $\# \circ v_{x_+}$ is zero on the scalars $c \in C^{\times}$ and functions of the form $f = 1 + \sum_{i\geq 1} a_i T^i$ with $a_i \in \mathfrak{m}$. Thanks to Proposition 5.1.2 and Lemma 5.1.12, this yields the vanishing of $\# \circ v_{x_+}$ on $\mathrm{H}^1(\mathbf{D}^{1,c}, \mu_n)$. As a consequence, we obtain the analytic trace morphism $t_{\mathbf{D}^1} \colon \mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \to \mathbf{Z}/n\mathbf{Z}$.

5.2. **Construction of the analytic trace: first proof.** In this subsection, we give the first proof of Theorem 5.1.9. The idea is to use the Noether Normalization Lemma to reduce the case of a general smooth affinoid curve X to the case of the closed unit disk which was already treated in Example 5.1.15.

In order to implement this strategy, we will need to verify certain technical lemmas about the trace morphisms (see Theorem 2.5.6) for finite flat morphisms of smooth rigid-analytic curves over C. This will occupy the most part of this subsection. As an application of these methods, we also show that the analytic trace is compatible with the finite flat trace defined in Theorem 2.5.6.

Setup 5.2.1. We fix a finite flat morphism $f \colon X = \mathrm{Spa}\,(B, B^\circ) \to Y = \mathrm{Spa}\,(A, A^\circ)$ of smooth affinoid curves over C with induced morphism $f^c \colon X^c \to Y^c$ between the universal compactifications. We also denote by $Z_Y = \{y_i\}_{i\in I}$ the finite complement $|Y^c| \smallsetminus |Y|$, and by $Z_i = f^{c,-1}(\{y_i\}) = \{x_{i,j_i}\}_{j_i\in J_i}$ the pre-image of y_i in X^c.

In Setup 5.2.1, Lemma 4.2.1, Lemma 4.2.4, and Lemma 4.2.5 ensure that Z_Y and Z_i are finite discrete sets consisting of rank-2 curve-like points and that $Z_X := \sqcup_{i\in I} Z_i = X^c \setminus X$. For each $i \in I$, we denote by

$$f^c_i \colon (X^c, Z_i) \to (Y^c, \{y_i\})$$

the morphism of pseudo-adic spaces induced by f^c. Similarly, for $i = 1, \ldots, r$, we denote by

$$g_i \colon \bigsqcup_{j_i\in J_i} \mathrm{Spec}\, \widehat{k(x_{i,j_i})}^{\mathrm{h}} \to \mathrm{Spec}\, \widehat{k(y_i)}^{\mathrm{h}}$$

the induced morphism of the henselized completed residue fields. These morphisms fit into the following commutative diagram of topoi:

(5.2.2)
$$\begin{tikzcd} X_{\text{ét}} \arrow[r, "j_{X,\text{ét}}"] \arrow[d, "f_{\text{ét}}"] & X^c_{\text{ét}} \arrow[d, "f^c_{\text{ét}}"] & (X^c, Z_X)_{\text{ét}} \simeq \prod\limits_{i\in I}(X^c, Z_i)_{\text{ét}} \arrow[l, "i_{X,\text{ét}}"'] \arrow[r, "\gamma_{Z_X}", "\sim"'] \arrow[d, "f'_{\text{ét}}:=\prod f^c_{i,\text{ét}}"] & \prod\limits_{i\in I, j_i\in J_i} \operatorname{Spec} \widehat{k(x_{i,j_i})}^{\text{h}}_{\text{ét}} \simeq \left(\coprod\limits_{i\in I, j_i\in J_i} \operatorname{Spec} \widehat{k(x_{j_i})}^{\text{h}}\right)_{\text{ét}} \arrow[d, "g_{\text{ét}}=(\sqcup_{i\in I} g_i)_{\text{ét}}"] \\ Y_{\text{ét}} \arrow[r, "j_{Y,\text{ét}}"] & Y^c_{\text{ét}} & (Y^c, Z_Y)_{\text{ét}} \simeq \prod\limits_{i\in I}(Y^c, \{y_i\})_{\text{ét}} \arrow[l, "i_{Y,\text{ét}}"'] \arrow[r, "\gamma_{Z_Y}", "\sim"'] & \prod\limits_{i\in I} \operatorname{Spec} \widehat{k(y_i)}^{\text{h}}_{\text{ét}} \simeq \left(\coprod\limits_{i\in I} \operatorname{Spec} \widehat{k(y_i)}^{\text{h}}\right)_{\text{ét}}, \end{tikzcd}$$

where the horizontal equivalences come from Lemma B.1.12 and Theorem B.2.1. Now we note that Lemma 2.4.6 ensures that f^c is finite flat, so Theorem 2.5.6 provides us with trace morphisms $\mathrm{tr}_{f,\mu_n} \colon f_* \mu_{n,X} \to \mu_{n,Y}$ and $\mathrm{tr}_{f^c,\mu_n} \colon f^c_* \mu_{n,X^c} \to \mu_{n,Y^c}$. Furthermore, Corollary 2.3.12 implies that

$$g \colon \coprod_{i\in I, j_i\in J_i} \operatorname{Spec} \widehat{k(x_{j_i})}^{\text{h}} \to \coprod_{i\in I} \operatorname{Spec} \widehat{k(y_i)}^{\text{h}}$$

is a finite flat morphism (of schemes). Therefore, using the horizontal equivalences in Diagram (5.2.2), we can define the trace morphism

$$\mathrm{tr}_{f',\mu_n} \colon f'_* \mu_{n,Z_X} \to \mu_{n,Z_Y}$$

as $\mathrm{tr}_{f',\mu_n} = \gamma^*_{Z_Y}(\mathrm{tr}_{g,\mu_n})$, where tr_{g,μ_n} is the algebraic finite flat trace map constructed in [AGV71, Exp. XVII, Th. 6.2.3] and [Sta22, Tag 0GKI]. Next lemma will be the key to our (first) proof of Theorem 5.1.9:

Lemma 5.2.3. *Let $f \colon X \to Y$ and $f^c \colon X^c \to Y^c$ be as in Setup 5.2.1. Then the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & j_{Y,!} f_* \mu_{n,X} \arrow[r] \arrow[d, "j_{Y,!}(\mathrm{tr}_{f,\mu_n})"] & f^c_* \mu_{n,X^c} \arrow[r] \arrow[d, "\mathrm{tr}_{f^c,\mu_n}"] & i_{Y,*} f'_* \mu_{n,Z_X} \arrow[r] \arrow[d, "i_{Y,*}(\mathrm{tr}_{f',\mu_n})"] & 0 \\ 0 \arrow[r] & j_{Y,!} \mu_{n,Y} \arrow[r] & \mu_{n,Y^c} \arrow[r] & i_{Y,*} \mu_{n,Z_Y} \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. It suffices to show that each square commutes separately. In order to check that the left square commutes, it suffices to show that $j^*_Y(\mathrm{tr}_{f^c,\mu_n}) = \mathrm{tr}_{f,\mu_n}$. This follows from the fact that tr_{f^c,μ_n} commutes with arbitrary base change (see Theorem 2.5.6).

In order to show that the right square commutes, it suffices to show that $i^*_Y(\mathrm{tr}_{f^c,\mu_n}) = \mathrm{tr}_{f',\mu_n}$. It will be more convenient to check this equality after applying the equivalence γ_{Z_Y} to both sides. For this, we recall that the discussion before Construction 2.5.4 ensures that we have a commutative diagram of topoi

$$\begin{tikzcd} X^c_{\text{ét}} \arrow[r, "c_X"] \arrow[d, "f^c_{\text{ét}}"] & (\operatorname{Spec} B)_{\text{ét}} \arrow[d, "f^{c,\text{alg}}_{\text{ét}}"] \\ Y^c_{\text{ét}} \arrow[r, "c_Y"] & (\operatorname{Spec} A)_{\text{ét}}, \end{tikzcd}$$

where c_A and c_B are the analytification morphisms. Furthermore, this fits into a bigger commutative diagram:

(5.2.4)
$$\begin{tikzcd}
(X^c, Z_X)_{\text{ét}} \arrow[rr, "\gamma_{Z_X}", "\sim"'] \arrow[dr, "i_{X,\text{ét}}"] \arrow[dd, "f'_{\text{ét}}"'] & & \left(\sqcup_{i\in I, j_i\in J_i} \operatorname{Spec} \widehat{k(x_{i,j_i})}^{\mathrm{h}}\right)_{\text{ét}} \arrow[dr, "i^{\mathrm{alg}}_{X,\text{ét}}"] \arrow[dd, "g_{\text{ét}}"] & \\
& (X^c)_{\text{ét}} \arrow[rr, "c_X"] \arrow[dd, "f^c_{\text{ét}}"] & & (\operatorname{Spec} B)_{\text{ét}} \arrow[dd, "f^{c,\mathrm{alg}}_{\text{ét}}"] \\
(Y^c, Z_Y)_{\text{ét}} \arrow[rr, "\gamma_{Z_Y}", "\sim"'] \arrow[dr, "i_{Y,\text{ét}}"] & & \left(\sqcup_{i\in I} \operatorname{Spec} \widehat{k(y_i)}^{\mathrm{h}}\right)_{\text{ét}} \arrow[dr, "i^{\mathrm{alg}}_{Y,\text{ét}}"] & \\
& (Y^c)_{\text{ét}} \arrow[rr, "c_Y"] & & (\operatorname{Spec} A)_{\text{ét}} .
\end{tikzcd}$$

We recall that tr_{f',μ_n} is defined as $\gamma^*_{Z_Y}(\mathrm{tr}_{g,\mu_n})$, and $\mathrm{tr}_{f^c,\mu_n} = c^*_Y(\mathrm{tr}_{f^{c,\mathrm{alg}},\mu_n})$ (see Theorem 2.5.6 (6)). Therefore, it suffices to show that $i_Y^{\mathrm{alg},*}(\mathrm{tr}_{f^{c,\mathrm{alg}},\mu_n}) = \mathrm{tr}_{g,\mu_n}$. Now we note that [Sta22, Tag 0GKI] guarantees that the algebraic finite flat trace map commutes with arbitrary base change. Therefore, it suffices to show that the natural morphism

$$\widehat{k(y_i)}^{\mathrm{h}} \otimes_A B \to \prod_{j_i\in J_i} \widehat{k(x_{i,j_i})}^{\mathrm{h}}$$

is an isomorphism for any $i \in I$. This follows directly from the combination of Lemma 4.2.6, Lemma 2.3.14, and Theorem 2.3.10. □

For each $i \in I$ and $j_i \in J_i$, we define the morphism $\mathrm{Nm}_{x_{i,j_i}/y_i} \colon \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} / \left(\widehat{k(x_{i,j_i})}^{\mathrm{h},\times}\right)^n \to \widehat{k(y_i)}^{\mathrm{h},\times} / \left(\widehat{k(y_i)}^{\mathrm{h},\times}\right)^n$ be the morphism induces by the norm morphism $\mathrm{Nm}\colon \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} \to \widehat{k(y_i)}^{\mathrm{h},\times}$ (we note that the Norm map is well-defined due to Corollary 2.3.12).

Corollary 5.2.5. *Let $f\colon X \to Y$ and $f^c\colon X^c \to Y^c$ be as in Setup 5.2.1. Then there is the following diagram of exact sequences:*

$$\begin{tikzcd}
\mathrm{H}^1(X^c, \mu_n) \arrow[r] \arrow[d, "\mathrm{H}^1(\mathrm{tr}_{f^c,\mu_n})"] & \bigoplus\limits_{i\in I, j\in J_i} \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} / \left(\widehat{k(x_{i,j_i})}^{\mathrm{h},\times}\right)^n \arrow[r, "\partial_X"] \arrow[d, "\oplus_{i\in I}\left(\sum_{j_i\in J_i} \mathrm{Nm}_{x_{i,j_i}/y_i}\right)"] & \mathrm{H}^2_c(X, \mu_n) \arrow[r] \arrow[d, "\mathrm{H}^2_c(\mathrm{tr}_{f,\mu_n})"] & 0 \\
\mathrm{H}^1(Y^c, \mu_n) \arrow[r] & \bigoplus\limits_{i\in I} \widehat{k(y_i)}^{\mathrm{h},\times} / \left(\widehat{k(y_i)}^{\mathrm{h},\times}\right)^n \arrow[r, "\partial_Y"] & \mathrm{H}^2_c(Y, \mu_n) \arrow[r] & 0.
\end{tikzcd}$$

Proof. Exactness of horizontal sequences follows directly from Proposition 5.1.5. Now let g_{i,j_i} be the natural morphism $\operatorname{Spec} \widehat{k(x_{i,j_i})}^{\mathrm{h}} \to \operatorname{Spec} \widehat{k(y_i)}^{\mathrm{h}}$. Then [AGV71, Exp. XVII, Diagram (6.3.18.2) on p. 198] implies that, under the identification $\mathrm{H}^1\big(\widehat{k(x_{i,j_i})}^{\mathrm{h}}, \mu_n\big) \simeq \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} / \big(\widehat{k(x_{i,j_i})}^{\mathrm{h},\times}\big)^n$ and under the similar identification for y_i, the trace map $\mathrm{H}^1(\mathrm{tr}_{g_{i,j_i},\mu_n})$ becomes equal to $\mathrm{Nm}_{x_{i,j_i}/y_i}$. Therefore, the result follows directly from Lemma 5.2.3. □

Now we are finally ready to give the first proof of Theorem 5.1.9:

First Proof of Theorem 5.1.9. In this proof, we use the notation from Setup 5.1.1; we warn readers that the notation is slightly different from Setup 5.2.1. We start the proof by noting that [Bos14, Prop. 3.1/2], [Sta22, Tag 00OK], and Lemma 4.1.2 allow us to find a finite flat morphism $f\colon X \to \mathbf{D}^1$. We denote by $f^c\colon X^c \to \mathbf{D}^{1,c}$ the induced morphism of universal compactifications.

Now we consider the open immersion $j\colon X \hookrightarrow X^c$ with $|X^c| \smallsetminus |X|$ consisting of finitely many rank-2 points $\{x_1, \dots, x_r\}$. Then Corollary 5.2.5, Lemma 2.2.10, and Lemma 4.2.5 imply that the following diagram

$$\begin{tikzcd}
\mathrm{H}^1(X^c, \mu_n) \arrow[r] \arrow[d, "\mathrm{H}^1(\mathrm{tr}_{f^c,\mu_n})"] & \bigoplus\limits_{i=1}^{r} \widehat{k(x_i)}^{\mathrm{h},\times} / \left(\widehat{k(x_i)}^{\mathrm{h},\times}\right)^n \arrow[r, "\sum_{i=1}^r \#\circ v_{x_i}"] \arrow[d, "\sum_{i=1}^r \mathrm{Nm}_{x_i/x_+}"'] & \mathbf{Z}/n\mathbf{Z} \\
\mathrm{H}^1(\mathbf{D}^{1,c}, \mu_n) \arrow[r] & \widehat{k(x_+)}^{\mathrm{h},\times} / \left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^n \arrow[ur, "\#\circ v_{x_+}"'] &
\end{tikzcd} \tag{5.2.6}$$

commutes. Therefore it suffices to show that the composition

$$\mathrm{H}^1(X^c, \mu_n) \xrightarrow{\mathrm{H}^1(\mathrm{tr}_{f^c,\mu_n})} \mathrm{H}^1(\mathbf{D}^{1,c}, \mu_n) \to \widehat{k(x_+)}^{\times,\mathrm{h}} / \left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^n \xrightarrow{\#\circ v_{x_+}} \mathbf{Z}/n\mathbf{Z}$$

is zero. To that end, we just note that Example 5.1.15 implies that the composition of the last two maps is already zero. □

As an application of our methods, we also show that the analytic trace morphism is compatible with the finite flat trace morphisms:

Theorem 5.2.7. *Let $f\colon X \to Y$ be a finite flat morphism of smooth affinoid rigid-analytic C-curves. Then the diagram*

$$\begin{tikzcd}
\mathrm{H}^2_c(X, \mu_n) \arrow[r, "t_X"] \arrow[d, "\mathrm{H}^2_c(\mathrm{tr}_{f,\mu_n})"'] & \mathbf{Z}/n\mathbf{Z} \\
\mathrm{H}^2_c(Y, \mu_n) \arrow[ur, "t_Y"'] &
\end{tikzcd}$$

commutes, where tr_f is the finite flat trace morphism from Theorem 2.5.6 and t_X, t_Y are analytic traces from Definition 5.1.10.

Proof. Keeping the notation of Setup 5.2.1, Corollary 5.2.5 ensures that the following diagram

$$\begin{tikzcd}
\bigoplus\limits_{i\in I, j\in J_i} \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} / \left(\widehat{k(x_{i,j_i})}^{\mathrm{h},\times}\right)^n \arrow[r, "\partial_X"] \arrow[d, "\bigoplus_{i\in I}\left(\sum_{j_i\in J_i} \mathrm{Nm}_{x_{i,j_i}/y_i}\right)"] & \mathrm{H}^2_c(X, \mu_n) \arrow[d, "\mathrm{H}^2_c(\mathrm{tr}_{f,\mu_n})"] \\
\bigoplus\limits_{i\in I} \widehat{k(y_i)}^{\mathrm{h},\times} / \left(\widehat{k(y_i)}^{\mathrm{h},\times}\right)^n \arrow[r, "\partial_Y"] & \mathrm{H}^2_c(Y, \mu_n)
\end{tikzcd}$$

commutes. Now we use that both ∂_X and ∂_Y are surjective and the definition of the analytic trace map (see Definition 5.1.10) to conclude that it suffices to show that the diagram

$$\begin{tikzcd}
\bigoplus\limits_{i\in I, j\in J_i} \widehat{k(x_{i,j_i})}^{\mathrm{h},\times} / \left(\widehat{k(x_{i,j_i})}^{\mathrm{h},\times}\right)^n \arrow[r, "\sum \#\circ v_{x_{j_i}}"] \arrow[d, "\bigoplus_{i\in I}\left(\sum_{j_i\in J_i} \mathrm{Nm}_{x_{i,j_i}/y_i}\right)"'] & \mathbf{Z}/n\mathbf{Z} \\
\bigoplus\limits_{i\in I} \widehat{k(y_i)}^{\mathrm{h},\times} / \left(\widehat{k(y_i)}^{\mathrm{h},\times}\right)^n \arrow[ur, "\sum \#\circ v_{y_i}"'] &
\end{tikzcd}$$

commutes. This now follows directly from Lemma 2.2.10 and Lemma 4.2.5. □

5.3. **Construction of the analytic trace: second proof.** Now we give another proof of Theorem 5.1.9 which does not resort to Noether normalization, finite flat traces or the results of Section 5.2. Instead, we use the interpretation of $\mathrm{H}^1(X^c,\mu_n)$ as isomorphism classes of μ_n-torsors to generalize Example 5.1.15 to the setting of smooth affinoid curves. This strategy necessitates a more detailed analysis of $\mathrm{H}^1(X^c,\mu_n)\simeq \mathrm{H}^1(X,\mu_n)$.

Construction 5.3.1. Recall that (5.1.3) induces a canonical identification

$$\mathrm{H}^1(X^c,\mu_n)\simeq\{(L,s)\mid L\in\mathrm{Pic}(X^c),\ s\colon\mathcal{O}\xrightarrow{\sim}L^{\otimes n}\}/\sim$$

with isomorphism classes of line bundles together with a trivialization of their n-th power. With this interpretation, we can attach to any point $x\in|X^c|$ a natural homomorphism

$$\rho_x\colon\ \mathrm{H}^1(X^c,\mu_n)\to k(x)^\times/\big(k(x)^\times\big)^n$$

as follows: Given $(L,s)\in\mathrm{H}^1(X^c,\mu_n)$, choose an open affinoid neighborhood $U\subseteq X^c$ of x on which the restricted line bundle $L\big|_U$ admits a trivialization $a\colon\mathcal{O}_U\xrightarrow{\sim}L\big|_U$. Then the image of (L,s) under $\mathrm{H}^1(X^c,\mu_n)\to\mathrm{H}^1(U,\mu_n)$ lies in the image of the boundary map $\mathcal{O}(U)^\times/\big(\mathcal{O}(U)^\times\big)^n\to\mathrm{H}^1(U,\mu_n)$ of the Kummer sequence; concretely, it is the well-defined (independent of the choice of a) element of $\mathcal{O}(U)^\times/\big(\mathcal{O}(U)^\times\big)^n$ determined by the isomorphism

$$a^{-n}\circ s_U\colon\mathcal{O}_U\xrightarrow{\sim}\big(L\big|_U\big)^{\otimes n}\xrightarrow{\sim}\mathcal{O}_U^{\otimes n}\simeq\mathcal{O}_U.$$

We define $\rho_x(L,s)$ to be the image of this element under the natural map

$$\mathcal{O}(U)^\times/\big(\mathcal{O}(U)^\times\big)^n\to\mathcal{O}_{U,x}^\times/\big(\mathcal{O}_{U,x}^\times\big)^n\to k(x)^\times/\big(k(x)^\times\big)^n.$$

By passing to common open affinoids trivializing several line bundles, one checks that this defines a group homomorphism.

Variant 5.3.2. By Theorem B.2.1 and Hilbert 90, we have an identification $\mathrm{H}^1(\{x\},\mu_n)\simeq\mathrm{H}^1(\mathrm{Spec}\,\widehat{k(x)}^{\mathrm{h}},\mu_n)\simeq\widehat{k(x)}^{\mathrm{h},\times}/\Big(\widehat{k(x)}^{\mathrm{h},\times}\Big)^n$. The functoriality of the Kummer sequence then shows that under this isomorphism, the composition

$$\mathrm{H}^1(X^c,\mu_n)\xrightarrow{\rho_x}k(x)^\times/\big(k(x)^\times\big)^n\longrightarrow\widehat{k(x)}^{\mathrm{h},\times}/\Big(\widehat{k(x)}^{\mathrm{h},\times}\Big)^n$$

is given by the natural map $\mathrm{H}^1(X^c,\mu_n)\to\mathrm{H}^1(\{x\},\mu_n)$. Concretely, it can again be described as in Construction 5.3.1 using trivializations of the pullback of L along the map of ringed étale topoi $\big(\mathrm{Spec}\,\big(k(x)^{\mathrm{h}}\big)_{\text{ét}},\mathcal{O}\big)\to\big(X^c_{\text{ét}},\mathcal{O}\big)$ induced by the map of étale topoi $\mathrm{Spec}\,\big(k(x)^{\mathrm{h}}\big)_{\text{ét}}\xleftarrow[\gamma]{\sim}\big(\mathrm{Spa}\,\big(k(x)^{\mathrm{h}},k(x)^{+,\mathrm{h}}\big),\{x\}\big)_{\text{ét}}\to X^c_{\text{ét}}$ from Theorem B.2.1. We still denote this map by ρ_x when there is no risk for confusion.

If $x\in|X|$, then ρ_x factors through $\mathrm{H}^1(X,\mu_n)$ and we can also work with line bundles L on X instead of X^c in Construction 5.3.1 and Variant 5.3.2. While the ultimate construction of the analytic trace in Definition 5.1.10 only uses rank-2 valuations for points in $|X^c|\smallsetminus|X|$, the second proof of Theorem 5.1.9 also uses Construction 5.3.1 for points in $|X|$. We then have the following compatibility with the inclusion of residue fields from [Hub96, Lem. 1.1.10.iii)]:

Lemma 5.3.3. *Let $z\in|X|$ be point of rank 1 and $y\in\overline{\{z\}}\subset X^c$ a specialization of z. Then the composition*

$$\mathrm{H}^1(X^c,\mu_n)\xrightarrow{\rho_y}k(y)^\times/\big(k(y)^\times\big)^n\longrightarrow k(z)^\times/\big(k(z)^\times\big)^n$$

with the morphism induced by the inclusion of residue fields $k(y)\to k(z)$ is equal to ρ_z.

Proof. Unwinding Construction 5.3.1, this follows from commutativity of the diagram of natural maps

$$\begin{tikzcd}
 & \mathcal{O}_{U,y} \arrow[r] \arrow[d] & k(y) \arrow[d] \\
\mathcal{O}(U) \arrow[ur] \arrow[dr] & & \\
 & \mathcal{O}_{U,z} \arrow[r] & k(z)
\end{tikzcd}$$

for any open affinoid neighborhood $U\subseteq X^c$ of y (and thus also z). ∎

Lemma 5.3.4. *Let $\mathscr{X}$ be an admissible formal $\mathcal{O}_C$-model of X whose special fiber $\mathscr{X}_s$ is a reduced separated scheme of pure dimension 1. Let $z \in |X|$ be point of rank 1 whose specialization $\zeta = \mathrm{sp}(z) \in |\mathscr{X}_s|$ is a generic point. Then there is a pushout diagram of groups*

$$\begin{tikzcd} (\mathcal{O}_C)^\times \arrow[r] \arrow[d] & (k(z)^+)^\times \arrow[d] \\ C^\times \arrow[r] & k(z)^\times. \end{tikzcd}$$

In particular, using Lemma 4.3.2. (iii) we get the well-defined "further specialization" *map*

$$\begin{tikzcd} k(z)^\times/n \arrow[rrr, bend left, "\mathrm{sp}_z"] & (k(z)^+)^\times/n \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{O}^\times_{\mathscr{X},\zeta}/n \arrow[r, two heads] & k(\zeta)^\times/n. \end{tikzcd}$$

Proof. The vertical maps are injective with cokernels the respective value groups. Thus, for the first statement we only need to show that the induced map of value groups is a bijection, which follows from Lemma 4.3.2. (v). The second statement then follows directly from the Snake Lemma and the fact that Γ_C is divisible (see [BGR84, Obs. 3.6/10]). $\square$

Notation 5.3.5. In the situation of Lemma 5.3.4, let $(L, s) \in \mathrm{H}^1(X^c, \mu_n)$. Then we denote the further specialization of $\rho_z(L, s)$ under the map $\mathrm{sp}_z : k(z)^\times/n \to k(\zeta)^\times/n$ by $\mathrm{sp}_z(L, s)$.

We recall that, for a smooth irreducible k-curve Y, a point $y \in Y(k)$, and a rational function $f \in k(Y)$, we denote by $\mathrm{ord}_y(f) \in \mathbf{Z}$ the order of vanishing of f at the point y.

Lemma 5.3.6. *In the situation of Lemma 5.3.4, let $y \in \overline{\{z\}} \subset |X^c|$ be a specialization of z. Then for any $\overline{f} \in k(y)^\times/n$, we have*

$$(\# \circ v_y)\big(\overline{f}\big) \equiv \mathrm{ord}_{\mu_\zeta(y)}\big(\mathrm{sp}_z(\overline{f})\big) \mod n \tag{5.3.7}$$

under the identification $\mathcal{O}_{\mathscr{X},\zeta} \xrightarrow{\sim} k(z)^+$ from Lemma 4.3.2. (iii) and the correspondence μ_ζ from Lemma 4.3.6. In particular, for any $(L, s) \in \mathrm{H}^1(X^c, \mu_n)$, we have

$$(\# \circ v_y)\big(\rho_y(L, s)\big) \equiv \mathrm{ord}_{\mu_\zeta(y)}\big(\mathrm{sp}_z(L, s)\big) \mod n$$

Proof. We choose a lift $f \in k(y)^\times$ of $\overline{f}$. Throughout this proof, we will freely identify $f \in k(y)$ with its image under the (injective) map $k(y) \to k(z)$. Lemma 5.3.4 ensures that there is an element $c \in C^\times$ such that $c \cdot f \in (k(z)^+)^\times$. Since $C^\times$ is n-divisible, we conclude that

$$\mathrm{sp}_z\big(\overline{c \cdot f}\big) = \mathrm{sp}_z\big(\overline{f}\big) \in k(\zeta)^\times/n.$$

We note that also $(\# \circ v_y)(c) = 0$ for any $c \in C^\times$, so we may and do replace f with $c \cdot f$ to assume that $f \in k(y)^\times \cap (k(z)^+)^\times$. In this case, Lemma 4.3.6 (iii) implies that $(\# \circ v_y)\big(\overline{f}\big) = \mathrm{ord}_{\mu_\zeta(y)}\big(\mathrm{sp}_z(\overline{f})\big)$.

The "in particular" part now follows directly from (5.3.7), Lemma 5.3.3, and Notation 5.3.5. $\square$

Next, we show that $(\# \circ v_x)(\rho_x(L, s))$ vanishes for rank-2 points $x \in |X|$ such that $\mathrm{sp}_{\mathscr{X}}(x)$ is a smooth point of some admissible formal $\mathcal{O}_C$-model $\mathscr{X}$ of X.

Lemma 5.3.8. *Let $\mathscr{X} = \mathrm{Spf}\, R$ be a smooth formal $\mathcal{O}_C$-scheme with irreducible special fiber. Let (L, s) be a μ_n-torsor on $\mathscr{X}_\eta = \mathrm{Spa}\,(R[\frac{1}{\varpi}], R)$ and let $\mathscr{L} \in \mathrm{Pic}(R)$ such that $\mathscr{L}[\frac{1}{\varpi}] \simeq L$. Then there exists an isomorphism $\sigma : \mathcal{O}_{\mathscr{X}} \xrightarrow{\sim} \mathscr{L}^{\otimes n}$ such that $\sigma[\frac{1}{\varpi}] = c \cdot s$ for some $c \in C^\times$.*

Proof. Let v be the supremum semi-norm on $R[\frac{1}{\varpi}]$ [BGR84, § 3.8]. Since $\mathrm{Spf}(R)$ is smooth and connected (thus irreducible), v is a valuation on $R[\frac{1}{\varpi}]$ due to [BGR84, Prop. 6.2.3/5]; this is exactly the unique rank-1 valuation corresponding to the generic point η of $\mathscr{X}_s$ under Lemma 4.3.2. (i) (see the proof of Lemma 4.3.2. (v) for the justification). Then Lemma 4.3.2 (v) implies that there is a scalar $c \in C^\times$ such that $v(\rho_v(L, c \cdot s)) = 1$. Now, we claim that $c \cdot s$ can be extended to an isomorphism σ.

We note that if an extension σ exists, it is unique. We may therefore localize on $\operatorname{Spf} R$ and assume that $\mathscr{L} \simeq \mathcal{O}_{\mathscr{X}}$. In this case, s just corresponds to an element of $R[\frac{1}{\varpi}]^\times$ and we wish to show that $c \cdot s$ lies in $R^\times$. By our assumption on s and c, we have $v(c \cdot s) = 1$. Now using [Lüt16, Prop. 3.4.1] and [BGR84, Prop. 6.2.3/1], we conclude that $R = \{r \in R[\frac{1}{\varpi}] \mid v(r) \leq 1\}$. Therefore, using multiplicativity of v, we conclude that $v(c \cdot s) = 1$ implies that $c \cdot s \in R^\times$ finishing the proof. $\square$

We are ready to prove the promised above vanishing:

Proposition 5.3.9. *Let $\mathscr{X}$ be an admissible formal $\mathcal{O}_C$-scheme such that the special fiber $\mathscr{X}_s$ is a reduced separated scheme of pure dimension 1. Let ζ be a generic point of $\mathscr{X}_s$ corresponding to $z \in X$ via Lemma 4.3.6. Let $y \in \overline{\{z\}}$ be a point given by a rank-2 valuation v_y. Assume that $\mathrm{sp}_{\mathscr{X}}(y) \in \mathscr{X}_s$ is a smooth point. Then for any $(L, s) \in \mathrm{H}^1(X^c, \mu_n)$, we have $\big(\# \circ v_y\big)\big(\rho_y(L, s)\big) = 0$.*

Proof. The question is Zariski-local on $\mathscr{X}$, so we may and do assume that $\mathscr{X} = \operatorname{Spf} R$ is smooth affine formal $\mathcal{O}_C$-scheme with irreducible special fiber. By Lemma 4.5.2, the line bundle L can then be extended to a line bundle $\mathscr{L}$ on $\mathscr{X}$. Lemma 5.3.8 guarantees that s can be extended to an isomorphism $\sigma \colon \mathcal{O}_{\mathscr{X}} \xrightarrow{\sim} \mathscr{L}^{\otimes n}$ after scaling by some $c \in C^\times$. Thus, for the purpose of showing that $\big(\# \circ v_y\big)\big(\rho_y(L, s)\big) = 0$, we can replace s with $c \cdot s$ to assume that s extends to an isomorphism $\sigma \colon \mathcal{O}_{\mathscr{X}} \xrightarrow{\sim} \mathcal{L}^{\otimes n}$.

Over the local ring $\mathcal{O}_{\mathscr{X}, \mathrm{sp}(y)}$, we choose a trivialization $a \colon \mathcal{O}_{\mathscr{X}, \mathrm{sp}(y)} \xrightarrow{\sim} \mathscr{L}\big|_{\mathcal{O}_{\mathscr{X}, \mathrm{sp}(y)}}$ and consider $a^{-n} \circ \sigma_x$ as an element of $\mathcal{O}^\times_{\mathscr{X}, \mathrm{sp}(y)}$. Using Lemma 5.3.6, we can identify $(\# \circ v_x)(\rho_x(L, s))$ with $\mathrm{ord}_{\mu_{\zeta}(y)}\big(a^{-n} \circ \sigma_x\big)$. Since $a^{-n} \circ \sigma_x \in \mathcal{O}^\times_{\mathscr{X}, \mathrm{sp}(y)}$, we conclude that this valuation is zero. $\square$

Next, we prove a vanishing statement for the nodes in the special fiber.

Proposition 5.3.10. *Let $\mathscr{X}$ be a separated semistable formal $\mathcal{O}_C$-curve; see Definition 4.1.3. Let $q \in \mathscr{X}_s$ be a node and $q_1, q_2 \in \mathscr{X}_s^n$ be the two points in the normalization lying above q. Let y_1 and y_2 be the points of $|X|$ with rank-2 valuation v_{y_1} and v_{y_2} which correspond to q_1 and q_2 under Lemma 4.3.6, respectively. Then $(\# \circ v_{y_1})(\rho_{y_1}(L, s)) + (\# \circ v_{y_2})(\rho_{y_2}(L, s)) = 0$.*

The proof relies on the following statement:

Proposition 5.3.11. *Fix an element $\pi \in \mathfrak{m}_C \smallsetminus \{0\}$ and set $\widetilde{R} := \frac{\mathcal{O}_C[\![S,T]\!]}{(ST - \pi)}$. Let $f \in \widetilde{R}[\frac{1}{\varpi}]^\times$ such that both f and f^{-1} are regular in the sense of Definition 4.4.3. (ii). Then*

$$v_S(f) + v_T(f) = 0,$$

where v_S and v_T are defined in Definition 4.4.9. (ii).

Proof. The multiplicativity of both v_S and v_T (see Lemma 4.4.10) implies that it suffices to show that $v_S(f) + v_T(f) \geq 0$ for any non-zero regular $f \in \widetilde{R}[\frac{1}{\varpi}]$.

If $f = S^n$, we see that $v_S(S^r) + v_T(S^r) = v_S(S^r) + v_T(\frac{\pi^r}{T^r}) = r - r = 0$. Therefore, we may replace f with $f \cdot S^r$ for any integer r. Since $v_S(S) = 1$, we can use the above observation to reduce to the case when $v_S(f) = 0$. Thus, we conclude that the S-adic expansion $f = \sum_{i \in \mathbf{Z}} a_i S^i$ inside $\widetilde{R}[\frac{1}{\varpi}]$ satisfies the property that $|a_i| < |a_0|$ if $i < 0$ and $|a_i| \leq |a_0|$ if $i > 0$.

Now if we look at the T-adic expansion of f, we have $f = \sum_{i \in \mathbf{Z}} a_{-i} \pi^{-i} T^i$. Thus, the coefficients of negative powers of T are of the form $a_{>0} \cdot$ (positive powers of π). In particular, their norm is strictly less than $|a_0|$. Hense, the very definition of $v_T(f)$ implies that $v_T(f) \geq 0$. $\square$

Proof of Proposition 5.3.10. Lemma 4.5.4 and the definition of semi-stable formal $\mathcal{O}_C$-curves imply that we can find a diagram of pointed admissible formal $\mathcal{O}_C$-schemes

$$\begin{tikzcd} & (\mathscr{U}, u) \arrow[dl, "h"'] \arrow[dr, "g"] & \\ (\mathscr{X}, q) & & (\mathscr{Y}_\pi, 0) := \Big(\operatorname{Spf}\big(\dfrac{\mathcal{O}_C\langle S, T\rangle}{(ST - \pi)}\big), (0,0)\Big) \end{tikzcd}$$

such that g and h are étale, $\pi \in \mathfrak{m}_C \smallsetminus \{0\}$, and there is a trivialization $a\colon \mathcal{O}_X \xrightarrow{\sim} L$.

Let $U := \mathscr{U}_\eta$ be the rigid generic fiber of $\mathscr{U}$, let u_i be the unique (due to the étaleness) points of $\mathscr{U}_s^n$ lying above q_i, and let $w_i \in |U|$ be the rank-2 points corresponding to u_i under Lemma 4.3.6, respectively. Using Lemma 5.3.6 and Lemma 4.3.6 (iii), we see that

$$(\# \circ v_{y_i})(\rho_{y_i}(L,s)) = (\# \circ v_{w_i})\big(\rho_{w_i}\big((L,s)|_U\big)\big).$$

Hence we may replace $(\mathscr{X}, q)$ by $(\mathscr{U}, u)$ and assume that our pointed semistable formal $\mathcal{O}_C$-curve admits an étale map $g\colon (\mathscr{X}, q) \to (\mathscr{Y}_\pi, 0)$ and there is a chosen trivialization $a\colon \mathcal{O}_X \xrightarrow{\sim} L$. By shrinking $(\mathscr{X}, q)$ we may assume that $\mathscr{X} = \operatorname{Spf} A_0$ is affine. We set $f := a^{-n} \circ s \in (A_0[\frac{1}{\varpi}])^\times$. Then it suffices to show that

$$(\# \circ v_{y_1})(f) + (\# \circ v_{y_2})(f) = 0.$$

Now we note that the map g induces an isomorphism

$$R := \Big(\big(\frac{\mathcal{O}_C\langle S,T\rangle}{(ST-\pi)}\big)^{\mathrm{h}}_{(S,T)}\Big)^\wedge_\varpi \xrightarrow{\sim} \big(\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q}\big)^\wedge_\varpi.$$

Following Notation 4.4.1, we have the natural morphism $A_0[\frac{1}{\varpi}] = \mathcal{O}_{\mathscr{X}}(\mathscr{X})[\frac{1}{\varpi}] \to \big(\mathcal{O}^{\mathrm{h}}_{\mathscr{X},q}\big)^\wedge_\varpi[\frac{1}{\varpi}] \xrightarrow{\sim} R[\frac{1}{\varpi}] \to \widetilde{R}[\frac{1}{\varpi}] := R^\wedge_{(S,T,\varpi)}[\frac{1}{\varpi}]$. We denote by $\widetilde{f}$ the image of f in $\widetilde{R}[\frac{1}{\varpi}]$. Then Proposition 4.4.11 ensures that $\# \circ v_{y_1}(f) = v_S(\widetilde{f})$ and $\# \circ v_{y_2} = v_T(\widetilde{f})$. Thus, we reduced the question to showing that

$$v_S(\widetilde{f}) + v_T(\widetilde{f}) = 0.$$

This follows immediately from Lemma 4.4.6 and Proposition 5.3.11. □

Second Proof of Theorem 5.1.9. Let $(L,s) \in \mathrm{H}^1(X, \mu_n)$. By Proposition 4.1.6 (5), we may choose a semistable $\mathcal{O}_C$-formal model $\mathscr{X}$ of X. Let $\mathscr{X}_s^c$ be the compactification of $\mathscr{X}_s$ such that $\mathscr{X}_s \subset \mathscr{X}_s^c$ is schematically dense and contains all the singular points of $\mathscr{X}_s^c$, and let $\nu\colon \mathscr{X}_s^{c,n} \to \mathscr{X}_s^c$ be its normalization. Now Lemma 4.3.6. (iv) and the assumption that all singular points of $\mathscr{X}^c$ are contained in $\mathscr{X}_s$ imply that the specialization map induces a bijection $|X^c| \smallsetminus |X| \xrightarrow{\sim} |\mathscr{X}_s^{c,n}| \smallsetminus |\mathscr{X}_s^n| \xrightarrow{\sim} |\mathscr{X}_s^c| \smallsetminus |\mathscr{X}_s|$. For any $q \in |\mathscr{X}_s^c|$, let ζ_q be the generic point of the irreducible component containing q. By Lemma 4.3.2, the set $\operatorname{sp}_{\mathscr{X}}^{-1}(\zeta_q) = \{z_q\}$ for some rank-1 point z_q. Then we have

$$\begin{aligned}
\tilde{t}_X(L,s) &\overset{\text{Lem. 4.3.6,}}{\underset{\text{Def. 5.1.8}}{\equiv}} \sum_{x_i \in |X^c| \smallsetminus |X|} (\# \circ v_i)\big(\rho_{x_i}(L,s)\big) \\
&\overset{\text{Lem. 5.3.6, Prop. 5.3.9,}}{\underset{\text{Prop. 5.3.10}}{\equiv}} \sum_{q \in |\mathscr{X}_s^c| \smallsetminus |\mathscr{X}_s|} \operatorname{ord}_q\big(\operatorname{sp}_{z_q}(L,s)\big) + \sum_{q \in |\mathscr{X}_s| \text{ smooth}} \operatorname{ord}_q\big(\operatorname{sp}_{z_q}(L,s)\big) \\
&\qquad + \sum_{\substack{q \in |\mathscr{X}_s| \text{ node} \\ \text{with } \nu^{-1}(q) = \{q_1, q_2\}}} \operatorname{ord}_{q_1}\big(\operatorname{sp}_{z_q}(L,s)\big) + \operatorname{ord}_{q_2}\big(\operatorname{sp}_{z_q}(L,s)\big) \\
&\overset{\text{combining}}{\underset{\text{terms}}{\equiv}} \sum_{\substack{Y \subseteq \mathscr{X}_s^{c,n} \\ \text{connected component}}} \sum_{q \in |Y| \text{ closed}} \operatorname{ord}_q\big(\operatorname{sp}_{z_q}(L,s)\big) \mod n.
\end{aligned}$$

Now fix a connected component $Y_\zeta \subseteq \mathscr{X}_s^{c,n}$ with generic point ζ. This is an algebraic curve over k on which $\operatorname{sp}_{z_q}(L,s) \in k(\zeta)^\times / \big(k(\zeta)^\times\big)^n$ defines a principal divisor (mod n). Thus,

$$\sum_{q \in |Y| \text{ closed}} \operatorname{ord}_q\big(\operatorname{sp}_{z_q}(L,s)\big) \equiv \deg\big(\operatorname{Div}(\operatorname{sp}_{z_q}(L,s))\big) \equiv 0 \mod n$$

and we win. □

5.4. **Compatibility with the algebraic trace map.** The main goal of this subsection is to formulate the statement that the analytic trace map constructed in Definition 5.1.10 is compatible with the algebraic one. Its proof is the content of the next two subsections. We begin by recalling the algebraic trace map.

Definition 5.4.1. Let $\overline{X}$ be a smooth proper rigid-analytic curve over C and let $\overline{X}^{\mathrm{alg}}$ be its unique algebraization (see Proposition 4.1.6. (ii)). The *algebraic trace map* on $\overline{X}$ is the homomorphism

$$t^{\mathrm{alg}}_{\overline{X}} \colon \mathrm{H}^2\big(\overline{X}, \mu_n\big) \simeq \mathrm{H}^2\big(\overline{X}^{\mathrm{alg}}, \mu_n\big) \xrightarrow{t_{\overline{X}^{\mathrm{alg}}}} \mathbf{Z}/n\mathbf{Z}$$

which is obtained as the composition of the (inverse of the) isomorphism from Proposition 4.1.6. (vi) and the schematic trace map (see [AGV71, Exp. XVIII, Th. 2.9]).

Then the main statement of this subsection is the following:

Theorem 5.4.2. *Let $\overline{X}$ be a smooth proper rigid-analytic curve over C and let $X \subset \overline{X}$ be a quasi-compact open affinoid subspace. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^2_c(X, \mu_n) \arrow[rr, "\mathrm{can}"] \arrow[dr, "t_X"'] & & \mathrm{H}^2(\overline{X}, \mu_n) \arrow[dl, "t^{\mathrm{alg}}_{\overline{X}}"] \\ & \mathbf{Z}/n\mathbf{Z} & \end{tikzcd}$$

Before we start the proof of Theorem 5.4.2, we record an application of the statement.

Corollary 5.4.3. *Let $f \colon X \to Y$ be an étale morphism of smooth affinoid curves over C. Then the diagram*

$$\begin{tikzcd} \mathrm{H}^2_c(X, \mu_n) \arrow[r, "t_X"] \arrow[d, "\mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_f(1))"'] & \mathbf{Z}/n \\ \mathrm{H}^2_c(Y, \mu_n) \arrow[ur, "t_Y"'] & \end{tikzcd}$$

commutes, where t_X and t_Y denote the analytic traces from Definition 5.1.10 and $\mathrm{tr}^{\text{ét}}_f(1)$ is the étale trace from Definition 2.5.10 and Notation 2.5.11.

Proof. First, when f is finite étale, the statement follows from Theorem 5.2.7 and property (5) in Theorem 2.5.6. Next, we contemplate the case where f is an open immersion. By Proposition 4.1.6. (iii), we may choose an open immersion $g \colon Y \hookrightarrow Z$ into a smooth proper curve Z over C. Consider the following diagram:

$$\begin{tikzcd} \mathrm{H}^2_c(X, \mu_n) \arrow[r, "\mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_f(1))"] \arrow[rrr, bend right, "t_X"'] & \mathrm{H}^2_c(Y, \mu_n) \arrow[r, "\mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_g(1))"] \arrow[rr, bend right, "t_Y"] & \mathrm{H}^2_c(Z, \mu_n) \arrow[r, "t^{\mathrm{alg}}_Z"] & \mathbf{Z}/n \end{tikzcd}$$

Since both X and Y are quasicompact open affinoid subspaces of Z via $g \circ f$ and g, respectively, and $\mathrm{H}^2_c\left(\mathrm{tr}^{\text{ét}}_g(1)\right) \circ \mathrm{H}^2_c\left(\mathrm{tr}^{\text{ét}}_f(1)\right) = \mathrm{H}^2_c\left(\mathrm{tr}^{\text{ét}}_{g\circ f}(1)\right)$, Theorem 5.4.2 gives

$$t_Y \circ \mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_f) = t^{\mathrm{alg}}_Z \circ \mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_g(1)) \circ \mathrm{H}^2_c(\mathrm{tr}^{\text{ét}}_f(1)) = t_X.$$

Now we can treat the general case. By [Hub96, Lem. 2.2.8] (cf. also [dJvdP96, Prop. 3.1.4]), we can choose a finite open affinoid cover $Y = \bigcup_{j\in J} V_j$ and factorizations

$$\begin{tikzcd} f^{-1}(V_j) \arrow[r, "h_j"] \arrow[dr, "f"'] & X_j \arrow[d, "g_j"] \\ & V_j \end{tikzcd}$$

such that the h_j are open immersions and the g_j are finite étale. They give rise to the diagram

$$\begin{tikzcd} \bigoplus_{j\in J} \mathrm{H}^2_c(f^{-1}(V_j),\mu_n) \arrow[r] \arrow[dr] & \mathrm{H}^2_c(X,\mu_n) \arrow[r, "t_X"] \arrow[d] & \mathbf{Z}/n \\ & \mathrm{H}^2_c(Y,\mu_n) \arrow[ur, "t_Y"] & \end{tikzcd}$$

in which all unlabeled arrows are the étale traces given by the respective counits of adjunction. The compatiblity of adjunction counits under composition then guarantees that the left triangle commutes. Moreover, the top horizontal composition is given by $\bigoplus_j t_{f^{-1}(V_j)}$ thanks to the already established case of open immersions.

The natural map $\bigoplus_{j\in J} i_{j,!}\mu_n \to \mu_n$ induced by the open cover $\{i_j\colon f^{-1}(V_j) \hookrightarrow X\}_{j\in J}$ is an epimorphism. Since $\mathrm{H}^2_c(f^{-1}(V_j),-) \simeq \mathrm{H}^2_c\big(X, i_{j,!}(-)\big)$ and $\mathrm{H}^2_c(X,-)$ is right exact [Hub96, Prop. 5.5.6, Prop. 5.5.8], the top left horizontal arrow in the diagram above is then also an epimorphism. To prove that the right triangle commutes, it therefore suffices to show that the outer triangle commutes, which can be checked on each factor $\mathrm{H}^2_c(f^{-1}(V_j),\mu_n)$ separately. Since $f^{-1}(V_j) \to Y$ factors into a composition of open immersions and finite étale morphisms, this follows from the first paragraph. $\square$

5.5. **Compatibility with the algebraic trace map. Closed unit disk.** The main goal of this subsection is to prove Theorem 5.4.2 in the case of the standard open immersion $\mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$. We recall that Proposition 5.1.5 gives some control over $\mathrm{H}^i_c(\mathbf{D}^1,\mu_n)$; however, the main drawback of this description is that it seems difficult to relate $\mathrm{H}^2_c(\mathbf{D}^1,\mu_n)$ to the cohomology of $\mathbf{P}^{1,\mathrm{an}}$.

For this reason, we take a different approach in this subsection and relate the compactly supported cohomology of $\mathbf{D}^1$ and the cohomology of $\mathbf{P}^{1,\mathrm{an}}$ directly. An explicit understanding of their difference will also be the key input in our proof of Theorem 5.4.2.

5.5.1. *Preliminaries. The ring* $A(Z)$. To start the proof, we denote by $j\colon \mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$ the usual open immersion and by $Z \coloneqq |\mathbf{P}^{1,\mathrm{an}}| \smallsetminus |\mathbf{D}^1|$ the closed complement of $\mathbf{D}^1$ inside $\mathbf{P}^{1,\mathrm{an}}$. This space does *not* admit any structure of an analytic adic space; instead, we consider Z as a pseudo-adic space $(\mathbf{P}^{1,\mathrm{an}}, Z)$ (see Appendix B) which we will, by abuse of notation, simply call Z. Remark B.1.9 implies that we have an exact triangle

$$\mathrm{R}\Gamma_c(\mathbf{D}^1,\mu_n) \to \mathrm{R}\Gamma(\mathbf{P}^{1,\mathrm{an}},\mu_n) \to \mathrm{R}\Gamma(Z,\mu_n). \tag{5.5.1}$$

To put it plainly, the difference between $\mathrm{R}\Gamma_c(\mathbf{D}^1,\mu_n)$ and $\mathrm{R}\Gamma(\mathbf{P}^{1,\mathrm{an}},\mu_n)$ is exactly controlled by the étale cohomology of Z.

To study these cohomology groups, we need to study the geometry of j in more detail. For this, we view $\mathbf{P}^{1,\mathrm{an}}$ as the glueing of two closed unit disks[19]

$$\mathbf{D}^1(0) = \mathrm{Spa}\left(C[T], \mathcal{O}_C[T]\right) \text{ and } \mathbf{D}^1(\infty) = \mathrm{Spa}\left(C[S], \mathcal{O}_C[S]\right)$$

along the torus

$$\mathrm{Spa}\left(C[T^{\pm 1}], \mathcal{O}_C[T^{\pm 1}]\right) \simeq \mathrm{Spa}\left(C[S^{\pm 1}], \mathcal{O}_C[S^{\pm 1}]\right)$$

via $T = S^{-1}$. Inclusion j just becomes the inclusion $\mathbf{D}^1(0) \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$. The complement of this inclusion is a special closed subset (see Example B.1.4)

$$Z = \mathbf{D}^1(\infty)\big(|S| < 1\big) \subset \mathbf{D}^1(\infty).$$

Now, when we realize Z as a special closed subset inside an affinoid $\mathbf{D}^1(\infty)$, Theorem B.3.5 (see also Definition B.3.1) and [Hub96, Cor. 2.3.8] ensure that there are canonical isomorphisms

$$\mathrm{R}\Gamma(Z,\mu_n) \simeq \mathrm{R}\Gamma((\mathbf{D}^1(\infty), Z),\mu_n) \simeq \mathrm{R}\Gamma(\mathrm{Spec}\, A(Z),\mu_n),$$

where $A(Z) = \mathcal{O}_C[S]^{\mathrm{h}}_{(\varpi,S)}\left[\frac{1}{\varpi}\right]$. For the notational convenience, we introduce the following notation:

Notation 5.5.2. We put $A(Z)^+ \coloneqq \mathcal{O}_C[S]^{\mathrm{h}}_{(\varpi,S)}$ and $A(Z) \coloneqq A(Z)^+\left[\frac{1}{\varpi}\right]$.

---

[19] In the formulas below, we endow $\mathcal{O}_C[T]$ and $\mathcal{O}_C[S]$ with the ϖ-adic topology.

So we reduce the question of studying cohomology of Z to the question of understanding alegbraic cohomology $\mathrm{R}\Gamma(\operatorname{Spec} A(Z), \mu_n)$. For this, we start we establishing certain algebraic properties of the ring Z. We stary by studying the Picard group of $A(Z)$:

Proposition 5.5.3. *Let $A(Z)$ be as above. Then* $\operatorname{Pic}(A(Z)) = 0$.

Proof. We recall that the ring $A(Z)^+ = \mathcal{O}_C[S]^{\mathrm{h}}_{(\varpi,S)}$ is a filtered colimit of rings which are étale over $\mathcal{O}_C[S]$. Since any rank-1 projective module L on $A(Z) = A(Z)^+[\frac{1}{\varpi}]$ must come from a rank-1 projective module on the $[\frac{1}{\varpi}]$-fibre of one of these algebras in the filtered colimit, we summon Lemma 4.5.2 to see that L must be the base change of a rank-1 projective module on $A(Z)^+$. So it suffices to show that $\operatorname{Pic}(A(Z)^+) = 0$. Now [Sta22, Tag 0F0L] implies that $A(Z)^+ = \mathcal{O}_C[S]^{\mathrm{h}}_{(\varpi,S)} \simeq \mathcal{O}_C[S]^{\mathrm{h}}_{(\mathfrak{m}_C,S)}$. But the latter algebra is local, hence its Picard group vanishes. $\square$

Our next goal is to understand the unit group $A(Z)^\times$. For this, we will need some preliminary lemmas:

Lemma 5.5.4. *The ring $A(Z)^+$ is ϖ-adically separated, i.e. $\bigcap_{n\geq 1} \varpi^n A(Z)^+ = \{0\}$. In particular, every nonzero element $f \in A(Z)$ can be scaled by a power of ϖ so that it lies in $A(Z)^+ \smallsetminus \varpi \cdot A(Z)^+$.*

Proof. We note that [Sta22, Tag 0F0L] ensures that $A(Z)^+$ is isomorphic to the $(\mathfrak{m}_C, S)$-adic henselization of $\mathcal{O}_C[S]$. In particular, $A(Z)^+$ is a local ring and can be written as a filtered colimit $A(Z)^+ = \operatorname{colim}_{i\in I} B_i$ such that each B_i is the localization of an étale $\mathcal{O}_C[S]_{(\mathfrak{m}_C,S)}$-algebra at a maximal ideal lying over the maximal ideal of $\mathcal{O}_C[S]_{(\mathfrak{m}_C,S)}$.

Suppose $0 \neq f \in \cap_{n\geq 1}\varpi^n A(Z)^+$, then f comes from an element $0 \neq f_i \in B_i$ for some $i \in I$. Since $B_i \to A(Z)^+$ is faithfully flat, we conclude that $\varpi^n A(Z)^+ \cap B_i = \varpi^n B_i$. Therefore, $0 \neq f_i \in \cap_{n\geq 1}\varpi^n B_i$. So it suffices to show that each B_i is ϖ-adically separated. We pick one and rename it as B.

Now B is a localization of an étale $\mathcal{O}_C[S]$-algebra, so $B\left[\frac{1}{\varpi}\right]$ is noetherian. Therefore, [FK18, Cor. 0.9.2.7 and Prop. 0.8.5.10] imply that B is ϖ-adically adhesive (see [FK18, Def. 0.8.5.1]). We note that $J \coloneqq \cap_{n\geq 1}\varpi^n B$ is a saturated ideal of B (because ϖ is a non-zero divisor in B). Thus [FK18, Prop. 0.8.5.3(c)] implies that J is finitely generated. Since also $J = \varpi \cdot J$ and ϖ lies inside the maximal ideal of B_i, Nakayama's lemma [Sta22, Tag 00DV] ensures that $J = 0$.

For the last sentence, first let us scale f by multiples of ϖ so that it lies in $A(Z)^+$. Then $\max\{n \mid f \in \varpi^n \cdot A(Z)^+\}$ exists because $A(Z)^+$ is ϖ-adically separated. Denote this number by n_0, then $\varpi^{-n_0} f$ does the job. $\square$

The following lemma is certainly well-known to the experts, however, it seems difficult to find a reference in the existing literature. For this reason, we spell out the proof below:

Lemma 5.5.5. *Let $\pi \in \mathcal{O}_C$ be a pseudo-uniformizer such that $\mathcal{O}_C/(\pi)$ shares the same characteristic*[20] *as the residue field k_C. Then the natural surjection $\rho\colon \mathcal{O}_C/(\pi) \to k_C$ admits a section.*

Proof. Let $\mathbf{F} \subset k_C$ be the prime field, [Sta22, Tag 030F] implies that we can choose a set of transcendental basis $\{x_i\}_{i\in I}$, so $\mathbf{F}(\underline{x}) \subset k_C$ is an algebraic extension. Let A be the perfection[21] of $\mathbf{F}[\underline{x}]$ which can be realized as a $\mathbf{F}[\underline{x}]$-subalgebra inside k_C. This induces a further inclusion $\operatorname{Frac}(A) \subset k_C$ of A-algebras.

We choose some lifts $\widetilde{x}_i \in \mathcal{O}_C/(\pi)$ of $x_i \in k_C$. This defines a morphism $\alpha\colon \mathbf{F}[\underline{x}] \to \mathcal{O}_C/(\pi)$ such that $\rho \circ \alpha$ is equal to the natural inclusion $\mathbf{F}[\underline{x}] \hookrightarrow k_C$. Now we wish to construct morphisms β, γ, and δ such that the diagram

$$\begin{tikzcd}
 & & & \mathcal{O}_C/(\pi) \arrow[d, "\rho"'] \\
\mathbf{F}[\underline{x}] \arrow[r, hook] \arrow[urrr, "\alpha"] & A \arrow[r, hook] \arrow[urr, dotted, "\beta"] & \operatorname{Frac}(A) \arrow[r, hook] \arrow[ur, dotted, "\gamma"] & k_C \arrow[u, dotted, bend right, "\delta"']
\end{tikzcd}$$

commutes and $\rho \circ \delta = \mathrm{id}$. We do this step by step.

First, we extend α to a morphism β. This is only an issue when k_C has characteristic p, in which case $A = \mathbf{F}[\underline{x}^{1/p^\infty}]$ and $\mathcal{O}_C/(\pi)$ is a semi-perfect algebra. Therefore, we can choose compatible p-power roots $\widetilde{x}_{i,r}$

[20]This condition is only relevant in the mixed characteristic situation, in which case we are just saying $(p) \subset (\pi)$.

[21]If $\operatorname{char} k_C = p > 0$, then $A = \mathbf{F}[\underline{x}^{1/p^\infty}]$. If $\operatorname{char} k_c = 0$, then we put $A = \mathbf{F}[\underline{x}]$.

of $\widetilde{x}_i$ and let β to be the unique ring homomorphism that sends x_i^{1/p^r} to $\widetilde{x}_{i,r}$. Then, in order to extend β to γ, we need to check that $\beta(a) \in \big(\mathcal{O}_C/(\pi)\big)^\times$ for each nonzero $a \in A$. This follows from the observation that the kernel of ρ is locally nilpotent and $\rho \circ \beta$ is the natural inclusion $A \hookrightarrow k_C$. Thus, β admits a unique extension γ.

Finally, we construct δ. For this, we notice that $\mathrm{Frac}(A)$ is a perfect field, so the algebraic extension $\mathrm{Frac}(A) \subset k_C$ is ind-étale. Now the maximal ideal in $\mathcal{O}_C/(\pi)$ is locally nilpotent, hence by [Sta22, Tag 0ALI] we know it is a henselian local ring. Finally applying [Sta22, Tag 08HR] with the $(R \to S, R \to A)$ there being our $(\mathrm{Frac}(A) \to \mathcal{O}_C/(\pi), \mathrm{Frac}(A) \subset k_C)$ here, we see that the section δ exists and uniquely depends on γ. ☐

Finally, we are ready to get the desired control over the units in $A(Z)$, in analogy with Lemma 5.1.13:

Lemma 5.5.6. *We have an equality*

$$A(Z)^\times = C^\times \cdot A(Z)^{+,\times}.$$

Proof. We first choose a pseudo-uniformizer $\pi \in \mathcal{O}_C$ such that $\pi \mid p$ if $\mathcal{O}_C$ is of mixed characteristic $(0,p)$.

Now we claim that $\min_{c\in C}\big\{|c| \mid f/c \in A(Z)^+\big\}$ exists for any nonzero $f \in A(Z)$. First, Lemma 5.5.4 implies that we may replace f by f/π^N for some N to assume that $f \in A(Z)^+ \smallsetminus \pi A(Z)^+$. Therefore in order to show that the desired minimum exists, it suffices to show that $A(Z)^+/(\pi)$ is a free $\mathcal{O}_C/(\pi)$-modules.

For this, we choose a section $k_C \to \mathcal{O}_C/(\pi)$ of the natural projection $\mathcal{O}_C/(\pi) \to k_C$, which exists due to Lemma 5.5.5. Since the maximal ideal of $\mathcal{O}_C/(\pi)$ is locally nilpotent, we conclude that the section $k_C \to \mathcal{O}_C/(\pi)$ is integral. Therefore, [Sta22, Tag 0DYE] implies that

$$A(Z)^+/(\pi) = \mathcal{O}_C[S]^{\mathrm{h}}_{(\pi,S)}/(\pi) \simeq \big(\mathcal{O}_C/(\pi)[S]\big)^{\mathrm{h}}_{(S)} \simeq \mathcal{O}_C/(\pi) \otimes_{k_C} k_C[S]^{\mathrm{h}}_{(S)}.$$

Since any k_C-module is free, we conclude that $A(Z)^+/(\pi)$ is a free $\mathcal{O}_C/(\pi)$-module as well.

Now let us define a function $|.|_\eta \colon A(Z) \to \Gamma_C \cup \{0\}$ by the rule

$$|f|_\eta = \min_{c\in C}\big\{|c| \mid f/c \in A(Z)^+\big\}.$$

A standard argument using that $A(Z)^+/\mathfrak{m}_C A(Z)^+ \simeq k_C[S]^{\mathrm{h}}_{(S)}$ is a domain shows that $|.|_\eta$ is multiplicative (see [Bos14, p. 13] for a version of this argument). Now let $f \in A(Z)^\times$, we choose some $c \in C$ such that $|c| = |f|_\eta$. It suffices to show that $f' = f/c$ is a unit in $A(Z)^+$. By construction, $|f'|_\eta = 1$ and f' is invertible in $A(Z)$. Thus, multiplicativity of $|.|_\eta$ implies that $|f'^{-1}|_\eta = 1$ so $(f')^{-1} \in A(Z)^+$ finishing the proof. ☐

Corollary 5.5.7. *We have*

$$\mathrm{H}^i(Z,\mu_n) \simeq \mathrm{H}^i(\mathrm{Spec}\, A(Z),\mu_n) \simeq \begin{cases} \mu_n(C) \cong \mathbf{Z}/n\mathbf{Z} & i=0 \\ A(Z)^\times/(A(Z)^\times)^n = A(Z)^{+,\times}/(A(Z)^{+,\times})^n & i=1 \\ 0 & i\geq 2 \end{cases}.$$

Proof. The proof is essentially the same as that of Proposition 5.1.2. One uses that $A(Z)$ is ind-étale over $C[S]$ and the Artin-Grothendieck vanishing theorem (see [Sta22, Tag 0F0V]) to get vanishing in higher degrees. Then one uses the Kummer exact sequence and Proposition 5.5.3 to get the calculation in lower degrees. ☐

Corollary 5.5.8. *We have $\mathrm{H}^i_c(\mathbf{D}^1,\mu_n) = 0$ for $i \neq 2$ and a natural exact sequence*

$$0 \to A(Z)^\times/(A(Z)^\times)^n \to \mathrm{H}^2_c(\mathbf{D}^1,\mu_n) \to \mathrm{H}^2(\mathbf{P}^{1,\mathrm{an}},\mu_n) \to 0$$

Proof. The first claim follows directly from Proposition 5.1.5 and Corollary 5.1.7. The second claim follows directly from (5.5.1), Proposition 4.1.6 (vi), and Corollary 5.5.7. ☐

5.5.2. *Beginning of the proof.* In this subsubsection, we show that Theorem 5.4.2 holds for the open immersion $j \colon \mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$ up to an invertible constant $\lambda \in \mathbf{Z}/n\mathbf{Z}^\times$. In the next subsubsection, we will show that this constant must be 1 due to some cycle class considerations.

We start the proof by relating the short exact sequence in Corollary 5.5.8 to the one in Proposition 5.1.5. For this, we recall that Z can be realized as the closed subset of $\mathbf{D}^1(\infty)\big(|S| < 1\big) \subset \mathbf{D}^1(\infty)$ and we denote by x_+ the unique (rank-2) point of $|\mathbf{D}^{1,c}(0)| \smallsetminus |\mathbf{D}^1(0)|$ (see Lemma 4.2.2) . We now consider the following commutative diagram[22] of pseudo-adic spaces:

$$\begin{tikzcd}
\left(\operatorname{Spa}\left(\widehat{k(x_+)}, \widehat{k(x_+)}^+\right), x_+\right) \arrow[r] \arrow[d, "\alpha"] & \left(\mathbf{D}^{1,c}(0), x_+\right) \arrow[d, "\beta"] \\
\left(\mathbf{D}^1(\infty), Z\right) \arrow[r] & \left(\mathbf{P}^{1,\mathrm{an}}, Z\right).
\end{tikzcd} \tag{5.5.9}$$

By [Hub96, Prop. 2.3.7], the horizontal morphisms are equivalences on the associated étale topoi, so they do not change cohomology.

Now the morphism α (due to Example B.3.3) induces the natural morphism $A(Z) \to \widehat{k(x_+)}^{\mathrm{h}}$ such that the image of $A(Z)^+$ lands inside $\widehat{k(x_+)}^{+,\mathrm{h}}$. After inverting ϖ, we denote the induced morphism by

$$\operatorname{Res}\colon A(Z) \to \widehat{k(x_+)}^{\mathrm{h}}.$$

Proposition 5.5.10. *There is a commutative diagram between two natural exact sequences*

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & A(Z)^\times/(A(Z)^\times)^n \arrow[r] \arrow[d, "\mathrm{Res}"] & \mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}^2(\mathbf{P}^{1,\mathrm{an}}, \mu_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C\langle T\rangle^\times/(C\langle T\rangle^\times)^n \arrow[r] & \widehat{k(x_+)}^{\mathrm{h},\times}/\left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^n \arrow[r] & \mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \arrow[r] & 0. &
\end{tikzcd}$$

Proof. The upper short exact sequence comes from Corollary 5.5.8. The lower short exact sequence comes from Proposition 5.1.2, Lemma 5.1.12, Proposition 5.1.5, and Corollary 5.1.7. Now the diagram of étale topoi below

$$\begin{tikzcd}
\mathbf{D}^1_{\text{ét}} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbf{D}^{1,c}_{\text{ét}} \arrow[d, "\mathrm{incl}"] & \left[\left(\mathbf{D}^{1,c}, x_+\right)_{\text{ét}} \arrow[l] \arrow[d, "\alpha"] & \left(\operatorname{Spa}\left(\widehat{k(x_+)}, \widehat{k(x_+)}^+\right), x_+\right)_{\text{ét}}\right] \arrow[l, "\sim"'] \arrow[d, "\beta"] \\
\mathbf{D}^1_{\text{ét}} \arrow[r] & \mathbf{P}^{1,\mathrm{an}}_{\text{ét}} & \left[\left(\mathbf{P}^{1,\mathrm{an}}, Z\right)_{\text{ét}} \arrow[l] & \left(\mathbf{D}^1_2, Z\right)_{\text{ét}}\right]. \arrow[l, "\sim"']
\end{tikzcd}$$

and Theorem B.3.5 gives rise to the following commutative diagram of exact triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma_c(\mathbf{D}^1, \mu_n) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{R}\Gamma(\mathbf{P}^{1,\mathrm{an}}, \mu_n) \arrow[r] \arrow[d, "\mathrm{res}"] & \mathrm{R}\Gamma(\operatorname{Spec} A(Z), \mu_n) \arrow[d, "\mathrm{res}"] \\
\mathrm{R}\Gamma_c(\mathbf{D}^1, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathbf{D}^{1,c}, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\operatorname{Spec} \widehat{k(x_+)}^{\mathrm{h}}, \mu_n).
\end{tikzcd}$$

This implies the desired commutative diagram by passing to cohomology and the observation that $\mathrm{R}\Gamma(\mathbf{D}^{1,c}, \mu_n) \simeq \mathrm{R}\Gamma(\mathbf{D}^1, \mu_n)$ (see Proposition 5.1.2). □

Now we are finally ready to start the proof of Theorem 5.4.2 for the open immersion $\mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$.

Lemma 5.5.11. *The composition* $A(Z)^\times \xrightarrow{\mathrm{Res}} \widehat{k(x_+)}^{\mathrm{h},\times} \xrightarrow{\# \circ v_{x_+}} \mathbf{Z}$ *is zero.*

Proof. By Lemma 5.5.6, it suffices to show that the composition is zero on $C^\times$ and $A(Z)^{+,\times}$. The composition is zero on $C^\times$ by construction. To deal with the elements of $A(Z)^{+,\times}$, we observe that Res maps them to the elements in $\left(\widehat{k(x_+)}^{+,\mathrm{h}}\right)^\times$ (see the discussion before Proposition 5.5.10). Therefore, the whole valuation v_{x_+} vanishes on these elements. □

[22] Here, we implicitly use [AGV22, Lem. 4.2.5 and Prop. 4.2.11] that ensures that $|\mathbf{D}^{1,c}|$ coincides with the topological closure of $\mathbf{D}^1$ inside $\mathbf{P}^{1,\mathrm{an}}$.

Corollary 5.5.12. *Let $j\colon \mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$ be the standard immersion. Then there is an invertible constant $\lambda \in (\mathbf{Z}/n\mathbf{Z})^\times$ such that the diagram*

$$\begin{tikzcd} \mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \arrow[rr, "\mathrm{can}"] \arrow[dr, "\lambda\cdot t_{\mathbf{D}^1}"'] & & \mathrm{H}^2(\mathbf{P}^{1,\mathrm{an}}, \mu_n) \arrow[dl, "t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}}"] \\ & \mathbf{Z}/n\mathbf{Z} & \end{tikzcd}$$

commutes.

Proof. First, we note that Remark 5.1.11 ensures that $t_{\mathbf{D}^1}$ is surjective. Furthermore, Proposition 5.5.10 ensures that can is surjective, while classical algebraic theory ensures that $t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}}$ is an isomorphism. This implies that $t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}} \circ \mathrm{can}$ is also surjective.

Now Proposition 5.5.10 and Lemma 5.5.11 imply that $t^{\mathrm{an}}_{\mathbf{D}^1}$ vanishes on the image of $A(Z)^\times$ in $\mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$. Therefore, Proposition 5.5.10 ensures that the analytic trace map factors through the surjection

$$\mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \xrightarrow{\mathrm{can}} \mathrm{H}^2(\mathbf{P}^{1,\mathrm{an}}, \mu_n) \simeq \mathbf{Z}/n\mathbf{Z}.$$

Since both $t_{\mathbf{D}^1}$ and $t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}} \circ \mathrm{can}$ are surjective and factor through can, we formally conclude that they must differ by an element $\mathrm{Aut}(\mathbf{Z}/n\mathbf{Z}) = (\mathbf{Z}/n\mathbf{Z})^\times$. This finishes the proof. □

5.5.3. *End of the proof.* In this subsubsection, we finally finish the proof of Theorem 5.4.2 in the case of the open immersion $j\colon \mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$.

We note that Corollary 5.5.12 implies that the only thing we are left to do is to pin down the constant λ. This will be done via cycle class considerations. For this, we recall that, for each classical point $a \in \mathcal{O}_C = \mathbf{D}^1(C)$, we can attach the localized cycle class in $\mathrm{H}^2_a(\mathbf{D}^1, \mu_n)$ and the compactly supported cycle class in $\mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$ (see Definition 3.1.4 and Definition 3.5.2 respectively). To clarify the exposition in this subsubsection, we denote the localized cycle class by $c\ell^{\mathrm{loc}}_{\mathbf{D}^1}(a) \in \mathrm{H}^2_a(\mathbf{D}^1, \mu_n)$ and the compactly supported cycle class by $c\ell_{\mathbf{D}^1}(a) \in \mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$; they are related via the natural map $\mathrm{H}^2_a(\mathbf{D}^1, \mu_n) \to \mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$.

In order to verify $\lambda = 1$, we will show that $t_{\mathbf{D}^1}$ and $t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}} \circ \mathrm{can}$ are both equal to 1 when evaluated on the cycle class of any point $a \in \mathbf{D}^1(C) = \mathcal{O}_C$.

Proposition 5.5.13. *Following the notation of Corollary 5.5.12, we have $t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}}\Big(\mathrm{can}\big(c\ell_{\mathbf{D}^1}(a)\big)\Big) = 1$ for any $a \in \mathcal{O}_C = \mathbf{D}^1(C)$.*

Proof. Lemma 3.4.1 implies that it suffices to show that $t_{\mathbf{P}^1}\left(c\ell_{\mathbf{P}^1}(a)\right) = 1$, where $t_{\mathbf{P}^1}$ is the schematic trace map and $c\ell_{\mathbf{P}^1}$ is the schematic cycle class. This is classical and follows from the equality

$$t_{\mathbf{P}^1}\left(c\ell_{\mathbf{P}^1}(a)\right) = \deg \mathcal{O}_{\mathbf{P}^1}(a) = \deg \mathcal{O}_{\mathbf{P}^1}(1) = 1.$$ □

Now we compute the analytic trace map applied to $c\ell_{\mathbf{D}^1}(a)$. We start with the following preliminary lemma:

Lemma 5.5.14. *The cycle class $c\ell_{\mathbf{D}^1}(a)$ is the image of $(T-a)^{-1}$ under the map $\widehat{k(x_+)}^{\mathrm{h},\times}/\left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^n \xrightarrow{\partial_{\mathbf{D}^1}} \mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$ from Proposition 5.5.10.*

Proof. We consider the open immersion $j\colon \mathbf{D}^1 \smallsetminus \{a\} \hookrightarrow \mathbf{D}^1$ and the closed complement $i\colon \{a\} \hookrightarrow \mathbf{D}^1$. Then we apply [Sta22, Tag 05R0] to the following commutative diagram (with distinguished rows and columns)

$$\begin{tikzcd} i_*\mathrm{R}i^!\mu_n \arrow[r] \arrow[d] & \mu_n \arrow[r] \arrow[d] & \mathrm{R}j_*\mu_n \arrow[d] \\ i_*\mathrm{R}i^!\mathbf{G}_m \arrow[r] \arrow[d] & \mathbf{G}_m \arrow[r] \arrow[d] & \mathrm{R}j_*\mathbf{G}_m \arrow[d] \\ i_*\mathrm{R}i^!\mathbf{G}_m \arrow[r] & \mathbf{G}_m \arrow[r] & \mathrm{R}j_*\mathbf{G}_m \end{tikzcd}$$

to conclude that the diagram

$$\begin{tikzcd}
\mathrm{H}^0(\mathbf{D}^1 \smallsetminus \{a\}, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathrm{H}^1_a(\mathbf{D}^1, \mathbf{G}_m) \arrow[d] \\
\mathrm{H}^1(\mathbf{D}^1 \smallsetminus \{a\}, \mu_n) \arrow[r] & \mathrm{H}^2_a(\mathbf{D}^1, \mu_n)
\end{tikzcd}$$

is *anti*-commutative. Following Definition 3.1.4, we see that the localized cycle class $c\ell^{\mathrm{loc}}_{\mathbf{D}^1}(a) \in \mathrm{H}^2_a(\mathbf{D}^1, \mu_n)$ is the image of $(T-a)$ going through the right top corner. Hence if we instead go through the left bottom corner, it then becomes the image of $(T-a)^{-1}$.

We denote by $j^c \colon \mathbf{D}^1 \hookrightarrow \mathbf{D}^{1,c}$ the open immersion of $\mathbf{D}^1$ into its universal compactification, we denote its closed complement by $i^c \colon \{x_+\} \hookrightarrow \mathbf{D}^{1,c}$. We also denote by $\overline{j} \colon \mathbf{D}^{1,c} \smallsetminus \{a\} \hookrightarrow \mathbf{D}^{1,c}$ the natural open immersion, and its closed complement by $\overline{i} \colon \{a\} \hookrightarrow \mathbf{D}^{1,c}$. Then we have the following commutative diagram of pseudo-adic spaces:

$$\begin{tikzcd}
\{a\} \arrow[r, hook, "i"] \arrow[d, equal] & \mathbf{D}^1 \arrow[d, hook, "j^c"] & \mathbf{D}^1 \smallsetminus \{a\} \arrow[l, hook', "j"'] \arrow[d, hook] \\
\{a\} \arrow[r, hook, "\overline{i}"] \arrow[d, hook, "i"] & \mathbf{D}^{1,c} \arrow[d, equal] & \mathbf{D}^{1,c} \smallsetminus \{a\} \arrow[l, hook', "\overline{j}"'] \\
\mathbf{D}^1 \arrow[r, hook, "j"] & \mathbf{D}^{1,c} & \{x_+\}. \arrow[l, hook', "i^c"'] \arrow[u, hook]
\end{tikzcd}$$

This induces the following commutative diagram of distinguished triangles in $D(\mathbf{D}^{1,c}_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$:

$$\begin{tikzcd}
\mathrm{R}j^c_* \, i_* \, \mathrm{R}i^! \, j^{c,*} \mu_n \arrow[r] & \mathrm{R}j^c_* \, j^{c,*} \mu_n \arrow[r] & \mathrm{R}j^{c,*} \, \mathrm{R}j_* \, j^* \, j^{c,*} \mu_n \\
\overline{i}_* \mathrm{R}\overline{i}^! \mu_n \arrow[r] \arrow[u, "\wr"] \arrow[d] & \mu_n \arrow[r] \arrow[u] \arrow[d, "\mathrm{id}"] & \mathrm{R}\overline{j}_* \overline{j}^* \mu_n \arrow[u] \arrow[d] \\
j^c_! j^{c,*} \mu_n \arrow[r] & \mu_n \arrow[r] & i^c_* i^{c,*} \mu_n,
\end{tikzcd} \tag{5.5.15}$$

where the top left vertical map is an isomorphism due to the observation that $\mathrm{R}\overline{i}^! \simeq \mathrm{R}i^! j^{c,*}$ and $\mathrm{R}j^c_* \, i_* \simeq \overline{i}_*$. Now we apply the derived global sections to (5.5.15) to get the following commutative diagram of distinguished triangles:

$$\begin{tikzcd}
\mathrm{R}\Gamma_a(\mathbf{D}^1, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathbf{D}^1, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathbf{D}^1 \smallsetminus \{a\}, \mu_n) \\
\mathrm{R}\Gamma_a(\mathbf{D}^{1,c}, \mu_n) \arrow[r] \arrow[u, "\mathrm{res}", "\wr"'] \arrow[d] & \mathrm{R}\Gamma(\mathbf{D}^{1,c}, \mu_n) \arrow[r] \arrow[u, "\mathrm{res}", "\wr"'] \arrow[d, "\mathrm{id}"'] & \mathrm{R}\Gamma(\mathbf{D}^{1,c} \smallsetminus \{a\}, \mu_n) \arrow[u, "\wr", "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \\
\mathrm{R}\Gamma_c(\mathbf{D}^1, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathbf{D}^{1,c}, \mu_n) \arrow[r] & \mathrm{R}\Gamma(\{x_+\}, \mu_n).
\end{tikzcd} \tag{5.5.16}$$

(5.5.15) implies that the left top vertical map in (5.5.16) is an isomorphism. Furthermore, Proposition 5.1.2 ensures that the middle top vertical map is an isomorphism. Therefore, the same holds for the right top vertical map as well.

One checks easily that the composition of the left column agrees with the map $\mathrm{R}\Gamma_a(\mathbf{D}^1, \mu_n) \to \mathrm{R}\Gamma_c(\mathbf{D}^1, \mu_n)$ appeared before Definition 3.5.2 (with the c being 1). Using the Kummer exact sequence and (5.5.16), we get the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^0(\mathbf{D}^1 \smallsetminus \{a\}, \mathbf{G}_m) \arrow[r] & \mathrm{H}^1_{\text{ét}}(\mathbf{D}^1 \smallsetminus \{a\}, \mu_n) \arrow[r] & \mathrm{H}^2_{\{a\}}(\mathbf{D}^1, \mu_n) \\ \mathrm{H}^0(\mathbf{D}^{1,c} \smallsetminus \{a\}, \mathbf{G}_m) \arrow[r] \arrow[u, "\mathrm{res}", "\wr"'] \arrow[d, "\mathrm{res}"] & \mathrm{H}^1(\mathbf{D}^{1,c} \smallsetminus \{a\}, \mu_n) \arrow[r] \arrow[u, "\mathrm{res}", "\wr"'] \arrow[d, "\mathrm{res}"] & \mathrm{H}^2_{\{a\}}(\mathbf{D}^{1,c}, \mu_n) \arrow[u, "\mathrm{res}", "\wr"'] \arrow[d] \\ \mathrm{H}^0(\operatorname{Spec} \widehat{k(x_+)}^{\mathrm{h}}, \mathbf{G}_m) \arrow[r] & \mathrm{H}^1(\{x_+\}, \mu_n) \arrow[r] & \mathrm{H}^2_c(\mathbf{D}^1, \mu_n). \end{tikzcd} \tag{5.5.17}$$

Here the left squares are the boundary maps coming from the corresponding Kummer exact sequences, and the right squares are the corresponding boundary maps of the distinguished triangles in (5.5.16). In particular, using the inverse of the top right vertical map, the right column sends localized cycle class $c\ell^{\mathrm{loc}}_{\mathbf{D}^1}(a) \in \mathrm{H}^2_{\{a\}}(\mathbf{D}^1, \mu_n)$ to the compactly supported cycle class $c\ell_{\mathbf{D}^1}(a) \in \mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$.

Using first paragraph, we see that the compactly supported cycle class $c\ell_{\mathbf{D}_C}(\{a\})$ is the image of $(T-a)^{-1}$ under the composite of maps where we start at the top left corner of (5.5.17) and go right-right-down-down to the bottom right corner. Since the invertible function $(T-a)^{-1}$ on $\mathbf{D}^1 \smallsetminus \{a\}$ extends to the invertible function $(T-a)^{-1}$ on $\mathbf{D}^{1,c} \smallsetminus \{a\}$, (5.5.17) shows that $c\ell_{\mathbf{D}^1}(\{a\})$ can be obtained from $(T-a)^{-1} \in \mathrm{H}^0(\operatorname{Spec} \widehat{k(x_+)}^{\mathrm{h}}, \mathbf{G}_m) = \widehat{k(x_+)}^{\mathrm{h},\times}$ by composing the two bottom horizontal arrows. This finishes the proof. □

Corollary 5.5.18. *For any $a \in \mathcal{O}_C = \mathbf{D}^1(C)$, we have $t_{\mathbf{D}^1}\big(c\ell_{\mathbf{D}^1}(a)\big) = 1$.*

Proof. Using Lemma 5.5.14 and the definition of $t_{\mathbf{D}^1}$ (see Definition 5.1.10), we conclude that

$$t_{\mathbf{D}^1}\big(c\ell_{\mathbf{D}^1}(a)\big) = \# \circ v_{x_+}((T-a)^{-1}).$$

Using the explicit formula for v_{x_+} from Lemma 4.2.2 and the implicit negative sign in the definition of $\#$ (see Warning 2.2.9), we easily conclude that $t_{\mathbf{D}^1}\big(c\ell_{\mathbf{D}^1}(a)\big) = 1$ for any $a \in \mathcal{O}_C$. □

We have finally arrived at the following statement.

Theorem 5.5.19. *Following the notation of Corollary 5.5.12, we have $\lambda = 1$.*

Proof. Corollary 5.5.12 implies that it suffices to show that there exists an element $x \in \mathrm{H}^2_c(\mathbf{D}^1, \mu_n)$ such that

$$t_{\mathbf{D}^1}(x) = 1 = t_{\mathbf{P}^{1,\mathrm{an}}}(\mathrm{can}(x)). \tag{5.5.20}$$

Now we note that the combination of Proposition 5.5.13 and Corollary 5.5.18 ensures that (5.5.20) holds for $x = c\ell_{\mathbf{D}^1}(a)$ for any $a \in \mathcal{O}_C$. □

We recall that, for an algebraic smooth connected curve X over C, the cycle class of a point $c\ell_X(a) \in \mathrm{H}^2_c(X, \mu_n)$ generates $\mathrm{H}^2_c(X, \mu_n)$ and is independent of the point $a \in X(C)$. It is natural to wonder if the same thing happens for a smooth affinoid connected curves over C. We show that this hope fails already for the closed unit disk:

Lemma 5.5.21. *Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0,p)$. Let $a, b \in \mathbf{D}^1(C) = \mathcal{O}_C$ be two classical point with corresponding cycle classes $c\ell_{\mathbf{D}^1}(a), c\ell_{\mathbf{D}^1}(b) \in \mathrm{H}^2_c(\mathbf{D}^1, \mu_{p^r})$ for some integer r. Then*

(i) *If $|b-a| = 1$, then $c\ell_{\mathbf{D}^1}(a) \neq c\ell_{\mathbf{D}^1}(b)$.*
(ii) *If $|b-a| < |p^r \cdot (\zeta_p - 1)|$, then $c\ell_{\mathbf{D}^1}(a) = c\ell_{\mathbf{D}^1}(b)$.*

In particular, $\mathrm{H}^2_c(\mathbf{D}^1, \mu_p)$ is infinite and its cardinality is at least cardinality of the residue field $k_C \coloneqq \mathcal{O}_C/\mathfrak{m}_C$.

Proof. Since the statements are unchanged under automorphisms of $\mathbf{D}^1$, we can assume that $b = 0$. Then Lemma 5.5.14 ensures that $c\ell_{\mathbf{D}^1}(b) - c\ell_{\mathbf{D}^1}(a)$ is given by the image of $\frac{T-a}{T} = 1 - \frac{a}{T}$ under the map

$\widehat{k(x_+)}^{\mathrm{h},\times}/\left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^{p^r} \to \mathrm{H}^2_c(\mathbf{D}^1, \mu_{p^r})$. Proposition 5.1.5 and Lemma 5.1.13 imply that we have the exact sequence

$$\left(1+\mathfrak{m}_C T\langle T\rangle\right) \times \left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^{p^r} \longrightarrow \widehat{k(x_+)}^{\mathrm{h},\times} \longrightarrow \mathrm{H}^2_c(\mathbf{D}_C, \mu_{p^r}) \longrightarrow 0.$$

We are now reduced the question when $1-\frac{a}{T}$ comes from an element in $\left(1+\mathfrak{m}_C T\langle T\rangle\right) \times \left(\widehat{k(x_+)}^{\mathrm{h},\times}\right)^{p^r}$, depending on the norm of a.

The explicit description of the valuation v_{x_+} for the point x_+ (see Lemma 4.2.2 and Lemma 4.3.6. (ii)) implies that $1-\frac{a}{T} \in \left(\widehat{k(x_+)}^{+,\mathrm{h}}\right)^{\times}$ and $1+\mathfrak{m}_C T\langle T\rangle \subset \left(\widehat{k(x_+)}^{+,\mathrm{h}}\right)^{\times}$. Combining these two observations, we see that the question is now further reduced to when the element $1-\frac{a}{T}$ comes from an element in $\left(1+\mathfrak{m}_C T\langle T\rangle\right) \times \left(\widehat{k(x_+)}^{+,\mathrm{h}}\right)^{\times, p^r}$.

For statement (i), note that Lemma 4.3.6 (ii) and [Sta22, Tag 0DYE] imply that

$$\widehat{k(x_+)}^{+,\mathrm{h}}/\mathfrak{m}_C\widehat{k(x_+)}^{+,\mathrm{h}} \simeq \mathcal{O}^{\mathrm{h}}_{\mathbf{P}^1_{k_C},\infty} \simeq k_C[T^{-1}]^{\mathrm{h}}_{(T^{-1})}.$$

Under this quotient, the set $\left(1+\mathfrak{m}_C T\langle T\rangle\right)$ is projected to 1. Thus, it suffices to know that $1-\frac{\overline{a}}{T}$ is not a p-th power in $k_C[T^{-1}]^{\mathrm{h}}_{(T^{-1})}$ whenever $\overline{a} \neq 0$ in k_C, which can be seen once one further completes with respect to T^{-1}.

To prove statement (ii), we use that $\frac{1}{T} \in \widehat{k(x_+)}^{+}$ by virtue of Lemma 4.3.6 (ii). Therefore, it suffices to show that the power series

$$\left(1-\frac{a}{T}\right)^{p^{-r}} := \sum_{i\geq 0} \binom{p^{-r}}{i} \cdot \left(-\frac{a}{T}\right)^i = \sum_{i\geq 0} \left(\binom{p^{-r}}{i}(-a)^i\right) \cdot \left(\frac{1}{T}\right)^i$$

converges p-adically to an element in $\widehat{k(x_+)}^{+,\mathrm{h}}$. In other words, we need to show that the additive p-adic valuation $\mathrm{ord}_p\left(\binom{p^{-r}}{i}(-a)^i\right) \to \infty$ as $i \to \infty$. For this, we note that this p-adic valuation equals

$$\mathrm{ord}_p(a)i - ri - \mathrm{ord}_p(i!) = \mathrm{ord}_p(a)i - ri - \sum_{m\geq 1} \lfloor i/p^m \rfloor > \left(\mathrm{ord}_p(a) - r - \frac{1}{p-1}\right)i.$$

Our assumption on a implies that $\mathrm{ord}_p(a) > \mathrm{ord}_p\left(p^r \cdot (\zeta_p - 1)\right) = r + \frac{1}{p-1}$. Therefore, $\left(\mathrm{ord}_p(a) - r - \frac{1}{p-1}\right)i \to \infty$ as $i \to \infty$. This finishes the proof. $\square$

Remark 5.5.22. It would be interesting to know if the distance inequality in Lemma 5.5.21. (ii) is sharp. Another puzzle is whether these compactly supported cycle classes of points can generate the whole $\mathrm{H}^2_c(\mathbf{D}_C, \mu_{p^r})$.

5.6. **Compatibility with the algebraic trace map. General case.** The main goal of this subsection is to prove Theorem 5.4.2 in full generality. We recall that Theorem 5.5.19 already proves the result for the standard open immersion $\mathbf{D}^1 \hookrightarrow \mathbf{P}^{1,\mathrm{an}}$. In the general case, our strategy is to use the refined version of Noether normalization from Lemma 4.1.7 to reduce the general case to the case of the disk. In order to run these reductions, it will be convenient to introduce some definitions:

Definition 5.6.1. A *pointed semi-stable formal $\mathcal{O}_C$-curve* $(\mathscr{X}, \{x_1, \dots, x_n\})$ is a pair consisting of a rig-smooth connected semi-stable proper formal $\mathcal{O}_C$-curve $\mathscr{X}$ and a finite non-empty set of closed point $\{x_1, \dots, x_n\} \subset |\mathscr{X}| = |\mathscr{X}_s|$. We denote by $\mathscr{U}_{\underline{x}} \subset \mathscr{X}$ the unique open formal $\mathcal{O}_C$-subscheme with special fiber $\mathscr{U}_{\underline{x},s} = \mathscr{X}_s \smallsetminus \{x_1, \dots, x_n\}$.

Remark 5.6.2. We note that [Zav21b, Lem. B.12] ensures that $\mathscr{X}_\eta$ is connected for any pointed semi-stable formal $\mathcal{O}_C$-curve $(\mathscr{X}, \{x_1, \dots, x_n\})$. Furthermore, Proposition 4.1.6. (i) then implies that $\mathscr{U}_{\underline{x},\eta}$ is always affinoid.

Definition 5.6.3. A pointed semi-stable formal $\mathcal{O}_C$-curve $(\mathscr{X}, \{x_1, \dots, x_n\})$ is *trace-friendly* if the diagram

$$\begin{tikzcd} \mathrm{H}^2_c(\mathscr{U}_{\underline{x},\eta}, \mu_n) \arrow[rr, "\mathrm{can}"] \arrow[dr, "t_{\mathscr{U}_{\underline{x},\eta}}"'] & & \mathrm{H}^2(\mathscr{X}_\eta, \mu_n) \arrow[dl, "t^{\mathrm{alg}}_{\mathscr{X}_\eta}"] \\ & \mathbf{Z}/n\mathbf{Z} & \end{tikzcd}$$

commutes.

Example 5.6.4 (Theorem 5.5.19)**.** The pointed semi-stable curve $(\widehat{\mathbf{P}}^1_{\mathcal{O}_C}, \{\infty\})$ is trace-friendly.

Our first goal is to show that every pointed semi-stable curve is trace-friendly (Theorem 5.6.13). This will be the hardest part in our proof of Theorem 5.4.2. Before we start showing this claim, we need to introduce some further notation:

Notation 5.6.5. Let $(\mathscr{X}, \{x_1, \dots, x_n\})$ be a pointed semi-stable formal $\mathcal{O}_C$-curve. Then, for each smooth (resp. nodal) point x_i, let $Z_{x_i} \subset \mathscr{U}^c_{\underline{x},\eta}$ be the pseudo-adic space consisting of the unique rank-2 point $\{u_i\}$ in $\mathscr{U}^c_{\underline{x},\eta}$ (resp. the two rank-2 points $\{v_i\} \sqcup \{w_i\}$ in $\mathscr{U}^c_{\underline{x},\eta}$) from Remark 4.3.10.

We note that Lemma 4.3.6. (iv) together with Lemma B.1.12 and Theorem B.2.1 imply that $|\mathscr{U}^c_{\underline{x},\eta}| \smallsetminus |\mathscr{U}_{\underline{x},\eta}| = \bigsqcup_{i=1}^n Z_{x_i}$ and that

$$\begin{aligned} \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x},\eta} \smallsetminus \mathscr{U}_{\underline{x},\eta}, \mu_n) &\simeq \mathrm{R}\Gamma\Big(\bigsqcup_{i=1}^n Z_{x_i}, \mu_n\Big) \simeq \bigoplus_{i=1}^n \mathrm{R}\Gamma(Z_{x_i}, \mu_n) \\ &\simeq \Bigg(\bigoplus_{i \mid x_i \in \mathscr{X}_s^{\mathrm{sm}}} \mathrm{R}\Gamma\Big(\operatorname{Spec} \widehat{k(u_i)}^{\mathrm{h}}, \mu_n\Big)\Bigg) \oplus \Bigg(\bigoplus_{i \mid x_i \in \mathscr{X}_s^{\mathrm{sing}}} \mathrm{R}\Gamma\Big(\operatorname{Spec} \widehat{k(v_i)}^{\mathrm{h}}, \mu_n\Big) \oplus \mathrm{R}\Gamma\Big(\operatorname{Spec} \widehat{k(w_i)}^{\mathrm{h}}, \mu_n\Big)\Bigg), \end{aligned}$$

where we treat $\mathscr{U}^c_{\underline{x},\eta} \smallsetminus \mathscr{U}_{\underline{x}}$ and Z_{x_i} as pseudo-adic spaces inside $\mathscr{U}^c_{\underline{x},\eta}$. With this notation, the boundary morphism $\partial_{\mathscr{U}_{\underline{x},\eta}}$ from Proposition 5.1.5 has the form $\partial_{\mathscr{U}_{\underline{x},\eta}} \colon \bigoplus_{i=1}^n \mathrm{H}^1(Z_{x_i}, \mu_n) \to \mathrm{H}^2_c(\mathscr{U}_{\underline{x},\eta}, \mu_n)$.

The next two results form the core of our proof that every pointed semi-stable formal $\mathcal{O}_C$-curve is trace-friendly.

Lemma 5.6.6. *Let $(\mathscr{X}, \underline{x}) = (\mathscr{X}, \{x_1, \dots, x_n\})$ be a pointed semi-stable $\mathcal{O}_C$-curve. For each $1 \le m \le n$, let $(\mathscr{X}, \underline{x}') = (\mathscr{X}, \{x_1, \dots, x_m\})$ be a pointed semi-stable $\mathcal{O}_C$-curve obtained by forgetting the last $n - m$ marked points. Then the diagram*

$$\begin{tikzcd} \bigoplus_{i=1}^m \mathrm{H}^1(Z_{x_i}, \mu_n) \arrow[r, hook, "\mathrm{incl}"] \arrow[d, "\partial_{\mathscr{U}_{\underline{x}',\eta}}"] & \bigoplus_{i=1}^n \mathrm{H}^1(Z_{x_i}, \mu_n) \arrow[d, "\partial_{\mathscr{U}_{\underline{x},\eta}}"] \\ \mathrm{H}^2_c(\mathscr{U}_{\underline{x}',\eta}, \mu_n) & \mathrm{H}^2_c(\mathscr{U}_{\underline{x},\eta}, \mu_n), \arrow[l, "\mathrm{can}"] \end{tikzcd}$$

commutes, where incl *is the evident inclusion morphism and* can *is the canonical morphism coming from the open immersion $\mathscr{U}_{\underline{x},\eta} \subset \mathscr{U}_{\underline{x}',\eta}$.*

Proof. In this proof, all spaces are regarded as pseudo-adic spaces; see Appendix B for the necessary definitions and results. By the discussion before this lemma, we have canonical isomorphisms

$$\bigoplus_{i=1}^n \mathrm{H}^1(Z_{x_i}, \mu_n) \simeq \mathrm{H}^1(\mathscr{U}^c_{\underline{x},\eta} \smallsetminus \mathscr{U}_{\underline{x},\eta}, \mu_n), \text{ and } \bigoplus_{i=1}^m \mathrm{H}^1(Z_{x_i}, \mu_n) \simeq \mathrm{H}^1\big(\mathscr{U}^c_{\underline{x}',\eta} \smallsetminus \mathscr{U}_{\underline{x}',\eta}, \mu_n\big).$$

Step 1. We recall the definitions of $\partial_{\mathscr{U}_{\underline{x}',\eta}}$, $\partial_{\mathscr{U}_{\underline{x},\eta}}$, and can. Consider the following diagram of inclusions:

$$\begin{tikzcd} \mathscr{U}_{\underline{x},\eta} \arrow[r, hook] \arrow[d, hook] & \mathscr{U}_{\underline{x}',\eta} \arrow[d, hook] \\ \mathscr{U}^c_{\underline{x},\eta} \arrow[r, hook] & \mathscr{U}^c_{\underline{x}',\eta}. \end{tikzcd}$$

Using the triangles for both compositions from the top left to the bottom right, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x},\eta},\mu_n) \arrow[r] \arrow[d, equal] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x},\eta},\mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x},\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n) \arrow[r, "\delta_1"] & \mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x},\eta},\mu_n)[1] \arrow[d, equal] \\
\mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x},\eta},\mu_n) \arrow[r] \arrow[d, "\widetilde{\mathrm{can}}"'] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta},\mu_n) \arrow[r] \arrow[u] \arrow[d, equal] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n) \arrow[r, "\delta_2"] \arrow[u, "\alpha"] \arrow[d, "\beta"'] & \mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x},\eta},\mu_n)[1] \arrow[d, "\widetilde{\mathrm{can}}{[1]}"'] \\
\mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x}',\eta},\mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta},\mu_n) \arrow[r] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta},\mu_n) \arrow[r, "\delta_3"] & \mathrm{R}\Gamma_c(\mathscr{U}_{\underline{x}',\eta},\mu_n)[1]
\end{tikzcd} \tag{5.6.7}$$

By the very definition, we have

$$\partial_{\mathscr{U}_{\underline{x},\eta}} = \mathrm{H}^1\big(\delta_1\big), \quad \partial_{\mathscr{U}_{\underline{x}',\eta}} = \mathrm{H}^1\big(\delta_3\big), \quad \text{and} \quad \mathrm{can} = \mathrm{H}^2\big(\widetilde{\mathrm{can}}\big) = \mathrm{H}^1\big(\widetilde{\mathrm{can}}[1]\big).$$

Step 2. We express incl *in more geometric terms.* Next, in order to get our hands on the morphism incl, we need to understand the space $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}$ better.

First, we note that $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}$ decomposes as a set into the disjoint union $\big(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta}\big)\sqcup\big(\mathscr{U}_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}\big)$. Clearly, $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta}$ is closed in $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}$, but we claim that it is also open. Indeed, [Sta22, Tag 0903] ensures that it suffices to show that $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta}$ is stable under generalizations in $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}$. Equivalently, we need to prove that for each $i=1,\dots,m$, the (unique) rank-1 generalization of points in Z_{x_i} lies in $\mathscr{U}_{\underline{x},\eta}$. This follows from the fact that these rank-1 generalizations correspond to generic points of the special fiber (see Lemma 4.3.6).Therefore, we conclude that $\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}$ has a clopen decomposition $\big(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta}\big)\sqcup\big(\mathscr{U}_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta}\big)$.

Thanks to this decomposition, Lemma B.1.12 ensures that we have a canonical isomorphism

$$\mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n) \simeq \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta},\mu_n)\oplus\mathrm{R}\Gamma(\mathscr{U}_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n).$$

This implies that the map β from (5.6.7) admits a canonical section

$$\iota\colon \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta},\mu_n)\to\mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta},\mu_n)\oplus\mathrm{R}\Gamma(\mathscr{U}_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n)\simeq\mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n)$$

such that the composition $\alpha\circ\iota$ coincides with the map

$$\begin{tikzcd}
\bigoplus\limits_{i=1}^{m}\mathrm{R}\Gamma(Z_{x_i},\mu_n) \arrow[r, hook, "\widetilde{\mathrm{incl}}"] \arrow[d, phantom, "\wr"] & \bigoplus\limits_{i=1}^{n}\mathrm{R}\Gamma(Z_{x_i},\mu_n) \arrow[d, phantom, "\wr"] \\
\mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x}',\eta}\smallsetminus\mathscr{U}_{\underline{x}',\eta},\mu_n) \arrow[r, hook, "\alpha\circ\iota"] & \mathrm{R}\Gamma(\mathscr{U}^c_{\underline{x},\eta}\smallsetminus\mathscr{U}_{\underline{x},\eta},\mu_n),
\end{tikzcd}$$

where $\widetilde{\mathrm{incl}}$ is induced by the inclusion $\bigsqcup_{i=1}^m Z_{x_i}\to\bigsqcup_{i=1}^n Z_{x_i}$. In particular, we conclude that the composition

$$\mathrm{H}^1(\alpha)\circ\mathrm{H}^1(\iota)=\mathrm{incl},$$

where incl is from (5.6.7).

Step 3. Finish the proof. Now the result follows by applying H^1 to the following sequence of equalities:

$$\delta_3=\delta_3\circ\beta\circ\iota=\widetilde{\mathrm{can}}[1]\circ\delta_2\circ\iota=\widetilde{\mathrm{can}}[1]\circ\delta_1\circ\alpha\circ\iota. \qquad \square$$

Corollary 5.6.8. *Let $\mathscr{X}$ be a rig-smooth connected semi-stable proper formal $\mathcal{O}_C$-curve, and let $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_m\}$ be finite non-empty sets of closed points in $\mathscr{X}_s$. Then the pair $(\mathscr{X},\{x_1,\dots,x_n,y_1,\dots,y_m\})$ is trace-friendly if and only if both $(\mathscr{X},\{x_1,\dots,x_n\})$ and $(\mathscr{X},\{y_1,\dots,y_m\})$ are so.*

Proof. Let us denote by $\mathscr{U}_{\underline{x}}$ (resp. $\mathscr{U}_{\underline{y}}$, resp. $\mathscr{U}_{\underline{x,y}}$) the open that is the “complement” of $\{x_1,\dots,x_m\}$ (resp. $\{y_1,\dots,y_m\}$, resp. $\{x_1,\dots,x_m,y_1,\dots,y_n\}$) in $\mathscr{X}$. We also denote by Z_{x_i} (resp. Z_{y_i}) the pseudo-adic spaces from Notation 5.6.5 applied to $(\mathscr{X},\{x_1,\dots,x_n\})$ (resp. $(\mathscr{X},\{y_1,\dots,y_m\})$).

Then Lemma 5.6.6 implies that the following diagram

$$
\begin{tikzcd}
\bigoplus_{i=1}^{n} \mathrm{H}^1(Z_{x_i}, \mu_n) \arrow[r, "\mathrm{incl}_1"] \arrow[d, "\partial_{\mathscr{U}_{\underline{x},\eta}}"] & \bigoplus_{i=1}^{n} \mathrm{H}^1(Z_{x_i}, \mu_n) \oplus \bigoplus_{j=1}^{m} \mathrm{H}^1(Z_{y_j}, \mu_n) \arrow[d, "\partial_{\mathscr{U}_{\underline{x},\underline{y},\eta}}"] & \bigoplus_{j=1}^{m} \mathrm{H}^1(Z_{y_j}, \mu_n) \arrow[l, "\mathrm{incl}_2"'] \arrow[d, "\partial_{\mathscr{U}_{\underline{y},\eta}}"] \\
\mathrm{H}^2_c(\mathscr{U}_{\underline{x},\eta}, \mu_n) \arrow[dr, "\mathrm{can}_{\underline{x}}"'] & \mathrm{H}^2_c(\mathscr{U}_{\underline{x},\underline{y},\eta}, \mu_n) \arrow[l, "\mathrm{can}_1"'] \arrow[r, "\mathrm{can}_2"] \arrow[d, "\mathrm{can}_{\underline{x},\underline{y}}"] & \mathrm{H}^2_c(\mathscr{U}_{\underline{y},\eta}, \mu_n) \arrow[dl, "\mathrm{can}_{\underline{y}}"] \\
& \mathrm{H}^2(\mathscr{X}_\eta, \mu_n) &
\end{tikzcd} \tag{5.6.9}
$$

commutes. Since the top vertical arrows in (5.6.9) are surjective (see Proposition 5.1.5), and the images of incl_1 and incl_2 generate the group $\bigoplus_{i=1}^n \mathrm{H}^1(Z_{x_i}, \mu_n) \bigoplus \bigoplus_{j=1}^m \mathrm{H}^1(Z_{y_j}, \mu_n)$, we conclude that $(\mathscr{X}, \{x_1, \dots, x_n, y_1, \dots, y_m\})$ is trace-friendly if and only if

$$t^{\mathrm{alg}}_{\mathscr{X}_\eta} \circ \mathrm{can}_{\underline{x},\underline{y}}(f) = t_{\mathscr{U}_{\underline{x},\underline{y},\eta}}(f) \text{ for} \tag{5.6.10}$$

$$f \in \mathrm{Im}(\partial_{\mathscr{U}_{\underline{x},\underline{y},\eta}} \circ \mathrm{incl}_1) \text{ and} \tag{5.6.11}$$

$$f \in \mathrm{Im}(\partial_{\mathscr{U}_{\underline{x},\underline{y},\eta}} \circ \mathrm{incl}_2). \tag{5.6.12}$$

Using (5.6.9) and the definition of the analytic trace map (see Definition 5.1.10), we conclude that Equation (5.6.10) for f as in (5.6.11) is equivalent to $(\mathscr{X}, \{x_1, \dots, x_n\})$ being trace-friendly, while Equation (5.6.10) for f as in (5.6.12) is equivalent to $(\mathscr{X}, \{y_1, \dots, y_m\})$ being trace-friendly. Combining these results, we get that $(\mathscr{X}, \{x_1, \dots, x_n, y_1, \dots, y_m\})$ is trace-friendly if and only if both $(\mathscr{X}, \{x_1, \dots, x_n\})$ and $(\mathscr{X}, \{y_1, \dots, y_m\})$ are so. □

Theorem 5.6.13. *Let $(\mathscr{X}, \{x_1, \dots, x_n\})$ be a pointed semi-stable formal $\mathcal{O}_C$-curve. Then it is trace-friendly.*

Proof. By Corollary 5.6.8, it suffices to prove the claim after replacing $(\mathscr{X}, \{x_1, \dots, x_n\})$ with $(\mathscr{X}, \{x_1, \dots, x_n, x_{n+1}, \dots, x_m\})$ for any set of closed points $x_{n+1}, \dots, x_m \in \mathscr{X}_s$. Therefore, we may and do assume that each irreducible component of $\mathscr{X}_s$ contains at least one point from the set $\{x_1, \dots, x_n\}$. In this situation, Lemma 4.1.7 and Lemma 4.1.2 imply that we can find a finite flat morphism $h\colon \mathscr{X}_\eta \to \mathbf{P}^{1,\mathrm{an}}$ such that $h^{-1}(\mathbf{D}^1) = \mathscr{U}_{\underline{x},\eta}$. We denote by $h'\colon \mathscr{U}_{\underline{x},\eta} \to \mathbf{D}^1$ the restriction of h, and consider the diagram

$$
\begin{tikzcd}
\mathrm{H}^2_c(\mathscr{U}_{\underline{x},\eta}, \mu_n) \arrow[r, "\mathrm{can}_{\mathscr{X}_\eta}"] \arrow[d, "\mathrm{H}^2_c(\mathrm{tr}_{h'})"] & \mathrm{H}^2(\mathscr{X}_\eta, \mu_n) \arrow[r, "t^{\mathrm{alg}}_{\mathscr{X}_\eta}"] \arrow[d, "\mathrm{H}^2(\mathrm{tr}_h)"] & \mathbf{Z}/n\mathbf{Z}. \\
\mathrm{H}^2_c(\mathbf{D}^1, \mu_n) \arrow[r, "\mathrm{can}_{\mathbf{P}^1}"] & \mathrm{H}^2(\mathbf{P}^{1,\mathrm{an}}, \mu_n), \arrow[ur, "t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}}"'] &
\end{tikzcd} \tag{5.6.14}
$$

where $\mathrm{tr}_h \coloneqq \mathrm{tr}_{h,\mu_n}$ and $\mathrm{tr}_{h'} \coloneqq \mathrm{tr}_{h',\mu_n}$ are the finite flat trace maps from Theorem 2.5.6. Then Theorem 2.5.6. (3) implies that the left square in (5.6.14) commutes, while Theorem 2.5.6. (7) and [AGV71, Exp. XVIII, Th. 2.9 and Prop. 2.10] imply that the right triangle commutes. Therefore, we conclude that Theorem 5.5.19, Theorem 5.2.7, and (5.6.14) imply that

$$t^{\mathrm{alg}}_{\mathscr{X}_\eta} \circ \mathrm{can}_{\mathscr{X}_\eta} = t^{\mathrm{alg}}_{\mathbf{P}^{1,\mathrm{an}}} \circ \mathrm{can}_{\mathbf{P}^1} \circ \mathrm{H}^2_c(\mathrm{tr}_{h'}) = t_{\mathbf{D}^1} \circ \mathrm{H}^2_c(\mathrm{tr}_{h'}) = t_{\mathscr{U}_{\underline{x},\eta}}.$$

In other words, $(\mathscr{X}, \{x_1, \dots, x_n\})$ is trace-friendly. □

Finally, we are ready to give the full proof of Theorem 5.4.2:

Proof of Theorem 5.4.2. First, we can clearly assume that $\overline{X}$ is connected. Then Proposition 4.1.6 (iv) implies that $X \subset \overline{X}$ is the adic generic fiber of $\mathscr{X} \subset \mathscr{X}^c$, where $\mathscr{X}^c$ is a semistable connected proper $\mathcal{O}_C$-curve and $\mathscr{X}$ is an open formal subscheme of $\mathscr{X}^c$.

Now we consider the open subscheme $\mathscr{Y}_s := \mathscr{X}^c_s \smallsetminus \overline{\mathscr{X}_s}$, and denote the corresponding open formal $\mathcal{O}_C$-subscheme of $\mathscr{X}^c$ by $\mathscr{Y}$. We also denote its rigid generic fiber Y. By construction, $\mathscr{Y}$ and $\mathscr{X}$ are disjoint in $\mathscr{X}^c$ (so X and Y are disjoint as well), and $(\mathscr{Y} \coprod \mathscr{X})_s \subset \mathscr{X}^c_s$ is dense.

Now we note that $X \coprod Y = (\mathscr{X} \coprod \mathscr{Y})_\eta$ is affinoid due to Proposition 4.1.6. Therefore, validity of Theorem 5.4.2 for the inclusion $X \coprod Y \subset \overline{X}$ implies validity of Theorem 5.4.2 for both $X \subset \overline{X}$ and $Y \subset \overline{X}$. Therefore, we can replace X with $X \coprod Y$ to assume that $\mathscr{X}_s \subset \mathscr{X}^c_s$ is *dense*.

Let $(x_1, \dots, x_n)$ be the finite non-empty set of points of $|\mathscr{X}^c_s| \smallsetminus |\mathscr{X}_s|$. Then $(\mathscr{X}^c, \{x_1, \dots, x_n\})$ is a pointed semi-stable $\mathcal{O}_C$-curve and $\mathscr{U}_{\underline{x},\eta} = X$ (see Definition 5.6.1). Therefore, Theorem 5.4.2 for the inclusion $X \subset \overline{X}$ follows directly from Theorem 5.6.13. $\square$

6. The trace map for smooth morphisms

In this section, we discuss the trace map for separated taut smooth morphisms between locally noetherian analytic adic spaces for constant coefficients and use it to revisit the behavior of lisse complexes under smooth proper pushforwards. In Section 7, we will then deal with traces for dualizing complexes along maps of rigid-analytic spaces. Throughout, we fix a positive integer n and set $\Lambda := \mathbf{Z}/n$.

6.1. Construction.

We begin with the construction of smooth traces, loosely following the strategy in [AGV71, Exp. XVII] and [Ber93, § 7.2] by reducing to the case of curves. Even though our eventual Poincaré duality statement in Theorem 6.4.1 uses smooth proper morphisms, for our construction of the smooth trace it will be vital to allow nonproper morphisms as well.

Theorem 6.1.1. *There is a unique way to assign to any separated taut smooth of equidimension d morphism $f\colon X \to Y$ of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$ a trace map $\mathrm{tr}_f \colon \mathrm{R}f_!\, \underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$ of complexes on $Y_{\text{ét}}$, satisfying the following properties:*

1. *(compatibility with compositions) For any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ as above of equidimension d and e, respectively, the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{R}(g\circ f)_!\, \underline{\Lambda}_X(d+e)[2(d+e)] \arrow[rr, "\mathrm{tr}_{g\circ f}"] \arrow[d, "\wr"] & & \underline{\Lambda}_Z \\ \mathrm{R}g_!\Big((\mathrm{R}f_!\, \underline{\Lambda}_X(d)[2d]) \otimes \underline{\Lambda}_Y(e)[2e]\Big) \arrow[rr, "\mathrm{R}g_!\,(\mathrm{tr}_f(e)[2e])"] & & \mathrm{R}g_!\, \underline{\Lambda}_Y(e)[2e] \arrow[u, "\mathrm{tr}_g"'] \end{tikzcd}$$

2. *(compatibility with pullbacks) For any pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

   in which f and f' are separated taut smooth of equidimension d as above, the following diagram is commutative (with the top row induced by the base change map from [Hub96, Th. 5.4.6][23]*):*

$$\begin{tikzcd} g^*\mathrm{R}f_!\, \underline{\Lambda}_X(d)[2d] \arrow[r] \arrow[d, "g^*\mathrm{tr}_f"] & \mathrm{R}f'_!\, \underline{\Lambda}_{X'}(d)[2d] \arrow[d, "\mathrm{tr}_{f'}"] \\ g^*\underline{\Lambda}_Y \arrow[r, "\sim"] & \underline{\Lambda}_{Y'} \end{tikzcd}$$

3. *(compatibility with the étale traces from Definition 2.5.10) If f is étale, then tr_f is given by the counit*

$$\mathrm{R}f_!\, \underline{\Lambda}_X \simeq f_! f^* \underline{\Lambda}_Y \to \underline{\Lambda}_Y$$

   of the adjunction between $f_!$ and f^*.

[23] We warn the reader that the base change map is not always an isomorphism unless n is invertible in $\mathcal{O}^+$.

(4) *(compatibility with algebraic traces) If f is the structure morphism $\mathbf{P}^{1,\mathrm{an}}_C \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ for some complete, algebraically closed nonarchimedean field C, then tr_f is identified with the algebraic trace from Definition 5.4.1.*

For the general construction of trace maps, the following lemmas will turn out to be useful.

Lemma 6.1.2. *Let $f\colon X \to Y$ be a separated taut smooth of dimension d morphism between locally noetherian analytic adic space. Then $\mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d]$ lies in $D^{\leq 0}(Y_{\text{ét}}; \Lambda)$. In particular, $\mathrm{R}\mathscr{H}om(\mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d], \underline{\Lambda}_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$, and every morphism $\mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$ uniquely factors as the composition*

$$\mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d] \xrightarrow{\tau^{\geq 0}} \mathrm{R}^{2d}f_!\,\underline{\Lambda}_X(d) \to \underline{\Lambda}_Y.$$

Proof. Note that by [Hub96, Prop. 1.8.7.ii)] $\dim.\mathrm{tr}(f) = d$. The lemma now simply follows from the fact that $\mathrm{R}f_!$ has cohomological dimension $\leq 2d$; see [Hub96, Prop. 5.5.8]. □

Lemma 6.1.3 (Gluing trace maps locally on the source)**.** *Let $f\colon X \to Y$ be a separated taut smooth of dimension d morphism between locally noetherian analytic adic spaces. Let $X = \bigcup_{i\in I} U_i$ be an open cover for which the corresponding open immersions $U_i \hookrightarrow X$ are taut. For all $i, i' \in I$, let $j_i\colon U_i \hookrightarrow X$ and $j_{i'i}\colon U_{i,i'} := U_i \cap U_{i'} \hookrightarrow U_i$ be the natural open immersions. Let $\mathrm{tr}^{\text{ét}}_{j_{i'i}}$ be the corresponding étale traces in the sense of Definition 2.5.10 and let $f_i := f|_{U_i}$ be the restriction of f to U_i. Assume there exist maps $\tau_i\colon \mathrm{R}f_{i,!}\,\underline{\Lambda}_{U_i}(d)[2d] \to \underline{\Lambda}_Y$ such that for all $i, i' \in I$, the diagram*

$$\begin{tikzcd}
(\mathrm{R}f_{i,!} \circ j_{i'i,!})\underline{\Lambda}_{U_{i,i'}}(d)[2d] \arrow[r, "\mathrm{R}f_{i,!}(\mathrm{tr}^{\text{ét}}_{j_{i'i}}(d)[2d])"] \arrow[d, equal] & \mathrm{R}f_{i,!}\,\underline{\Lambda}_{U_i}(d)[2d] \arrow[dr, "\tau_i"] & \\
& & \underline{\Lambda}_Y \\
(\mathrm{R}f_{i',!} \circ j_{ii',!})\underline{\Lambda}_{U_{i',i}}(d)[2d] \arrow[r, "\mathrm{R}f_{i',!}(\mathrm{tr}^{\text{ét}}_{j_{ii'}}(d)[2d])"'] & \mathrm{R}f_{i',!}\,\underline{\Lambda}_{U_{i'}}(d)[2d] \arrow[ur, "\tau_{i'}"'] &
\end{tikzcd} \tag{6.1.4}$$

commutes. Then there exists a unique map $\tau\colon \mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$ such that for all $i \in I$ the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{R}f_{i,!}\,\underline{\Lambda}_{U_i}(d)[2d] \arrow[rr, "\mathrm{R}f_!\,(\mathrm{tr}^{\text{ét}}_{j_i}(d)[2d])"] \arrow[dr, "\tau_i"] & & \mathrm{R}f_!\,\underline{\Lambda}_X(d)[2d] \arrow[dl, "\tau"'] \\
& \underline{\Lambda}_Y &
\end{tikzcd}$$

Proof. For any finite subset $J \subseteq I$, let $U_J := \bigcap_{i\in J} U_i$ and $f_J := f|_{U_J}$ be the restriction of f to U_J. Note that giving a map $\mathrm{R}f_{J,!}\,\underline{\Lambda}_{U_J}(d)[2d] \to \underline{\Lambda}_Y$ is equivalent to giving a map $\mathrm{R}^{2d}f_{J,!}\,\underline{\Lambda}_{U_J}(d) \to \underline{\Lambda}_Y$ due to Lemma 6.1.2. By [Hub96, Rmk. 5.5.12.iii)], we have a spectral sequence

$$\mathrm{E}_1^{pq} := \bigoplus_{\substack{J\subseteq I \\ |J|=-p+1}} \mathrm{R}^q f_{J,!}\,\underline{\Lambda}_{U_J} \Longrightarrow \mathrm{R}^{p+q}f_!\,\underline{\Lambda}_Y.$$

Since the $\mathrm{R}f_{J,!}$ have cohomological dimension $\leq 2d$ [Hub96, Prop. 5.5.8], the associated abutment filtration for the antidiagonal $p + q = 2d$ reduces to an isomorphism

$$\mathrm{R}^{2d}f_!\,\underline{\Lambda}_X(d) \simeq \mathrm{coker}\Big(\bigoplus_{\{i,i'\}\subseteq I} \mathrm{R}^{2d}f_{ii',!}\,\underline{\Lambda}_{U_{i,i'}}(d) \to \bigoplus_{i\in I} \mathrm{R}^{2d}f_{i,!}\,\underline{\Lambda}_{U_i}(d)\Big)$$

and the τ_i assemble to a map $\tau\colon \bigoplus_{i\in I} \mathrm{R}^{2d}f_{i,!}\,\underline{\Lambda}_{U_i}(d) \to \underline{\Lambda}_Y$. On the other hand, the assumption on the commutativity of (6.1.4) guarantees that τ factors uniquely through the cokernel, so we win. □

Lemma 6.1.5. *Let Y be a locally noetherian analytic adic space and let $a, b\colon \mathcal{F} \to \mathcal{G}$ be two morphisms of étale sheaves on Y. Assume that $\mathcal{G}$ is overconvergent (in the sense of* [Hub96, Def. 8.2.1]*) and that $a_{\overline{\eta}} = b_{\overline{\eta}}$ for every geometric point $\overline{\eta}\colon \mathrm{Spa}\,(C_{\overline{\eta}}, \mathcal{O}_{C_{\overline{\eta}}}) \to Y$ of rank 1. Then $a = b$.*

Proof. Any $y \in |Y|$ has an associated geometric point ([Hub96, (2.5.2)])

$$\overline{y}\colon \left(\mathrm{Spa}\,(\widehat{\overline{k(y)}}, \widehat{\overline{k(y)}}^{+}), \{y\}\right) \to Y,$$

where the source is the (strongly) pseudo-adic space whose underlying topological space is the closed point $\{y\}$ of $\mathrm{Spa}\left(\widehat{\overline{k(y)}}, \widehat{\overline{k(y)}}^{+}\right)$. By [Hub96, Prop. 2.5.5], it suffices to check that the induced maps on stalks $a_{\overline{y}}, b_{\overline{y}}\colon F_{\overline{y}} \to G_{\overline{y}}$ coincide. The rank-1 generalization of $\overline{y}$ is given by the geometric point

$$\overline{\eta}\colon \mathrm{Spa}\left(\widehat{\overline{k(y)}}, \widehat{\overline{k(y)}}^{\circ}\right) \to Y.$$

The resulting specialization maps for $\mathcal{F}$ and $\mathcal{G}$ from [Hub96, (2.5.16)] fit into the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{\overline{y}} \arrow[r, shift left, "a_{\overline{y}}"] \arrow[r, shift right, "b_{\overline{y}}"'] \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] & \mathcal{G}_{\overline{y}} \arrow[d, "\mathrm{sp}_{\mathcal{G}}"] \\
\mathcal{F}_{\overline{\eta}} \arrow[r, shift left, "a_{\overline{\eta}}"] \arrow[r, shift right, "b_{\overline{\eta}}"'] & \mathcal{G}_{\overline{\eta}}
\end{tikzcd}$$

and the overconvergence assumption on G guarantees that sp_G is an isomorphism. Thus, we obtain the desired

$$a_{\overline{y}} = \mathrm{sp}_{\mathcal{G}}^{-1} \circ a_{\overline{\eta}} \circ \mathrm{sp}_{\mathcal{F}} = \mathrm{sp}_{\mathcal{G}}^{-1} \circ b_{\overline{\eta}} \circ \mathrm{sp}_{\mathcal{F}} = b_{\overline{y}}. \qquad \square$$

This finishes the sequence of preliminary lemmas. We are ready to show the uniqueness part of Theorem 6.1.1. For the proof, recall that for any locally noetherian analytic adic space Y, the d-dimensional unit disk over Y is defined as $\mathbf{D}^d_Y := \mathrm{Spa}\,(\mathbf{Z}[T_1, \ldots, T_d], \mathbf{Z}[T_1, \ldots, T_d]) \times_{\mathrm{Spa}\,(\mathbf{Z},\mathbf{Z})} Y$.

Proof of Theorem 6.1.1, uniqueness. Suppose there are two ways to assign to any separated taut smooth of equidimension d morphism $f\colon X \to Y$ of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$ trace morphisms tr_f and tr'_f satisfying the properties of the statement. We need to show that $\mathrm{tr}_f = \mathrm{tr}'_f$ for all f.

First, for any such f, we may pick an open affinoid cover $Y = \bigcup_{j\in J} V_j$ and an affinoid open cover $f^{-1}(V_j) = \bigcup_{i \in I_j} U_{ji}$ such that $f_{ji} := f|_{U_{ji}}\colon U_{ji} \to V_j$ factors as

$$\begin{tikzcd}
U_{ji} \arrow[r, "g_{ji}"] \arrow[dr, "f_{ji}"'] & \mathbf{D}^d_{V_j} \arrow[d, "\pi_{V_j}"] \\
& V_j
\end{tikzcd}$$

with g_{ji} étale [Hub96, Cor. 1.6.10]. Further, since the U_{ji} are qcqs, the open immersions $U_{ji} \hookrightarrow X$ are taut due to [Hub96, Lem. 5.1.3]. By the uniqueness assertions in Lemma 6.1.3 and Lemma 6.1.2, it suffices to show that $\mathrm{tr}_{f_{ji}} = \mathrm{tr}'_{f_{ji}}$ for all f_{ji}.

Moreover, the compatibility with étale traces gives $\mathrm{tr}_{g_{ji}} = \mathrm{tr}^{\text{ét}}_{g_{ji}} = \mathrm{tr}'_{g_{ji}}$. Thanks to the compatibility with compositions, it then suffices to show that $\mathrm{tr}_{\pi_{V_j}} = \mathrm{tr}'_{\pi_{V_j}}$. In fact, by writing π_{V_j} as a composition of projections $\pi_n\colon \mathbf{D}^n_{V_j} \simeq \mathbf{D}^1_{\mathbf{D}^{n-1}_{V_j}} \to \mathbf{D}^{n-1}_{V_j}$ away from the last coordinate for $n = 1, \ldots, d$, it is enough to check that $\mathrm{tr}_{\pi_n} = \mathrm{tr}'_{\pi_n}$ for all n. By Lemma 6.1.2 and Lemma 6.1.5, this can be checked on stalks at every geometric point of $\mathbf{D}^{n-1}_{V_j}$ of rank 1. The compatibility with pullbacks and (weak) proper base change [Hub96, Cor. 5.4.8] guarantee that the stalks of the trace maps at these points are just the trace maps of the fibers. In conclusion, we are therefore reduced to the verification that tr and tr$'$ agree on the closed unit disk $\mathbf{D}^1_C$ over a complete, algebraically closed nonarchimedean field C. This is a consequence of property (4), the compatibility with compositions, and the compatibility with the étale trace for the open immersion $j\colon \mathbf{D}^1_C \hookrightarrow \mathbf{P}^{1,\mathrm{an}}_C$:

$$\mathrm{tr}_{\mathbf{D}^1_C} = \mathrm{tr}_{\mathbf{P}^1_C} \circ \mathrm{tr}_j(1)[2] = \mathrm{tr}_{\mathbf{P}^1_C} \circ \mathrm{tr}^{\text{ét}}_j(1)[2] = \mathrm{tr}'_{\mathbf{P}^1_C} \circ \mathrm{tr}^{\text{ét}}_j(1)[2] = \mathrm{tr}'_{\mathbf{P}^1_C} \circ \mathrm{tr}'_j(1)[2] = \mathrm{tr}'_{\mathbf{D}^1_C}. \qquad \square$$

The uniqueness proof already suggests that we should define the trace of a smooth morphism by locally factoring it into an étale morphism and a relative disk. The main work is to show that this is well-defined, that is, independent of the factorization. For technical reasons (cf. Remark 6.1.8), it will be advantageous to work with affine spaces instead of disks; recall that the d-dimensional affine space over any locally noetherian analytic adic space Y is defined as $\mathbf{A}^{d,\mathrm{an}}_Y := \mathrm{Spa}\,(\mathbf{Z}[T_1,\dots,T_d],\mathbf{Z}) \times_{\mathrm{Spa}\,(\mathbf{Z},\mathbf{Z})} Y$. We begin by constructing the trace map for such families of affine spaces, following again the strategy in the uniqueness part of the proof of Theorem 6.1.1.

Lemma 6.1.6. *For any a locally noetherian analytic adic space Y with $n \in \mathcal{O}_Y^\times$ and structure map $\pi_Y \colon \mathbf{A}^{d,\mathrm{an}}_Y \to Y$, there exist maps $\mathrm{tr}_{\pi_Y} \colon \mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \to \underline{\Lambda}$ with the following properties:*

(1) *tr_{π_Y} is compatible with pullbacks in Y in the sense of Theorem 6.1.1.*
(2) *tr_{π_Y} is invariant under permutations; that is, if*
$$\sigma_Y \colon \mathbf{A}^{d,\mathrm{an}}_Y \xrightarrow{\sim} \mathbf{A}^{d,\mathrm{an}}_Y, \quad (y_1,\dots,y_d) \mapsto (y_{\sigma(1)},\dots,y_{\sigma(d)})$$
is the isomorphism permuting the coordinates according to some $\sigma \in \mathfrak{S}_d$, then the map
$$\mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \simeq \mathrm{R}(\pi_Y\circ\sigma_Y)_!\underline{\Lambda}(d)[2d] \simeq \mathrm{R}\pi_{Y,!}(\sigma_{Y,!}\underline{\Lambda})(d)[2d] \xrightarrow[\sim]{\mathrm{R}\pi_{Y,!}(\mathrm{tr}_{\sigma_Y}(d)[2d])} \mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \xrightarrow{\mathrm{tr}_{\pi_Y}} \underline{\Lambda}$$
agrees with tr_{π_Y} in $\mathrm{Hom}\big(\mathrm{R}\pi_{Y,!}\,\underline{\Lambda}(d)[2d],\underline{\Lambda}\big)$.
(3) *Assume that $d=1$ and $Y=\mathrm{Spa}\,(C,\mathcal{O}_C)$ for some complete, algebraically closed nonarchimedean field C. Denote by $\tilde{j}\colon \mathbf{D}^1_Y \hookrightarrow \mathbf{A}^{1,\mathrm{an}}_Y$ the canonical open immersion. Then the map $\mathrm{H}^0\big(\mathrm{tr}_{\pi_Y}\circ R\pi_{Y,!}(\mathrm{tr}^{\text{ét}}_{\tilde{j}}(1)[2])\big)$ is the analytic trace map from Example 5.1.15.*

Proof. Step 1. *We assume that $d=1$.* Let $\overline{\pi}_Y \colon \mathbf{P}^{1,\mathrm{an}}_Y \to Y$ be the structure map of the relative (analytic) projective line; see e.g. [Zav23a, § 6] for an account of the latter in the locally noetherian adic context. Below, we will describe a trace map $\mathrm{tr}_{\overline{\pi}_Y}\colon \mathrm{R}\overline{\pi}_{Y,!}\underline{\Lambda}(1)[2] \to \underline{\Lambda}$. Granted the existence of $\mathrm{tr}_{\overline{\pi}_Y}$, we can use the commutative diagram
$$\begin{tikzcd} \mathbf{A}^{1,\mathrm{an}}_Y \arrow[r, hook, "\overline{j}"] \arrow[d, "\pi_Y"'] & \mathbf{P}^{1,\mathrm{an}}_Y \arrow[dl, "\overline{\pi}_Y"] \\ Y & \end{tikzcd}$$
and the trace map $\mathrm{tr}^{\text{ét}}_{\overline{j}}\colon \overline{j}_!\underline{\Lambda} \to \underline{\Lambda}$ for the (étale) open immersion $\overline{j}$ from Definition 2.5.10 to define tr_{π_Y} as the composition
$$\mathrm{tr}_{\pi_Y}\colon \mathrm{R}\pi_{Y,!}\,\underline{\Lambda}(1)[2] \simeq \mathrm{R}\overline{\pi}_{Y\,!}(\overline{j}_!\,\underline{\Lambda}(1)[2]) \xrightarrow{\mathrm{R}\overline{\pi}_{Y\,!}(\mathrm{tr}^{\text{ét}}_{\overline{j}}(1)[2])} \mathrm{R}\overline{\pi}_{Y\,!}\,\underline{\Lambda}(1)[2] \xrightarrow{\mathrm{tr}_{\overline{\pi}_Y}} \underline{\Lambda}.$$

One way to define $\mathrm{tr}_{\overline{\pi}_Y}$ comes from algebraic geometry: By Lemma 6.1.2, it suffices to do it locally on Y. So we may and do assume that $Y=\mathrm{Spa}\,(A,A^+)$ for a strongly noetherian Tate–Huber pair (A,A^+) with $n\in A^\times$. Then $\overline{\pi}_Y$ is the relative analytification of the morphism of schemes $\mathbf{P}^1_A \to \mathrm{Spec}\,(A)$ along the map of locally ringed spaces $\mathrm{Spa}\,(A,A^+)\to\mathrm{Spec}\,(A)$ as in Construction 2.5.4. Thus we can construct $\mathrm{tr}_{\overline{\pi}_Y}$ following Section 5.4 from the algebraic trace map via [Hub96, Th. 3.7.2]. However, in order to avoid any reliance on the algebraic trace map, we can alternatively proceed as follows: By [Zav23b, Prop. 6.1.6], the étale first Chern class of the universal line bundle $\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_Y}(1)$ (defined in the analytic context in [Zav23b, Def. 6.1.2]) induces an isomorphism
$$c_1^{\text{ét}}\big(\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_Y}(1)\big)\colon \underline{\Lambda} \xrightarrow{\sim} \mathrm{R}^2\overline{\pi}_{Y,*}\underline{\Lambda}(1) = \mathrm{R}^2\overline{\pi}_{Y,!}\underline{\Lambda}(1).$$
Using [Hub96, Prop. 5.5.8], one then sets
$$\mathrm{tr}_{\overline{\pi}_Y}\colon \mathrm{R}\overline{\pi}_{Y,!}\underline{\Lambda}(1)[2] \to \tau^{\geq 0}\mathrm{R}\overline{\pi}_{Y,!}\underline{\Lambda}(1)[2] \simeq \mathrm{R}^2\overline{\pi}_{Y,!}\underline{\Lambda}(1) \xrightarrow[\sim]{c_1^{\text{ét}}\left(\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_Y}(1)\right)^{-1}} \underline{\Lambda}.$$

Since the formation of the adjunction map $\overline{j}_!\underline{\Lambda}\to\underline{\Lambda}$ and of the first Chern classes for $\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_Y}(1)$ commutes with arbitrary base change in Y (Lemma 2.5.12, Remark 3.1.10), we conclude (1) for $d=1$. Further, (2) is an empty statement for $d=1$, so we are only left to show (3). For this, let $j\colon \mathbf{D}^1_Y \hookrightarrow \mathbf{P}^{1,\mathrm{an}}_Y$ be the canonical open immersion. Since $j=\overline{j}\circ\tilde{j}$, Lemma 2.5.12 guarantees that the composition
$$\mathrm{tr}_{\pi_Y}\circ \mathrm{R}\pi_{Y,!}\big(\mathrm{tr}^{\text{ét}}_{\tilde{j}}(1)[2]\big)\colon \mathrm{R}(\overline{\pi}_Y\circ j)_!\underline{\Lambda}(1)[2] \simeq \mathrm{R}(\pi_Y\circ\tilde{j})_!\underline{\Lambda}(1)[2] \to \underline{\Lambda}$$

is given by $\mathrm{tr}_{\overline{\pi}_Y} \circ \mathrm{R}\overline{\pi}_{Y,!}\big(\mathrm{tr}_j^{\text{ét}}(1)[2]\big)$. By Section 5.5, especially Theorem 5.5.19, and Remark 3.1.9, the latter map agrees with the one from Example 5.1.15 when $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$, yielding (3).

Step 2. *Now let $d \in \mathbf{Z}_{\geq 1}$ be general.* By successive projections away from the last coordinate, one can factor π_Y as

$$\mathbf{A}_Y^{d,\mathrm{an}} \simeq \mathbf{A}_Y^{d-1,\mathrm{an}} \times_Y \mathbf{A}_Y^{1,\mathrm{an}} \simeq \mathbf{A}^{1,\mathrm{an}}_{\mathbf{A}_Y^{d-1,\mathrm{an}}} \xrightarrow{\pi_d} \mathbf{A}_Y^{d-1,\mathrm{an}} \simeq \mathbf{A}_Y^{d-2,\mathrm{an}} \times_Y \mathbf{A}_Y^{1,\mathrm{an}} \simeq \mathbf{A}^{1,\mathrm{an}}_{\mathbf{A}_Y^{d-2,\mathrm{an}}} \xrightarrow{\pi_{d-1}} \cdots \xrightarrow{\pi_2} \mathbf{A}_Y^{1,\mathrm{an}} \xrightarrow{\pi_1} Y.$$

We set

$$\mathrm{tr}_{\pi_Y} := \mathrm{tr}_{\pi_1} \circ \mathrm{R}\pi_{1,!}\big(\mathrm{tr}_{\pi_2}(1)[2]\big) \circ \cdots \circ \mathrm{R}(\pi_1 \circ \cdots \pi_{d-1})_!\big(\mathrm{tr}_{\pi_d}(d-1)[2d-2]\big) \colon \mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \to \underline{\Lambda}. \tag{6.1.7}$$

Then tr_{π_Y} satisfies again (1) because the tr_{π_i} do so separately by the case $d = 1$. Thus, it remains to verify property (2) that the definition of tr_{π_Y} is independent of the coordinates under permutation.

Since $\mathfrak{S}_d$ is generated by adjacent transpositions, we may assume for this that $d = 2$ and $\sigma \in \mathfrak{S}_2$ is the nontrivial element. Thanks to Lemma 6.1.2, it suffices to verify that

$$\mathrm{R}^4\pi_{Y,!}(\sigma_{Y,!}\underline{\Lambda})(2) \xrightarrow[\sim]{\mathrm{R}^4\pi_{Y,!}(\mathrm{tr}_{\sigma_Y}(2))} \mathrm{R}^4\pi_{Y,!}\underline{\Lambda}(2) \xrightarrow{\mathrm{H}^0(\mathrm{tr}_{\pi_Y})} \underline{\Lambda}$$

agrees with $\mathrm{H}^0(\mathrm{tr}_{\pi_Y})$. The sheaf $\underline{\Lambda}$ is overconvergent, so Lemma 6.1.5 allows us to check this on stalks at geometric points of rank 1. Since the formation of derived proper pushforwards commutes with talking stalks [Hub96, Th. 5.4.6] and tr_{π_Y} is compatible with pullbacks by (1), we may therefore further assume that $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for some algebraically closed complete nonarchimedean field C.

In this case, we note that it suffices to show a stronger claim that $\mathfrak{S}_2$ acts trivially on $\mathrm{H}^4_c\big(\mathbf{A}_C^{2,\mathrm{an}}, \Lambda(2)\big)$. In order to justify this, we prove an even stronger claim that $\mathrm{GL}_2(C)$ acts trivially on $\mathrm{H}^4_c\big(\mathbf{A}_C^{2,\mathrm{an}}, \Lambda(2)\big)$. For this, we note that $\mathrm{H}^4_c\big(\mathbf{A}_C^{2,\mathrm{an}}, \Lambda(2)\big) \simeq \Lambda$: this follows from the analogous statement for the algebraic compactly supported cohomology $\mathrm{H}^4_c\big(\mathbf{A}_C^{2,\mathrm{an}}, \Lambda(2)\big)$ by Huber's comparison theorem [Hub96, Th. 5.7.2]. Alternatively (if one wants to avoid using the nontrivial [Hub96, Th. 3.2.10]), one can adapt the proof of [Ber93, Th. 7.1.1] to the adic context. Now we observe that the action of $\mathrm{GL}_2(C)$ on Λ^* has to factor through the maximal abelian quotient

$$\mathrm{GL}_2(C)/\big[\mathrm{GL}_2(C), \mathrm{GL}_2(C)\big] \simeq \mathrm{GL}_2(C)/\,\mathrm{SL}_2(C) \xrightarrow[\det]{\sim} C^*,$$

which is divisible and therefore cannot admit any nontrivial maps to the torsion group Λ^*. □

Remark 6.1.8. As we explain below, the action of $\mathfrak{S}_2$ on $\mathrm{H}^4_c(\mathbf{D}^2_C, \mu_p^{\otimes 2})$ is nontrivial when the ground field C is of mixed characteristic $(0, p)$. Therefore, in the *proof* of Lemma 6.1.6, it is crucial to use $\mathbf{A}_Y^{d,\mathrm{an}}$ instead of $\mathbf{D}^d_Y$.

To see that the action of $\mathfrak{S}_2$ is nontrival, we set $x_1 := (0,1) \in \mathbf{D}^2$ and $x_2 := (1,0) \in \mathbf{D}^2$. Then Lemma 3.3.4 ensures that $\sigma^* c\ell_{\mathbf{D}^2}(x_1) = c\ell_{\mathbf{D}^2}(\sigma(x_1)) = c\ell_{\mathbf{D}^2}(x_2)$, so it suffices to show that

$$c\ell_{\mathbf{D}^2}(x_1) \neq c\ell_{\mathbf{D}^2}(x_2) \in \mathrm{H}^4_c(\mathbf{D}^2_C, \mu_p^{\otimes 2}).$$

For this, we consider the hyperplane sections $\{x_i\} \xhookrightarrow{\iota_{x_i}} \{x_i\} \times \mathbf{D}^1 \xhookrightarrow{\iota'_{x_i}} \mathbf{D}^2$ for $i = 1, 2$ and the projection onto the second factor $\mathrm{pr}_2 \colon \mathbf{D}^2 \to \mathbf{D}^1$. Corollary 3.3.8 and Lemma 6.2.2 below (whose proof does not use this remark) imply that $\mathrm{tr}_{\mathrm{pr}_2}\big(c\ell_{\mathbf{D}^2}(x_1)\big) = c\ell_{\mathbf{D}^1}\big(\{0\}\big)$ and $\mathrm{tr}_{\mathrm{pr}_2}\big(c\ell_{\mathbf{D}^2}(x_2)\big) = c\ell_{\mathbf{D}^1}\big(\{1\}\big)$. On the other hand,

$$c\ell_{\mathbf{D}^1}(\{0\}) \neq c\ell_{\mathbf{D}^1}(\{1\}) \in \mathrm{H}^2_c(\mathbf{D}^1, \mu_p).$$

thanks to Lemma 5.5.21. (i), yielding the claim.

The trace map for affine spaces from Lemma 6.1.6 is related to the trace maps for smooth affinoid curves from Section 5:

Lemma 6.1.9. *Let X be a smooth affinoid curve over an algebraically closed complete nonarchimedean field C and $n \in C^\times$. Assume that the structure morphism $f\colon X \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ factors as*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & \mathbf{A}^{1,\mathrm{an}}_C \arrow[d, "\pi_C"] \\ & \mathrm{Spa}\,(C, \mathcal{O}_C) \end{tikzcd}$$

with g étale. Then the map $\mathrm{H}^0\big(\mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_g(1)[2])\big)$ is the analytic trace $t_X\colon \mathrm{H}^2_c(X, \mu_n) \to \mathbf{Z}/n$ from Definition 5.1.10.

Proof. Since X is quasicompact, g factors through a closed unit disk $\mathbf{D}^1(r)$ of some radius r. By the same argument as in the last paragraph of the proof of Lemma 6.1.6, the natural action of $C^* = \mathrm{GL}_1(C)$ on $\mathrm{H}^2_c(\mathbf{A}^1_C, \mu_n)$ is trivial. After a renormalization, we therefore end up in the situation

$$\begin{tikzcd} U \arrow[r, "\tilde{g}"] \arrow[rr, bend left, "g"] \arrow[drr, bend right, "f"'] & \mathbf{D}^1_C \arrow[r, hook, "\tilde{j}"] \arrow[dr] & \mathbf{A}^{1,\mathrm{an}}_C \arrow[d, "\pi_C"] \\ & & \mathrm{Spa}\,(C, \mathcal{O}_C), \end{tikzcd}$$

where $\tilde{j}\colon \mathbf{D}^1_C \hookrightarrow \mathbf{A}^{1,\mathrm{an}}_C$ is the canonical open immersion. Then Lemma 6.1.6. (3) guarantees that $\mathrm{H}^0\big(\mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_{\tilde{j}}(1)[2])\big)$ is the analytic trace morphism $t_{\mathbf{D}^1_C}$ from Example 5.1.15. Moreover, the analytic trace for smooth affinoid curves in Definition 5.1.10 is compatible with étale morphisms (Corollary 5.4.3) and the étale trace is compatible with compositions (Lemma 2.5.12), so we conclude that

$$\mathrm{H}^0\big(\mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_g(1)[2])\big) = \mathrm{H}^0\big(\mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_{\tilde{j}}(1)[2]) \circ (\mathrm{R}\pi_{C,!} \circ \tilde{j}_!)(\mathrm{tr}^{\text{ét}}_{\tilde{g}}(1)[2])\big) = t_{\mathbf{D}^1_C} \circ \mathrm{H}^2_c\big(\mathrm{tr}^{\text{ét}}_{\tilde{g}}(1)\big) = t_X. \quad \square$$

Lemma 6.1.9 leads to the following uniqueness statement.

Lemma 6.1.10. *Let $f\colon X \to Y$ be a separated taut smooth of equidimension d morphism of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$. Let*

$$\begin{tikzcd} X \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, "f"] & \mathbf{A}^{d,\mathrm{an}}_Y \arrow[d, "\pi_Y"] \\ \mathbf{A}^{d,\mathrm{an}}_Y \arrow[r, "\pi_Y"] & Y \end{tikzcd}$$

be a commutative diagram of factorizations of f such that g_i is étale for $i = 1, 2$. Then the induced diagram

$$\begin{tikzcd} & (\mathrm{R}\pi_{Y,!} \circ g_{1,!})\underline{\Lambda}(d)[2d] \arrow[r, "\mathrm{R}\pi_{Y,!}(\mathrm{tr}^{\text{ét}}_{g_1}(d)[2d])"] & \mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \arrow[dr, "\mathrm{tr}_{\pi_Y}"] & \\ \mathrm{R}f_!\underline{\Lambda}(d)[2d] \arrow[ur, equal] \arrow[dr, equal] & & & \underline{\Lambda} \\ & (\mathrm{R}\pi_{Y,!} \circ g_{2,!})\underline{\Lambda}(d)[2d] \arrow[r, "\mathrm{R}\pi_{Y,!}(\mathrm{tr}^{\text{ét}}_{g_1}(d)[2d])"'] & \mathrm{R}\pi_{Y,!}\underline{\Lambda}(d)[2d] \arrow[ur, "\mathrm{tr}_{\pi_Y}"'] & \end{tikzcd}$$

involving the maps $\mathrm{tr}^{\text{ét}}_{g_i}$ from Definition 2.5.10 and the map tr_{π_Y} from Lemma 6.1.6 commutes as well.

Proof. Since the sheaf $\underline{\Lambda}$ is overconvergent, Lemma 6.1.2 and Lemma 6.1.5 allow us to check the commutativity of the diagram on stalks at geometric points of rank 1. Again, the formation of derived proper pushforwards commutes with taking such stalks [Hub96, Th. 5.4.6] and the trace maps $\mathrm{tr}^{\text{ét}}_{g_i}$ and tr_{π_Y} are compatible with pullbacks by Lemma 2.5.12 and Lemma 6.1.6. (1). Thus, we may assume that $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for some algebraically closed complete nonarchimedean field C and that X is a separated taut smooth rigid space of equidimension d over C.

The two functions $g_i\colon X \to \mathbf{A}_C^{d,\mathrm{an}}$ are given by tuples $(g_i^{(1)}, \dots, g_i^{(d)})$ with $g_i^{(n)} \in \mathcal{O}(X)$. Let $x \in X$. Since the g_i are étale at x, the differentials $dg_i^{(1)}, \dots, dg_i^{(d)}$ reduce to a basis of the fiber $\Omega^1_{X/C} \otimes k(x)$ for $i = 1, 2$ [Hub96, Prop. 1.6.9.iii)]. By the Steinitz exchange lemma, we can find $\sigma \in \mathfrak{S}_d$ such that for each $n = 1, \dots, d$, the differentials $dg_1^{(\sigma(1))}, \dots, dg_1^{(\sigma(n-1))}, dg_2^{(n)}, \dots, dg_2^{(d)}$ reduce to a basis of $\Omega^1_{X/C} \otimes k(x)$. Since the zero locus of $\left(dg_1^{(\sigma(1))} \wedge \cdots \wedge dg_1^{(\sigma(n-1))} \wedge dg_2^{(n)} \wedge \cdots \wedge dg_2^{(d)}\right) \in \Omega^d_X(X)$ is Zariski-closed as the section of a line bundle, its complement then gives a Zariski-open neighborhood U of x over which each

$$\left(g_1^{(\sigma(1))}, \dots, g_1^{(\sigma(n-1))}, g_2^{(n)}, \dots, g_2^{(d)}\right)\colon U \to \mathbf{A}_C^{d,\mathrm{an}}$$

is étale (again thanks to [Hub96, Prop. 1.6.9.iii)]). As the Zariski-open neighborhoods for various points x cover the adic space X and the corresponding Zariski-open embeddings are taut [Hub96, Lem. 5.1.4.i), Lem. 5.1.3.iii)], it suffices to show that the diagram commutes over each U by the uniqueness assertion in Lemma 6.1.3. Combined with Lemma 6.1.6. (2), we may therefore assume that $(g_1^{(1)}, \dots, g_1^{(n-1)}, g_2^{(n)}, \dots, g_2^{(d)})\colon U \to \mathbf{A}_C^{d,\mathrm{an}}$ is étale for all $n = 1, \dots, d$.

Arguing one coordinate at a time, it now suffices to show the statement when $\pi_n \circ g_1 = \pi_n \circ g_2 =: f'$, where $\pi_n\colon \mathbf{A}_C^{d,\mathrm{an}} \to \mathbf{A}_C^{d-1,\mathrm{an}}$ is the projection away from the n-th coordinate for some $1 \le n \le d$. In that case, we are in the situation of the following commutative diagram:

$$\begin{tikzcd}
U \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, "f'"] & \mathbf{A}_C^{d,\mathrm{an}} \arrow[d, "\pi_n"] \arrow[ddr, bend left, "\pi_Y"] & \\
\mathbf{A}_C^{d,\mathrm{an}} \arrow[r, "\pi_n"] \arrow[drr, bend right, "\pi_Y"'] & \mathbf{A}_C^{d-1,\mathrm{an}} \arrow[dr, "\pi'_Y"] & \\
& & \operatorname{Spa}(C, \mathcal{O}_C)
\end{tikzcd}$$

By the definition of the trace morphisms in Lemma 6.1.6 (and the invariance under permutation of coordinates), we have $\mathrm{tr}_{\pi_Y} = \mathrm{tr}_{\pi'_Y} \circ \mathrm{R}\pi'_{Y,!}(\mathrm{tr}_{\pi_n}(d-1)[2d-2])$. Therefore, we only need to prove that

$$\mathrm{tr}_{\pi_n} \circ \mathrm{R}\pi_{n,!}\big(\mathrm{tr}^{\text{ét}}_{g_1}(1)[2]\big) = \mathrm{tr}_{\pi_n} \circ \mathrm{R}\pi_{n,!}\big(\mathrm{tr}^{\text{ét}}_{g_2}(1)[2]\big).$$

As in the first paragraph of the proof, we may check this statement on stalks at geometric rank-1 points and use (weak) proper base change [Hub96, Th. 5.4.6] to reduce to the case where $d = n = 1$.

Now we note that for each affinoid open $j\colon V \hookrightarrow U$, the morphism is taut due to [Hub96, Lem. 5.1.3.(i),(iii)]. Therefore, we can shrink U even further to assume that U is a smooth affinoid curve over C. In that case, we have

$$\mathrm{H}^0\big(\mathrm{tr}_{\pi_1} \circ \mathrm{R}\pi_{1,!}(\mathrm{tr}^{\text{ét}}_{g_1}(1)[2])\big) = t_U = \mathrm{H}^0\big(\mathrm{tr}_{\pi_1} \circ \mathrm{R}\pi_{1,!}(\mathrm{tr}^{\text{ét}}_{g_2}(1)[2])\big)$$

by Lemma 6.1.9. The desired statement follows from Lemma 6.1.2. □

We are finally ready to discuss the trace for smooth morphisms in general. The following construction, which was forced upon us by the uniqueness part of the proof, is essentially independent of the construction of the analytic trace map in Section 5. However, in order to see that it does not depend on any of the choices made in the process, the existence of an *a priori* well-defined analytic trace map, which was used in the proof of Lemma 6.1.10, is indispensable.

Proof of Theorem 6.1.1, existence. As in the uniqueness part of the proof, we may pick open affinoid (and thus also taut) covers $Y = \bigcup_{j \in J} V_j$ and $f^{-1}(V_j) = \bigcup_{i \in I_j} U_{ji}$ such that $f_{ji} := f\big|_{U_{ji}}\colon U_{ji} \to V_j$ factors as

$$\begin{tikzcd}
U_{ji} \arrow[r, "g_{ji}"] \arrow[dr, "f_{ji}"'] & \mathbf{A}_{V_j}^{d,\mathrm{an}} \arrow[d, "\pi_{V_j}"] \\
& V_j
\end{tikzcd} \tag{6.1.11}$$

with g_{ji} étale. Set $\mathrm{tr}_{f_{ji}} := \mathrm{tr}_{\pi_{V_j}} \circ \mathrm{R}\pi_{V_j,!}(\mathrm{tr}^{\text{ét}}_{g_{ji}}(d)[2d])$. This is independent of the factorization by Lemma 6.1.10; in particular, all the various traces agree on intersections and we may glue them to a trace

$$\mathrm{tr}_f \colon \mathrm{R}f_! \, \underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$$

thanks to Lemma 6.1.3 and Lemma 6.1.2.

It remains to verify that tr_f satisfies the desired properties. In the situation of (1), pick open affinoid covers $Z = \bigcup_{k\in K} W_k$, $g^{-1}(W_k) = \bigcup_{j\in J_k} V_{kj}$, and $f^{-1}(V_{kj}) = \bigcup_{i\in I_{kj}} U_{kji}$ which fit into the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[r, phantom, "\supseteq"] & U_{kji} \arrow[d, "f_{kji}"] \arrow[r, "h''_{kji}"] & \mathbf{A}^{d,\mathrm{an}}_{V_{kj}} \simeq \mathbf{A}^{d,\mathrm{an}}_{W_k} \times_{W_k} V_{kj} \arrow[dl, "\pi_{V_{kj}}"'] \arrow[r, "h'_{kj}"] & \mathbf{A}^{d+e,\mathrm{an}}_{W_k} \arrow[dl, "\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k}}"] \\
Y \arrow[d, "g"'] \arrow[r, phantom, "\supseteq"] & V_{kj} \arrow[d, "g_{kj}"] \arrow[r, "h_{kj}"] & \mathbf{A}^{e,\mathrm{an}}_{W_k} \arrow[dl, "\pi_{W_k}"] & \\
Z \arrow[r, phantom, "\supseteq"] & W_k & &
\end{tikzcd} \tag{6.1.12}$$

with h_{kj} and h''_{kji} étale. The construction of the traces in the first paragraph makes it clear that $\mathrm{tr}_{\pi_{V_{kj}}} \circ \mathrm{R}\pi_{V_{kj},!}(\mathrm{tr}^{\text{ét}}_{h''_{kji}}(d)[2d]) = \mathrm{tr}_{f_{kji}}$ and $\mathrm{tr}_{\pi_{W_k}} \circ \mathrm{R}\pi_{W_k,!}(\mathrm{tr}^{\text{ét}}_{h_{kj}}(e)[2e]) = \mathrm{tr}_{g_{kj}}$. Therefore, the compatibility of traces under composition boils down to the verification that the diagram

$$\begin{tikzcd}
(\mathrm{R}\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k},!} \circ \mathrm{R}h'_{kj,!})\underline{\Lambda}(e)[2e] \arrow[r, "\mathrm{R}\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k},!}(\mathrm{tr}^{\text{ét}}_{h'_{kj}}(e)[2e])"] \arrow[d, equal] & \mathrm{R}\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k},!}\underline{\Lambda}(e)[2e] \arrow[dr, "\mathrm{tr}_{\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k}}}"] & \\
(\mathrm{R}h_{kj,!} \circ \mathrm{R}\pi_{V_{kj},!})\underline{\Lambda}(e)[2e] \arrow[r, "\mathrm{R}h_{kj,!}(\mathrm{tr}_{\pi_{V_{kj}}})"] & \mathrm{R}h_{kj,!}\underline{\Lambda} \arrow[r, "\mathrm{tr}^{\text{ét}}_{h_{kj}}"] & \underline{\Lambda}
\end{tikzcd}$$

of traces in the parallelogram commutes. As before, [Hub96, Prop. 2.5.5], Lemma 6.1.2 and Lemma 6.1.5 allow us to check this on stalks at geometric points of rank 1 of $\mathbf{A}^{e,\mathrm{an}}_{W_k}$. Moreover, the base change isomorphism of derived pushfowards with compact support from [Hub96, Th. 5.4.6] is compatible with the formation of $\mathrm{tr}_{\pi_{V_{kj}}}$ and $\mathrm{tr}_{\pi_{\mathbf{A}^{e,\mathrm{an}}_{W_k}}}$ (Lemma 6.1.6. (1)) as well as $\mathrm{tr}^{\text{ét}}_{h_{kj}}$ and $\mathrm{tr}^{\text{ét}}_{h'_{kj}}$ (Lemma 2.5.12). Therefore, we may check the commutativity of traces in the parallelogram after pulling back along a geometric point of rank 1 of $\mathbf{D}^e_{W_k}$, where the statement is clear because the pullback of V_{kj} decomposes as finite disjoint union of geometric points of rank 1.

Next, the $\mathrm{tr}_{\pi_{V_j}}$ are compatible with pullbacks by Lemma 6.1.6. (1) and the $\mathrm{tr}^{\text{ét}}_{g_{ji}}$ are compatible with pullbacks by Lemma 2.5.12, so tr_f satisfies property (2). Property (3) holds by definition.

Lastly, we show (4). Consider the morphism $f\colon \mathbf{P}^1_C \to \operatorname{Spec} C$. Let $\mathbf{P}^1_C = \mathbf{A}^1_C(0) \cup \mathbf{A}^1_C(\infty)$ be the open affine cover of $\mathbf{P}^1_C$ by the affine lines around 0 and ∞ and denote the restricted structure morphisms by $f_i\colon \mathbf{A}^1_C(i) \to \operatorname{Spa}(C, \mathcal{O}_C)$ for $i \in \{0, \infty\}$. Since both algebraic and analytic trace maps are compatible with open immersions, and the morphism $\mathrm{H}^2_c\left(\mathbf{A}^{1,\mathrm{an}}_C(0), \Lambda(1)\right) \oplus \mathrm{H}^2_c\left(\mathbf{A}^{1,\mathrm{an}}_C(\infty), \Lambda(1)\right) \to \mathrm{H}^2\left(\mathbf{P}^{1,\mathrm{an}}_C, \Lambda(1)\right)$ is surjective, it suffices to show that the following diagrams

$$\begin{tikzcd}
\mathrm{H}^2_c(\mathbf{A}^1_C(i), \Lambda(1)) \arrow[r, "\sim"] \arrow[d, "\mathrm{H}^0(\mathrm{tr}_{f_i})"] & \mathrm{H}^2_c(\mathbf{A}^{1,\mathrm{an}}_C(i), \Lambda(1)) \arrow[dl, "\mathrm{H}^0(\mathrm{tr}_{f_i^{\mathrm{an}}})"] \\
\Lambda &
\end{tikzcd}$$

commute for $i \in \{0, \infty\}$. This follows directly from the definition of the analytic trace map on the analytic affine line and Lemma 2.5.12. $\square$

6.2. **Properties of the smooth trace.** In this subsection, we establish some properties of the smooth trace with constant coefficients constructed in Theorem 6.1.1. We begin with the compatibility with the analytic trace for affinoid curves constructed in Definition 5.1.10.

Lemma 6.2.1. *Let $f\colon X \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ be a smooth affinoid curve over an algebraically closed complete nonarchimedean field C and $n \in C^\times$. Then the analytic trace $t_X\colon \mathrm{H}^2_c(X, \mu_n) \to \mathbf{Z}/n$ from Definition 5.1.10 agrees with $\mathrm{H}^0(\mathrm{tr}_f)$ for the smooth trace tr_f constructed in Theorem 6.1.1.*

Proof. Recall the construction of the smooth trace tr_f: we pick an affinoid open cover $X = \bigcup_{i\in I} U_i$ and diagrams

$$\begin{tikzcd} U_i \arrow[r, "g_i"] \arrow[dr, "f_i"'] & \mathbf{A}^{1,\mathrm{an}}_C \arrow[d, "\pi_C"] \\ & \mathrm{Spa}\,(C, \mathcal{O}_C) \end{tikzcd}$$

with g_i étale, set $\mathrm{tr}_{f_i} := \mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_{g_i}(1)[2])$ (using Lemma 6.1.6 and the étale traces $\mathrm{tr}^{\text{ét}}_{g_i}$ from Definition 2.5.10), and descend $\bigoplus_i \mathrm{tr}_{f_i}$ to a morphism $\mathrm{tr}_f\colon \mathrm{R}f_!\mu_n[2] \to \mathbf{Z}/n$ via Lemma 6.1.2 and the epimorphism $\bigoplus_i \mathrm{H}^2_c(U_i, \mu_n) \twoheadrightarrow \mathrm{H}^2_c(X, \mu_n)$ from the proof of Lemma 6.1.3.

By Lemma 6.1.9, $\mathrm{H}^0\big(\mathrm{tr}_{\pi_C} \circ \mathrm{R}\pi_{C,!}(\mathrm{tr}^{\text{ét}}_{g_i}(1)[2])\big)$ is given by the analytic trace $t_{U_i}\colon \mathrm{H}^2_c(U_i, \mu_n) \to \mathbf{Z}/n$ from Definition 5.1.10. On the other hand, Corollary 5.4.3 applied to the open immersions $U_i \hookrightarrow X$ guarantees that the composition $\bigoplus_i \mathrm{H}^2_c(U_i, \mu_n) \twoheadrightarrow \mathrm{H}^2_c(X, \mu_n) \xrightarrow{t_X} \mathbf{Z}/n$ is given by $\bigoplus_i t_{U_i}$. Since the first map is an epimorphism, we can conclude the desired identity $t_X = \mathrm{H}^0(\mathrm{tr}_f)$. □

Lemma 6.2.2. *Let $f\colon X \to Y$ be a separated taut smooth of equidimension d morphism of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$. Assume that f has a section $s\colon Y \hookrightarrow X$. Then s is an lci immersion of pure codimension d and the composition*

$$\underline{\Lambda}_Y = \mathrm{R}f_! \circ s_*\underline{\Lambda}_Y \xrightarrow{\mathrm{R}f_!(\mathrm{cl}_s)} \mathrm{R}f_!\underline{\Lambda}_X(d)[2d] \xrightarrow{\mathrm{tr}_f} \underline{\Lambda}_Y,$$

of the induced cycle class map from Variant 3.3.3 with the smooth trace of f from Theorem 6.1.1 is the identity.

Proof. The first statement that s is an lci immersion of pure codimension d is proven in [Zav23a, Cor. 5.10]. To verify the second statement, we proceed in two steps:

Step 1. Case when f is the structure morphism $X = \mathbf{A}^{d,\mathrm{an}}_Y \to Y$ and s is the zero section. We argue by induction on d. If $d = 0$, the claim is trivial. If $d = 1$, the claim essentially follows from the construction of the trace map in Lemma 6.1.6.

Now we fix $d > 1$ and assume that the claim has been proven in dimensions $< d$. Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[rr, hook, bend left, "s"] \arrow[dr, hook, "i"'] \arrow[ddr, equal] & \mathbf{A}^{d-1,\mathrm{an}}_Y \arrow[r, hook, "j"] \arrow[d, equal] & \mathbf{A}^{d,\mathrm{an}}_Y \arrow[dl, "g"'] \arrow[ddl, "f"] \\ & \mathbf{A}^{d-1,\mathrm{an}}_Y \arrow[d, "h"] & \\ & Y, & \end{tikzcd}$$

where $i\colon Y \hookrightarrow \mathbf{A}^{d-1,\mathrm{an}}_Y$ is the zero section, $j\colon \mathbf{A}^{d-1,\mathrm{an}}_Y \hookrightarrow \mathbf{A}^{d,\mathrm{an}}_Y$ is the natural inclusion as the vanishing locus of the last coordinate, g is the projection onto the first $d-1$ factors, and h is the structure morphism. Then Theorem 6.1.1. (1), Corollary 3.3.8, and the induction hypothesis guarantee that

$$\begin{aligned} \mathrm{tr}_f \circ \mathrm{R}f_!(\mathrm{cl}_s) = \mathrm{tr}_h \circ \mathrm{R}h_!\big(\mathrm{tr}_g(d-1)[2d-2]\big) \circ \mathrm{R}f_!\big(\mathrm{cl}_j(d-1)[2d-2]\big) \circ \mathrm{R}h_!(\mathrm{cl}_i) = \\ = \mathrm{tr}_h \circ \mathrm{R}h_!(\mathrm{id}) \circ \mathrm{R}h_!(\mathrm{cl}_i) = \mathrm{tr}_h \circ \mathrm{R}h_!(\mathrm{cl}_i) = \mathrm{id}\,. \end{aligned}$$

Step 2. General case. Since the sheaf $\underline{\Lambda}_Y$ is overconvergent, the equality of two endomorphisms may be checked on stalks at geometric points attached to rank-1 points of $|Y|$ (Lemma 6.1.2 and Lemma 6.1.5). Moreover, the formation of $\mathrm{R}f_!$ and $\mathrm{cl}_X(Y)$ is compatible with pullbacks to these geometric points by [Hub96, Cor. 5.4.8] and Lemma 3.3.4, respectively. Thus, we are reduced to the case where $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for some algebraically closed complete nonarchimedean field C and s is given by a rational point $y \in X(C)$.

Since the morphism $\mathrm{cl}_s\colon s_*\underline{\Lambda}_Y \to \underline{\Lambda}_X$ in Variant 3.3.3 is constructed from the cycle class $c\ell_s$ via the adjunction $(s_*, Rs^!)$, the image of $1 \in \Lambda$ under the compactly supported pushforward

$$\mathrm{R}\Gamma_c\big(\mathrm{cl}_X(y)\big)\colon \Lambda \simeq \mathrm{R}\Gamma_c(X, s_*\Lambda_y) \to \mathrm{R}\Gamma_c\big(X, \Lambda(d)[2d]\big) \to \mathrm{H}^{2d}_c\big(X, \Lambda(d)\big)$$

is the compactly supported cohomology cycle class $c\ell_X(y)$ from Definition 3.5.2, which similarly arises from composing with the counit of adjunction. In conclusion, it suffices to prove that for any separated taut smooth rigid space $f\colon X \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ and any rational point $y \in X(C)$, we have $\mathrm{tr}_f\big(c\ell_X(y)\big) = 1 \in \Lambda$ for the compactly supported cycle class $c\ell_X(y) \in \mathrm{H}^{2d}_c\big(X, \Lambda(d)\big)$.

By [Hub96, Cor. 1.6.10], the point y has a quasicompact open neighborhood $U \subseteq X$ such that $f\big|_U$ factors through an étale map $U \to \mathbf{D}^d_C$; we may assume that it sends y to the origin $0 \in \mathbf{D}^d_C(C)$. Lemma 3.5.3, Theorem 6.1.1. (1) and Theorem 6.1.1. (3) applied to the diagram of pointed étale maps

$$(X, y) \longleftarrow (U, y) \longrightarrow (\mathbf{D}^d_C, 0) \longrightarrow \big(\mathbf{A}^{d,\mathrm{an}}_C, 0\big)$$

then allow us to deduce the general statement from that for $0 \in \mathbf{A}^{d,\mathrm{an}}_C$. This case was already treated in Step 1. □

Lemma 6.2.3. *Let $f\colon X \to Y$ be a separated taut smooth of equidimension d morphism of locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$ such that the fibers of f are nonempty. Then the (truncation of the) trace map $\mathrm{tr}_f\colon \mathrm{R}^{2d}f_!\underline{\Lambda}_X(d) \to \underline{\Lambda}_Y$ is an epimorphism of étale sheaves on Y.*

Proof. By [Hub96, Prop. 2.5.5], it suffices to check that for any $y \in |Y|$, the induced maps on stalks $\big(\mathrm{tr}_f\big)_{\overline{y}}\colon \big(\mathrm{R}^{2d}f_!\underline{\Lambda}_X(d)\big)_{\overline{y}} \to \underline{\Lambda}_{Y,\overline{y}}$ at the geometric point $\overline{y}\colon \Big(\mathrm{Spa}\,\big(\widehat{\overline{k(y)}}, \widehat{\overline{k(y)}}^+\big), \{y\}\Big) \to Y$ attached to y is an epimorphism. This would follow from the assertion that tr_f becomes an epimorphism after pullback along the natural map $\mathrm{Spa}\,\Big(\widehat{\overline{k(y)}}, \widehat{\overline{k(y)}}^+\Big) \to Y$ through which $\overline{y}$ factors. Thanks to the proper base change isomorphism for morphisms of transcendence dimension 0 [Hub96, Th. 5.4.6] and the compatibility of the smooth trace under pullbacks (Theorem 6.1.1. (2)), we may therefore assume that $Y = \mathrm{Spa}\,(C, C^+)$ for some algebraically closed nonarchimedean field C with $n \in C^\times$ and some open and bounded valuation subring $C^+ \subset C$.

By [Hub96, Cor. 1.6.10] applied to a point lying over the closed point of $Y = \mathrm{Spa}\,(C, C^+)$, we can pick an open $U \subseteq X$ for which $f\big|_U$ is still surjective and factors as

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f|_U"'] & \mathbf{D}^d_Y \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

with g étale. Since $g(U) \subseteq \mathbf{D}^d_Y$ is open, $U \to g(U)$ is surjective, $\mathrm{tr}^{\text{ét}}_g$ is given by summing over fibers and $\mathrm{R}^{2d}\pi_{Y,!}$ is right exact, it suffices to prove the statement for $\pi_Y\big|_{g(U)}\colon g(U) \to Y$. But $\pi_Y\big|_{g(U)}$ has a section by [Sch17, Lem. 9.5] because $g(U)$ still surjects onto Y; an application of Lemma 6.2.2 then finishes the proof. □

Next, we formulate and prove the comparison between our smooth trace and the "usual" one coming from algebraic geometry. First, we fix some notation. Let $S = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid, let $f\colon X \to Y$ be a separated finite type morphism between locally finite type A-schemes, and let relative analytification $f^{\mathrm{an}/S}\colon X^{\mathrm{an}/S} \to Y^{\mathrm{an}/S}$ be its relative analytification (see Construction 2.5.4). Then by [Hub96, (3.2.8)], we have a commutative diagram of étale topoi

$$
\begin{tikzcd}
X^{\mathrm{an}/S}_{\mathrm{\acute{e}t}} \arrow[r, "c_{X/S}"] \arrow[d, "f^{\mathrm{an}/S}_{\mathrm{\acute{e}t}}"] & X_{\mathrm{\acute{e}t}} \arrow[d, "f_{\mathrm{\acute{e}t}}"] \\
Y^{\mathrm{an}/S}_{\mathrm{\acute{e}t}} \arrow[r, "c_{Y/S}"] \arrow[d] & Y_{\mathrm{\acute{e}t}} \arrow[d] \\
\mathrm{Spa}\,(A, A^+)_{\mathrm{\acute{e}t}} \arrow[r, "c_S"] & (\mathrm{Spec}\,A)_{\mathrm{\acute{e}t}}.
\end{tikzcd}
$$

The algebraic pushforward with compact support and the analytic pushforward with compact support are related via the natural isomorphism of functors $c^*_{Y/S}\big(\mathrm{R}f_!(-)\big) \xrightarrow{\sim} \mathrm{R}f^{\mathrm{an}/S}_!\big(c^*_{X/S}(-)\big)$, see [Hub96, Th. 5.7.2]. If the map f is in addition smooth of equidimension d, it comes equipped with a trace map $\mathrm{tr}_f\colon \mathrm{R}f_!\underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_Y$ (see [AGV71, Exp. XVIII, Th. 2.9]).

Proposition 6.2.4 (Compatibility with algebraic geometry)**.** *Let $S = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid, let X and Y be locally finite type A-schemes, and let $f\colon X \to Y$ be a finite type, smooth, separated morphism of equidimension d. Then we have $c^*_{Y/S}\big(\mathrm{tr}_f\big) = \mathrm{tr}_{f^{\mathrm{an}/S}}$.*

Proof. First, we note that Lemma 2.5.12 proves the claim when f is étale. Then we recall that Zariski-open immersions are taut (see [Hub96, Lem. 5.1.4]). Therefore, the established above case of étale maps and Lemma 6.1.3 imply that the statement is local on X, so we can further assume that the morphism $f\colon X \to Y$ factors as a composition of an étale map $g\colon X \to \mathbf{A}^d_Y$ followed by the projection $\pi_Y\colon \mathbf{A}^d_Y \to Y$. Using that both the algebraic trace maps are compatible with compositions and the established above case of an étale map, we conclude that it suffices to show the claim when f is the relative affine line $\mathbf{A}^1_Y \to Y$.

In either case, we note that Lemma 6.1.2 and Lemma 6.1.5 ensure that it suffices to check equality $c^*_{Y/S}\big(\mathrm{tr}_f\big) = \mathrm{tr}_{f^{\mathrm{an}/S}}$ on stalks at rank-1 geometric points. Using algebraic proper base change and weak analytic proper base change (see [Hub96, Th. 5.4.6]), we can assume that $A = C$, $A^+ = \mathcal{O}_C$ for an algebraically closed non-archimedean field C, and $Y = \mathrm{Spec}\,C$.

In this case, we can use the case of étale morphisms again to reduce the question to the case of the projective line $f\colon \mathbf{P}^1_C \to \mathrm{Spec}\,C$. Then the result follows from Theorem 6.1.1 (4). $\square$

Lastly, we show that our trace map is compatible with other constructions in the context of rigid geometry. We begin with a comparison with Berkovich's trace, whenever both are defined. Fix a complete nonarchimedean field K with a valuation of rank 1. Recall that Huber constructed an equivalence of categories

$$
(6.2.5)\qquad u\colon (A)' := \left\{\begin{matrix}\text{taut adic spaces locally of finite}\\ \text{type over } \mathrm{Spa}\,(K, \mathcal{O}_K)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Hausdorff strictly } K\text{-analytic}\\ \text{Berkovich spaces}\end{matrix}\right\} =: (An)
$$

which roughly sends an adic space X in $(A)'$ to its maximal Hausdorff quotient; see [Hub96, Rmk. 8.3.2]. By [Hub96, p. 427, (a)], a morphism f in $(A)'$ is partially proper and étale if and only if $u(f)$ is étale in the sense of Berkovich [Ber93, Def. 3.3.4]. Thus, for any $X \in (A)'$, the equivalence u induces a morphism of topoi

$$\theta_X\colon X_{\mathrm{\acute{e}t}} \longrightarrow u(X)_{\mathrm{\acute{e}t}}.$$

where $X_{\mathrm{\acute{e}t}}$ denotes Huber's étale topos used in our paper and $u(X)_{\mathrm{\acute{e}t}}$ the étale topos defined by Berkovich [Ber93, § 4.1]; cf. [Hub96, p. 426].[24] This morphism is fully faithful, with essential image the overconvergent sheaves [Hub96, Th. 8.3.5]. Moreover, it is functorial in the following sense: for any partially proper morphism $f\colon X \to Y$ in $(A)'$, there is a natural isomorphism of functors on $D^+_{\mathrm{\acute{e}t}}(u(X), \mathbf{Z})$

$$\alpha_f\colon \mathrm{R}f_! \circ \theta^*_X \xrightarrow{\sim} \theta^*_Y \circ \mathrm{R}u(f)_!,$$

[24] The étale site from [Ber93, § 4.1] uses étale covers by arbitrary (not neccessarily *stricly* K-analytic) Berkovich spaces, which are not included in the equivalence u. Therefore, Huber first considers the morphism $\theta_X\colon \mathrm{\acute{E}t}_{/X} \to s.\,\mathrm{\acute{E}t}_{/u(X)}$ to the strict étale site of $u(X)$, a slightly modified version of the étale site $\mathrm{\acute{E}t}_{/u(X)}$ that only includes covers in (An). Since the natural morphism $\mathrm{\acute{E}t}_{/u(X)} \to s.\,\mathrm{\acute{E}t}_{/u(X)}$ induces an equivalence of topoi [Hub96, Cor. A.5], this yields the mentioned comparison of the adic étale and the Berkovich étale topos.

where $\mathrm{R}u(f)_!$ is Berkovich's derived pushforward with compact support from [Ber93, § 5.1]; cf. [Hub96, Prop. 8.3.6] and [Zav21b, Th. A.15].

Proposition 6.2.6 (Compatibility with the Berkovich trace)**.** *Let $f\colon X \to Y$ be a partially proper, smooth of equidimension d morphism of taut adic spaces that are locally of finite type over $\mathrm{Spa}\,(K, \mathcal{O}_K)$. Let $u(f)\colon u(X) \to u(Y)$ be the separated smooth of equidimension d morphism of Hausdorff Berkovich spaces that is associated with f under the equivalence (6.2.5) and denote by $\mathrm{tr}_{u(f)}\colon \mathrm{R}u(f)_!\underline{\Lambda}_{u(X)}(d)[2d] \to \mathrm{R}^{2d}u(f)_!\underline{\Lambda}_{u(X)}(d) \to \underline{\Lambda}_{u(Y)}$ the trace morphism defined in [Ber93, Th. 7.2.1]. Then the following natural diagram commutes:*

$$\begin{tikzcd}
\mathrm{R}f_!\underline{\Lambda}_X(d)[2d] \simeq \mathrm{R}f_!\theta_X^*\underline{\Lambda}_{u(X)}(d)[2d] \arrow[rr, "\alpha_f", "\sim"'] \arrow[dr, "\mathrm{tr}_f"'] & & \theta_Y^*\mathrm{R}u(f)_!\underline{\Lambda}_{u(X)}(d)[2d] \arrow[dl, "\theta_Y^*\mathrm{tr}_{u(f)}"] \\
& \underline{\Lambda}_Y \simeq \theta_Y^*\underline{\Lambda}_{u(Y)} &
\end{tikzcd}$$

Note that partially proper morphisms of locally noetherian analytic adic spaces are automatically separated and taut [Hub96, Def. 1.3.3.ii), Lem. 5.1.10.i)], so tr_f is indeed defined. Moreover, we use that the induced morphism $u(f)$ of Berkovich spaces is smooth in the sense of Berkovich because f is partially proper and smooth [Duc18, Cor. 5.4.8] (cf. also [Zav21b, Cor. A.11]).

Proof. First, we note that [Zav21b, Lem. A.19] proves the claim when g is partially proper and étale. Thus Lemma 6.1.3 suffices that we can argue locally on X^B. Since the induced morphism $u(f)$ of Berkovich spaces is smooth in the sense of Berkovich, so any $x \in u(X)$ has an open neighborhood $V \subset u(X)$ such that $u(f)\big|_V$ factors as

$$\begin{tikzcd}
V \arrow[r, "h"] \arrow[dr, "u(f)|_V"'] & \mathbf{A}_Y^{d,B} \arrow[d, "u(\pi)"] \\
& Y
\end{tikzcd}$$

such that h is étale (cf. [Ber93, Def. 3.5.1]). Applying u^{-1}, one obtains a factorization

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[dr, "f|_U"'] & \mathbf{A}_Y^{d,\mathrm{an}} \arrow[d, "\pi"] \\
& Y
\end{tikzcd}$$

with g being partially proper and étale (see [Hub96, p. 427, (a)]). Moreover, $U = u^{-1}(V) \subseteq X$ is open and partially proper inside X because u is defined by passing to the maximal Hausdorff quotient. Since both the analytic and the Berkovich trace are compatible with compositions, we conclude that it suffices to prove the claim when f is partially proper étale or the relative affine line $\mathbf{A}_Y^{1,\mathrm{an}} \to Y$. The case of a partially proper étale morphism was solved above, so we only need to consider the case of the relative affine line.

Since $\underline{\Lambda}_Y$ is overconvergent, we may check the equality of the two trace morphisms after passing to stalks at geometric points of rank 1 (Lemma 6.1.2 and Lemma 6.1.5). Thanks to [Hub96, Cor. 5.4.8] and [Ber93, Th. 5.3.1], we can therefore assume that $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ for some algebraically closed nonarchimedean field C, and $X = \mathbf{A}_C^{1,\mathrm{an}}$. In this case, both the analytic trace and the Berkovich trace comes as an analytification of the algebraic trace for the schematic affine line $\mathbf{A}_C^1 \to \mathrm{Spec}\, C$. This finishes the proof. □

Corollary 6.2.7. *Let $f\colon X \to Y$ be a partially proper smooth of equidimension d morphism between rigid-analytic spaces over $\mathrm{Spa}\,(K, \mathcal{O}_K)$. Then $\mathcal{H}^{2d}(\mathrm{tr}_f)\colon \mathrm{R}^{2d}f_!\underline{\Lambda}_X(d) \to \underline{\Lambda}_Y$ coincides with the Berkovich trace t_f from [Zav21b, Th. 5.3.3].*

Proof. The question is local on Y, so we can assume that Y is an affinoid. Then the result follows directly from Proposition 6.2.6. □

Assume now that K is a p-adic field (i.e., a complete discrete valuation field of mixed characteristic $(0, p)$ whose residue field is perfect). Set $C := \widehat{\overline{K}}$. Recall that for every Zariski-compactifiable smooth rigid-analytic space U of equidimension d over K, Lan–Liu–Zhu also constructed in [LLZ23, Th. 1.3] a p-adic rational trace

$$t_{U,\text{ét}}\colon \mathrm{H}^{2d}_c\big(U_C,\mathbf{Q}_p(d)\big) := \Big(\lim_r \mathrm{H}^{2d}_c\big(U_C,\mathbf{Z}/p^r\mathbf{Z}(d)\big)\Big)\Big[\frac{1}{p}\Big] \longrightarrow \mathbf{Q}_p.$$

The trace maps from Theorem 6.1.1 gives rise to another p-adic rational trace map $\mathrm{tr}_U\colon \mathrm{H}^{2d}_c\big(U_C,\mathbf{Q}_p(d)\big)\to\mathbf{Q}_p$ via the formula $\mathrm{tr}_U := \big(\lim_r \mathrm{H}^{2d}(\mathrm{tr}_{\mathbf{Z}/p^r\mathbf{Z}})\big)\big[\frac{1}{p}\big]$. We claim that these two maps coincide:

Lemma 6.2.8. *In the situtation described above, we have $t_{U,\text{ét}} = \mathrm{tr}_U\colon \mathrm{H}^{2d}_c\big(U_C,\mathbf{Q}_p(d)\big)\to\mathbf{Q}_p$.*

Sketch of the proof. Using resolution of singularities (see [Tem12, Th. 5.2.2]), we can assume that U admits a smooth proper (Zariski-)compactification X. By virtue of [LLZ23, Th. 4.4.1(1)] and Theorem 6.1.1. (3), it then suffices to prove the statement for the smooth and proper X. After passing to a finite extension of K and a component of X, we may further assume that X is geometrically connected and admits a rational point $x\in X(K)$.

Now we argue by induction on $d=\dim X$. If $d=0$, the claim is obvious. Therefore, we assume that $d=\dim X>0$ and that the claim is known in dimensions $<d$. Theorem 6.4.1 below (whose proof does not use Lemma 6.2.8) and Lemma 6.2.2 imply that $\mathrm{H}^{2d}\big(X_C,\mathbf{Q}_p(d)\big)$ is one-dimensional and that $\mathrm{tr}_X\big(c\ell_X(x)\big)=1$. Thus, it suffices to prove that $t_{X,\text{ét}}\big(c\ell_X(x)\big)=1$.

An inspection of the proof of [LLZ23, Th. 4.4.1] shows that we can further assume that there is an effective Cartier divisor $x\in E\overset{i}{\hookrightarrow} X$. In this case, the proof of *loc. cit.* guarantees that the composition[25] $\mathrm{H}^{2d-2}\big(E_C,\mathbf{Q}_p(d-1)\big)\xrightarrow{\mathrm{H}^{2d}\left(X_C,\mathrm{cl}_i(d-1)\right)}\mathrm{H}^{2d}\big(X_C,\mathbf{Q}_p(d)\big)\xrightarrow{t_{X,\text{ét}}}\mathbf{Q}_p$ is equal to $t_{E,\text{ét}}$. Combined with Lemma 3.5.4, the statement is therefore reduced to proving that $t_{E,\text{ét}}\big(c\ell_E(x)\big)=1$. This follows immediately from the induction hypothesis. □

Remark 6.2.9. In [Man22, Cor. 3.10.22], Mann proves Poincaré duality for smooth proper morphisms of rigid-analytic spaces over a nonarchimedean field extension K of $\mathbf{Q}_p$ along the following lines:

(1) For any analytic adic space X over $\mathbf{Q}_p$ (or more generally small v-stack), he defines an ∞-category $\mathcal{D}^{\mathrm{a}}_{\square}(\mathcal{O}^+_X/p)^{\varphi}$ of "almost quasicoherent solid φ-modules over $\mathcal{O}^+_X/p$" [Man22, Th. 3.9.10.(b)] together with a fully faithful "Riemann–Hilbert functor" from overconvergent étale $\mathbf{F}_p$-sheaves

$$-\otimes^L\mathcal{O}^{+,\mathrm{a}}_X/p\colon \mathcal{D}_{\text{ét}}(X,\mathbf{F}_p)^{\mathrm{oc}}\hookrightarrow\mathcal{D}^{\mathrm{a}}_{\square}(\mathcal{O}^+_X/p)^{\varphi}. \tag{6.2.10}$$

This functor admits a right adjoint, which is (locally in the v-topology) roughly given by taking φ-invariants and induces an equivalence on perfect objects [Man22, Th. 1.2.7].

(2) He develops a 6-functor formalism for $\mathcal{D}^{\mathrm{a}}_{\square}(\mathcal{O}^+_X/p)^{\varphi}$ [Man22, Th. 1.2.4].

(3) For any smooth morphism $f\colon X\to Y$ of equidimension d between analytic adic spaces over $\mathbf{Q}_p$, he proves that $\mathrm{R}f^!\big(\mathcal{O}^{+,\mathrm{a}}_Y/p\big)\simeq\mathcal{O}^{+,\mathrm{a}}_X/p(d)[2d]$ [Man22, Th. 1.2.8]. When f is in addition proper, this leads, under the equivalence from (6.2.10), to a Poincaré duality for perfect complexes of overconvergent étale $\mathbf{F}_p$-sheaves; the resulting trace map corresponds to the counit $\mathrm{R}f_!\mathrm{R}f^!\big(\mathcal{O}^{+,\mathrm{a}}_Y/p\big)\simeq\mathrm{R}f_*\big(\mathcal{O}^{+,\mathrm{a}}_X/p(d)[2d]\big)\to\mathcal{O}^{+,\mathrm{a}}_Y/p$.

However, it does not seem straightforward to compare Mann's trace map and Poincaré duality isomorphism with the one from this paper. In fact, the 6-functor formalism in (2) is not compatible with Huber's functors from [Hub96] when the latter are defined. On the one hand, Huber's $\mathrm{R}f_!$ need not preserve overconvergent sheaves unless f is partially proper, hence cannot be given as the φ-invariants of Mann's compactly supported pushforward functor.

On the other hand, Mann's compactly supported pushforward functor is not the image of Huber's compactly supported pushforward under (6.2.10): For instance, when $f\colon X:=\mathring{\mathbf{D}}^1_C\to\operatorname{Spa}(C,\mathcal{O}_C)$ is the structure

[25] We implicitly use that the Gysin map $i_*\underline{\mathbf{Q}}_p\to\underline{\mathbf{Q}}_p(1)[2]$ defined in [LLZ23, Rmk. 4.3.12] coincides with cl_i. For this, we note that [BH22, Th. 3.21] implies that $\mathrm{R}\mathscr{H}om\big(i_*\underline{\mathbf{Q}}_p,\underline{\mathbf{Q}}_p(1)[2]\big)=i_*\underline{\mathbf{Q}}_p$, so the question whether two maps between these complexes coincide is étale local on X. Thus, it suffices to prove the claim for $\mathbf{A}^{d-1,\mathrm{an}}_k\hookrightarrow\mathbf{A}^{d,\mathrm{an}}_k$. In this case, both maps coincide with the analytification of the algebraic cycle class map due to Lemma 3.4.1 and the claim that the construction in [LLZ23, Rmk. 4.3.12] coincides with the construction in [Fal02, § 4].

morphism of the open unit disk over an algebraically closed nonarchimedean field C of mixed characteristic $(0, p)$, Mann proves that in his formalism the natural transformation of functors $\mathrm{R}f^!\big(\mathcal{O}_X^{+,\mathrm{a}}/p\big) \otimes^L Lf^*(-) \to \mathrm{R}f^!(-)$ is an equivalence [Man22, Th. 3.10.17, Prop. 3.8.4.(i)]. In particular, $\mathrm{R}f^!(-)$ preserves filtered colimits, so its left adjoint $\mathrm{R}f_!(-)$ preserves almost compact objects. If Mann's $\mathrm{R}f_!\big(\mathcal{O}_X^{+,\mathrm{a}}/p\big)$ was isomorphic to the image of Huber's $\mathrm{R}f_!(\underline{\mathbf{F}}_p)$ under the fully faithful functor (6.2.10), it would also be discrete and thus a perfect object of $\mathcal{D}^{\mathrm{a}}_{\square}(\mathcal{O}_C/p)^{\varphi}$; cf. [Man22, Prop. 3.7.5, Def. 3.9.15]. However, since (6.2.10) induces an equivalence of perfect objects, this would imply that Huber's $\mathrm{H}^2_c\big(\mathring{\mathbf{D}}^1_C, \mathbf{F}_p\big)$ is finite-dimensional, contradicting Example 6.3.3. 2.

6.3. **Digression: Künneth Formula.** In this subsection, we establish a version of the Künneth Formula that we will crucially use in our proof of Poincaré duality in Section 6.4. We recall that we fix a positive integer n and put $\Lambda := \mathbf{Z}/n\mathbf{Z}$.

Lemma 6.3.1 (Proper Base Change)**.** *Consider a cartesian diagram of locally noetherian analytic adic spaces*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with $n \in \mathcal{O}_Y^\times$ and f proper. Let $\mathcal{F} \in D(X_{\text{ét}}; \Lambda)$ be a complex of étale sheaves such that, for each geometric point $\overline{y} \to Y$ of rank 1, the restriction $\mathcal{F}\big|_{X_{\overline{y}}}$ lies in $D_{zc}(X_{\overline{y},\text{ét}}; \Lambda)$. Then the natural morphism

$$\mathrm{BC}_{f,g} \colon g^*\mathrm{R}f_*\mathcal{F} \to \mathrm{R}f'_* g'^*\mathcal{F}$$

is an isomorphism.

Proof. First, the claim is étale local on Y and Y', so we can assume that they are both affinoid. Then [Zav23a, Lem. 9.1.(1)] implies that f has bounded cohomological dimension. Thus, a standard argument allows us to reduce to the case $\mathcal{F} \in \mathrm{Shv}(X_{\text{ét}}; \Lambda)$. In this case, [Hub96, Prop. 8.2.3] ensures that all cohomology sheaves of $g^*\mathrm{R}f_*\mathcal{F}$ and $\mathrm{R}f'_*g'^*\mathcal{F}$ are overconvergent. Therefore, it suffices to show that $\mathrm{BC}_{f,g}$ is an isomorphism at geometric rank-1 points. Thanks to [Hub96, Prop. 2.6.1], we may thus assume that $Y' = \mathrm{Spa}\,(C', \mathcal{O}_{C'})$ and $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$ are both geometric points of rank 1. Now if $\mathrm{char}\, C > 0$, the result follows from [Hub96, Th. 4.4.1(a)]. If $\mathrm{char}\, C = 0$, then the result follows from [BH22, Th. 3.15]. □

Remark 6.3.2. We note that [BH22, Th. 3.15] crucially uses perfectoid spaces in its proof. However, this is the only instance where we need to use perfectoid spaces in this paper.

We note that proper base change fails for more general coefficients:

Example 6.3.3. Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0, p)$.

(1) (constructible example[26]) Let $j \colon \mathbf{D}^1_C \hookrightarrow \mathbf{P}^{1,\mathrm{an}}_C$ be the natural inclusion of the closed unit disk into the analytic projective line. Then proper base change for the sheaf $\mathcal{F} = j_!\mu_p$ would, in particular, imply that $\mathrm{R}\Gamma_c(\mathbf{D}^1_C, \mu_p)$ does not depend on the choice of the algebraically closed ground field C. However, $\mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p)$ does depend on C due to Lemma 5.5.21: one can take $C \subset C'$ such that the cardinality of the residue field of C' is bigger than $\mathrm{H}^2_c(\mathbf{D}^1_C, \mu_p)$.

(2) (overconvergent example) Let $j \colon \mathring{\mathbf{D}}^1_C \hookrightarrow \mathbf{P}^{1,\mathrm{an}}_C$ be the natural inclusion of the open unit disk into the analytic projective line. Similarly, proper base change for the sheaf $\mathcal{F} = j_!\mu_p$ would, in particular, imply that $\mathrm{R}\Gamma_c(\mathring{\mathbf{D}}^1_C, \mu_p)$ does not depend on the choice of the algebraically closed ground field C. The closed complement to j is equal to $\mathbf{D}^{1,c}_C$, the universal compactification of the closed unit disc. Therefore, the excision sequence and Proposition 5.1.2 imply that we have the following exact triangle:

$$\mathrm{R}\Gamma_c(\mathring{\mathbf{D}}^1_C, \mu_p) \to \mathrm{R}\Gamma(\mathbf{P}^{1,\mathrm{an}}_C, \mu_p) \to \mathrm{R}\Gamma(\mathbf{D}^1_C, \mu_p)$$

[26]Here, we use the word "constructible" in the sense of [Hub96, Def. 2.7.2]. In particular, a sheaf which is both constructible and Zariski-constructible must be a local system due to [Hub96, Lem. 2.7.10].

Hence, proper base change for $j_!\mu_p$ would, in particular, imply that $\mathrm{R\Gamma}(\mathbf{D}^1_C, \mu_p)$ is independent of the choice of algebraically closed C. However, $\mathrm{H}^1(\mathbf{D}^1_C, \mu_p)$ does depend on C due to Remark 5.1.14: one can take $C \subset C'$ such that the cardinality of $\mathfrak{m}_{C'}/p\mathfrak{m}_{C'}$ is bigger than $\mathrm{H}^1(\mathbf{D}^1_C, \mu_p)$).

Construction 6.3.4 (Künneth map)**.** Consider the following commutative diagram of locally noetherian analytic adic spaces:

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ X' \arrow[r, "f'"] & Y \end{tikzcd} \tag{6.3.5}$$

Let $\mathcal{E} \in D(X_{\text{ét}}; \Lambda)$ and $\mathcal{E}' \in D(X'_{\text{ét}}; \Lambda)$. We define the *Künneth map*

$$\mathrm{KM}\colon \mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f'_*\mathcal{E}' \to \mathrm{R}h_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}')$$

as the adjoint to the map

$$h^*\big(\mathrm{R}f_*(\mathcal{E}) \otimes^L \mathrm{R}f'_*(\mathcal{E}')\big) \simeq (g^*f^*\mathrm{R}f_*\mathcal{E}) \otimes^L (g'^*f'^*\mathrm{R}f'_*\mathcal{E}') \xrightarrow{g^*(\epsilon_f)\otimes^L g'^*(\epsilon_{f'})} g^*\mathcal{E} \otimes^L g'^*\mathcal{E}',$$

where ϵ_f (resp. $\epsilon_{f'}$) denotes the counit of the $(f^*, \mathrm{R}f_*)$-adjunction (resp. the $(f'^*, \mathrm{R}f'_*)$-adjunction).

The Künneth map is functorial in both variables $\mathcal{E}$ and $\mathcal{E}'$, and in Diagram (6.3.5).

Remark 6.3.6. We note that the Künneth map boils down to the cup-product map (see [Sta22, Tag 0B6C]) when $W = X = X'$, $f = f'$, and $g = g' = \mathrm{id}_X$.

Remark 6.3.7. Now consider the case $X = X'$, $f = f'$, and $W = X \times_Y X$ with g and g' being the natural projection maps. In this situation, the functoriality in W (with respect to the morphism $\Delta\colon X \to X \times_Y X$) implies that the composition

$$\mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f_*\mathcal{E}' \xrightarrow{\mathrm{KM}} \mathrm{R}h_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}') \xrightarrow{\mathrm{R}h_*(\eta_\Delta)} \mathrm{R}h_*\big(\mathrm{R}\Delta_*\Delta^*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}')\big) =$$
$$= \mathrm{R}f_*(\Delta^*\left(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}'\right)) \simeq \mathrm{R}f_*(\mathcal{E} \otimes^L \mathcal{E}')$$

is equal to the cup-product morphism, where η_Δ denotes the unit of the $(\Delta^*, \mathrm{R}\Delta_*)$-adjunction.

Now we wish to prove that under some assumptions, the Künneth map is an isomorphism. For this, we will need the following very general lemma:

Lemma 6.3.8. *Consider a commutative diagram of locally noetherian analytic adic spaces*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ X' \arrow[r, "f'"] & Y. \end{tikzcd}$$

Let $\mathcal{E} \in D(X_{\text{ét}}; \Lambda)$ and $\mathcal{E}' \in D(X'_{\text{ét}}; \Lambda)$. Then the following diagram commutes:

$$\begin{tikzcd} \mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f'_*\mathcal{E}' \arrow[r, "\mathrm{PF}_f"] \arrow[d, "\mathrm{KM}"] & \mathrm{R}f_*\big(\mathcal{E} \otimes^L (f^*\mathrm{R}f'_*\mathcal{E}')\big) \arrow[d, "\mathrm{R}f_*(\mathrm{id}\otimes \mathrm{BC}_{f',f})"] \\ \mathrm{R}h_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}') \simeq \mathrm{R}f_*\mathrm{R}g_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}') & \mathrm{R}f_*\big(\mathcal{E} \otimes^L (\mathrm{R}g_*g'^*\mathcal{E}')\big). \arrow[l, "\mathrm{R}f_*(\mathrm{PF}_g)"'] \end{tikzcd}$$

Here, BC *stands for the base change morphism, and* PF *stands for the projection formula morphism (see* [Sta22, Tag 07A7] *and* [Sta22, Tag 0B56]*).*

Proof. In this proof, we denote the counit of the $(f^*, \mathrm{R}f_*)$-adjunction by ϵ_f. Then we consider the following diagram:

$$\begin{tikzcd}
h^*(\mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f'_*\mathcal{E}') \arrow[r, "h^*(\mathrm{PF}_f)"] \arrow[dr, "g^*(\epsilon_f \otimes \mathrm{id})"'] & h^*\mathrm{R}f_*(\mathcal{E} \otimes^L f^*\mathrm{R}f'_*\mathcal{E}') \arrow[r, "h^*\mathrm{R}f_*(\mathrm{id} \otimes \mathrm{BC}_{f',f})"] \arrow[d, "g^*(\epsilon_f)"] & h^*\mathrm{R}f_*(\mathcal{E} \otimes^L \mathrm{R}g_*g'^*\mathcal{E}') \arrow[r, "h^*(\mathrm{R}f_*(\mathrm{PF}_g))"] \arrow[d, "g^*(\epsilon_f)"] & h^*\mathrm{R}f_*\mathrm{R}g_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}') \arrow[d, "g^*(\epsilon_f)"] \\
& g^*(\mathcal{E} \otimes^L f^*\mathrm{R}f'_*\mathcal{E}') \arrow[r, "g^*(\mathrm{id} \otimes \mathrm{BC}_{f',f})"] \arrow[dr, "\sim"'] & g^*(\mathcal{E} \otimes^L \mathrm{R}g_*g'^*\mathcal{E}') \arrow[r, "g^*(\mathrm{PF}_g)"] \arrow[dr, "\mathrm{id} \otimes \epsilon_g"] & g^*\mathrm{R}g_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}') \arrow[d, "\epsilon_g"] \\
& & g^*\mathcal{E} \otimes^L g'^* f'^* \mathrm{R}f'_*\mathcal{E}' \arrow[r, "\mathrm{id} \otimes g'^*(\epsilon_{f'})"] & g^*\mathcal{E} \otimes^L g'^*\mathcal{E}',
\end{tikzcd}$$

where we implicitly identify $g^* \circ f^* \simeq h^* \simeq g'^* \circ f'^*$. Note that the diagram above commutes: indeed, the top left and bottom right triangles commute due to the definition of the projection formula morphism, the bottom parallelogram commutes due to the definition of the base change morphism, and the top two squares commute due to the functoriality of ϵ_f. Now it only remains to observe that if we go from the top left to the bottom right corner in the clockwise direction in the above diagram, we get the $(h^*, \mathrm{R}h_*)$-adjoint of the composition $\mathrm{R}f_*(\mathrm{PF}_g) \circ \mathrm{R}f_*(\mathrm{id} \otimes \mathrm{BC}_{f',f}) \circ \mathrm{PF}_f$, whereas if we go counterclockwise, we get the morphism $g^*(\epsilon_f) \otimes g'^*(\epsilon_{f'})$. By definition, the latter is the $(h^*, \mathrm{R}h_*)$-adjoint of the Künneth map from Construction 6.3.4. $\square$

Corollary 6.3.9 (Künneth formula)**.** *Consider a cartesian diagram of locally noetherian analytic adic spaces*

$$\begin{tikzcd}
X \times_Y X' \arrow[r, "g"] \arrow[d, "g'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\
X' \arrow[r, "f'"] & Y
\end{tikzcd}$$

with $n \in \mathcal{O}_Y^\times$. Let $\mathcal{E} \in D(X_{\text{ét}}; \Lambda)$ and $\mathcal{E}' \in D(X'_{\text{ét}}; \Lambda)$ such that, for each geometric point geometric point $\overline{y} \to Y$ of rank 1, the restriction $\mathcal{E}\big|_{X_{\overline{y}}}$ lies in $D_{zc}(X_{\overline{y},\text{ét}}; \Lambda)$. If f and f' are proper, then the Künneth map

$$\mathrm{KM}\colon \mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f'_*\mathcal{E}' \to \mathrm{R}h_*(g^*\mathcal{E} \otimes^L g'^*\mathcal{E}')$$

is an isomorphism.

Proof. This follows directly from Lemma 6.3.8, proper base change (see Lemma 6.3.1), and the projection formula (see [Zav23a, Prop. 9.3.(2)]). $\square$

6.4. **Poincaré duality for locally constant coefficients, revisited.** In this subsection, we prove Poincaré duality for smooth proper morphisms and lisse sheaves. Our proof will simultaneously show that higher direct images along smooth proper morphisms preserve lisse sheaves. Somewhat surprisingly, our proofs will be essentially formal and diagrammatic in nature. As before, we fix a positive integer n and set $\Lambda := \mathbf{Z}/n$.

We first verify both claims for the subclass of perfect lisse complexes, and then extend the results to the general case. For this, we recall that perfect lisse complexes have the following categorical description: for a locally noetherian analytic adic space X, the category of perfect complexes in $D(X_{\text{ét}}; \Lambda)$ coincides with the category dualizable[27] objects and this category is contained in the full subcategory $D^{(b)}_{\text{lis}}(X_{\text{ét}}; \Lambda)$ of locally bounded complexes of étale sheaves with lisse cohomology sheaves; see e.g. [Sta22, Tag 0FPU, Tag 0FPV], and [Zav23a, Lem. 11.1].

Theorem 6.4.1 (cf. [Zav21b, Th. 1.1.2] and [Man22, Cor. 1.2.9])**.** *Let X and Y be locally noetherian analytic adic spaces such that $n \in \mathcal{O}_Y^\times$, let $f\colon X \to Y$ be a smooth proper morphism of equidimension d, and let $\mathcal{E} \in D(X_{\text{ét}}; \Lambda)$ be a dualizable object with left dual $\mathcal{E}^\vee := \mathrm{R}\mathscr{H}om_X(\mathcal{E}, \underline{\Lambda}_X)$. Then the evaluation and coevaluation maps from Construction 6.4.2 below make $\mathrm{R}f_*(\mathcal{E}^\vee(d)[2d])$ into a left dual of $\mathrm{R}f_*\mathcal{E}$. In particular $\mathrm{R}f_*\mathcal{E}$ is a dualizable object of $D(Y_{\text{ét}}; \Lambda)$ and there is a natural isomorphism*

$$\mathrm{PD}_f(\mathcal{E})\colon \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_Y(\mathrm{R}f_*\mathcal{E}, \underline{\Lambda}_Y).$$

[27] For the notion of dualizable objects (or objects having "left dual"), we refer the reader to [Sta22, Tag 0FFP].

We first explain how to construct the evaluation map, the coevalution map, and the duality map $\mathrm{PD}_f(\mathbf{E})$ in the statement of Theorem 6.4.1. Later, we check that these maps indeed define the structure of a dualizable object on $\mathrm{R}f_*\mathcal{E}$.

Construction 6.4.2. Let $f\colon X \to Y$ be as in Theorem 6.4.1. Let $\mathcal{E} \in D_{\mathrm{lis}}(X;\Lambda)$ and let $\mathcal{E}^\vee := \mathrm{R}\mathscr{H}om_X(\mathcal{E}, \underline{\Lambda}_X)$ be its (naive) dual. We denote by $ev_{\mathcal{E}}\colon \mathcal{E}^\vee \otimes \mathcal{E} \to \underline{\Lambda}_X$ the natural evaluation map. If $\mathcal{E}$ is dualizable, we also denote by $coev_{\mathcal{E}}\colon \underline{\Lambda}_X \to \mathcal{E}^\vee \otimes \mathcal{E}$ its coevaluation map. Moreover, we use the notation of the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, "h"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(1) (*Evaluation map*) We define the *evaluation map* $e(f,\mathcal{E})\colon \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \otimes^L \mathrm{R}f_*(\mathcal{E}) \to \underline{\Lambda}_Y$ as the composition

$$\mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \otimes^L \mathrm{R}f_*(\mathcal{E}) \xrightarrow{\cup} \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d] \otimes^L \mathcal{E}) \xrightarrow{\mathrm{R}f_*(ev_{\mathcal{E}}(d)[2d])} \mathrm{R}f_*(\underline{\Lambda}_X(d)[2d]) \xrightarrow{\mathrm{tr}_f} \underline{\Lambda}_Y,$$

where $\cup$ is the cup-product map from [Sta22, Tag 0B6C] (Remark 6.3.6) and tr_f is the trace morphism from Theorem 6.1.1.

(2) (*Duality map*) We define the *duality map* $\mathrm{PD}_f(\mathcal{E})\colon \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \to \mathrm{R}\mathscr{H}om_Y(\mathrm{R}f_*\mathcal{E}, \underline{\Lambda}_Y)$ as the map adjoint to $e(f,\mathcal{E})$ under the tensor-hom adjunction. In other words, $\mathrm{PD}_f(\mathcal{E})$ is the composition

$$\mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \longrightarrow \mathrm{R}\mathscr{H}om_Y(\mathrm{R}f_*\mathcal{E}, \mathrm{R}f_*\underline{\Lambda}_X(d)[2d]) \xrightarrow{\mathrm{tr}_f\circ -} \mathrm{R}\mathscr{H}om_Y(\mathrm{R}f_*\mathcal{E}, \underline{\Lambda}_X),$$

where the first map comes from [Sta22, Tag 0B6D].

(3) (*Coevaluation map*) Now we also assume that $\mathcal{E}$ is dualizable. We define the *coevaluation map*

$$c(f,\mathcal{E})\colon \underline{\Lambda}_Y \to \mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d])$$

as the composition

$$\begin{tikzcd}
\underline{\Lambda}_Y \arrow[r, "\eta_f"] & \mathrm{R}f_*\underline{\Lambda}_X \arrow[r, "\mathrm{R}f_*(coev_{\mathcal{E}})"] & \mathrm{R}f_*(\mathcal{E} \otimes^L \mathcal{E}^\vee) \arrow[r, "\sim"] & \mathrm{R}h_*(\mathrm{R}\Delta_*\Delta^*(\pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee)) \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})", "\wr"'] \\
\mathrm{R}f_*(\mathcal{E}) \otimes^L \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) & \mathrm{R}h_*\left(\pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee(d)[2d]\right) \arrow[l, "\sim", "\mathrm{KM}^{-1}"'] & & \mathrm{R}h_*\left((\pi_1^*\mathcal{E} \otimes^L \pi_2^*\mathcal{E}^\vee) \otimes^L \Delta_*(\underline{\Lambda}_X)\right), \arrow[ll, "\mathrm{R}h_*(\mathrm{id}\otimes^L \mathrm{cl}_\Delta)"]
\end{tikzcd}$$

where $\mathrm{cl}_\Delta := \mathrm{cl}_{X\times_Y X}(X)$ is the cycle class map[28] introduced in Variant 3.3.3, PF is the projection formula map, and KM is the Künneth map introduced in Construction 6.3.4. We crucially use [Zav23a, Prop. 9.3.(2)] and Corollary 6.3.9 to invert the projection formula map and the Künneth map, respectively.

Given the evaluation and coevaluation maps, the proof of Theorem 6.4.1 essentially boils down to verifying that some diagrams commute. To do this, we need some preliminary lemmas:

Lemma 6.4.3. *Keep the notation of Construction 6.4.2. Then the diagram*

$$\begin{tikzcd}
\mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f_*\underline{\Lambda}_X(d)[2d] \arrow[r, "\mathrm{id}\otimes^L \mathrm{tr}_f"] \arrow[d, "\mathrm{KM}"] & \mathrm{R}f_*\mathcal{E} \\
\mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes^L \underline{\Lambda}_{X\times_Y X}(d)[2d]) \arrow[r, "\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1})"] & \mathrm{R}f_*(\mathcal{E} \otimes^L \mathrm{R}\pi_{1,*}\Lambda_{X\times_Y X}(d)[2d]) \arrow[u, "\mathrm{R}f_*(\mathrm{id}\otimes^L \mathrm{tr}_{\pi_1})"']
\end{tikzcd}$$

commutes.

[28]Note that Δ is an lci immersion of pure codimension d due to [Zav23a, Cor. 5.11].

Proof. First, we note that it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}f_*\mathcal{E} \arrow[d, equal] & \mathrm{R}f_*\mathcal{E} \otimes^L \mathrm{R}f_*\underline{\Lambda}_X(d)[2d] \arrow[l, "\mathrm{id}\otimes^L \mathrm{tr}_f"'] \arrow[r, "\mathrm{KM}"] \arrow[d, "\mathrm{PF}_f"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes^L \underline{\Lambda}_{X\times_Y X}(d)[2d]) \\
\mathrm{R}f_*\mathcal{E} & \mathrm{R}f_*(\mathcal{E} \otimes f^*\mathrm{R}f_*\underline{\Lambda}_X(d)[2d]) \arrow[l, "\mathrm{R}f_*(\mathrm{id}\otimes^L f^* \mathrm{tr}_f)"'] \arrow[r, "\mathrm{R}f_*(\mathrm{id}\otimes^L \mathrm{BC}_{f,f})"] & \mathrm{R}f_*(\mathcal{E} \otimes^L \mathrm{R}\pi_{1,*}\underline{\Lambda}_{X\times_Y X}(d)[2d]) \arrow[u, "\mathrm{R}f_*(\mathrm{PF}_{\pi_1})"'] \arrow[ll, bend left, "\mathrm{R}f_*(\mathrm{id}\otimes^L \mathrm{tr}_{\pi_1})"]
\end{tikzcd}$$

For this, one can easily check that the left square commutes using the very definition of the projection formula morphism (see [Sta22, Tag 0B56]). Lemma 6.3.8 ensures that the right square commutes and Theorem 6.1.1 (2) guarantees that the bottom triangle commutes. $\square$

For the next lemma, we need to introduce a new construction:

Construction 6.4.4. Let $i\colon X \hookrightarrow Y$ be an lci closed immersion of pure codimension c and let $\mathcal{F}$ be an object of $D(Y_{\text{ét}};\Lambda)$. Then we define the *cycle class morphism* $\mathrm{cl}_i(\mathcal{F})\colon i_*i^*\mathcal{F} \to \mathcal{F} \otimes \underline{\Lambda}_Y(c)[2c]$ as the composition

$$i_*i^*\mathcal{F} \xrightarrow[\sim]{\mathrm{PF}_i^{-1}} \mathcal{F} \otimes^L i_*\underline{\Lambda}_X \xrightarrow{\mathrm{id}\otimes^L \mathrm{cl}_i} \mathcal{F} \otimes^L \underline{\Lambda}_Y(c)[2c],$$

where cl_i is the cycle class map from Variant 3.3.3.

Lemma 6.4.5. *Let X and Y be locally noetherian analytic adic spaces with $n \in \mathcal{O}_Y^\times$, let $f\colon X \to Y$ be a smooth proper morphism of equidimension d. Assume that f has a section $s\colon Y \to X$. Then the compositions*

$$\underline{\Lambda}_Y \simeq \mathrm{R}f_*(s_*\underline{\Lambda}_Y) \xrightarrow{\mathrm{cl}_s} \mathrm{R}f_*(\underline{\Lambda}_X(d)[2d]) \xrightarrow{\mathrm{tr}_f} \underline{\Lambda}_Y \quad \textit{and}$$

$$\begin{aligned}\underline{\Lambda}_Y(d)[2d] \simeq \mathrm{R}f_*\big(s_*\underline{\Lambda}_Y(d)[2d]\big) &\xrightarrow{\mathrm{R}f_*(\mathrm{cl}_s(\underline{\Lambda}(d)[2d]))} \mathrm{R}f_*\big(\underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_X(d)[2d]\big) \\ &\xrightarrow[\sim]{\mathrm{PF}_f^{-1}} \mathrm{R}f_*\big(\underline{\Lambda}_X(d)[2d]\big) \otimes^L \underline{\Lambda}_Y(d)[2d] \xrightarrow{\mathrm{tr}_f \otimes^L \mathrm{id}} \underline{\Lambda}_Y(d)[2d]\end{aligned}$$

are equal to the identity morphisms.

Proof. For brevity, we denote the first composition by α and the second composition by β. For any objects $\mathcal{F}, \mathcal{G} \in D(X_{\text{ét}};\Lambda)$, we denote by $\sigma\colon \mathcal{F} \otimes^L \mathcal{G} \xrightarrow{\sim} \mathcal{G} \otimes^L \mathcal{F}$ the morphism that "swaps factors". Then Lemma 6.2.2 directly implies that $\alpha = \mathrm{id}$.

To see that $\beta = \mathrm{id}$, we now prove the stronger assertion that $\beta = \alpha(d)[2d]$. Since $\underline{\Lambda}_Y(d)[2d]$ is a locally free sheaf concentrated in degree $2d$, our sign conventions for the commutativity constraint (see [Sta22, Tag 0GWN]) imply that the morphism $\sigma\colon \underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_X(d)[2d] \to \underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_X(d)[2d]$ is given by multiplication with $(-1)^{(2d)\cdot(2d)} = 1$; that is, it is the identity morphism. This observation implies that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}f_*s_*\underline{\Lambda}_Y(d)[2d] \arrow[r, "\mathrm{R}f_*(\mathrm{PF}_s^{-1})"] & \mathrm{R}f_*\big(\underline{\Lambda}_X(d)[2d] \otimes^L s_*\underline{\Lambda}_Y\big) \arrow[r, "\mathrm{R}f_*(\mathrm{id}\otimes^L \mathrm{cl}_s)"] \arrow[d, "\mathrm{PF}_f^{-1}"] & \mathrm{R}f_*\big(\underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_X(d)[2d]\big) \arrow[r, "\mathrm{R}f_*(\sigma)=\mathrm{id}", "\sim"'] \arrow[d, "\mathrm{PF}_f^{-1}"] & \mathrm{R}f_*\big(\underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_X(d)[2d]\big) \arrow[d, "\mathrm{PF}_f^{-1}"] \\
\underline{\Lambda}_Y(d)[2d] \arrow[u, "\wr"] \arrow[r, "\sim"] & \underline{\Lambda}_Y(d)[2d] \otimes^L \mathrm{R}f_*s_*\underline{\Lambda}_Y \arrow[r, "\mathrm{id}\otimes^L \mathrm{cl}_s"] & \underline{\Lambda}_Y(d)[2d] \otimes^L \mathrm{R}f_*\underline{\Lambda}_X(d)[2d] \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}\otimes^L \mathrm{tr}_f"] & \mathrm{R}f_*\underline{\Lambda}_X(d)[2d] \otimes^L \underline{\Lambda}_Y(d)[2d] \arrow[dl, "\mathrm{tr}_f \otimes^L \mathrm{id}"] \\
 & & \underline{\Lambda}_Y(d)[2d]. &
\end{tikzcd}$$

Therefore, the map $\beta\colon \underline{\Lambda}_Y(d)[2d] \to \underline{\Lambda}(d)[2d]$ (given by the blue arrows in the diagram above) is equal to $\mathrm{id} \otimes \alpha = \alpha(d)[2d]$ (given by the red arrows), as desired. $\square$

Proof of Theorem 6.4.1. The only thing we need to check is that the compositions[29]

$$\mathrm{R}f_*\mathcal{E} \xrightarrow{\text{“}c(f,\mathcal{E})\otimes^L \mathrm{id}\text{”}} \mathrm{R}f_*\mathcal{E}\otimes^L \mathrm{R}f_*\mathcal{E}^\vee(d)[2d]\otimes^L \mathrm{R}f_*\mathcal{E} \xrightarrow{\text{“}\mathrm{id}\otimes^L e(f,\mathcal{E})\text{”}} \mathrm{R}f_*\mathcal{E},$$

$$\mathrm{R}f_*\mathcal{E}^\vee(d)[2d] \xrightarrow{\text{“}\mathrm{id}\otimes^L c(f,\mathcal{E})\text{”}} \mathrm{R}f_*\mathcal{E}^\vee(d)[2d]\otimes^L \mathrm{R}f_*\mathcal{E}\otimes^L \mathrm{R}f_*\mathcal{E}^\vee(d)[2d] \xrightarrow{\text{“}e(f,\mathcal{E})\otimes^L \mathrm{id}\text{”}} \mathrm{R}f_*\mathcal{E}^\vee(d)[2d]$$

are equal to the identity morphisms. For brevity, we denote the first composition by $\varphi(f,\mathcal{E})$ and the second composition by $\psi(f,\mathcal{E})$. We give a full justification for why $\varphi(f,\mathcal{E}) = \mathrm{id}$, and then only describe the necessary changes to justify that $\psi(f,\mathcal{E}) = \mathrm{id}$.

Recall that we denote by η the unit of the (derived) pullback-pushforward adjunction, by KM the Künneth map from Construction 6.3.4, by PF the projection formula map, and, for any objects $\mathcal{F},\mathcal{G}\in D(X_{\acute{e}t};\Lambda)$, by $\sigma\colon \mathcal{F}\otimes^L\mathcal{G}\xrightarrow{\sim}\mathcal{G}\otimes^L\mathcal{F}$ the morphism that “swaps factors”. In what follows, we also freely use Corollary 6.3.9 and [Zav23a, Prop. 9.3] which guarantee that the Künneth map and the projection formula map are isomorphisms under some assumptions that are always satisfied in this proof.

That being said, we resume the notation of Construction 6.4.2 and consider the diagram in Fig. 1. Using the definitions of PF, KM, and basic properties of adjunctions, one can check that this diagram commutes. For the most part, the verification is very similar to that in the proof of Lemma 6.3.8 with the following two exceptions: triangle (6.4.6) commutes due to the assumption $ev_{\mathcal{E}}$ and $coev_{\mathcal{E}}$ define a duality datum on $\mathcal{E}$, and trapezoid (6.4.7) commutes due to Lemma 6.4.3.

The map $\Lambda\otimes^L \mathrm{R}f_*\mathcal{E}\to \mathrm{R}f_*\mathcal{E}\otimes^L\Lambda$ obtained by going down the entire left column is equal to

$$\big(\mathrm{id}\otimes^L e(f,\mathcal{E})\big)\circ\big(c(f,\mathcal{E})\otimes^L \mathrm{id}\big)$$

by its very construction. The commutativity of the diagram in Fig. 1 implies that this composition can be computed by going around the outer diagram from the top left corner to the bottom left corner in a clockwise direction. Furthermore, we see that Lemma 6.4.5 and the formula $\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1})\circ \mathrm{R}h_*(\mathrm{PF}_{\Delta}^{-1}) = \mathrm{id}$ imply that

$$\big(\mathrm{id}\otimes^L e(f,\mathcal{E})\big)\circ\big(c(f,\mathcal{E})\otimes^L \mathrm{id}\big) = \mathrm{PF}_f^{-1}\circ \mathrm{R}f_*(\sigma)\circ \mathrm{PF}_f = \sigma\colon \Lambda\otimes^L \mathrm{R}f_*\mathcal{E}\to \mathrm{R}f_*\mathcal{E}\otimes^L\Lambda.$$

This formally implies that $\varphi(f,\mathcal{E}) = \mathrm{id}\colon \mathrm{R}f_*\mathcal{E}\to \mathrm{R}f_*\mathcal{E}$.

To see that $\psi(f,\mathcal{E}) = \mathrm{id}$, we need to use a diagram similar to that of Fig. 1; we leave it to the reader to figure out the exact shape of the diagram. We only mention that every instance of π_1 should be replaced with π_2 (and vice versa) and one needs to use the second part of Lemma 6.4.5 (as opposed to the first part used in the proof above). ☐

As the first application of Theorem 6.4.1, we show that derived pushforwards along smooth and proper morphisms preserve lisse sheaves.

Corollary 6.4.8. *Let $f\colon X\to Y$ be a smooth proper morphism of locally noetherian analytic adic spaces with $n\in\mathcal{O}_Y^\times$. Let $\mathcal{E}\in D_{\mathrm{lis}}(X_{\acute{e}t};\Lambda)$ be a lisse complex. Then $\mathrm{R}f_*\mathcal{E}$ lies in $D_{\mathrm{lis}}(Y_{\acute{e}t};\Lambda)$. If $\mathcal{E}$ is locally bounded (resp. perfect), then so is $\mathrm{R}f_*\mathcal{E}$.*

Proof. The statement is local on Y, so we may assume that Y is affinoid. As the property of being smooth of equidimension d is open on the source, we have a finite disjoint decomposition $X=\bigsqcup_{i=0}^n X_i$ such that $X_i\to Y$ is smooth of equidimension i. Since each X_i is clopen in X, we conclude that each X_i is also proper over Y. Therefore, we can replace X with each X_i separately to assume that f is smooth proper of equidimension d for some $d\in\mathbf{Z}_{\geq 0}$. Then Theorem 6.4.1 and [Zav23a, Lem. 11.1] show that $\mathrm{R}f_*$ preserves perfect complexes. Furthermore, the cohomological dimension of $\mathrm{R}f_*$ is bounded by $2d$ due to [Hub96, Prop. 5.3.11], so it only remains to show that $\mathrm{R}f_*$ preserves lisse complexes.

Using again the finite cohomological dimension of $\mathrm{R}f_*$, we may assume that $\mathcal{E}$ is a bounded below lisse complex. Then a standard argument using [Sta22, Tag 093U] and the Leray spectral sequence from [Sta22, Tag 0732] implies that we can assume that $\mathcal{E}$ is a lisse Λ-module on $X_{\acute{e}t}$. The Chinese remainder theorem implies that we can assume that $\Lambda=\mathbf{Z}/p^m$ for some prime number $p\in\mathcal{O}_Y^\times$. By considering the p-adic filtration on $\mathcal{E}$ and arguing one graded piece at a time, we reduce to the case when $\Lambda=\mathbf{F}_p$ and $\mathcal{E}$ is a lisse

[29]In these formulas, we implicitly make the usual identifications $\mathrm{R}f_*\mathcal{E}\otimes^L\underline{\Lambda}_Y\simeq \mathrm{R}f_*\mathcal{E}$, $\underline{\Lambda}_Y\otimes^L \mathrm{R}f_*\mathcal{E}\simeq \mathrm{R}f_*\mathcal{E}$, etc. To indicate this subtlety, we use quotation marks in the maps which implicitly use these identifications.

$$
\begin{tikzcd}[column sep=small]
\underline{\Lambda} \otimes \mathrm{R}f_*\mathcal{E} \arrow[rr, "\mathrm{PF}_f"] \arrow[d, "\eta_f \otimes \mathrm{id}"] & & \mathrm{R}f_*(\underline{\Lambda} \otimes \mathcal{E}) \arrow[d, "\mathrm{R}f_*(\eta_{\pi_2})"] \arrow[dr, bend left, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})"] & & \\
\mathrm{R}f_*\underline{\Lambda} \otimes \mathrm{R}f_*\mathcal{E} \arrow[rr, "\mathrm{KM}"] \arrow[d, "\mathrm{R}f_*(coev_{\mathcal{E}})\otimes\mathrm{id}"] & & \mathrm{R}h_*(\underline{\Lambda} \otimes \pi_2^*\mathcal{E}) \arrow[r, "\mathrm{R}h_*(\eta_\Delta\otimes\mathrm{id})"] \arrow[d, "\mathrm{R}h_*\left(\pi_1^*(coev_{\mathcal{E}})\otimes\mathrm{id}\right)"] & \mathrm{R}h_*(\Delta_*\underline{\Lambda} \otimes \pi_2^*\mathcal{E}) \arrow[d, "\mathrm{R}h_*\left(\Delta_*(coev_{\mathcal{E}})\otimes\mathrm{id}\right)"] \arrow[dr, bend left, "\mathrm{R}h_*(\mathrm{PF}_\Delta)"] & \\
\mathrm{R}f_*(\mathcal{E} \otimes \mathcal{E}^\vee) \otimes \mathrm{R}f_*\mathcal{E} \arrow[rr, "\mathrm{KM}"] \arrow[d, equal] & & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E}) \arrow[r, "\mathrm{R}h_*(\eta_\Delta\otimes\mathrm{id})"] \arrow[d, equal] & \mathrm{R}h_*\left(\Delta_*(\mathcal{E}\otimes\mathcal{E}^\vee) \otimes \pi_2^*\mathcal{E}\right) \arrow[dd, "\mathrm{R}h_*(\mathrm{PF}_\Delta)"] \arrow[ddr, phantom, "(6.4.6)"] & \mathrm{R}f_*(\underline{\Lambda} \otimes \mathcal{E}) \arrow[ddl, "\mathrm{R}f_*(coev_{\mathcal{E}}\otimes\mathrm{id})"'] \arrow[dd, "\mathrm{R}f_*(\sigma)"] \\
\mathrm{R}f_*\Delta^*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee) \otimes \mathrm{R}f_*\mathcal{E} \arrow[r, "\mathrm{R}f_*(\eta_{\pi_1})\otimes\mathrm{R}f_*(\eta_{\pi_2})"] \arrow[d, equal] & \mathrm{R}h_*\pi_1^*\Delta^*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee) \otimes \mathrm{R}h_*\pi_2^*\mathcal{E} \arrow[r, "\cup"] \arrow[d, "\mathrm{R}h_*(\eta_\Delta)\otimes\mathrm{id}"] & \mathrm{R}h_*\left(\pi_1^*(\Delta^*\pi_1^*\mathcal{E} \otimes \Delta^*\pi_2^*\mathcal{E}^\vee) \otimes \pi_2^*\mathcal{E}\right) \arrow[d, "\mathrm{R}h_*(\eta_\Delta\otimes\mathrm{id})"] & & \\
\mathrm{R}h_*\Delta_*\Delta^*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee) \otimes \mathrm{R}f_*\mathcal{E} \arrow[r, "\mathrm{id}\otimes\mathrm{R}f_*(\eta_{\pi_2})"] \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})\otimes\mathrm{id}"] & \mathrm{R}h_*\left(\Delta_*\Delta^*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee)\right) \otimes \mathrm{R}h_*\pi_2^*\mathcal{E} \arrow[r, "\cup"] \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})\otimes\mathrm{id}"] & \mathrm{R}h_*\left(\Delta_*\Delta^*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee) \otimes \pi_2^*\mathcal{E}\right) \arrow[r, "\mathrm{R}h_*(\mathrm{PF}_\Delta)"] \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1}\otimes\mathrm{id})"] & \mathrm{R}f_*(\mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{E}) \arrow[r, "\mathrm{R}f_*(\mathrm{id}\otimes ev_{\mathcal{E}})"] \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})"] & \mathrm{R}f_*(\mathcal{E} \otimes \underline{\Lambda}) \arrow[d, "\mathrm{R}h_*(\mathrm{PF}_\Delta^{-1})"] \\
\mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee \otimes \Delta_*\underline{\Lambda}) \otimes \mathrm{R}f_*\mathcal{E} \arrow[r, "\mathrm{id}\otimes\mathrm{R}f_*(\eta_{\pi_2})"] \arrow[d, "\mathrm{R}h_*(\mathrm{id}\otimes\mathrm{cl}_\Delta)\otimes\mathrm{id}"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee \otimes \Delta_*\underline{\Lambda}) \otimes \mathrm{R}h_*\pi_2^*\mathcal{E} \arrow[r, "\cup"] \arrow[d, "\mathrm{R}h_*(\mathrm{id}\otimes\mathrm{cl}_\Delta)\otimes\mathrm{id}"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee \otimes \Delta_*\underline{\Lambda} \otimes \pi_2^*\mathcal{E}) \arrow[r, "\mathrm{R}h_*(\mathrm{id}\otimes\mathrm{PF}_\Delta)"] \arrow[d, "\mathrm{R}h_*(\mathrm{id}\otimes\mathrm{cl}_\Delta\otimes\mathrm{id})"'] \arrow[rr, bend right, "\mathrm{R}h_*\left(\mathrm{id}\otimes\pi_2^*(ev_{\mathcal{E}})\otimes\mathrm{id}\right)"'] & \mathrm{R}h_*\left(\pi_1^*\mathcal{E} \otimes \Delta_*(\mathcal{E}^\vee \otimes \mathcal{E})\right) \arrow[r, "\mathrm{R}h_*\left(\mathrm{id}\otimes\Delta_*(ev_{\mathcal{E}})\right)"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \Delta_*\underline{\Lambda}) \arrow[ddd, "\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1})"] \arrow[dddl, bend left, "\mathrm{R}h_*(\mathrm{id}\otimes\mathrm{cl}_\Delta)"'] \\
\mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee(d)[2d]) \otimes \mathrm{R}f_*\mathcal{E} \arrow[r, "\mathrm{id}\otimes\mathrm{R}f_*(\eta_{\pi_2})"] \arrow[d, "\mathrm{KM}^{-1}\otimes\mathrm{id}"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee(d)[2d]) \otimes \mathrm{R}h_*\pi_2^*\mathcal{E} \arrow[r, "\cup"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee(d)[2d] \otimes \pi_2^*\mathcal{E}) \arrow[dd, equal] & & \\
\mathrm{R}f_*\mathcal{E} \otimes \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d]) \otimes \mathrm{R}f_*\mathcal{E} \arrow[d, "\mathrm{id}\otimes\cup"] & & & & \\
\mathrm{R}f_*\mathcal{E} \otimes \mathrm{R}f_*(\mathcal{E}^\vee(d)[2d] \otimes \mathcal{E}) \arrow[rr, "\mathrm{KM}"] \arrow[d, "\mathrm{id}\otimes\mathrm{R}f_*\left(ev_{\mathcal{E}}(d)[2d]\right)"] & & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^\vee(d)[2d] \otimes \pi_2^*\mathcal{E}) \arrow[r, "\mathrm{R}h_*\left(\mathrm{id}\otimes\pi_2^*(ev_{\mathcal{E}}(d)[2d])\right)"] \arrow[d, "\mathrm{R}h_*\left(\mathrm{id}\otimes\pi_2^*(ev_{\mathcal{E}}(d)[2d])\right)"] & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \underline{\Lambda}(d)[2d]) \arrow[d, "\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1})"] & \mathrm{R}f_*(\mathcal{E} \otimes \underline{\Lambda}) \arrow[dl, "\mathrm{R}f_*\left(\mathrm{id}\otimes\mathrm{R}\pi_{1,*}(\mathrm{cl}_\Delta)\right)"] \\
\mathrm{R}f_*\mathcal{E} \otimes \mathrm{R}f_*\underline{\Lambda}(d)[2d] \arrow[rr, "\mathrm{KM}"] \arrow[d, "\mathrm{id}\otimes\mathrm{tr}_f"] \arrow[drr, phantom, "(6.4.7)"] & & \mathrm{R}h_*(\pi_1^*\mathcal{E} \otimes \underline{\Lambda}(d)[2d]) \arrow[r, "\mathrm{R}f_*(\mathrm{PF}_{\pi_1}^{-1})"] & \mathrm{R}f_*(\mathcal{E} \otimes \mathrm{R}\pi_{1,*}\underline{\Lambda}(d)[2d]) \arrow[dl, "\mathrm{R}f_*(\mathrm{id}\otimes\mathrm{tr}_{\pi_1})"] & \\
\mathrm{R}f_*\mathcal{E} \otimes \underline{\Lambda} & & \mathrm{R}f_*(\mathcal{E} \otimes \underline{\Lambda}) \arrow[ll, "\mathrm{PF}_f^{-1}"'] & &
\end{tikzcd}
$$

FIGURE 1. The gigantic diagram. Beware that $\otimes$ denotes the derived tensor product so that the figure fits into the page.

$\mathbf{F}_p$-module on $X_{\text{ét}}$. In this case, $\mathcal{E}$ is a dualizable object of $D(X_{\text{ét}};\Lambda)$ by virtue of [Zav23a, Lem. 11.1]. Then Theorem 6.4.1 and another application of *loc. cit.* imply that $\mathrm{R}f_*\mathcal{E} \in D^b_{\text{lis}}(Y_{\text{ét}};\Lambda)$. □

Remark 6.4.9. It seems that Corollary 6.4.8 is a new result in this level of generality. However, it was certainly known under some additional assumptions. If Y admits a map to $\mathrm{Spa}\,(K,\mathcal{O}_K)$ for a nonarchimedean field K and $n \in (\mathcal{O}_Y^+)^\times$, this result was shown in [Hub96, Cor. 6.2.3] by an extremely elaborate argument. The assumption that Y admits a map to $\mathrm{Spa}\,(K,\mathcal{O}_K)$ was recently removed in [Zav23b, App. 1.3.4(4)]. Now if Y is a rigid-analytic space over $\mathrm{Spa}\,(K,\mathcal{O}_K)$ and p is equal to the characteristic of the residue field of $\mathcal{O}_K$, this result was shown in [SW20, Th. 10.5.1] using the full strength of the perfectoid and diamond machinery. In contrast to these two proofs, our proof is uniform in n, is essentially formal, and remains largely in the world of locally noetherian analytic adic spaces.[30]

The main goal of the rest of this subsection is to extend Poincaré duality to general (not necessarily dualizable) lisse sheaves. The essential difficulty comes from the fact that the constant sheaf $\underline{\mathbf{Z}/p\mathbf{Z}}$ is not dualizable in $D(X_{\text{ét}};\mathbf{Z}/p^2\mathbf{Z})$ for any prime p. Nevertheless, our extension of Poincaré duality to this kind of coefficients will be essentially formal.

Theorem 6.4.10. *Let X and Y be locally noetherian analytic adic spaces such that $n \in \mathcal{O}_Y^\times$, let $f\colon X \to Y$ be a smooth proper morphism of equidimension d, and let $\mathcal{E} \in D_{\text{lis}}(X_{\text{ét}};\Lambda)$ be a complex with lisse cohomology sheaves. Then the duality morphism*

$$\mathrm{PD}_f(\mathcal{E})\colon \mathrm{R}f_*(\mathrm{R}\mathscr{H}om_\Lambda(\mathcal{E},\underline{\Lambda}_X(d)[2d])) \to \mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*\mathcal{E},\underline{\Lambda}_Y)$$

from Construction 6.4.2. (2) is an isomorphism.

Proof. The strategy of this proof is to reduce to the case where $\Lambda = \mathbf{Z}/p^r\mathbf{Z}$ for some prime number $p \in \mathcal{O}_Y^\times$ and $\mathcal{E} = \underline{\mathbf{F}}_p$. In this case, we deduce the result from Theorem 6.4.1.

Step 0. We reduce to the case when X is qcqs and connected and $\Lambda = \mathbf{Z}/p^r\mathbf{Z}$ for a prime number $p \in \mathcal{O}_Y^\times$. First, the question is clearly local on Y, so we can assume that Y is an affinoid. Furthermore, [Zav23a, Cor. 2.3] shows that connected components of X are clopen, so we may and do assume that X is qcqs and connected. Then the Chinese Remainder Theorem implies that we can assume that $\Lambda = \mathbf{Z}/p^r\mathbf{Z}$ for some prime number $p \in \mathcal{O}_Y^\times$ and some integer $r > 0$.

In the rest of the proof, we will freely use the following two basic “reduction principles”:

(a) (“two-out-of-three”) If we have a triangle $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3$ in $D_{\text{lis}}(X_{\text{ét}};\Lambda)$ and $\mathrm{PD}_f(\mathcal{E}_i)$ is an isomorphism for two of the three $\mathcal{E}_i$, then $\mathrm{PD}_f(\mathcal{E}_i)$ is an isomorphism for all three $\mathcal{E}_i$.

(b) (“closure under retracts”) If $\mathcal{E}$ is a direct summand of $\mathcal{G}$ and $\mathrm{PD}_f(\mathcal{G})$ is an isomorphism, then $\mathrm{PD}_f(\mathcal{E})$ is an isomorphism as well.

Step 1. We reduce to the case when $\mathcal{E}$ is a lisse sheaf of Λ-modules. First, we note that $\mathrm{R}f_*$ commutes with sequential homotopy colimits (e.g., as defined in [Sta22, Tag 0A5K]) due to [Zav23a, Lem. 9.1]. This implies that both the source and target of PD_f (viewed as functors in $\mathcal{E}$) transform sequential homotopy colimits into sequential homotopy limits (e.g., as defined in [Sta22, Tag 08TB]). Since the natural morphism $\mathrm{hocolim}_N\, \tau^{\leq N}\mathcal{E} \to \mathcal{E}$ is an isomorphism (this can be deduced from [Sta22, Tag 0CRK]), we reduce to the case when $\mathcal{E} \in D^-_{\text{lis}}(X_{\text{ét}};\Lambda)$. In this case, we consider the exact triangle

$$\tau^{\leq -N}(\mathcal{E}) \to \mathcal{E} \to \tau^{>-N}\mathcal{E}.$$

Recall that $\mathrm{R}f_*$ has cohomological dimension $2d$ by virtue of [Hub96, Prop. 5.3.11]. As a consequence, both $\mathrm{R}f_*\Big(\mathrm{R}\mathscr{H}om_\Lambda\big(\tau^{\leq -N}(\mathcal{E}),\underline{\Lambda}_X(d)[2d]\big)\Big)$ and $\mathrm{R}\mathscr{H}om_\Lambda\big(\mathrm{R}f_*\tau^{\leq -N}(\mathcal{E}),\underline{\Lambda}_Y\big)$ lie in $D^{\geq N-2d}(Y_{\text{ét}};\Lambda)$. Given an integer q, the map on cohomology sheaves $\mathcal{H}^q\big(\mathrm{PD}_f(\mathcal{E})\big)$ is therefore an isomorphism if and only if $\mathcal{H}^q\big(\mathrm{PD}_f(\tau^{>-(q+1+2d)}\mathcal{E})\big)$ is so. In particular, if $\mathrm{PD}_f(\tau^{>-N}\mathcal{E})$ is an isomorphism for all N, then $\mathrm{PD}_f(\mathcal{E})$ is an isomorphism as well. Thus, we reduce to the case when $\mathcal{E}$ is bounded. In this case, the two-out-of-three reduction principle reduces the question further to the case when $\mathcal{E}$ is a lisse sheaf of Λ-modules on $X_{\text{ét}}$.

[30] The only exception to this occurs in the proof of Lemma 6.3.1; see Remark 6.3.2.

Step 2. We reduce to the case $\mathcal{E} = \underline{\mathbf{F}}_p$. First, the two-out-of-three reduction principle implies that it suffices to prove the claim for $p^k\mathcal{E}/p^{k+1}\mathcal{E}$ for each $0 \le k \le r-1$. Therefore, we can assume that $\mathcal{E}$ is an $\mathbf{F}_p$-lisse sheaf (considered as a $\mathbf{Z}/p^r\mathbf{Z}$-lisse sheaf). The "méthode de la trace"[31] then implies that there is a finite étale morphism $\pi\colon X' \to X$ of constant degree prime to p such that $\mathcal{E}' := \mathcal{E}\big|_{X'}$ is a finite successive extension of constant sheaves $\underline{\mathbf{F}}_p$. The composition

$$\mathcal{E} \to \pi_*\mathcal{E}' \xrightarrow{\mathrm{tr}_{\pi,\mathcal{E}}} \mathcal{E}$$

is equal to $\deg(\pi)$ (see Theorem 2.5.6. (4)). Since $\deg(\pi)$ is coprime to p, we conclude that $\mathcal{E}$ is a direct summand of $\pi_*\mathcal{E}'$. By the closure under retracts reduction principle, it suffices to show that $\mathrm{PD}_f\left(\pi_*\mathcal{E}'\right)$ is an isomorphism. Since the smooth trace is compatible with compositions (Theorem 6.1.1. (1)), we see that the composition

$$\mathrm{R}f_*\pi_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\mathcal{E}', \underline{\Lambda}_{X'}(d)[2d])\big) \xrightarrow{\mathrm{R}f_*(\mathrm{PD}_\pi(\mathcal{E}')(d)[2d])} \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\pi_*\mathcal{E}', \underline{\Lambda}_X(d)[2d])\big) \longrightarrow$$
$$\xrightarrow{\mathrm{PD}_f(\pi_*\mathcal{E}')} \mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*\pi_*\mathcal{E}', \underline{\Lambda}_Y)$$

is given by $\mathrm{PD}_{f\circ\pi}(\mathcal{E}')$. Hence, we are reduced to showing that both $\mathrm{PD}_\pi(\mathcal{E}')$ and $\mathrm{PD}_{f\circ\pi}(\mathcal{E}')$ are isomorphisms. In other words, we can assume[32] that $\mathcal{E}$ is a finite successive extension of constant sheaves $\mathbf{F}_p$. The two-out-of-three reduction principle then allows us to reduce to $\mathcal{E} = \underline{\mathbf{F}}_p$.

Step 3. End of proof. Now we prove the claim for $\Lambda = \mathbf{Z}/p^r\mathbf{Z}$ and $\mathcal{E} = \mathbf{F}_p$. In this case, the lisse sheaf of Λ-modules $\underline{\mathbf{F}}_{p,X}$ has the following free resolution:

$$C := \left(\dots \xrightarrow{p} \underline{\Lambda}_X \xrightarrow{p^{r-1}} \underline{\Lambda}_X \xrightarrow{p} \underline{\Lambda}_X\right) \xrightarrow{\sim} \underline{\mathbf{F}}_{p,X}.$$

For any integer i, denote the naive truncation of C by $C_i := \sigma^{\ge -i}C$. Then C_i fits into the exact sequence

$$\underline{\mathbf{F}}_{p,X}[i] \to C_i \to \underline{\mathbf{F}}_{p,X}.$$

This induces the following morphism of exact triangles

$$\begin{tikzcd}
\mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\underline{\mathbf{F}}_{p,X}, \underline{\Lambda}_X(d)[2d])\big) \arrow[r] \arrow[d, "\mathrm{PD}_f(\underline{\mathbf{F}}_{p,X})"'] & \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(C_i, \underline{\Lambda}_X(d)[2d])\big) \arrow[r] \arrow[d, "\mathrm{PD}_f(C_i)"'] & \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\underline{\mathbf{F}}_{p,X}[i], \underline{\Lambda}_X(d)[2d])\big) \arrow[d, "\mathrm{PD}_f(\underline{\mathbf{F}}_{p,X}[i])"] \\
\mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*\underline{\mathbf{F}}_{p,X}, \underline{\Lambda}_Y) \arrow[r] & \mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*C_i, \underline{\Lambda}_Y) \arrow[r] & \mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*\underline{\mathbf{F}}_{p,X}[i], \underline{\Lambda}_Y).
\end{tikzcd}$$

By construction, C_i is a perfect (hence dualizable) object in $D(X_{\text{ét}}; \Lambda)$ for each i, so Theorem 6.4.1 implies that $\mathrm{PD}_f(C_i)$ is an isomorphism for every integer i. Now as in Step 1, both $\mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\underline{\mathbf{F}}_{p,X}[i], \underline{\Lambda}_X(d)[2d])\big)$ and $\mathrm{R}\mathscr{H}om_\Lambda(\mathrm{R}f_*\underline{\mathbf{F}}_{p,X}[i], \underline{\Lambda}_Y)$ lie in $D^{\ge i-2d}(Y_{\text{ét}}; \Lambda)$. Therefore, we conclude that $\mathcal{H}^q\big(\mathrm{PD}_f(\underline{\mathbf{F}}_{p,X})\big)$ is an isomorphism for $q < i - 2d$. Since i was an arbitrary integer, $\mathrm{PD}_f(\underline{\mathbf{F}}_{p,X})$ is an isomorphism, finishing the proof. □

Remark 6.4.11. We point out that the example of the closed unit disk prevents any naïve form of "weak" Poincaré duality to hold. Namely, let C be an algebraically closed nonarchimedean field of mixed characteristic $(0,p)$, let $X = \mathbf{D}^1_C$, and let $n = p > 0$. Then Theorem 6.1.1 induces a pairing

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathrm{H}^{2-i}_c(X, \mu_p) \xrightarrow{\cup} \mathrm{H}^2_c(X, \mu_p) \xrightarrow{\mathrm{H}^2(\mathrm{tr}_X)} \mathbf{F}_p$$

for each integer $0 \le i \le 2$. One may wonder whether, for a fixed i, the duality map from one of these two $\mathbf{F}_p$-vector spaces to the dual of the other can be an isomorphism (or at least injective or surjective). It turns out that none of these options holds:

(1) Lemma 5.5.21 guarantees that $\mathrm{H}^2_c(X, \mu_p)$ is infinite, while Proposition 5.1.2 guarantees that $\mathrm{H}^0(X, \mathbf{F}_p) \simeq \mathbf{F}_p$. Thus, the map $\mathrm{H}^2_c(X, \mu_p) \to \mathrm{H}^0(X, \mathbf{F}_p)^\vee$ cannot be injective, and the map $\mathrm{H}^0(X, \mathbf{F}_p) \to \mathrm{H}^2_c(X, \mu_p)^\vee$ cannot be surjective.

[31]To make this precise, one can argue as in the proof of [Sta22, Tag 0A3R].

[32]We note that we replace X with X', which might be disconnected. This will not be important for the rest of the argument. At any rate, we can further replace X' with its connected component to preserve the assumption that X is connected.

(2) On the other hand, Remark 5.1.14 implies that $\mathrm{H}^1(X, \mathbf{Z}/p) \cong \mathrm{H}^1(X, \mu_p)$ is infinite, while Corollary 5.1.7 implies that $\mathrm{H}^1_c(X, \mu_p) = 0$. Thus, the map $\mathrm{H}^1_c(X, \mu_p) \to \mathrm{H}^1(X, \mathbf{F}_p)^\vee$ is not surjective, whereas the map $\mathrm{H}^1(X, \mathbf{F}_p) \to \mathrm{H}^1_c(X, \mu_p)^\vee$ is not injective.

A similar computation can be adapted to the *open* unit disk $X = \mathring{\mathbf{D}}^1$ showing that no form of "weak" Poincaré duality could hold in the partially proper case as well.

Remark 6.4.12. It seems plausible that there could be a more sophisticated version of Poincaré duality in the style of [CGN23]. For example, in the case of smooth connected affinoid curves, Proposition 5.1.2, Proposition 5.1.5, and Corollary 5.1.7 imply that, among the cohomology groups involved in Poincaré duality, only $\mathrm{H}^1(X, \mathbf{F}_p)$ and $\mathrm{H}^2_c(X, \mu_p)$ could be infinite. It is believable that the hugeness of $\mathrm{H}^1(X, \mathbf{F}_p)$ is "dual" to the hugeness of $\mathrm{H}^2_c(X, \mu_p)$, or rather to the hugeness of $\ker(\mathrm{tr}_X) \subset \mathrm{H}^2_c(X, \mu_p)$. Even though the numerology of the usual Poincaré duality does not allow this, a more elaborate form of duality (involving higher Ext groups) might "mix" degrees appropriately. A similar phenomenon occurs in [CGN23]. Unfortunately, we do not know how to make this precise.

7. The trace map for proper morphisms

In this section, we discuss the construction of a trace map for an arbitrary proper morphism of rigid-analytic spaces over a non-archimedean field of characteristic 0. Then we prove a version of Poincaré duality for an arbitrary proper morphism; this positively answers the question raised in [BH22, Rmk. 3.23]. In order to even formulate the notion of a trace map and of Poincaré duality for proper morphisms that are not necessarily smooth, we need to use the notions of Zariski-constructible sheaves and of dualizing complexes developed in [BH22, § 3.1-3.2] and [BH22, § 3.4], respectively. Since the theory has only been worked out for rigid-analytic spaces over a nonarchimedean field of 0, we always work in this setup in this section (in contrast to Section 6, where we considered general locally noetherian analytic adic spaces).

Throughout this section, we fix a non-archimedean field K of characteristic 0, an integer $n > 0$, and put $\Lambda := \mathbf{Z}/n\mathbf{Z}$.

7.1. Preliminaries on dualizing complexes.

We recall that [BH22, Th. 3.21] constructs a dualizing complex ω_X for any rigid-analytic space X over K. The main goal of this subsection is to record some basic facts about these dualizing complexes that are not addressed in [BH22]. Namely, we show that the formation of ω_X behaves well with respect to smooth morphisms and relative analytifications.

First, we start with the following lemma:

Lemma 7.1.1. *Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be a smooth morphism of affinoid rigid-analytic spaces over K. Then the morphism $f^\sharp\colon A \to B$ is regular. If f is of equidimension d and $\mathfrak{m} \subset A$ is a maximal ideal such that $B \otimes_A k(\mathfrak{m}) \neq 0$, then $B \otimes_A k(\mathfrak{m})$ has pure Krull dimension d.*

This lemma holds without the assumption that $\mathrm{char}\, K = 0$.

Proof. First, we show that $f^\sharp$ is regular. For this, we note that $f^\sharp\colon A \to B$ is flat due to flatness of f and [Zav24, Lem. B.4.3]. Furthermore, [BKKN67, Satz 3.3.3] and [Kie69, Th. 3.3] imply that the rings A and B are excellent. Therefore, [Sta22, Tag 07NQ] and [And74, Th. p.1] ensure that it suffices to show that $k(\mathfrak{m}) \otimes_A B$ is either zero or geometrically regular for any maximal ideal $\mathfrak{m} \subset A$. We put $C_\mathfrak{m} := \widehat{\overline{k(\mathfrak{m})}}$. Then [Con99, Lem. 1.1.5(i)] implies that it suffices to show that the $C_\mathfrak{m}$-algebra $C_\mathfrak{m} \widehat{\otimes}_A B$ is either regular or zero. For this, the maximal ideal $\mathfrak{m}$ uniquely defines a (classical) point $y \in Y$. Then the geometric fiber $X_{\overline{y}}$ of f over y is given by $\mathrm{Spa}\left(C_\mathfrak{m} \widehat{\otimes}_A B, \left(C_\mathfrak{m} \widehat{\otimes}_A B\right)^\circ\right)$. Since f is smooth, we conclude that $X_{\overline{y}}$ is smooth over $\mathrm{Spa}\,(C_\mathfrak{m}, C_\mathfrak{m}^\circ)$, so [FvdP04, Th. 3.6.3] implies that $C_\mathfrak{m} \widehat{\otimes}_A B$ is regular or zero.

We are left to show that $B \otimes_A k(\mathfrak{m})$ has pure dimension d if it is non-zero and f is of equidimension d. We keep the notation of the previous paragraph and observe that the fiber X_y is given by $\mathrm{Spa}\left(B \otimes_A k(\mathfrak{m}), \left(B \otimes_A k(\mathfrak{m})\right)^\circ\right)$. Then [Hub96, Lem. 1.8.6(ii)] implies that each connected component of $\mathrm{Spec}\left(B \otimes_A k(\mathfrak{m})\right)$ is of Krull dimension d. Since we already know that $B \otimes_A k(\mathfrak{m})$ is regular, we conclude that it is of pure Krull dimension d. $\square$

Corollary 7.1.2. *Let $f\colon X \to Y$ be a smooth morphism of equidimension d between rigid-analytic spaces over K. Then there is a canonical isomorphism*

$$\alpha_f\colon f^*\omega_Y(d)[2d] \xrightarrow{\sim} \omega_X.$$

Proof. After unraveling the definition of the dualizing complex in [BH22, Th. 3.21], we reduce the question to showing that for a smooth morphism $f\colon \mathrm{Spa}\,(B, B^\circ) \to \mathrm{Spa}\,(A, A^\circ)$ of equidimension d with associated morphism $f^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, there is a unique isomorphism (compatible with the pinnings) of potential dualizing complexes[33] $\alpha_{f^{\mathrm{alg}}}\colon f^{\mathrm{alg},*}\omega_A(d)[2d] \xrightarrow{\sim} \omega_B$. This follows from Lemma 7.1.1 and [BH22, Lem. 3.22]. □

Remark 7.1.3 (Algebraic version of Corollary 7.1.2)**.** (1) A proof similar to that of Corollary 7.1.2 (in fact, easier), shows that for any K-affinoid algebra A and any smooth morphism $f\colon X \to Y$ of equidimension d between locally finite type A-schemes, there is a canonical isomorphism

$$\alpha_f^{\mathrm{alg}}\colon f^*\omega_Y(d)[2d] \xrightarrow{\sim} \omega_X.$$

(2) In the proof of Corollary 7.1.2, we also used the following fact: for a smooth morphism $f\colon \mathrm{Spa}\,(B, B^\circ) \to \mathrm{Spa}\,(A, A^\circ)$ of equidimension d between rigid-analytic spaces over K and the corresponding morphism $f^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ of affine schemes (which is not necessarily of finite type), there is a canonical isomorphism

$$\alpha_{f^{\mathrm{alg}}}\colon f^{\mathrm{alg},*}\omega_{\mathrm{Spec}\, A}(d)[2d] \xrightarrow{\sim} \omega_{\mathrm{Spec}\, B}.$$

Both of these isomorphisms essentially come from [BH22, Lem. 3.22] (or [ILO14, Exp. XVII, Prop. 4.1.1]).

Now we discuss the behavior of dualizing complexes with respect to relative analytifications. Again, we first need to verify an algebra result:

Lemma 7.1.4. *Let A be a K-affinoid algebra, let B be a finite type A-algebra, let $X = \mathrm{Spec}\, B$, and let $U = \mathrm{Spa}\,(R, R^\circ) \subset X^{\mathrm{an}/A}$ be an open affinoid in the relative analytification of X (see Construction 2.5.4). Then the natural morphism $B \to R$ is regular, and $\dim R \otimes_B k(\mathfrak{m}) = 0$ for any maximal ideal $\mathfrak{m} \subset B$ such that $R \otimes_B k(\mathfrak{m}) \neq 0$.*

This lemma holds without the assumption that $\mathrm{char}\, K = 0$.

Proof. First, we note that A is a Jacobson ring by virtue of [Bos14, Prop. 3.1/3]. Therefore, [Sta22, Tag 00GB] implies that $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ sends closed points to closed points. Since $X^{\mathrm{an}/A} \to \mathrm{Spa}\,(A, A^\circ)$ sends classical points to classical points, [Con99, Lem. 5.1.2] implies that $|c_{X/A}|\colon |X^{\mathrm{an}/A}| \to |X|$ defines a bijection between classical points of $X^{\mathrm{an}/A}$ and closed points of X. As a consequence, the natural morphism $r\colon \mathrm{Spec}\, R \to \mathrm{Spec}\, B$ sends closed points to closed points and is injective on closed points. Since R is Jacobson, we conclude that $r^{-1}(\{s\})$ consists of at most one closed point for any closed point $s \in \mathrm{Spec}\, B$. Combining these results with [Bos14, Prop. 4.1/2] and [Con99, Lem. 5.1.2(2)], we conclude that, for every maximal ideal $\mathfrak{m} \subset B$ such that $k(\mathfrak{m}) \otimes_B R \neq 0$, the ideal $\mathfrak{m}R \subset R$ is maximal, and the morphism $B_\mathfrak{m} \to R_\mathfrak{m}$ induces an isomorphism on residue fields.

Now we recall that B is excellent due to [Kie69, Th. 3.3] and [Gro65, Scholie 7.8.3(ii)]. Therefore, [Sta22, Tag 07NQ], [And74, Th. p.1], and the conclusion of the previous paragraph imply that, in order to obtain both claims of the lemma, it suffices to show that, for every maximal ideal $\mathfrak{m} \subset B$ such that $k(\mathfrak{m}) \otimes_B R \neq 0$, the natural morphism

$$B_\mathfrak{m} \to R_\mathfrak{m}$$

is flat and induces an isomorphism on residue fields. The latter claim was already verified in the previous paragraph. The first claim follows from [Zav23a, Lem. 6.4], [Sta22, Tag 0523], and [Bos14, Prop. 4.1/2]. □

Corollary 7.1.5. *Let A be a K-affinoid algebra, let X be a locally finite type A-scheme with the relative analytification $c_{X/A}\colon X^{\mathrm{an}/A}_{\text{ét}} \to X_{\text{ét}}$. Then there is a canonical isomorphism*

$$\beta_{X/A}\colon c^*_{X/A}\omega_X \xrightarrow{\sim} \omega_{X^{\mathrm{an}/A}},$$

where ω_X is a potential dualizing complex on X (see [BH22, Th. 3.19] and [ILO14, Exp. XVII, Th. 5.1.1]).

[33] See [ILO14, Exp. XVII, § 2] for the detailed and self-contained discussion of potential dualizing complexes.

Proof. The proof is completely analogous to that of [BH22, Th. 3.21(7)] using Lemma 7.1.4 and [BH22, Lem. 3.22]. □

Lemma 7.1.6. *Let A be a K-affinoid algebra, let $f^{\mathrm{alg}}\colon X \to \operatorname{Spec} A$ be a smooth morphism of equidimension d, and let $f\colon X^{\mathrm{an}/A} \to \operatorname{Spa}(A, A^\circ)$ be its relative analytification. Then the diagram*

$$\begin{tikzcd} c^*_{X/A} f^{\mathrm{alg},*}\omega_{\operatorname{Spec} A} \simeq f^* c^*_A \omega_{\operatorname{Spec} A} \arrow[r, "f^*(\beta_{\operatorname{Spec} A/A})", "\sim"'] \arrow[d, "c^*_{X/A}\alpha_{f^{\mathrm{alg}}}", "\wr"'] & f^*\omega_{\operatorname{Spa}(A,A^\circ)} \arrow[d, "\alpha_f", "\wr"'] \\ c^*_{X/A}\omega_X \arrow[r, "\beta_{X/A}", "\sim"'] & \omega_{X^{\mathrm{an}/A}} \end{tikzcd} \tag{7.1.7}$$

commutes, where the β's are the isomorphisms from Corollary 7.1.5, α_f is the isomorphism from Corollary 7.1.2, and $\alpha_{f^{\mathrm{alg}}}$ is the isomorphism from Remark 7.1.3.

Proof. We note that $c^*_{X/A} f^{\mathrm{alg},*}\omega_{\operatorname{Spec} A}$ is isomorphic to $\omega_{X^{\mathrm{an}/A}}$ via the composition $\alpha_f \circ f^*(\beta_{\operatorname{Spec} A/A})$. Therefore, [BH22, Th. 3.21(3)] implies that $\mathrm{R}\mathscr{H}om(c^*_{X/A} f^{\mathrm{alg},*}\omega_{\operatorname{Spec} A}, \omega_{X^{\mathrm{an}}/A}) \simeq \underline{\Lambda}_{X^{\mathrm{an}/A}}$ lies in $D^{\geq 0}(X^{\mathrm{an}/A}_{\text{ét}}; \Lambda)$. As a consequence, it suffices to check that Diagram (7.1.7) commutes étale locally on $X^{\mathrm{an}/A}$. After unraveling the definitions, the result then follows from Lemma 7.1.1, Lemma 7.1.4, and (most importantly) [ILO14, Exp. XVII, Rmq. 4.1.3]. □

7.2. **Smooth and closed traces.** The main goal of this subsection is to define versions of trace maps for closed immersions and smooth morphisms (with coefficients in dualizing sheaves). The first construction will essentially come from adjunction, while the second construction will essentially come from the smooth trace map of Theorem 6.1.1.

We start with the case of closed immersions. For this, we recall that [BH22, Th. 3.21.(1)] provides us with a canonical isomorphism $c_i\colon \omega_X \xrightarrow{\sim} \mathrm{R}i^!\omega_Y$ for any closed immersion $i\colon X \hookrightarrow Y$. This gives us the desired trace via the following construction:

Construction 7.2.1 (Closed trace)**.** Let $i\colon X \hookrightarrow Y$ be a closed immersion of rigid-analytic spaces over K. The *closed trace map* is the morphism $\mathrm{Tr}_i\colon i_*\omega_X \to \omega_Y$ defined as the composition

$$i_*\omega_X \xrightarrow{i_*(c_i)} i_*\mathrm{R}i^!\omega_Y \xrightarrow{\epsilon_i} \omega_Y$$

where ϵ_i is the counit of the $(i_*, \mathrm{R}i^!)$-adjunction. In other words, Tr_i is adjoint to the isomorphism c_i.

The closed trace has its obvious analog in algebraic geometry:

Construction 7.2.2 (Closed trace in algebraic geometry)**.** Let A be a K-affinoid algebra, and let $i\colon X \hookrightarrow Y$ be a closed immersion of locally finite type A-schemes. The *closed trace map* is the morphism $\mathrm{Tr}^{\mathrm{alg}}_i\colon i_*\omega_X \to \omega_Y$ defined as the composition

$$i_*\omega_X \xrightarrow[\sim]{i_*(c^{\mathrm{alg}}_i)} i_*\mathrm{R}i^!\omega_Y \xrightarrow{\epsilon_i} \omega_Y$$

where c^{alg}_i is the isomorphism induced by [ILO14, Exp. XVII, Prop. 4.1.2] and ϵ_i is the counit of the $(i_*, \mathrm{R}i^!)$-adjunction.

Remark 7.2.3. The closed trace map satisfies the following basic properties:

(1) the closed trace is compatible with compositions, i.e., for a pair of Zariski-closed immersions $i_1\colon X \hookrightarrow Y$ and $i_2\colon Y \hookrightarrow Z$, we have the following equality:

$$\mathrm{Tr}_{i_2} \circ i_{2,*}(\mathrm{Tr}_{i_1}) = \mathrm{Tr}_{i_2\circ i_1}\colon (i_2 \circ i_1)_*\omega_X \to \omega_Z;$$

(2) the closed trace is étale local, i.e., for a Zariski-closed immersion $i\colon X \hookrightarrow Y$, an étale morphism $g\colon Y' \to Y$, and the fiber product $X' \coloneqq Y' \times_Y X$ with the two projections $i'\colon X' \hookrightarrow Y'$ and $g'\colon X' \to X$, the diagram

$$\begin{tikzcd} i'_*\omega_{X'} \arrow[rr, "\mathrm{Tr}_{i'}"] & & \omega_{Y'} \\ i'_*g'^*\omega_X \arrow[u, "i'_*(\alpha_{g'})", "\wr"'] & g^*i_*\omega_X \arrow[l, "\mathrm{BC}", "\sim"'] \arrow[r, "g^*\mathrm{Tr}_i"] & g^*\omega_Y, \arrow[u, "\alpha_g", "\wr"'] \end{tikzcd}$$

commutes, where the α's are the isomorphism from Corollary 7.1.2;

(3) the closed trace is compatible with relative analytification, i.e., for a K-affinoid algebra A and a closed immersion $i\colon X \to Y$ of locally finite type A-schemes, the diagram

$$\begin{tikzcd} c^*_{Y/A} i_*\omega_X \arrow[r, "\sim"] \arrow[d, "c^*_{Y/A}(\mathrm{Tr}^{\mathrm{alg}}_i)"] & i^{\mathrm{an}/A}_* c^*_{X/A}\omega_X \arrow[r, "i^{\mathrm{an}/A}_*(\beta_{X/A})", "\sim"'] & i^{\mathrm{an}/A}_* \omega_{X^{\mathrm{an}/A}} \arrow[d, "\mathrm{Tr}_{i^{\mathrm{an}/A}}"] \\ c^*_{Y/A}\omega_Y \arrow[rr, "\beta_{Y/A}", "\sim"'] & & \omega_{Y^{\mathrm{an}}/A}, \end{tikzcd}$$

commutes, where the top left arrow is the isomorphism from [Hub96, Th. 5.7.2] and the β's are the isomorphisms from Corollary 7.1.5.

We do not justify these facts fully. Instead, we only mention the main ingredients and leave the details to the interested reader. Using the constructions of α and of the trace map (and the construction of c_i in [BH22, Th. 3.21(1)]), one reduces the first two claims to the analogous claims in algebraic geometry,[34] then using the $(g_!, g^*)$- and the $(i_*, \mathrm{R}i^!)$-adjunctions, one reduces the first claim to [ILO14, Exp. XVII, Rmq. 4.1.3] and the second claim to [ILO14, Exp. XVII, Lem. 4.3.2.3].[35] The last claim follows from the construction of closed traces, Lemma 7.1.4, and [ILO14, Exp. XVII, Lem. 4.3.2.3].

For future reference, we also record the following basic result:

Lemma 7.2.4. *Let $i\colon X \to Y$ be a nil-immersion of rigid-analytic spaces over K. Then $\mathrm{Tr}_i\colon i_*\omega_X \to \omega_Y$ is an isomorphism.*

Proof. After unraveling the definition, we see that it suffices to show that the counit morphism $i_*\mathrm{R}i^!\omega_Y \xrightarrow{\epsilon_i} \omega_Y$ is an isomorphism. For this, it suffices to show that $i_*\colon \mathrm{Shv}(X_{\text{ét}};\Lambda) \to \mathrm{Shv}(Y_{\text{ét}};\Lambda)$ is an equivalence. This follows directly from [Hub96, Prop. 2.3.7]. ☐

Now we wish to explicate this construction in some cases. For this, we assume that $i\colon X \to Y$ is a closed immersion, Y is smooth of equidimension d_Y, and X is smooth of equidimension d_X. Then [BH22, Th. 3.21(1)] ensures that there are canonical isomorphisms $\omega_X \simeq \underline{\Lambda}_X(d_X)[2d_X]$ and $\omega_Y \simeq \underline{\Lambda}_Y(d_Y)[2d_Y]$. Furthermore, [Zav23a, Lem. 5.9 and Lem. 5.6] ensure that i is an lci closed immersion of pure codimension $d_Y - d_X$.

Construction 7.2.5 (Cycle class)**.** With the notation above, we define the *cycle class* $\mathrm{cl}_i\colon i_*\omega_X \to \omega_Y$ as the composition

$$i_*\omega_X \simeq i_*\underline{\Lambda}_X(d_X)[2d_X] \xrightarrow{\mathrm{cl}_i(\underline{\Lambda}_Y(d_X)[2d_X])} \underline{\Lambda}_Y(d_Y)[2d_Y] \simeq \omega_Y,$$

where $\mathrm{cl}_i\big(\underline{\Lambda}_Y(d_X)[2d_X]\big)$ is the cycle class from Construction 6.4.4.

Note that the notation in Construction 7.2.5 leads to a slight ambiguity because cl_i already denoted the cycle class map $i_*\underline{\Lambda}_X \to \underline{\Lambda}_Y(d_Y - d_X)[2(d_Y - d_X)]$ in Variant 3.3.3. However, as we always consider cycle class morphisms for dualizing complexes in this section, this should not cause any confusion. Now we show that Construction 7.2.1 and Construction 7.2.5 agree with one another:

Lemma 7.2.6. *Let X and Y be smooth rigid-analytic spaces over K of equidimension d_X and d_Y, respectively, and let $i\colon X \to Y$ be a closed immersion. Then*

$$\mathrm{cl}_i = \mathrm{Tr}_i\colon i_*\omega_X \to \omega_Y.$$

Proof. First, [BH22, Th. 3.21(1),(3)] implies that $\mathrm{R}\mathscr{H}om(i_*\omega_X, \omega_Y) \simeq i_*\mathrm{R}\mathscr{H}om(\omega_X, \omega_X) \simeq i_*\underline{\Lambda}_X \in D^{\geq 0}(Y_{\text{ét}};\Lambda)$. Therefore, we can check locally on Y that cl_i and Tr_i coincide. So we may and do assume that $Y = \mathrm{Spa}\,(A, A^\circ)$ is affinoid, and then $X = \mathrm{Spa}\,(A/I, (A/I)^\circ)$ for some ideal $I \subset A$. In this case, [FvdP04, Th. 3.6.3] implies that both A and A/I are regular. The dualizing complexes of Y and X are constructed as

[34] For the second claim, note that the ring map $A \to B$ induced by an étale morphism $g\colon \mathrm{Spa}\,(B, B^\circ) \to \mathrm{Spa}\,(A, A^\circ)$ is in general not an étale ring morphism, but only a regular morphism for which all nonempty fibers over the closed points have pure dimension 0. Fortunately, [ILO14, Exp. XVII, § 4] is written in the generality of regular morphisms and can thus still be applied.

[35] The statement [ILO14, Exp. XVII, Lem. 4.3.2.3] imposes the additional assumption that the morphism g is surjective, but the proof does not use it.

an analytification of potential dualizing complexes on $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ respectively. Tracing through the proof of [BH22, Th. 3.21(1)] and using [ILO14, Exp. XVII, Lem. 2.4.3.4], we see that Tr_i is the analytification of the (appropriately twisted and shifted) algebraic cycle class map for $i^{\mathrm{alg}}\colon \operatorname{Spec} A/I \to \operatorname{Spec} A$. Thus, we reduce the question to showing that the analytic cycle class map is equal to the analytification of the algebraic one. This was already proven in Lemma 3.4.1. □

Next, we define a version of smooth trace maps with coefficients in dualizing complexes.

Construction 7.2.7 (Smooth trace)**.** Let $f\colon X \to Y$ be a separated taut smooth morphism of rigid-analytic spaces over K.

(1) Assume that f is of equidimension d. Then we define the *smooth trace map* $\mathrm{Tr}_f\colon \mathrm{R}f_!\omega_X \to \omega_Y$ as the composition

$$\mathrm{R}f_!\,\omega_X \xrightarrow{\mathrm{R}f_!(\alpha_f^{-1})} \mathrm{R}f_!\big(f^*\omega_Y(d)[2d]\big) \xrightarrow{\mathrm{PF}_f^{-1}} \omega_Y \otimes^L \mathrm{R}f_!\big(\underline{\Lambda}_X(d)[2d]\big) \xrightarrow{\mathrm{id}\otimes^L \mathrm{tr}_f} \omega_Y,$$

where α_f is the isomorphism from Corollary 7.1.2, PF_f is the projection formula isomorphism from [Hub96, Th. 5.5.9.(ii)], and tr_f is the trace morphism from Theorem 6.1.1.

(2) Now let f be a general separated taut smooth morphism. Then there exists a clopen decomposition $X = \bigsqcup_{d\in\mathbf{N}} X_d$ such that $f_d \coloneqq f\big|_{X_d}\colon X_d \to Y$ is of equidimension d. We define

$$\mathrm{Tr}_f\colon \mathrm{R}f_!\,\omega_X \simeq \bigoplus_{d\in\mathbf{N}} \mathrm{R}f_{d,!}\,\omega_{X_d} \xrightarrow{\sum \mathrm{Tr}_{f_d}} \omega_Y.$$

Construction 7.2.8 (Smooth trace in algebraic geometry)**.** Let A be a K-affinoid algebra and let $f\colon X \to Y$ be a separated smooth morphism of locally finite type A-schemes. Then one can define the trace map

$$\mathrm{Tr}_f^{\mathrm{alg}}\colon \mathrm{R}f_!\omega_X \to \omega_Y$$

similarly to Construction 7.2.7 (using [AGV71, Exp. XVIII, Th. 2.9] in place of the analytic trace map and the isomorphism α_f^{alg} from Remark 7.1.3 in place of α_f).

Remark 7.2.9. If f is separated taut étale, then Theorem 6.1.1 (3) implies that Tr_f is given by the composition $f_!\,\omega_{X'} \xrightarrow{f_!(\alpha_f^{-1})} f_!f^*\omega_X \xrightarrow{\epsilon_f} \omega_X$, where ϵ_f is the counit of the $(f_!, f^*)$-adjunction.

Remark 7.2.10. The smooth trace map satisfies the following properties:

(1) the smooth trace is compatible with compositions, i.e., for a pair of smooth separated taut morphisms $f_1\colon X \to Y$ and $f_2\colon Y \to Z$, we have the following equality:

$$\mathrm{Tr}_{f_2} \circ \mathrm{R}f_{2,!}(\mathrm{Tr}_{f_1}) = \mathrm{Tr}_{f_2\circ f_1}\colon \mathrm{R}(f_2\circ f_1)_!\omega_X \to \omega_Z;$$

(2) the smooth trace is étale local, i.e., for a smooth separated taut morphism $f\colon X \to Y$, an étale morphism $g\colon Y' \to Y$, and the fiber product $X' \coloneqq Y' \times_Y X$ with the two projections $f'\colon X' \to Y'$ and $g'\colon X' \to X$, the diagram

$$\begin{tikzcd}
\mathrm{R}f'_!\omega_{X'} \arrow[rr, "\mathrm{Tr}_{f'}"] & & \omega_{Y'} \\
\mathrm{R}f'_!g'^*\omega_X \arrow[u, "\mathrm{R}f'_!(\alpha_{g'})", "\wr"'] & g^*\mathrm{R}f_!\omega_X \arrow[l, "\mathrm{BC}_!"', "\sim"] \arrow[r, "g^*\mathrm{Tr}_f"] & g^*\omega_Y \arrow[u, "\alpha_g", "\wr"']
\end{tikzcd}$$

commutes, where the α's are the isomorphisms from Corollary 7.1.2 and $\mathrm{BC}_!$ is the base change map for compactly supported pushforward from [Hub96, Th. 5.4.6];

(3) the smooth trace is compatible with relative analytifications, i.e., for a K-affinoid algebra A and a smooth separated morphism $f\colon X \to Y$ of locally finite type A-schemes, the diagram

$$\begin{tikzcd}
c_{Y/A}^*\mathrm{R}f_!\omega_X \arrow[r, "\sim"] \arrow[d, "c_{Y/A}^*(\mathrm{Tr}_f^{\mathrm{alg}})"] & \mathrm{R}f_!^{\mathrm{an}/A}c_{X/A}^*\omega_X \arrow[r, "\mathrm{R}f_!^{\mathrm{an}/A}(\beta_{X/A})", "\sim"'] & \mathrm{R}f_!^{\mathrm{an}/A}\omega_{X^{\mathrm{an}/A}} \arrow[d, "\mathrm{Tr}_{f^{\mathrm{an}/A}}"] \\
c_{Y/A}^*\omega_Y \arrow[rr, "\beta_{Y/A}", "\sim"'] & & \omega_{Y^{\mathrm{an}}/A},
\end{tikzcd}$$

commutes, where the top left arrow is the isomorphism from [Hub96, Th. 5.7.2] and the β's are the isomorphism from Corollary 7.1.5.

The first two claims follow immediately from Theorem 6.1.1. (1) and Theorem 6.1.1. (2), respectively. The last claim follows from Lemma 7.1.6 and Proposition 6.2.4.

The rest of this subsection is devoted to showing that closed and smooth trace maps are compatible with each other in a precise way. We start with the following basic lemma:

Lemma 7.2.11. *Let X', X, Y, and Z be rigid-analytic spaces over K, let $h\colon X' \to X$ be a surjective separated taut étale morphism, let $g\colon X \to Y$ be a separated taut smooth morphism, and let $i\colon Y \to Z$ be a closed immersion. Set $f := i \circ g\colon X \to Z$ and $f' := i \circ g \circ h\colon X' \to Z$. Then:*

(i) *$\mathrm{R}\mathscr{H}om(\mathrm{R}f_!\,\omega_X, \omega_Z)$ lies in $D^{\geq 0}(Z_{\text{ét}}; \Lambda)$;*
(ii) *$\mathrm{Hom}(\mathrm{R}f_!\omega_X, \omega_Z) \simeq \mathrm{H}^0\big(Y, \mathscr{H}om(\mathrm{R}^{2d}g_!\underline{\Lambda}_X(d), \underline{\Lambda}_Y)\big)$ if g is of equidimension d;*
(iii) *the morphism $\mathrm{Hom}(\mathrm{R}f_!\omega_X, \omega_Z) \to \mathrm{Hom}(\mathrm{R}f'_!\,\omega_{X'}, \omega_Z)$ induced by Tr_h is injective.*

Proof. Using a similar clopen decomposition as in Construction 7.2.7 (2), we can reduce all three parts to the case where g is of relative equidimension d for some integer $d \geq 0$. Then Corollary 7.1.2 and the projection formula imply that $\mathrm{R}g_!\,\omega_X \simeq \mathrm{R}g_!\underline{\Lambda}_X(d)[2d] \otimes^L \omega_Y$. Therefore, we have

$$\begin{aligned}(7.2.12)\quad \mathrm{R}\mathscr{H}om\big(\mathrm{R}f_!\,\omega_X, \omega_Z\big) &\simeq \mathrm{R}\mathscr{H}om(i_*\mathrm{R}g_!\,\omega_X, \omega_Z) \simeq i_*\mathrm{R}\mathscr{H}om\big(\mathrm{R}g_!\,\underline{\Lambda}_X(d)[2d] \otimes^L \omega_Y, \omega_Y\big) \\ &\simeq i_*\mathrm{R}\mathscr{H}om\big(\mathrm{R}g_!\,\underline{\Lambda}_X(d)[2d], \mathrm{R}\mathscr{H}om(\omega_Y, \omega_Y)\big) \simeq i_*\mathrm{R}\mathscr{H}om\big(\mathrm{R}g_!\,\underline{\Lambda}_X(d)[2d], \underline{\Lambda}_Y\big) \in D^{\geq 0}(Z_{\text{ét}}; \Lambda),\end{aligned}$$

where the first isomorphism follows from $\mathrm{R}i_! \simeq i_*$, the second isomorphism follows from [BH22, Th. 3.21(1)] and $\mathrm{R}g_!\,\omega_X \simeq \mathrm{R}g_!\underline{\Lambda}_X(d)[2d] \otimes^L \omega_Y$, the third isomorphism follows from the (derived) tensor-hom adjunction, the fourth isomorphism follows from [BH22, Th. 3.21(3)], and the last containment follows from Lemma 6.1.2. This finishes the proof of (i).

Now (7.2.12) and [Hub96, Prop. 5.5.8] directly imply that

$$\mathrm{Hom}(\mathrm{R}f_!\omega_X, \omega_Z) \simeq \mathrm{H}^0\big(Y, \mathscr{H}om(\mathrm{R}^{2d}g_!\underline{\Lambda}_X(d), \underline{\Lambda}_Y)\big).$$

This proves (ii). Now we deal with (iii). Set $g' := g \circ h\colon X' \to Y$; this is also a separated smooth taut morphism of equidimension d. Thus, (ii) implies that

$$\mathrm{Hom}(\mathrm{R}f_!\omega_X, \omega_Z) \simeq \mathrm{H}^0\left(Y, \mathscr{H}om(\mathrm{R}^{2d}g_!\underline{\Lambda}_X(d), \underline{\Lambda}_Y)\right) \text{ and } \mathrm{Hom}(\mathrm{R}f'_!\omega_{X'}, \omega_Y) \simeq \mathrm{H}^0\left(Y, \mathscr{H}om(\mathrm{R}^{2d}g'_!\underline{\Lambda}_{X'}(d), \underline{\Lambda}_Y)\right).$$

Therefore, it suffices to show injectivity of the morphism $\mathscr{H}om(\mathrm{R}^{2d}g_!\underline{\Lambda}_X(d), \underline{\Lambda}_Y) \to \mathscr{H}om(\mathrm{R}^{2d}g'_!\underline{\Lambda}_{X'}(d), \underline{\Lambda}_Y)$, which is induced by

$$\mathrm{R}^{2d}g_!\big(\operatorname{tr}^{\text{ét}}_h(d)\big)\colon \mathrm{R}^{2d}g'_!\underline{\Lambda}_{X'}(d) \to \mathrm{R}^{2d}g_!\underline{\Lambda}_X(d)$$

thanks to Remark 7.2.9. For this, it suffices to prove that $\mathrm{R}^{2d}g_!\big(\operatorname{tr}^{\text{ét}}_h(d)\big)$ is surjective. This follows from [Hub96, Prop. 5.5.8] and (an easy case of) Lemma 6.2.3. □

The following technical lemma comes in handy later.

Lemma 7.2.13. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a Cartesian diagram of locally noetherian analytic adic spaces with f étale and g proper. Then:

(i) *the base change natural transformation $\mathrm{BC}\colon f^*\mathrm{R}g_* \to \mathrm{R}g'_*f'^{,*}$ and the reverse direction natural transformation $\mathrm{BC}'\colon \mathrm{R}g'_*f'^{,*} \simeq \mathrm{R}g'_!f'^{,!} \to f^!\mathrm{R}g_! \simeq f^*\mathrm{R}g_*$ are inverse to each other;*
(ii) *the two natural transformations*

$$\mathrm{R}g'_* \xrightarrow{\mathrm{R}g'_*(\eta_{f'})} \mathrm{R}g'_*f'^{,*}f'_! \xrightarrow{\mathrm{BC}^{-1}} f^*\mathrm{R}g_*f'_! \simeq f^*f_!\mathrm{R}g'_* \quad \text{and} \quad \mathrm{R}g'_* \xrightarrow{\eta_f} f^*f_!\mathrm{R}g'_*$$

agree, where η_f (resp. $\eta_{f'}$) denotes the unit of $(f_!, f^*)$ (resp. $(f'_!, f'^{,*})$).

Proof. To see (i), note that the functors involved are derived functors of functors defined at sheaf level. Since f is étale, the functors f^*, $f^{\prime,*}$, g_*, and g'_* preserve K-injective complexes. For each $\mathcal{F} \in D(Y_{\text{ét}};\Lambda)$, both $\mathrm{BC}(\mathcal{F})$ and $\mathrm{BC}'(\mathcal{F})$ are hence computed by applying them termwise to a fixed K-injective complex representing $\mathcal{F}$. Therefore, it suffices to check the claim for injective sheaves, and at sheaf level one can argue on stalks. Now the question is étale local on Y, so we may even assume that the étale map is an open immersion, in which case it follows directly from the definition that the two natural transformations are inverse to each other.

To deduce (ii) from (i), we consider the reverse base change maps BC' for the two diagrams

$$\begin{tikzcd} X' \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\mathrm{id}"] & X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\ X' \arrow[r, "g'"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Our claim then follows from the fact that the two outer diagrams are the same and the reverse base change maps are compatible with "concatenation" of diagrams. □

We can now study the compatibility of closed and smooth trace maps in cartesian diagrams:

Lemma 7.2.14 (Compatibility for Cartesian diagrams)**.** *Let $f\colon X \to Y$ be a separated smooth taut morphism of rigid-analytic spaces over K, let $i\colon Y' \to Y$ be a closed immersion of rigid-analytic spaces over K, and let*

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "f'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook, "i"] & Y \end{tikzcd}$$

be the resulting pullback square. Then the following diagram commutes:

$$\begin{tikzcd} \mathrm{R}h_!\,\omega_{X'} \arrow[r, "\mathrm{R}f_!(\mathrm{Tr}_{i'})"] \arrow[d, "i_*(\mathrm{Tr}_{f'})"] & \mathrm{R}f_!\,\omega_X \arrow[d, "\mathrm{Tr}_f"] \\ i_*\omega_{Y'} \arrow[r, "\mathrm{Tr}_i"] & \omega_Y. \end{tikzcd} \tag{7.2.15}$$

Proof. *Step 1. Proof for étale f.* We first establish the claim when f is additionally assumed to be étale. To do so, we verify the commutativity of the $(f_!, f^*)$-adjoint of (7.2.15). Explicitly, this adjoint is given by the red rectangle in the following diagram:

$$\begin{tikzcd}[column sep=small] & i'_*f'_*\omega_{Y'} \arrow[dddl, "i'_*(\eta_{f'})"'] & i'_*\omega_{X'} \arrow[l, "i'_*(\alpha_{f'}^{-1})"', "\sim"] \arrow[r, red, "\mathrm{Tr}_{i'}"] \arrow[dd, red, "\eta_f"] \arrow[dddl, "i'_*(\eta_{f'})"'] & \omega_X \arrow[dr, "\alpha_f^{-1}", "\sim"'] \arrow[dd, red, "\eta_f"] & \\ & & & & f^*\omega_Y \arrow[dd, "\eta_f"] \\ & & \text{(7.2.16)}\quad f^*f_!i'_*\omega_{X'} \arrow[r, red, "f^*f_!(\mathrm{Tr}_{i'})"] & f^*f_!\omega_X \arrow[dr, "f^*f_!(\alpha_f^{-1})", "\sim"'] \arrow[dd, red, "f^*(\mathrm{Tr}_f)"] & \\ i'_*f'^{,*}f'_!f'^{,*}\omega_{Y'} \arrow[dr, "i'_*f'^{,*}(\epsilon_{f'})"'] & i'_*f'^{,*}f'_!\omega_{X'} \arrow[l, "i'_*f'^{,*}f'_!(\alpha_{f'}^{-1})"', "\sim"] \arrow[r, "\mathrm{BC}^{-1}", "\sim"'] \arrow[d, "i'_*f'^{,*}(\mathrm{Tr}_{f'})"] & f^*i_*f'_!\omega_{X'} \arrow[u, phantom, "\wr"] \arrow[d, red, "f^*i_*(\mathrm{Tr}_{f'})"] & & f^*f_!f^*\omega_Y \arrow[dl, "f^*(\epsilon_f)"] \\ & i'_*f'^{,*}\omega_{Y'} \arrow[r, "\mathrm{BC}^{-1}", "\sim"'] & f^*i_*\omega_{Y'} \arrow[r, red, "f^*(\mathrm{Tr}_i)"] & f^*\omega_Y & \end{tikzcd}$$

Note that except for the lower red rectangle, every other part of this diagram commutes, thanks to the naturality of the base change maps BC^{-1} and the adjunction units η_f and $\eta_{f'}$, the description of étale trace maps (Remark 7.2.9), as well as Lemma 7.2.13. (ii) in the case of (7.2.16). Thus, it suffices to show that the outer diagram commutes. Since $f^*(\epsilon_f) \circ \eta_f = \mathrm{id}$ and $f'^{,*}(\epsilon_{f'}) \circ \eta_{f'} = \mathrm{id}$, this amounts to the commutativity of

$$\begin{tikzcd} i'_*\omega_{X'} \arrow[rrr, "\mathrm{Tr}_{i'}"] \arrow[d, "i'_*(\alpha_{f'}^{-1})", "\wr"'] & & & \omega_X \arrow[d, "\alpha_f^{-1}", "\wr"'] \\ i'_*f'^{,*}\omega_{Y'} \arrow[r, "\mathrm{BC}^{-1}", "\sim"'] & f^*i_*\omega_{Y'} \arrow[rr, "f^*(\mathrm{Tr}_i)"] & & f^*\omega_Y, \end{tikzcd}$$

which follows directly from the observation that the closed trace map is étale-local on the target (Remark 7.2.3).

Step 2. The statement is étale local on X and Y. Next, we prove that if $g' \colon U \twoheadrightarrow X$ and $g \colon V \twoheadrightarrow Y$ are separated taut étale covers and $\tilde{f} \colon U \to V$ is a separated smooth taut morphism fitting into a diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{f}"] \arrow[d, two heads, "g'"] & V \arrow[d, two heads, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

then it suffices to show the assertion for $\tilde{f} \colon U \to V$ and $\tilde{i} := (i, \mathrm{id}) \colon Y' \times_Y V \to V$ instead of f and i. To see this, note that we can extend (7.2.15) to the following diagram, in which we set $\tilde{f}' := (\mathrm{id}, \tilde{f}) \colon Y' \times_Y U \to Y' \times_Y V$, $\tilde{i}' := (i, \mathrm{id}) \colon Y' \times_Y U \to U$, and $\tilde{h} := g \circ \tilde{f} \circ \tilde{i}' = g \circ \tilde{i} \circ \tilde{f}'$ and all the diagonal arrows are étale trace maps:

$$\begin{tikzcd} & \mathrm{R}\tilde{h}_! \omega_{Y' \times_Y U} \arrow[rr, "g_! \mathrm{R}\tilde{f}_!(\mathrm{Tr}_{\tilde{i}'})"] \arrow[dl] \arrow[dd, "g_! \tilde{i}_*(\mathrm{Tr}_{\tilde{f}'})"] & & g_! \mathrm{R}\tilde{f}_! \omega_U \arrow[dl] \arrow[dd, "g_!(\mathrm{Tr}_{\tilde{f}})"] \\ \mathrm{R}h_! \omega_{X'} \arrow[rr, "\mathrm{R}f_!(\mathrm{Tr}_{i'})"] \arrow[dd, "i_*(\mathrm{Tr}_{f'})"] & & \mathrm{R}f_! \omega_X \arrow[dd, "\mathrm{Tr}_f"] & \\ & g_! \tilde{i}_* \omega_{Y' \times_Y V} \arrow[rr, "g_!(\mathrm{Tr}_{\tilde{i}})"] \arrow[dl] & & g_! \omega_V \arrow[dl] \\ i_* \omega_{Y'} \arrow[rr, "\mathrm{Tr}_i"] & & \omega_Y & \end{tikzcd}$$

The top and bottom side of the diagram are induced by cartesian squares and commute thanks to Step 1. The left and right sides commute by the compatibility of smooth traces with compositions (Remark 7.2.10). Since the morphism

$$\mathrm{Hom}(\mathrm{R}h_! \, \omega_{X'}, \omega_Y) \to \mathrm{Hom}(\mathrm{R}\tilde{h}_! \, \omega_{Y' \times_Y U}, \omega_Y)$$

induced by $\mathrm{Tr}_{(\mathrm{id}, g')}$ is injective by virtue of Lemma 7.2.11. (iii), a simple diagram chase shows that the commutativity of the back side of the diagram implies the commutativity of the front side.

Step 3. Reduce to the case of $X = \mathbf{P}^{d,\mathrm{an}}_Y$ and Y is affinoid. By Step 2 and [Hub96, Cor. 1.6.10, Lem. 5.1.3], we may assume that Y is affinoid, that f is of equidimension d for some integer $d \geq 0$, and that f factors as $X \xrightarrow{g} \mathbf{A}^{d,\mathrm{an}}_Y \to Y$ for some separated taut étale morphism $g \colon X \to \mathbf{A}^{d,\mathrm{an}}_Y$. Since the commutativity of (7.2.15) can be checked for the two morphisms separately and was verified for g in Step 1, we may assume that $X = \mathbf{A}^{d,\mathrm{an}}_Y$. Another application of Step 1 to the open immersion $\mathbf{A}^{d,\mathrm{an}}_Y \hookrightarrow \mathbf{P}^{d,\mathrm{an}}_Y$ reduces us to the case when $X = \mathbf{P}^{d,\mathrm{an}}_Y$ and f is the structure map $\mathbf{P}^{d,\mathrm{an}}_Y \to Y$.

Step 4. End of proof. Now we are in the situation where Y is affinoid and $X = \mathbf{P}^{d,\mathrm{an}}_Y$. We put $Y = \mathrm{Spa}\,(A, A^\circ)$ and $Y' = \mathrm{Spa}\,(A/I, (A/I)^\circ)$. Thanks to Remark 7.2.3. (3) and Remark 7.2.10. (3), it suffices to show that for the corresponding Cartesian diagram of finite type A-schemes

$$\begin{tikzcd} \mathbf{P}^d_{A/I} \arrow[r, hook, "i^{\prime,\mathrm{alg}}"] \arrow[d, "f^{\prime,\mathrm{alg}}"] \arrow[dr, "h^{\mathrm{alg}}"] & \mathbf{P}^d_A \arrow[d, "f^{\mathrm{alg}}"] \\ \mathrm{Spec}\, A/I \arrow[r, hook, "i^{\mathrm{alg}}"] & \mathrm{Spec}\, A, \end{tikzcd}$$

the induced diagram

$$\begin{tikzcd} \mathrm{R}h^{\mathrm{alg}}_* \, \omega_{\mathbf{P}^d_{A/I}} \arrow[r, "\mathrm{R}f^{\mathrm{alg}}_*(\mathrm{Tr}^{\mathrm{alg}}_{i^{\prime,\mathrm{alg}}})"] \arrow[d, "i^{\mathrm{alg}}_*(\mathrm{Tr}^{\mathrm{alg}}_{f^{\prime,\mathrm{alg}}})"] & \mathrm{R}f^{\mathrm{alg}}_* \, \omega_{\mathbf{P}^d_A} \arrow[d, "\mathrm{Tr}^{\mathrm{alg}}_{f^{\mathrm{alg}}}"] \\ i^{\mathrm{alg}}_* \omega_{\mathrm{Spec}\, A/I} \arrow[r, "\mathrm{Tr}^{\mathrm{alg}}_{i^{\mathrm{alg}}}"] & \omega_{\mathrm{Spec}\, A} \end{tikzcd} \tag{7.2.17}$$

commutes. Recall that the adjoint to the algebraic trace morphism $\mathrm{R}f^{\mathrm{alg}}_* f^{\mathrm{alg},*}\omega_{\mathrm{Spec}\,A}(d)[2d] \to \omega_{\mathrm{Spec}\,A}$ (see [AGV71, Exp. XVIII, Th. 2.9]) defines the Poincaré duality isomorphism $\mathrm{PD}_{f^{\mathrm{alg}}}\colon f^{\mathrm{alg},*}\omega_{\mathrm{Spec}\,A}(d)[2d] \xrightarrow{\sim} \mathrm{R}f^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A}$, and similarly for $f'^{,\mathrm{alg}}$. We denote by $c_{f^{\mathrm{alg}}}\colon \omega_{\mathbf{P}^d_A} \xrightarrow{\sim} \mathrm{R}f^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A}$ the isomorphism coming from [ILO14, Exp. XVII, Prop. 4.1.2] (and similarly for $f'^{,\mathrm{alg}}$, i^{alg}, and $i'^{,\mathrm{alg}}$) and remind the reader of the isomorphism $\alpha_{f^{\mathrm{alg}}}\colon f^{\mathrm{alg},*}\omega_{\mathrm{Spec}\,A}(d)[2d] \xrightarrow{\sim} \omega_{\mathbf{P}^d_A}$ from Remark 7.1.3. Then the second paragraph after [ILO14, Exp. XVII, Lem. 4.4.1] implies that the composition

$$f^{\mathrm{alg},*}\omega_{\mathrm{Spec}\,A}(d)[2d] \xrightarrow[\sim]{\alpha_{f^{\mathrm{alg}}}} \omega_{\mathbf{P}^d_A} \xrightarrow[\sim]{c_{f^{\mathrm{alg}}}} \mathrm{R}f^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A}$$

is equal to the Poincaré duality isomorphism $\mathrm{PD}_{f^{\mathrm{alg}}}$ defined above; the same claim holds for f'. Therefore, after unraveling the definition of $\mathrm{Tr}^{\mathrm{alg}}_{f^{\mathrm{alg}}}$, we conclude that $\mathrm{Tr}^{\mathrm{alg}}_{f^{\mathrm{alg}}}$ is given by the composition

$$\mathrm{R}f^{\mathrm{alg}}_*\omega_{\mathbf{P}^d_A} \xrightarrow[\sim]{\mathrm{R}f^{\mathrm{alg}}_*(c_{f^{\mathrm{alg}}})} \mathrm{R}f^{\mathrm{alg}}_*\mathrm{R}f^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A} \xrightarrow{\epsilon_{f^{\mathrm{alg}}}} \omega_{\mathrm{Spec}\,A},$$

where $\epsilon_{f^{\mathrm{alg}}}$ is the counit of the $(\mathrm{R}f^{\mathrm{alg}}_!, \mathrm{R}f^{\mathrm{alg},!})$-adjunction; a similar formula holds for $f'^{,\mathrm{alg}}$. After unraveling the definition of $\mathrm{Tr}_{i^{\mathrm{alg}}}$, we see that it is also given by the composition

$$\mathrm{R}i^{\mathrm{alg}}_*\omega_{\mathrm{Spec}\,A/I} \xrightarrow[\sim]{i^{\mathrm{alg}}_*(c_{i^{\mathrm{alg}}})} i^{\mathrm{alg}}_*\mathrm{R}i^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A} \xrightarrow{\epsilon_{i^{\mathrm{alg}}}} \omega_{\mathrm{Spec}\,A}.$$

In conclusion, for the purpose of proving the commutativity of Diagram (7.2.17), it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}h^{\mathrm{alg}}_*\omega_{\mathbf{P}^d_{A/I}} \arrow[r, "\mathrm{R}h^{\mathrm{alg}}_*(c_{i',\mathrm{alg}})"] \arrow[d, "\mathrm{R}h^{\mathrm{alg}}_*(c_{f',\mathrm{alg}})"] & \mathrm{R}h^{\mathrm{alg}}_*\mathrm{R}i'^{,\mathrm{alg},!}\omega_{\mathbf{P}^d_A} \arrow[r, "\mathrm{R}f^{\mathrm{alg}}_*(\epsilon_{i^{\mathrm{alg}}})"] \arrow[d, "\mathrm{R}h^{\mathrm{alg}}_*\mathrm{R}i'^{,\mathrm{alg},!}(c_{f^{\mathrm{alg}}})"] & \mathrm{R}f^{\mathrm{alg}}_*\omega_{\mathbf{P}^d_A} \arrow[d, "\mathrm{R}f^{\mathrm{alg}}_*(c_{f^{\mathrm{alg}}})"] \\
\mathrm{R}h^{\mathrm{alg}}_*\mathrm{R}f'^{,\mathrm{alg},!}\omega_{\mathrm{Spec}\,A/I} \arrow[r, "\mathrm{R}h^{\mathrm{alg}}_*\mathrm{R}f'^{,\mathrm{alg},!}(c_{i^{\mathrm{alg}}})"] \arrow[d, "i^{\mathrm{alg}}_*(\epsilon_{f',\mathrm{alg}})"] & \mathrm{R}h^{\mathrm{alg}}_*\mathrm{R}h^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A} \arrow[r, "\mathrm{R}f^{\mathrm{alg}}_*(\epsilon_{i',\mathrm{alg}}\mathrm{R}f^{\mathrm{alg},!})"] \arrow[d, "i^{\mathrm{alg}}_*(\epsilon_{f',\mathrm{alg}}i^{\mathrm{alg},!})"] & \mathrm{R}f^{\mathrm{alg}}_*\mathrm{R}f^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A} \arrow[d, "\epsilon_{f^{\mathrm{alg}}}"] \\
i^{\mathrm{alg}}_*\omega_{\mathrm{Spec}\,A/I} \arrow[r, "c_{i^{\mathrm{alg}}}"] & i^{\mathrm{alg}}_*\mathrm{R}i^{\mathrm{alg},!}\omega_{\mathrm{Spec}\,A} \arrow[r, "\epsilon_{i^{\mathrm{alg}}}"] & \omega_{\mathrm{Spec}\,A}
\end{tikzcd}$$

The top right square and the bottom left square commute due to functoriality of $\epsilon_{i^{\mathrm{alg}}}$ and $\epsilon_{f',\mathrm{alg}}$, respectively, the bottom right square commutes due to the compatibility of adjunction counits with compositions, and the top left square commutes due to [ILO14, Exp. XVII, Rmq. 4.1.3]. □

Now we are ready to show a version of Lemma 6.2.2 for traces with coefficients in dualizing complexes:

Lemma 7.2.18. *Let $f\colon X \to Y$ be a separated taut smooth morphism of rigid-analytic spaces over K and let $s\colon Y \hookrightarrow X$ be a section. Then the composition of trace maps*

$$\omega_Y \simeq (\mathrm{R}f_! \circ s_*)(\omega_Y) \xrightarrow{\mathrm{R}f_!(\mathrm{Tr}_s)} \mathrm{R}f_!\,\omega_X \xrightarrow{\mathrm{Tr}_f} \omega_Y$$

is the identity.

Proof. Without loss of generality, we may and do assume that Y is connected. Pick a classical point $y \in Y$. The canonical inclusions fit into a cartesian diagram

$$\begin{tikzcd}
X_y \arrow[r, hook, "i'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
y \arrow[r, hook, "i"] \arrow[u, bend left, "s'"] & Y. \arrow[u, bend left, "s"]
\end{tikzcd}$$

Thanks to Remark 7.2.3. (1) and Lemma 7.2.14, respectively, the left and right square in the following diagram commute:

$$\begin{tikzcd} i_*\omega_y \simeq i_*\mathrm{R}f'_!s'_*\omega_y \arrow[r, "i_*\mathrm{R}f'_!(\mathrm{Tr}_{s'})"] \arrow[d, "\mathrm{Tr}_i"] & i_*\mathrm{R}f'_!\omega_{X_y} \arrow[r, "i_*(\mathrm{Tr}_{f'})"] \arrow[d, "\mathrm{R}f_!(\mathrm{Tr}_{i'})"] & i_*\omega_y \arrow[d, "\mathrm{Tr}_i"] \\ \omega_Y \simeq \mathrm{R}f_!s_*\omega_Y \arrow[r, "\mathrm{R}f_!(\mathrm{Tr}_s)"] & \mathrm{R}f_!\omega_X \arrow[r, "\mathrm{Tr}_f"] & \omega_Y. \end{tikzcd}$$

It follows that it suffices to prove the assertion for f' instead of f: indeed, by [BH22, Th. 3.21.(3)] and the full faithfulness of i_*, the two horizontal compositions are given by scalar multiplication with an element of Λ and the commutativity of the diagram guarantees that both classes map to the same element in

$$\begin{tikzcd} \Lambda \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{Hom}(\omega_Y, \omega_Y) \arrow[d, "\wr", "-\circ \mathrm{Tr}_i"'] \\ \mathrm{Hom}(\omega_y, \omega_y) \arrow[r, "\mathrm{Tr}_i \circ i_*(-)", "\sim"'] & \mathrm{Hom}(i_*\omega_y, \omega_Y), \end{tikzcd}$$

hence they must be equal. In conclusion, it is enough to prove the statement when Y is a smooth rigid space over K (in fact, $Y = \mathrm{Spa}\,(K', \mathcal{O}_{K'})$ for some finite extension K'/K) and X is a separated taut smooth rigid-analytic space over K. In this case, the result follows directly from Lemma 6.2.2 and Lemma 7.2.6. □

Lastly, we extend Lemma 7.2.14 to commutative diagrams that are not necessarily cartesian.

Theorem 7.2.19 (Compatibility for commutative diagrams)**.** *Consider a commutative diagram of rigid-analytic spaces over K*

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "f'"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook, "i"] & Y. \end{tikzcd}$$

Suppose that f and f' are separated, taut and smooth, and that i and i' are closed immersions. Then the following diagram in $D(Y_{\text{ét}}; \Lambda)$ commutes:

$$\begin{tikzcd} \mathrm{R}h_!\,\omega_{X'} \arrow[r, "\mathrm{R}f_!(\mathrm{Tr}_{i'})"] \arrow[d, "i_*(\mathrm{Tr}_{f'})"] & \mathrm{R}f_!\,\omega_X \arrow[d, "\mathrm{Tr}_f"] \\ i_*\omega_{Y'} \arrow[r, "\mathrm{Tr}_i"] & \omega_Y. \end{tikzcd}$$

Proof. We first deal with two special cases in which one of the morphisms is the identity and then use them to deduce the general version.

Step 1. Proof when $X' = Y'$ and $f' = \mathrm{id}$. The fiber product $W := X \times_Y Y'$ comes with natural projections $g\colon W \to Y'$ and $j\colon W \hookrightarrow X$. Moreover, i' induces a natural section $s\colon Y' \to W$ of g. These maps fit into the commutative diagram

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[u, bend left, "s"] \arrow[ur, "i'"] \arrow[r, hook, "i"] & Y. \end{tikzcd}$$

An application of Remark 7.2.3. (1), Lemma 7.2.14, and Lemma 7.2.18, respectively, then gives the desired identity

$$\mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{i'}) = \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_j) \circ \mathrm{R}(f \circ j)_!(\mathrm{Tr}_s) = \mathrm{Tr}_i \circ i_*(\mathrm{Tr}_g) \circ \mathrm{R}(i \circ g)_!(\mathrm{Tr}_s) = \mathrm{Tr}_i\,.$$

Step 2. Proof when $Y' = Y$ and $i = \mathrm{id}$. The fiber product $W := X' \times_Y X$ comes with natural projections $g\colon W \to X'$ and $g'\colon W \to X$, which are again separated, taut and smooth. Moreover, i' induces a natural section $s\colon X' \to W$ of g. These maps fits into the commutative diagram

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[u, bend left, "s"] \arrow[ur, hook, "i'"] \arrow[r, "f'"] & Y. \end{tikzcd}$$

An application of Step 1, Remark 7.2.10. (1), and Lemma 7.2.18, respectively, then gives the desired identity

$$\mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{i'}) = \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{g'}) \circ \mathrm{R}(f\circ g')_!(\mathrm{Tr}_s) = \mathrm{Tr}_{f'} \circ \mathrm{R}f'_!(\mathrm{Tr}_g) \circ \mathrm{R}(f'\circ g)_!(\mathrm{Tr}_s) = \mathrm{Tr}_{f'}\,.$$

Step 3. Proof in the general case. The fiber product $W \coloneqq X' \times_Y Y$ fits into the commutative diagram

$$\begin{tikzcd} X' \arrow[dr, hook, "j'"] \arrow[drr, hook, bend left, "i'"] \arrow[ddr, bend right, "f'"] & & \\ & W \arrow[r, hook, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ & Y' \arrow[r, hook, "i"] & Y, \end{tikzcd}$$

where j and j' are closed immersions. An application of Remark 7.2.3. (1), Lemma 7.2.14, and Step 2 respectively, then gives the desired identity

$$\mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{i'}) = \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_j) \circ \mathrm{R}(f\circ j)_!(\mathrm{Tr}_{j'}) = \mathrm{Tr}_i \circ i_*(\mathrm{Tr}_g) \circ \mathrm{R}(i\circ g)_!(\mathrm{Tr}_{j'}) = \mathrm{Tr}_i \circ i_*(\mathrm{Tr}_{f'}). \qquad \square$$

7.3. **Smooth-source trace.** The main goal of this subsection is to construct a trace map for any separated taut morphism $f\colon X \to Y$ of rigid-analytic spaces over K with X smooth and Y separated and taut. In the next subsection, we will drop the assumptions on X and Y at the expense of assuming properness of f.

For the next construction, we fix a separated taut morphism $f\colon X \to Y$ as above. This automatically implies that X is separated and taut as well. Then we factor f as the composition

$$X \xhookrightarrow{\Gamma_f} X \times Y \xrightarrow{\pi_Y} Y$$

of the graph morphism Γ_f and the natural projection π_Y. Note that Γ_f is a closed immersion since Y is separated (see [Zav24, Cor. B.6.10 and B.7.4]) and that π_Y is a separated taut smooth morphism because X is separated taut and smooth over K.

Construction 7.3.1 (Smooth-source trace)**.** For f as above, the *smooth-source trace map* $\mathrm{Tr}_f\colon \mathrm{R}f_!\,\omega_X \to \omega_Y$ is the composition

$$\mathrm{R}f_!\,\omega_X \simeq (\mathrm{R}\pi_{Y,!} \circ \Gamma_{f,*})\omega_X \xrightarrow{\mathrm{R}\pi_{Y,!}(\mathrm{Tr}_{\Gamma_f})} \mathrm{R}\pi_{Y,!}\,\omega_{X\times Y} \xrightarrow{\mathrm{Tr}_{\pi_Y}} \omega_Y,$$

where Tr_{Γ_f} is the closed trace from Construction 7.2.1 and Tr_{π_Y} is the smooth trace from Construction 7.2.7.

We now verify some basic properties of smooth-source trace maps.

Proposition 7.3.2. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be separated taut morphisms of rigid-analytic spaces over K. Assume that X is smooth over K and that Z is separated and taut over K.*

(i) *(Compatibility with smooth trace) If f is smooth, then Tr_f is equal to the smooth trace map from Construction 7.2.7. In particular, $\mathrm{Tr}_f = \mathrm{id}$ when $f = \mathrm{id}$.*
(ii) *(Compatibility with closed trace) If f is a closed immersion, then Tr_f is equal to the closed trace map from Construction 7.2.1.*
(iii) *(Compatibility with smooth maps) If $h\colon Y' \to Y$ is a separated taut smooth morphism and*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

is a cartesian diagram, then $\mathrm{Tr}_h \circ \mathrm{R}h_!(\mathrm{Tr}_{f'}) = \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{h'})$*, where* Tr_h *and* $\mathrm{Tr}_{h'}$ *denote the smooth trace maps from Construction 7.2.7.*

(iv) *(Compatibility with compositions I) If Y is smooth, then* $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_!(\mathrm{Tr}_f)$.

(v) *(Compatibility with compositions II) If g is a closed immersion (resp. smooth), then* $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_!(\mathrm{Tr}_f)$ *where* Tr_g *is the closed trace from Construction 7.2.1 (resp. the smooth trace from Construction 7.2.7).*

Proof. Parts (i) and (ii) follow directly from Theorem 7.2.19 (more specifically, from the special cases treated in Step 2 and Step 1 of its proof, respectively).

Now we deal with (iii). For this, we consider the following commutative diagram:[36]

$$\begin{tikzcd}
X' \arrow[r, hook, "\Gamma_{f'}"] \arrow[d, "h'"] \arrow[rr, bend left, "f'"] & X' \times Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "h' \times h"] & Y' \arrow[d, "h"] \\
X \arrow[r, hook, "\Gamma_f"] \arrow[rr, bend right, "f"'] & X \times Y \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

Note that h, h', π_Y, and $\pi_{Y'}$ are separated, taut and smooth, and that Γ_f and $\Gamma_{f'}$ are closed immersions. Therefore, the assertion results from

$$\begin{aligned}
\mathrm{Tr}_h \circ \mathrm{R}h_!(\mathrm{Tr}_{f'}) &= \mathrm{Tr}_h \circ \mathrm{R}h_!(\mathrm{Tr}_{\pi_{Y'}}) \circ \mathrm{R}(h \circ \pi_{Y'})_!(\mathrm{Tr}_{\Gamma_{f'}}) \\
&= \mathrm{Tr}_{\pi_Y} \circ \mathrm{R}\pi_{Y,!}(\mathrm{Tr}_{h'\times h}) \circ \mathrm{R}\big(\pi_Y \circ (h' \times h)\big)_!(\mathrm{Tr}_{\Gamma_{f'}}) \\
&= \mathrm{Tr}_{\pi_Y} \circ \mathrm{R}\pi_{Y,!}\big(\mathrm{Tr}_{\Gamma_f} \circ \mathrm{R}\,\Gamma_{f,!}(\mathrm{Tr}_{h'})\big) \\
&= \mathrm{Tr}_{\pi_Y} \circ \mathrm{R}\pi_{Y,!}(\mathrm{Tr}_{\Gamma_f}) \circ \mathrm{R}f_!(\mathrm{Tr}_{h'}) \\
&= \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{h'}),
\end{aligned}$$

where the first equality follows from Construction 7.3.1, the second equality follows from Remark 7.2.10. (1), third equality follows from Theorem 7.2.19, the fourth equality follows from $f = \pi_Y \circ \Gamma_f$, and the last equality follows again from Construction 7.3.1.

To prove (iv), we first treat the case when f is a closed immersion. In this case, we consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, hook, "\Gamma_{g\circ f}"] \arrow[dr, hook, "f"'] & & X \times Z \arrow[d, hook, "f \times \mathrm{id}"] \arrow[dd, bend left=60, "\pi^X_Z"] \\
& Y \arrow[r, hook, "\Gamma_g"] \arrow[dr, "g"'] & Y \times Z \arrow[d, "\pi^Y_Z"] \\
& & Z
\end{tikzcd}$$

Note that f, $\Gamma_{g\circ f}$, $f \times \mathrm{id}$, and Γ_g are closed immersions. Therefore, the assertion results from

$$\begin{aligned}
\mathrm{Tr}_g \circ \mathrm{R}g_!(\mathrm{Tr}_f) &= \mathrm{Tr}_{\pi^Y_Z} \circ \mathrm{R}\pi^Y_{Z,!}(\mathrm{Tr}_{\Gamma_g}) \circ \mathrm{R}\pi^Y_{Z,!}\big(\mathrm{R}\,\Gamma_{g,!}(\mathrm{Tr}_f)\big) \\
&= \mathrm{Tr}_{\pi^Y_Z} \circ \mathrm{R}\pi^Y_{Z,!}(\mathrm{Tr}_{f\times \mathrm{id}}) \circ \mathrm{R}\pi^X_{Z,!}(\mathrm{Tr}_{\Gamma_{g\circ f}}) \\
&= \mathrm{Tr}_{\pi^X_Z} \circ \mathrm{R}\pi^X_{Z,!}(\mathrm{Tr}_{\Gamma_{g\circ f}}) \\
&= \mathrm{Tr}_{g\circ f},
\end{aligned}$$

where the first equality follows from Construction 7.3.1, the second equality follows from part (ii) and Remark 7.2.3. (1), the third equality follows from Theorem 7.2.19, and the last equality follows again from Construction 7.3.1.

[36] We warn the reader that the inner squares in the diagram below are not cartesian.

Now we prove (iv) for a general f. For this, we consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "\Gamma_f"] \arrow[rr, bend left, "\Gamma_{g\circ f}"] \arrow[dr, "f"'] & X \times Y \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\pi_Y"] & X \times Z \arrow[d, "\pi_Z"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then the assertion results from

$$\begin{aligned} \mathrm{Tr}_g \circ \mathrm{R}g_!(\mathrm{Tr}_f) &= \mathrm{Tr}_g \circ \mathrm{R}g_!(\mathrm{Tr}_{\pi_Y}) \circ \mathrm{R}g_!\big(\mathrm{R}\pi_{Y,!}(\mathrm{Tr}_{\Gamma_f})\big) \\ &= \mathrm{Tr}_{\pi_Z} \circ \mathrm{R}\pi_{Z,!}(\mathrm{Tr}_{\mathrm{id}\times g}) \circ \mathrm{R}\pi_{Z,!}\big(\mathrm{R}(\mathrm{id}\times g)_!(\mathrm{Tr}_{\Gamma_f})\big) \\ &= \mathrm{Tr}_{\pi_Z} \circ \mathrm{R}\pi_{Z,!}(\mathrm{Tr}_{\Gamma_{g\circ f}}) \\ &= \mathrm{Tr}_{g\circ f}, \end{aligned}$$

where the first equality follows from Construction 7.3.1, the second equality follows from Theorem 7.2.19, the third equality follows from the observation that $X \times Y$ is smooth and the case of closed immersions established above, and the last equality follows again from Construction 7.3.1.

The proof of (v) is similar to that of (iv). We leave the details to the interested reader. □

Proposition 7.3.2. (iii) formally implies that the smooth-source trace is étale local, yielding the following variant of Remark 7.2.3. (2) and Remark 7.2.10. (2):

Remark 7.3.3 (Smooth-source trace is étale-local on the target)**.** Let $f\colon X \to Y$ be a separated taut morphism and $h\colon Y' \to Y$ be a separated taut étale morphism of rigid-analytic spaces over K. Assume that X is smooth over K and that Y is separated and taut over K. Let $X' := Y' \times_Y X$ be the fiber product and $f'\colon X' \to Y'$ and $h'\colon X' \to X$ the two natural projections. Then the $(h_!, h^*)$-adjoint of the equality $\mathrm{Tr}_h \circ \mathrm{R}h_!(\mathrm{Tr}_{f'}) = \mathrm{Tr}_f \circ \mathrm{R}f_!(\mathrm{Tr}_{h'})$ from Proposition 7.3.2. (iii) amounts by virtue of Remark 7.2.9 to the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{R}f'_!\omega_{X'} \arrow[rr, "\mathrm{Tr}_{f'}"] & & \omega_{Y'} \\ \mathrm{R}f'_!h'^*\omega_X = h^*\mathrm{R}f_!\omega_X \arrow[u, "\mathrm{R}f'_!(\alpha_{h'})", "\wr"'] \arrow[rr, "h^*(\mathrm{Tr}_f)"] & & h^*\omega_Y. \arrow[u, "\alpha_h", "\wr"'] \end{tikzcd}$$

Our first application of the smooth-source trace will be a vanishing result for the Verdier dual of the derived pushforward of a dualizing sheaf (see Theorem 7.3.14). This vanishing result will play a crucial role in our construction of general proper trace in next subsection.

Before we start proving this vanishing result, we need a number of preliminary lemmas.

Lemma 7.3.4. *Let $f\colon X \to Y$ be a morphism of rigid-analytic spaces over K, let $i\colon Y' \hookrightarrow Y$ be a closed immersion of rigid-analytic spaces over K, and let*

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook, "i"] & Y \end{tikzcd}$$

be the resulting pullback square. Then the natural transformation of functors

$$\mathrm{R}f'_*\mathrm{R}i'^{,!}(-) \to \mathrm{R}i^!\mathrm{R}f_*(-)\colon D(X_{\text{ét}};\Lambda) \longrightarrow D(Y'_{\text{ét}};\Lambda), \tag{7.3.5}$$

given by the $(i_*, \mathrm{R}i^!)$-adjoint to $i_*\mathrm{R}f'_*\mathrm{R}i'^{,!}(-) \simeq \mathrm{R}f_*i'_*\mathrm{R}i'^{,!}(-) \xrightarrow{\mathrm{R}f_*(\epsilon_{i'})} \mathrm{R}f_*(-)$, is an equivalence.

Proof. Since both $\mathrm{R}f_*$ and $\mathrm{R}i^!$ are right adjoints, we may show instead that the natural transformation (7.3.5) is an equivalence after passing to left adjoints. In other words, it suffices to prove that the natural transformation $f^*i_*(-) \to i'_*f'^*(-)$ is an equivalence. This can be checked easily by arguing on stalks. □

Corollary 7.3.6. *Under the assumptions of Lemma 7.3.4, there is a canonical isomorphism*

$$c_{f,i}\colon \mathrm{R}f'_*\omega_{X'} \xrightarrow{\sim} \mathrm{R}i^!\mathrm{R}f_*\omega_X.$$

Proof. This follows directly from Lemma 7.3.4 and [BH22, Th. 3.21(1)]. □

Notation 7.3.7. Let $f\colon X \to Y$ and $i\colon Y' \hookrightarrow Y$ be as in Lemma 7.3.4. Assume that f is proper, X is smooth over K, and Y is taut and separated. Then we denote by

$$\mathrm{R}i^!(\mathrm{Tr}_f)\colon \mathrm{R}f'_*\omega_{X'} \to \omega_{Y'}$$

the following composition

$$\mathrm{R}f'_*\omega_{X'} \xrightarrow[\sim]{c_{f,i}} \mathrm{R}i^!\mathrm{R}f_*\omega_X \xrightarrow{\mathrm{R}i^!(\mathrm{Tr}_f)} \mathrm{R}i^!\omega_X \xrightarrow[\sim]{c_i^{-1}} \omega_Z,$$

where Tr_f is the smooth-source trace from Construction 7.3.1.

For future reference, it will be convenient to introduce the following definition:

Definition 7.3.8. Let X be a rigid-analytic space over K, $i\colon Z \hookrightarrow X$ a Zariski-closed immersion, and $U \subset X$ be its (Zariski) open complement.

(i) A *U-modification* $\pi\colon X' \to X$ is a proper morphism of rigid-analytic spaces over K such that $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$ is an isomorphism.
(ii) A *U-admissible modification* $\pi\colon X' \to X$ is a U-modification such that $U \subset X$ and $U \simeq \pi^{-1}(U) \subset X'$ are dense.
(iii) A *regular U-modification* $\pi\colon X' \to X$ is a U-modification such that X' is smooth over K.
(iv) A *regular U-admissible modification* is a regular U-modification which is U-admissible.

For a U-modification $\pi\colon X' \to X$, we will often denote by $i'\colon Z' \coloneqq \pi^{-1}(Z) = X' \times_X Z \hookrightarrow X'$ the preimage of Z along X.

There is an abundance of regular U-admissible modifications:

Proposition 7.3.9. *Let X be a quasicompact reduced rigid-analytic space over K. Then:*

(i) *The smooth locus of X is Zariski-open and dense.*
(ii) *(Temkin) For any Zariski-open and dense subspace $U \subseteq X$ that is contained in the smooth locus of X, there exists a regular U-admissible modification $\pi\colon X' \to X$.*

We remind the reader that we always (implicitly) assume that $\operatorname{char} K = 0$ in this section.

Proof. Part (i) follows from the fact that K-affinoid algebras are excellent; see e.g. the discussion after [Con99, Lem. 3.3.1]. Part (ii) is [Tem12, Th. 5.2.2] (cf. also [Tem12, Th. 1.2.1]). □

Lemma 7.3.10. *Let $U \subset X$ be a dense Zariski-open subspace, and let $\pi\colon X' \to X$ be a U-admissible modification. Then, for any classical point $x \in X$, we have $\dim \pi^{-1}(x) < \max(\dim X, 1)$.*

Proof. We denote by $Z \subset X$ Zariski-closed complement to U (with reduced adic space structure) and by $Z' \subset X'$ the fiber product $Z' \coloneqq Z \times_X X'$.

Now we start the proof. If $x \in U$, then $\pi^{-1}(x)$ is a singleton. In particular, $\dim \pi^{-1}(x) = 0 < \max(\dim X, 1)$. If $x \in Z$, then $\dim \pi^{-1}(x) \le \dim Z' < \dim X' = \dim U = \dim X \le \max(\dim X, 1)$, where the second inequality holds due to the assumption that $Z' \subset X'$ is nowhere dense. □

Lemma 7.3.11. *Let X be a rigid-analytic space over K, let $i\colon Z \hookrightarrow X$ be a Zariski-closed subspace over K with the open complement $U \subset X$, and let $\pi\colon X' \to X$ be a regular U-modification. Put $Z' \coloneqq X' \times_X Z$ and $i'\colon Z' \hookrightarrow X'$ and $\pi'\colon Z' \to Z$ be the natural projections and $h\colon Z' \to X$ the evident composition. Then there is an exact triangle*

$$\mathrm{R}h_*\omega_{Z'} \xrightarrow{\left(-\mathrm{R}\pi_*(\mathrm{Tr}_{i'}),\, i_*\mathrm{R}i^!(\mathrm{Tr}_\pi)\right)} \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z \xrightarrow{\mathrm{Tr}_\pi \oplus \mathrm{Tr}_i} \omega_X \to \mathrm{R}h_*\omega_{Z'}[1]$$

in $D(Y_{\text{ét}}; \Lambda)$.

Proof. For brevity, we denote by $\alpha\colon \mathrm{R}h_*\omega_{Z'} \to \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z$ the morphism $\big(-\mathrm{R}\pi_*(\mathrm{Tr}_{i'}), i_*\mathrm{R}i^!(\mathrm{Tr}_\pi)\big)$ and by $\beta\colon \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z \to \omega_X$ the morphism $\mathrm{Tr}_\pi \oplus \mathrm{Tr}_i$. We also set $C \coloneqq \mathrm{fib}(\beta) = \mathrm{cone}(\beta)[-1]$. It fits into an exact triangle

$$C \xrightarrow{\alpha'} \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z \xrightarrow{\beta} \omega_X \xrightarrow{\gamma} C[1].$$

Now, after unravelling all the definitions, we see that the following diagram commutes:

$$\begin{tikzcd}
i_*\mathrm{R}\pi'_*\omega_{Z'} \arrow[r, "i_*(c_{\pi,i})"] \arrow[d, equal] & i_*\mathrm{R}i^!\pi_*\omega_{X'} \arrow[r, "i_*\mathrm{R}i^!(\mathrm{Tr}_\pi)"] \arrow[d, "\epsilon_i"] & i_*\mathrm{R}i^!\omega_X \arrow[r, "i_*(c_i^{-1})"] \arrow[dr, "\epsilon_i"] & i_*\omega_Z \arrow[d, "\mathrm{Tr}_i"] \\
\mathrm{R}\pi_*i'_*\omega_{Z'} \arrow[r, "\mathrm{R}\pi_*(\mathrm{Tr}_{i'})"] & \mathrm{R}\pi_*\omega_{X'} \arrow[rr, "\mathrm{Tr}_\pi"] & & \omega_X,
\end{tikzcd}$$

where ϵ_i is the counit of the $(i_*, \mathrm{R}i^!)$-adjunction. Since the composition of red arrows is equal to $\mathrm{Tr}_i \circ i_*\mathrm{R}i^!(\mathrm{Tr}_\pi)$, we conclude that $\beta \circ \alpha = 0$. Then the axioms of triangulated categories imply that there is a morphism $A\colon \mathrm{R}h_*\omega_{Z'} \to C$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{R}h_*\omega_{Z'} \arrow[r, "\alpha"] \arrow[d, "A"] & \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"] & \omega_X \arrow[d, "\mathrm{id}"] \\
C \arrow[r, "\alpha'"] & \mathrm{R}\pi_*\omega_{X'} \oplus i_*\omega_Z \arrow[r, "\beta"] & \omega_X.
\end{tikzcd} \tag{7.3.12}$$

Therefore, it suffices to show that A is an isomorphism. For this, it suffices to check that both $A\big|_U$ and $\mathrm{R}i^!A$ are isomorphisms.

We first show that $A\big|_U$ is an isomorphism. Clearly, $\big(\mathrm{R}h_*\omega_{Z'}\big)\big|_U = 0$, so it suffices to show that $C\big|_U = 0$. This follows from the observations that $\big(i_*\omega_Z\big)\big|_U = 0$ and $(\mathrm{Tr}_\pi)\big|_U = \mathrm{id}$ (see Proposition 7.3.2. (i)).

Now we show that $\mathrm{R}i^!A$ is an isomorphism. In this case, we note that, after applying $\mathrm{R}i^!$ to (7.3.12), it becomes isomorphic to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\pi'_*\omega_{Z'} \arrow[r, "\mathrm{R}i^!\alpha=\left(-\mathrm{id},\mathrm{R}i^!(\mathrm{Tr}_\pi)\right)"] \arrow[d, "\mathrm{R}i^!A"] & \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z \arrow[r, "\mathrm{R}i^!\beta=\mathrm{R}i^!(\mathrm{Tr}_\pi)\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \omega_Z \arrow[d, "\mathrm{id}"] \\
\mathrm{R}i^!C \arrow[r, "\mathrm{R}i^!\alpha'"] & \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z \arrow[r, "\mathrm{R}i^!\beta=\mathrm{R}i^!(\mathrm{Tr}_\pi)\oplus\mathrm{id}"] & \omega_Z.
\end{tikzcd} \tag{7.3.13}$$

Now we note that the map $\mathrm{R}i^!\beta$ admits a section $(0, \mathrm{id})\colon \omega_Z \to \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z$. Therefore, we conclude that the boundary map $\mathrm{R}i^!(\gamma) = 0$. Likewise, we use that $\mathrm{R}i^!\beta$ admits a section to verify that the first row of (7.3.13) extends to a distinguished triangle $\mathrm{R}\pi'_*\omega_{Z'} \xrightarrow{\mathrm{R}i^!(\alpha)} \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z \xrightarrow{\mathrm{R}i^!\beta} \omega_Z \xrightarrow{0} \mathrm{R}\pi'_*\omega_{Z'}[1]$. Therefore, we conclude that we can extend (7.3.13) to a morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\pi'_*\omega_{Z'} \arrow[r, "\mathrm{R}i^!\alpha=(-\mathrm{id},\mathrm{R}i^!(\mathrm{Tr}_\pi))"] \arrow[d, "\mathrm{R}i^!A"] & \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z \arrow[r, "\mathrm{R}i^!\beta=\mathrm{R}i^!(\mathrm{Tr}_\pi)\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \omega_Z \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & \mathrm{R}\pi_*\omega_{Z'}[1] \arrow[d, "\mathrm{R}i^!A[1]"] \\
\mathrm{R}i^!C \arrow[r, "\mathrm{R}i^!\alpha'"] & \mathrm{R}\pi'_*\omega_{Z'} \oplus \omega_Z \arrow[r, "\mathrm{R}i^!\beta=\mathrm{R}i^!(\mathrm{Tr}_\pi)\oplus\mathrm{id}"] & \omega_Z \arrow[r, "\mathrm{R}i^!\gamma=0"] & \mathrm{R}i^!C[1].
\end{tikzcd}$$

Since this is a morphism of distinguished triangles and two-out-of-three vertical arrows are isomorphisms, we conclude that $\mathrm{R}i^!A$ must be an isomorphism as well. This finishes the proof. $\square$

Finally, we are ready to prove the desired vanishing result:

Theorem 7.3.14. *Let $f\colon X \to Y$ be a proper map of rigid-analytic spaces over K. Then $\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$.*

Proof. *Step 1. Reduce to the case when Y is a geometric point.* First, we note that $\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ lies in $D_{\mathrm{zc}}(Y_{\text{ét}}; \Lambda)$ [BH22, Th. 3.10, Th. 3.21(3)]. Therefore, it suffices to show that for every classical point

$i_y\colon y = \mathrm{Spa}\big(k(y), k(y)^\circ\big) \hookrightarrow Y$, the pullback $i_y^*\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ lies in $D^{\geq 0}(y_{\text{ét}}; \Lambda)$. To simplify this complex even further, we consider the following cartesian square:

$$\begin{tikzcd} X_y \arrow[r, "i'_y"] \arrow[d, "f_y"] & X \arrow[d, "f"] \\ y \arrow[r, "i_y"] & Y. \end{tikzcd}$$

Corollary 7.3.6 constructs a canonical isomorphism $\mathrm{R}f_{y,*}\omega_{X_y} \simeq \mathrm{R}i_y^!\mathrm{R}f_*\omega_X$. Combining this with (the proof of) [BH22, Th. 3.21(4)], we obtain

$$i_y^*\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y) = i_y^*\mathbf{D}_Y(\mathrm{R}f_*\omega_X) \simeq \mathbf{D}_y(\mathrm{R}i_y^!\mathrm{R}f_*\omega_X) \simeq \mathbf{D}_y(\mathrm{R}f_{y,*}\omega_{X_y}).$$

Therefore, it suffices to show that $\mathbf{D}_y(\mathrm{R}f_{y,*}\omega_{X_y})$ lies in $D^{\geq 0}(y_{\text{ét}}; \Lambda)$. Since the statement does not depend on K, we can replace K with $k(y)$ to assume that $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$. Then [BH22, Prop. 3.24] ensures that we can replace K with $\widehat{\overline{K}}$ to assume that K is algebraically closed.

Step 2. Proof when Y is a geometric point. In this case, we have $D(Y_{\text{ét}}; \Lambda) = D(\Lambda)$ and $\omega_Y = \Lambda$. Therefore, the question becomes equivalent to showing that $\mathrm{R}\Gamma(X, \omega_X)$ lies in $D^{\leq 0}(\Lambda)$. Thanks to Lemma 7.2.4, we have $\mathrm{R}\Gamma(X, \omega_X) \simeq \mathrm{R}\Gamma(X_{\mathrm{red}}, \omega_{X_{\mathrm{red}}})$. After replacing X with X_{red}, we may thus assume that X is reduced.

Since X is proper, it has finite dimension. We prove the claim by induction on $d = \dim X$. If $d \leq 0$, then X is either empty or a finite disjoint union of points, so the result is obvious. Therefore, we assume that $d \geq 1$ and that we know the result for all $\dim X \leq d-1$.

Let $U \subseteq X$ the smooth locus of X. By Proposition 7.3.9, U is Zariski-open and dense and there exists a regular U-admissible modification $\pi\colon X' \to X$. Set $Z := X \smallsetminus U$ (with the reduced adic space structure) and $Z' := Z \times_X X'$; we have $\dim Z < \dim X$ and $\dim Z' < \dim X' = \dim X$ since U is dense in X and $U \simeq \pi^{-1}(U)$ is dense in X'. We denote by $h\colon Z' \to X$ the natural composition. Then Lemma 7.3.11 implies that we have the following exact triangle

$$\mathrm{R}h_*\omega_{Z'} \to i_*\omega_Z \oplus \mathrm{R}\pi_*\omega_{X'} \to \omega_X.$$

Therefore, it suffices to show that the complexes $\mathrm{R}\Gamma(X, \mathrm{R}h_*\omega_{Z'}) = \mathrm{R}\Gamma(Z', \omega_{Z'})$, $\mathrm{R}\Gamma(X, i_*\omega_Z) = \mathrm{R}\Gamma(Z, \omega_Z)$, and $\mathrm{R}\Gamma(X, \mathrm{R}\pi_*\omega_{X'}) = \mathrm{R}\Gamma(X', \omega_{X'})$ lie in $D^{\leq 0}(\Lambda)$. Since $\dim Z < \dim X = d$ and $\dim Z' < \dim X' = d$, we conclude that the first two complexes lie in $D^{\leq 0}(\Lambda)$ by the induction hypothesis. Finally, Lemma 6.1.2 and smoothness of X' imply that $\mathrm{R}\Gamma(X', \omega_{X'})$ lies in $D^{\leq 0}(\Lambda)$ as well. This finishes the proof. □

Remark 7.3.15. It seems reasonable to expect that $\mathrm{R}\mathscr{H}om(\mathrm{R}f_!\omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}}; \Lambda)$ for any taut separated morphism f. For instance, Lemma 7.2.11 implies that this holds for smooth f, while Theorem 7.3.14 ensures it for proper maps. However, we cannot justify this more general expectation in full generality.

Corollary 7.3.16. *Keep the notation of Lemma 7.3.11. Let $f\colon X \to Y$ be a proper morphism of rigid-analytic spaces over K, and let $i_Y\colon Z \to Y$, $\pi_Y\colon X' \to Y$, and $h_Y\colon Z' \to Y$ be the natural compositions. Then the sequence*

$$0 \to \mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) \xrightarrow{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\pi), -\circ \mathrm{R}f_*(\mathrm{Tr}_i))} \mathrm{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big)$$
$$\xrightarrow{-\circ \mathrm{R}\pi_{Y,*}((-1)\cdot \mathrm{Tr}_{i'}) \oplus -\circ \mathrm{R}i_{Y,*}(\mathrm{R}i^!\,\mathrm{Tr}_\pi)} \mathrm{Hom}\big(\mathrm{R}h_*\omega_{Z'}, \omega_Y\big)$$

is exact.

Proof. Lemma 7.3.11 implies that there is an exact triangle

$$\mathrm{RHom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) \xrightarrow{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\pi), -\circ \mathrm{R}f_*(\mathrm{Tr}_i))} \mathrm{RHom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \mathrm{RHom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big)$$
$$\xrightarrow{-\circ \mathrm{R}\pi_{Y,*}((-1)\cdot \mathrm{Tr}_{i'}) \oplus -\circ \mathrm{R}i_{Y,*}(\mathrm{R}i^!\,\mathrm{Tr}_\pi)} \mathrm{RHom}\big(\mathrm{R}h_*\omega_{Z'}, \omega_Y\big)$$

Therefore, it suffices to show that $\mathrm{Ext}^{-1}(\mathrm{R}h_*\omega_{Z'}, \omega_Y) = 0$. This follows directly from Theorem 7.3.14. □

7.4. **Proper trace in general.** It is time to bootstrap Construction 7.3.1 to the case of general proper maps. We begin by stating the main goal for this subsection.

Theorem 7.4.1. *There is a unique way to assign to every proper morphism $f\colon X \to Y$ of rigid-analytic spaces over K a trace map $\mathrm{Tr}_f \in \mathrm{Hom}(\mathrm{R}f_*\omega_X, \omega_Y)$ in $D(Y_{\text{ét}};\Lambda)$, satisfying the following properties:*

(1) *(Compatibility with closed trace) When f is a closed immersion, then Tr_f equals the closed trace map from Construction 7.2.1.*
(2) *(Compatibility with smooth-source trace) When X is smooth and Y is separated and taut, then Tr_f equals the smooth-source trace map from Construction 7.3.1.*
(3) *(Compatibility with compositions) For any two proper morphisms $f\colon W \to X$ and $g\colon X \to Y$ of rigid-analytic spaces over K, we have $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f)$.*
(4) *(Étale-local on target) For any pullback diagram of rigid-analytic spaces over K*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "h"] & Y \end{tikzcd}$$

in which f and $\tilde{f}$ are proper and h and $\tilde{h}$ are étale, the following diagram commutes (with the vertical isomorphisms coming from Corollary 7.1.2):

$$\begin{tikzcd} \mathrm{R}\tilde{f}_*\omega_{\widetilde{X}} \arrow[rr, "\mathrm{Tr}_{\tilde{f}}"] & & \omega_{\widetilde{Y}} \\ \mathrm{R}\tilde{f}_*\tilde{h}^*\omega_X = h^*\mathrm{R}f_*\omega_X \arrow[u, "\mathrm{R}\tilde{f}_*(\alpha_{\tilde{h}})", "\wr"'] \arrow[rr, "h^*(\mathrm{Tr}_f)"] & & h^*\omega_Y \arrow[u, "\alpha_h", "\wr"'] \end{tikzcd}$$

Before embarking on the proof of Theorem 7.4.1, we explain the main idea: Assume that $f\colon X \to Y$ is a proper morphism of rigid-analytic spaces over K with X reduced, quasicompact, and separated. Let $U \subseteq X$ be a dense, Zariski-open subspace which is contained in the smooth locus of X (cf. Proposition 7.3.9. (i)). Let $Z := X \smallsetminus U$ be its complement, endowed with the canonical reduced adic space structure. Proposition 7.3.9. (ii) yields a regular U-admissible modification $\pi\colon X' \to X$, fitting into a commutative diagram

$$\begin{tikzcd} Z \times_X X' =: Z' \arrow[r, hook, "i'"] \arrow[d, "\pi'"] \arrow[dr, "h"] & X' \arrow[d, "\pi"] \arrow[ddr, bend left, "\pi_Y"] & \\ Z \arrow[r, hook, "i"] \arrow[drr, bend right, "i_Y"] & X \arrow[dr, "f"] & \\ & & Y. \end{tikzcd} \tag{7.4.2}$$

Since X' is smooth over K, Construction 7.3.1 provides us with a trace map Tr_{π_Y}, and since $\dim Z < \dim X$, we may assume (by induction on the dimension of the source) that we also have a trace map Tr_{i_Y}. Using Corollary 7.3.16, we will then check that these uniquely determine a map $\mathrm{Tr}_f\colon \mathrm{R}f_*\omega_X \to \omega_Y$. Afterward, we will show that this definition does not depend on the choice of U and π and satisfies the desired compatibilities (1), (2), (3), and (4). Now the details:

Proof of Theorem 7.4.1. *Step 1. It suffices to show the statement when all rigid-analytic spaces involved are quasicompact and separated.* Indeed, for any proper morphism $f\colon X \to Y$ of general rigid-analytic spaces over K, the coconnectivity result from Theorem 7.3.14 combined with the BBDG gluing lemma [BBDG18, Prop. 3.2.2] guarantees that $\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ forms a sheaf on the étale site $Y_{\text{ét}}$. For any quasicompact and separated open $U \subseteq Y$, the preimage $f^{-1}(U)$ under the proper map f is again quasicompact and separated. Assume that we have unique trace maps $\mathrm{Tr}_{f|_U} \in \mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)(U)$ which satisfy the desired compatibilities. Then the étale locality of traces from (4) allow us to glue the $\mathrm{Tr}_{f|_{U_i}}$ for some quasicompact separated (e.g., affinoid) covering $X = \bigcup_{i\in I} U_i$ to a map $\mathrm{Tr}_f\colon \mathrm{R}f_*\omega_X \to \omega_Y$; the uniqueness guarantees that this does not depend on the choice of the covering subspaces U_i.

Compatibilities (1), (2), and (3) can be immediately reduced to the case of quasicompact separated rigid-analytic spaces because $\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ is a sheaf (so we can check equality of two maps between $\mathrm{R}f_*\omega_X$ and ω_Y étale locally on Y), preimages of quasicompact and separated subspaces under proper maps are quasicompact and separated, and the facts that closed trace and smooth-source traces are étale local on the target (see Remark 7.2.3. (2) and Remark 7.3.3). Lastly, the Tr_f will still satisfy (4) because it is étale local by its very construction as a section of the étale sheaf $\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$.

Step 2. Induction setup. A quasicompact rigid-analytic space over K has finite dimension (see, for example, [Zav23a, Lem. 3.7]). By Step 1, it thus remains to prove the following statement, which we show via induction on d:

Let $d \in \mathbf{Z}_{\geq 0} \cup \{-\infty\}$. There is a unique way to assign to every proper morphism $f\colon X \to Y$ of quasicompact and separated rigid-analytic spaces over K with $\dim X \leq d$ a trace map $\mathrm{Tr}_f\colon \mathrm{R}f_*\omega_X \to \omega_Y$ satisfying properties (1), (2), (4) (with $\widetilde{Y}$ also quasicompact and separated), as well as

(3)' (Compatibility with compositions II) For any two proper morphisms $f\colon W \to X$ and $g\colon X \to Y$ of quasicompact and separated rigid-analytic spaces over K such that $\dim W \leq d$ and either X is smooth or $\dim X \leq d$, we have $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f)$, where Tr_g denotes the smooth-source trace from Construction 7.3.1 when X is smooth.

Note for (3)' that in the induction step Tr_g is not yet defined in general, so we need to assume that either X is smooth or $\dim X \leq d$ in order for the statement to have meaning. When X satisfies both assumptions, there will be no ambiguity by (2). On the other hand, it will be indispensable during the induction step that we allow for smooth X with $\dim X > d$.

In the base case $d = -\infty$, we have $X = \varnothing$, so there is nothing to show. We proceeed with the induction step.

Step 3. Induction step: construction and uniqueness. Assume that $d \geq 0$ and that the statement has been shown in dimensions $< d$. Let $f\colon X \to Y$ be a proper morphism of quasicompact and separated rigid-analytic spaces over K with $\dim X = d$.

First, consider the closed immersion $\iota\colon X_{\mathrm{red}} \hookrightarrow X$ from the maximal reduced closed subspace and set $f_{\mathrm{red}} := f \circ \iota$. By Lemma 7.2.4, precomposition with the closed trace map $\mathrm{Tr}_\iota\colon \iota_*\omega_{X_{\mathrm{red}}} \to \omega_X$ induces an isomorphism $\mathrm{Hom}(\mathrm{R}f_*\omega_X, \omega_Y) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{R}f_{\mathrm{red},*}\omega_{X_{\mathrm{red}}}, \omega_Y)$. In view of properties (1) and (3), any trace map $\mathrm{R}f_{\mathrm{red},*}\omega_{X_{\mathrm{red}}} \to \omega_Y$ hence pins down a unique trace map $\mathrm{R}f_*\omega_X \to \omega_Y$. As a consequence, it suffices to verify uniqueness of a proper trace satisfying (1), (2), (3)', and (4) for reduced X. Furthermore, a construction of a proper trace for reduced X canonically extends to all X. We will construct a proper trace map in this Step and verify all the desired compatibilities in Step 4. Note, however, that it is a priori not clear that Tr_f satisfies (1), (2), (3)', and (4) for all X if it satisfied these compatibilities for reduced X.

Now we implement the main idea described before the proof: We pick a dense, Zariski-open subspace $U \subseteq X$ which is contained in the smooth locus of X and let $Z := X \smallsetminus U$ be its reduced complement. Thanks to Proposition 7.3.9. (ii), we can find a regular U-admissible modification $\pi\colon X' \to X$ and consider the diagram (7.4.2). Since X' is smooth, we have smooth-source traces for all arrows in the diagram starting from X'. On the other hand, since $\dim Z < \dim X$ and $\dim Z' < \dim X' = \dim X$, the induction hypothesis provides us with unique traces subject to the desired properties for all (compositions of) arrows starting from either Z or Z'.

The traces Tr_i, $\mathrm{Tr}_{i'}$, and $\mathrm{Tr}_{\pi'}$ coming from induction and the smooth source trace Tr_π satisfy

$$\mathrm{Tr}_\pi \circ \mathrm{Tr}_{i'} = \mathrm{Tr}_h = \mathrm{Tr}_i \circ \mathrm{Tr}_{\pi'};$$

the first equality uses the modified compatibility (3)' from the induction hypothesis. Under the $(i_*, \mathrm{R}i^!)$-adjunction, this translates to

$$\mathrm{R}i^! \,\mathrm{Tr}_\pi = \mathrm{Tr}_{\pi'},$$

where we use Notation 7.3.7. A combination with Corollary 7.3.16 and property (1) yields the exact sequence

$$\begin{aligned} (7.4.3)\quad 0 \to \mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) &\xrightarrow{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\pi), -\circ \mathrm{R}f_*(\mathrm{Tr}_i))} \mathrm{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big) \\ &\xrightarrow{-\circ \mathrm{R}\pi_{Y,*}((-1)\cdot \mathrm{Tr}_{i'}) \oplus -\circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{\pi'})} \mathrm{Hom}\big(\mathrm{R}h_*\omega_{Z'}, \omega_Y\big). \end{aligned}$$

If $\mathrm{Tr}_f \in \mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big)$ is to satisfy property (3)', it needs to map to $\big(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y}\big)$ in this sequence. Hence, Tr_f is unique if it exists. Conversely, $\big(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y}\big)$ must be induced by an element of $\mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big)$ because

$$\mathrm{Tr}_{\pi_Y} \circ \mathrm{R}\pi_{Y,*}(\mathrm{Tr}_{i'}) = \mathrm{Tr}_{\pi_Y \circ i'} = \mathrm{Tr}_{i_Y \circ \pi'} = \mathrm{Tr}_{i_Y} \circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{\pi'})$$

by property (3)' from the induction hypothesis. This produces a unique trace map $\mathrm{Tr}_f \colon \mathrm{R}f_*\omega_X \to \omega_Y$, which a priori depends on the choice of U and π.

To finish the construction, we explain why Tr_f does in fact not depend on the choice of the dense, Zariski-open, smooth subspace $U \subseteq X$, nor on the choice of the regular U-admissible modification π. Let $U_j \subseteq X$ and $\pi_j \colon X'_j \to X$ for $j = 1, 2$ be two such choices. Set $U := U_1 \cap U_2$ and $Z := X \smallsetminus U$ (again with the reduced adic space structure). Then $U \subset X$ is a dense, Zariski-open, smooth subspace and $(\pi_1, \pi_2) \colon X'_1 \times_X X'_2 \to X$ is a U-admissible modification. Applying Proposition 7.3.9. (ii) to $(\pi_1, \pi_2)^{-1}(U) \subseteq (X'_1 \times_X X'_2)_{\mathrm{red}}$, we obtain a regular U-admissible modification $\pi \colon X' \to X$ that factors through both π_1 and π_2. It suffices to compare the trace morphisms obtained from the U_j-admissible modifications π_j for $j = 1, 2$ to the trace morphism obtained from the U-admissible modification π. Note that the π_j can be considered both as U_j-modification and as U-modification; for clarity, we write (π_j, U_j) and (π_j, U).

The modifications (π, U), (π_j, U_j), and (π_j, U) form the red-purple rectangle, the blue-purple rectangle, and the red-blue-purple square, respectively, in the commutative diagram

$$\begin{tikzcd}
& Z' \arrow[r, hook] \arrow[d] & X' \arrow[d, "\pi_{X'_j}"'] \arrow[dddr, bend left, "\pi_Y"] & \\
Z'_j \arrow[r, hook] \arrow[d] & X'_j \times_X Z \arrow[r, hook] \arrow[d] & X'_j \arrow[d, "\pi_j"'] \arrow[ddr, bend left, "\pi_{j,Y}"] & \\
Z_j \arrow[r, hook, "i_{j,Z}"] \arrow[drrr, bend right, "i_{j,Y}"'] & Z \arrow[r, hook, "i"] \arrow[drr, bend right, "i_Y"'] & X \arrow[dr, "f"'] & \\
& & & Y.
\end{tikzcd}$$

With this notation, the injections in the exact sequence (7.4.3) for (π, U), (π_j, U_j), and (π_j, U) fit into a square

$$\begin{tikzcd}
\mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) \arrow[r, hook, "{(-\circ \mathrm{R}f_*(\mathrm{Tr}_{\pi_j}), -\circ \mathrm{R}f_*(\mathrm{Tr}_{i_j}))}"] \arrow[d, hook, "{(-\circ \mathrm{R}f_*(\mathrm{Tr}_{\pi}), -\circ \mathrm{R}f_*(\mathrm{Tr}_{i}))}"'] \arrow[dr, hook, "{(-\circ \mathrm{R}f_*(\mathrm{Tr}_{\pi_j}), -\circ \mathrm{R}f_*(\mathrm{Tr}_{i}))}"] & \mathrm{Hom}\big(\mathrm{R}\pi_{j,Y,*}\omega_{X'_j}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{j,Y,*}\omega_{Z_j}, \omega_Y\big) \\
\mathrm{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{Y,*}\omega_{Z}, \omega_Y\big) & \mathrm{Hom}\big(\mathrm{R}\pi_{j,Y,*}\omega_{X'_j}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{Y,*}\omega_{Z}, \omega_Y\big). \arrow[l, hook', "{(-\circ \mathrm{R}\pi_{j,Y,*}(\mathrm{Tr}_{\pi_{X'_j}}), \mathrm{id})}"] \arrow[u, "{(\mathrm{id}, -\circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{i_{j,Z}}))}"']
\end{tikzcd} \tag{7.4.4}$$

The two triangles commute by virtue of the equalities $\mathrm{Tr}_\pi = \mathrm{Tr}_{\pi_j} \circ \mathrm{R}\pi_{j,*}(\mathrm{Tr}_{\pi_{X'_j}})$ and $\mathrm{Tr}_{i_j} = \mathrm{Tr}_i \circ i_*(\mathrm{Tr}_{i_{j,Z}})$ from Proposition 7.3.2. (iv) and Remark 7.2.3. (1), respectively. By construction, the three different choices of Tr_f for (π, U), (π_j, U_j), and (π_j, U) correspond to the elements $(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y})$, $(\mathrm{Tr}_{\pi_{j,Y}}, \mathrm{Tr}_{i_{j,Y}})$, and $(\mathrm{Tr}_{\pi_{j,Y}}, \mathrm{Tr}_{i_Y})$ in the bottom left, top right, and bottom right corner in Diagram (7.4.4), respectively. Another application of Proposition 7.3.2. (iv) and Remark 7.2.3. (1) shows that

$$\mathrm{Tr}_{\pi_Y} = \mathrm{Tr}_{\pi_{j,Y}} \circ \mathrm{R}\pi_{j,Y,*}(\mathrm{Tr}_{\pi_{X'_j}}) \quad \text{and} \quad \mathrm{Tr}_{i_{j,Y}} = \mathrm{Tr}_{i_Y} \circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{i_{j,Z}}),$$

so that $(\mathrm{Tr}_{\pi_{j,Y}}, \mathrm{Tr}_{i_Y})$ maps to $(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y})$ and $(\mathrm{Tr}_{\pi_{j,Y}}, \mathrm{Tr}_{i_{j,Y}})$ under the two maps $(-\circ \mathrm{R}\pi_{j,Y,*}(\mathrm{Tr}_{\pi_{X'_j}}), \mathrm{id})$ and $(\mathrm{id}, -\circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{i_{j,Z}}))$ of Diagram (7.4.4), respectively. Since the three maps emanating from $\mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big)$ are all injective, this shows that the three different choices of Tr_f must all coincide.

Step 4. Induction step: verification of properties. To finish, we show that the trace maps Tr_f constructed in Step 3 satisfy the compatibilities (1), (2), (3)', and (4). When X is smooth, it is in particular reduced and Tr_f is constructed via the modification diagram (7.4.2). Moreover, we may choose $U = X$ and $\pi = \mathrm{id}$, thanks to the independence of Tr_f from the choice of π. This immediately yields (2).

When f is a closed immersion, we consider again the modification diagram (7.4.2), but with $f\colon X \to Y$ replaced by $f_{\mathrm{red}}\colon X_{\mathrm{red}} \xhookrightarrow{\iota} X \xrightarrow{f} Y$. Denote the closed trace map of ι by Tr_ι. We defined Tr_f as the unique element mapping to $(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y})$ under the injection

$$\begin{aligned}\mathrm{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) &\xrightarrow[\sim]{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\iota))} \mathrm{Hom}\big(\mathrm{R}f_{\mathrm{red},*}\omega_{X_{\mathrm{red}}}, \omega_Y\big) \\ &\xhookrightarrow{(-\circ \mathrm{R}f_{\mathrm{red},*}(\mathrm{Tr}_\pi), -\circ \mathrm{R}f_{\mathrm{red},*}(\mathrm{Tr}_i))} \mathrm{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \mathrm{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big).\end{aligned}$$

On the other hand, Remark 7.2.3. (1) and Proposition 7.3.2. (v) ensure that the image of the closed trace of f under this injection has image $(\mathrm{Tr}_{\pi_Y}, \mathrm{Tr}_{i_Y})$. This shows (1).

Next, we verify (3)'. Let $f\colon W \to X$ and $g\colon X \to Y$ be two proper morphisms of quasicompact and separated rigid-analytic spaces over K such that $\dim W \leq d$ and either X is smooth or $\dim X \leq d$. Leaving d fixed, we show that $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f)$ by induction on $e := \dim W \leq d$. When $e = -\infty$ (so $W = \varnothing$), the statement is clear.

Now assume that $d \geq \dim W = e \geq 0$ and that (3)' has been proven in dimensions $< e$. We assume first that W and X are both reduced. Under this assumption, we may choose a dense, Zariski-open, smooth $V \subseteq W$ with reduced complement $A := W \smallsetminus V$ and a regular V-admissible modification $\rho\colon W' \to W$; see Proposition 7.3.9. Then the definition of Tr_f via the injection in (7.4.3) allows us again to check (3)' for $f\colon W \to X$ replaced by $f\big|_A\colon A \to X$ and by $f\circ\rho\colon W' \to X$. For the former morphism, we may apply the induction hypothesis because $\dim A < \dim W = e$. For the latter morphism, we may check (3)' separately on each connected (and hence irreducible) component of W'. The statement for connected components of dimension $< c$ is again covered by the induction hypothesis. Thus, after replacing W by one of the remaining connected components of W', it suffices to verify (3)' when W is smooth, irreducible and of equidimension $e \leq d$.

In this case, Tr_f and $\mathrm{Tr}_{g\circ f}$ are given by the smooth-source trace thanks to (2), which has already been verified. If X is smooth, then the compatibility $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f)$ follows from Proposition 7.3.2. (iv). Thus, we may assume that $\dim X \leq d$. Recall that $f(W) \subseteq X$ is a Zariski-closed subset [BGR84, Prop. 9.6.3/3]. Since W is irreducible, so is $f(W)$. Thus, we can pick an irreducible component $X_0 \subseteq X$ such that $f(W) \subseteq X_0$. If $\dim f(W) < \dim X_0$, there exists a dense, Zariski-open, smooth $U \subset X$ whose reduced Zariski-closed complement $i\colon Z \hookrightarrow X$ contains $f(W)$. Then Tr_g is computed using a modification square as in (7.4.2) for some regular U-admissible modification $\pi\colon X' \to X$. Denoting by $f_Z\colon W \to Z$ the map factoring $f\colon W \to X$ and setting $i_Y\colon Z \xhookrightarrow{i} X \xrightarrow{g} Y$, we conclude

$$\mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f) = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_i) \circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{f_Z}) = \mathrm{Tr}_{i_Y} \circ \mathrm{R}i_{Y,*}(\mathrm{Tr}_{f_Z});$$

here, the first equality follows from the fact that Tr_f is given by the smooth-source trace and Proposition 7.3.2. (v) and the second equality follows from the construction of Tr_g and property (1) of the induction hypothesis. Replacing f by f_Z, g by i_Y, and X_0 by the irreducible component of Z containing $f_Z(W)$, we may therefore reduce $\dim X_0$ by at least 1; repeating the same process finitely many times, we finally arrive at a situation where $f(W) = X_0$.

Choose once more a dense, Zariski-open, smooth subspace $U \subseteq X$ with reduced complement $Z \subset X$ and a regular U-admissible modification $\pi\colon X' \to X$ fitting into a diagram of the form (7.4.2). Since W is irreducible and $f(W) = X_0$, the preimage $V := f^{-1}(U) \subseteq W$ is still dense and Zariski-open. Let $W_0 \subseteq (W \times_X X')_{\mathrm{red}}$ be an irreducible component containing $V \times_X X'$. The induced map $\rho\colon W_0 \to W$ is a V-admissible modification. Proposition 7.3.9. (ii) yields a regular $\rho^{-1}(V)$-admissible modification $\rho'\colon W' \to W_0$. Then $\pi' := \rho \circ \rho'\colon W' \to W$ is a regular V-admissible modification, fitting into a commutative diagram

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "\pi'"] & X' \arrow[d, "\pi"] \arrow[dr, bend left, "\pi_Y"] & \\ W \arrow[r, "f"] & X \arrow[r, "g"] & Y. \end{tikzcd}$$

Let $A \subset W$ be the reduced complement of V. As before, we may check (3)' for $f\colon W \to X$ replaced by $f\big|_A\colon A \to X$ and by $f \circ \pi'\colon W' \to X$. Since $\dim A < \dim W \leq e$, the former follows from the induction hypothesis. The latter comes from the identity

$$\mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{f\circ\pi'}) = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{\pi\circ f'}) = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{\pi}) \circ \mathrm{R}\pi_{Y,*}(\mathrm{Tr}_{f'}) = \mathrm{Tr}_{\pi_Y} \circ \mathrm{R}\pi_{Y,*}(\mathrm{Tr}_{f'}) = \mathrm{Tr}_{\pi'_Y},$$

in which every trace map except Tr_g is the smooth-source trace and hence the second and last equality follow from Proposition 7.3.2. (iv), whereas the third equality follows from the construction of Tr_g. This finishes the verification of (3)' when W and X are both reduced.

In order to prove (3)' for $d \geq \dim W = e \geq 0$ in general, consider the commutative diagram

$$\begin{tikzcd}
W_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, hook, "\iota'"] & X_{\mathrm{red}} \arrow[d, hook, "\iota"] \arrow[dr, bend left, "g_{\mathrm{red}}"] & \\
W \arrow[r, "f"] & X \arrow[r, "g"] & Y
\end{tikzcd}$$

in which the top row consists of the maximal reduced closed subspaces and the vertical arrows of the canonical closed immersions. We still have $\dim X_{\mathrm{red}} = \dim X \leq d$ and $\dim W_{\mathrm{red}} = \dim W = e \leq d$. By the construction in Step 3 and property (1), the trace map $\mathrm{Tr}_{g\circ f}$ is uniquely determined by the property

$$\mathrm{Tr}_{g\circ f} \circ \mathrm{R}(g\circ f)_*(\mathrm{Tr}_{\iota'}) = \mathrm{Tr}_{g\circ f\circ\iota'} = \mathrm{Tr}_{g_{\mathrm{red}}\circ f_{\mathrm{red}}}.$$

On the other hand, we have

$$\begin{aligned}
\mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f) \circ \mathrm{R}(g\circ f)_*(\mathrm{Tr}_{\iota'}) &= \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{f\circ\iota'}) = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{\iota\circ f_{\mathrm{red}}}) \\
&= \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_{\iota}) \circ \mathrm{R}g_{\mathrm{red},*}(\mathrm{Tr}_{f_{\mathrm{red}}}) = \mathrm{Tr}_{g_{\mathrm{red}}} \circ \mathrm{R}g_{\mathrm{red},*}(\mathrm{Tr}_{f_{\mathrm{red}}}) = \mathrm{Tr}_{g_{\mathrm{red}}\circ f_{\mathrm{red}}}
\end{aligned}$$

where the first and fourth equality follow from the construction of Tr_f and Tr_g via $\mathrm{Tr}_{f_{\mathrm{red}}}$ and $\mathrm{Tr}_{g_{\mathrm{red}}}$, respectively, and the third and fifth equality hold because W_{red} and X_{red} are reduced of the right dimensions. As a consequence, $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{R}g_*(\mathrm{Tr}_f)$, yielding the general case of (3)'.

It remains to deal with property (4). Let $h\colon \widetilde{Y} \to Y$ be an étale map from a quasicompact and separated rigid space $\widetilde{Y}$ over K; in particular, h is in addition separated and taut [Hub96, Lem. 5.1.3]. Once more, we assume first that X is reduced. In that case, we can choose a dense, Zariski-open, smooth subspace $U \subseteq X$ with reduced complement $Z \subset X$ and a regular U-admissible modification $\pi\colon X' \to X$.

Set $\widetilde{U} := U \times_Y \widetilde{Y} \subseteq X \times_Y \widetilde{Y} =: \widetilde{X}$. The pullback $\tilde{\pi}\colon \widetilde{X}' \to \widetilde{X}$ of π along $\tilde{h}\colon \widetilde{X} \to X$ is again a regular $\widetilde{U}$-admissible[37] modification, giving rise to a commutative diagram

$$\begin{tikzcd}
& \widetilde{X}' \arrow[rr, "\tilde{h}'_\pi"] \arrow[dd, "\tilde{\pi}"] \arrow[dddd, bend left, "\tilde{\pi}_{\widetilde{Y}}"] & & X' \arrow[dd, "\pi"] \arrow[dddd, bend left, "\pi_Y"] \\
\widetilde{Z}' \arrow[ur] \arrow[rr, "\tilde{h}'_i"] \arrow[dd] & & Z' \arrow[ur] \arrow[dd] & \\
& \widetilde{X} \arrow[rr, "\tilde{h}"] \arrow[dd, "\tilde{f}"] & & X \arrow[dd, "f"] \\
\widetilde{Z} \arrow[ur, "\tilde{i}"] \arrow[rr] \arrow[dr, "\tilde{i}_{\widetilde{Y}}"] & & Z \arrow[ur, "i"] \arrow[dr, "i_Y"] & \\
& \widetilde{Y} \arrow[rr, "h"] & & Y,
\end{tikzcd}$$

in which all squares are pullback squares. Since the closed trace and the smooth-source trace are étale local on the target by Remark 7.2.3. (2) and Remark 7.3.3, respectively, the same is true for the short exact sequences (7.4.3) coming from Corollary 7.3.16. This yields the following commutative diagram, in which the upper

[37] To see the $\widetilde{U}$-admissibility, we observe that étale maps are open, hence the preimage of a dense open is again dense open.

vertical maps are given by pullback along h:

$$\begin{tikzcd}
\operatorname{Hom}\big(\mathrm{R}f_*\omega_X,\omega_Y\big) \arrow[r, hook, "{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\pi),-\circ\mathrm{R}f_*(\mathrm{Tr}_i))}"] \arrow[d] & \operatorname{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'},\omega_Y\big)\oplus \operatorname{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z,\omega_Y\big) \arrow[d] \\
\operatorname{Hom}\big(h^*\mathrm{R}f_*\omega_X,h^*\omega_Y\big) \arrow[r, hook, "{(-\circ h^*\mathrm{R}f_*(\mathrm{Tr}_\pi),-\circ h^*\mathrm{R}f_*(\mathrm{Tr}_i))}"] \arrow[d, "\alpha_h\circ-\circ\mathrm{R}\tilde{f}_*\left(\alpha_{\tilde{h}}^{-1}\right)", "\wr"'] & \operatorname{Hom}\big(h^*\mathrm{R}\pi_{Y,*}\omega_{X'},h^*\omega_Y\big)\oplus \operatorname{Hom}\big(h^*\mathrm{R}i_{Y,*}\omega_Z,h^*\omega_Y\big) \arrow[d, "\left(\alpha_h\circ-\circ\mathrm{R}\tilde{\pi}_{\widetilde{Y},*}\left(\alpha^{-1}_{\tilde{h}'_\pi}\right)\right)\oplus\left(\alpha_h\circ-\circ\mathrm{R}\tilde{i}_{\widetilde{Y},*}\left(\alpha^{-1}_{\tilde{h}'_i}\right)\right)"', "\wr"] \\
\operatorname{Hom}\big(\mathrm{R}\tilde{f}_*\omega_{\widetilde{X}},\omega_{\widetilde{Y}}\big) \arrow[r, hook, "{(-\circ \mathrm{R}\tilde{f}_*(\mathrm{Tr}_{\tilde{\pi}}),-\circ\mathrm{R}\tilde{f}_*(\mathrm{Tr}_{\tilde{i}}))}"] & \operatorname{Hom}\big(\mathrm{R}\tilde{\pi}_{\widetilde{Y},*}\omega_{\widetilde{X}'},\omega_{\widetilde{Y}}\big)\oplus \operatorname{Hom}\big(\mathrm{R}\tilde{i}_{\widetilde{Y},*}\omega_{\widetilde{Z}},\omega_{\widetilde{Y}}\big)
\end{tikzcd}$$

We need to show that $\mathrm{Tr}_f\in\operatorname{Hom}\big(\mathrm{R}f_*\omega_X,\omega_Y\big)$ maps to $\mathrm{Tr}_{\tilde{f}}\in\operatorname{Hom}\big(\mathrm{R}\tilde{f}_*\omega_{\widetilde{X}},\omega_{\widetilde{Y}}\big)$ under the composition of the left vertical maps. By the injectivity of the horizontal maps and the construction of Tr_f and $\mathrm{Tr}_{\tilde{f}}$ from Step 3, it suffices to see that $\big(\mathrm{Tr}_{\pi_Y},\mathrm{Tr}_{i_Y}\big)$ maps to $\big(\mathrm{Tr}_{\tilde{\pi}_{\widetilde{Y}}},\mathrm{Tr}_{\tilde{i}_{\widetilde{Y}}}\big)$ under the composition of the right vertical maps. For the first component, this follows from Remark 7.3.3. For the second component, one can use property (4) of the induction hypothesis. This shows property (4) when X is reduced.

For general X, the extended pullback diagram of rigid-analytic spaces over K

$$\begin{tikzcd}
\widetilde{X}_{\mathrm{red}} \arrow[r, hook, "\iota'"] \arrow[d, "\tilde{h}_{\mathrm{red}}"] \arrow[rr, bend left, "\tilde{f}_{\mathrm{red}}"] & \widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{h}"] & \widetilde{Y} \arrow[d, "h"] \\
X_{\mathrm{red}} \arrow[r, hook, "\iota"] \arrow[rr, bend right, "f_{\mathrm{red}}"'] & X \arrow[r, "f"] & Y,
\end{tikzcd}$$

where as before ι and ι' denote the closed immersions from the maximal reduced closed subspaces, induces the following extended diagram:

$$\begin{tikzcd}
\mathrm{R}\tilde{f}_{\mathrm{red},*}\omega_{\widetilde{X}_{\mathrm{red}}} \arrow[r, "\mathrm{R}f_*(\mathrm{Tr}_{\iota'})", "\sim"'] \arrow[rr, bend left, "\mathrm{Tr}_{\tilde{f}_{\mathrm{red}}}"] & \mathrm{R}\tilde{f}_*\omega_{\widetilde{X}} \arrow[r, "\mathrm{Tr}_{\tilde{f}}"] & \omega_{\widetilde{Y}} \\
\mathrm{R}\tilde{f}_{\mathrm{red},*}\tilde{h}^*_{\mathrm{red}}\omega_{X_{\mathrm{red}}} \arrow[u, "\mathrm{R}\tilde{f}_{\mathrm{red},*}(\alpha_{\tilde{h}_{\mathrm{red}}})", "\wr"'] \arrow[d, equal] & & \\
\mathrm{R}\tilde{f}_*\tilde{h}^*\iota_*\omega_{X_{\mathrm{red}}} \arrow[r, "\mathrm{R}\tilde{f}_*\tilde{h}^*(\mathrm{Tr}_\iota)", "\sim"'] \arrow[d, equal] & \mathrm{R}\tilde{f}_*\tilde{h}^*\omega_X \arrow[uu, "\mathrm{R}\tilde{f}_*(\alpha_{\tilde{h}})", "\wr"'] \arrow[d, equal] & \\
h^*\mathrm{R}f_{\mathrm{red},*}\omega_{X_{\mathrm{red}}} \arrow[r, "h^*\mathrm{R}f_*(\mathrm{Tr}_\iota)", "\sim"'] \arrow[rr, bend right, "h^*(\mathrm{Tr}_{f_{\mathrm{red}}})"'] & h^*\mathrm{R}f_*\omega_X \arrow[r, "h^*(\mathrm{Tr}_f)"] & h^*\omega_Y \arrow[uuu, "\alpha_h", "\wr"']
\end{tikzcd}$$

In this diagram, the upper and lower "triangles" commute by the construction of $\mathrm{Tr}_{\tilde{f}}$ and Tr_f via $\mathrm{Tr}_{\tilde{f}_{\mathrm{red}}}$ and $\mathrm{Tr}_{f_{\mathrm{red}}}$, the upper left rectangle commutes because the closed trace map is étale local (Remark 7.2.3. (2)), the lower left rectangle commutes by the naturality of the base change isomorphism, and the outer diagram commutes thanks to the special case treated above when the source is reduced. On the other hand, Lemma 7.2.4 and Corollary 7.1.2 guarantee that all arrows on the left side of the diagram are isomorphisms, so the right square must commute as well. This establishes property (4) in general. We can finally declare victory in the proof of Theorem 7.4.1! □

We now check some compatibilities of the proper trace map beyond those mentioned in statement of Theorem 7.4.1.

Proposition 7.4.5. *Let $f\colon X\to Y$ be a smooth, proper morphism of rigid-analytic spaces over K. Then the smooth trace map of f from Construction 7.2.7 agrees with the proper trace map of f from Theorem 7.4.1.*

Proof. Since $\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\omega_X, \omega_Y)$ lies in $D^{\geq 0}(Y_{\text{ét}};\Lambda)$ (see Theorem 7.3.14), we may check the statement étale locally on Y and thus assume that Y is quasicompact and separated. The closed immersions of the maximal reduced closed subspaces fit into a commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, hook, "\iota'"] \arrow[d, "f_{\mathrm{red}}"] \arrow[dr, "h"] & X \arrow[d, "f"] \\ Y_{\mathrm{red}} \arrow[r, hook, "\iota"] & Y. \end{tikzcd}$$

Since f is smooth, this diagram is even Cartesian; in particular, f_{red} is also smooth and proper. By virtue of Theorem 7.2.19 (or even Lemma 7.2.14) and a combination of Theorem 7.4.1. (1) and Theorem 7.4.1. (3), respectively, the induced diagram

$$\begin{tikzcd} \mathrm{R}h_*\omega_{X_{\mathrm{red}}} \arrow[r, "\iota_*(\mathrm{Tr}_{f_{\mathrm{red}}})"] \arrow[d, "\wr"', "\mathrm{R}f_*(\mathrm{Tr}_{\iota'})"] & \iota_*\omega_{Y_{\mathrm{red}}} \arrow[d, "\wr"', "\mathrm{Tr}_\iota"] \\ \mathrm{R}f_*\omega_X \arrow[r, "\mathrm{Tr}_f"] & \omega_Y \end{tikzcd}$$

in $D(Y_{\text{ét}};\Lambda)$ commutes for both the smooth and the proper traces of f_{red} and f, where in both cases $\mathrm{Tr}_{\iota'}$ and Tr_ι denote the closed trace maps. Moreover, these closed trace maps are isomorphisms by Lemma 7.2.4, so the smooth and proper trace for f agree if and only they agree for f_{red}. As a consequence, we may also assume that both X and Y are reduced.

After these reductions, we proceed by induction on $d := \dim(Y)$. When $d = 0$, the reducedness of Y together with the standing assumption that $\operatorname{char} K = 0$ means that Y is smooth. Since f is smooth, X is also smooth and we win thanks to Proposition 7.3.2. (i) and Theorem 7.4.1. (2).

Now assume that $\dim Y = d > 0$ and the statement has been proven in dimensions $\leq d-1$. Proposition 7.3.9 allows us to pick a dense, Zariski-open subspace $V \subseteq Y$ which is contained in the smooth locus of Y and a regular V-admissible modification $Y' \to Y$. Denote by $A \subset Y$ the complement of V endowed with the canonical reduced adic space structure; we have $\dim A < \dim Y$.

Let $U := f^{-1}(V)$ be the preimage, which is automatically Zariski-open. Consider the following diagram with Cartesian squares:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "\bar{f}"] \arrow[dr, "i_Y"] & X \arrow[d, "f"] & X' \arrow[l, "\pi"'] \arrow[d, "f'"] \arrow[dl, "\pi_Y"] \\ A \arrow[r, hook, "j"] & Y & Y' \arrow[l, "\rho"] \end{tikzcd}$$

Since the pullback $\pi\colon X' \to X$ of ρ is still a regular U-modification,[38] Corollary 7.3.16 gives an injection

$$\operatorname{Hom}\big(\mathrm{R}f_*\omega_X, \omega_Y\big) \xhookrightarrow{(-\circ \mathrm{R}f_*(\mathrm{Tr}_\pi), -\circ \mathrm{R}f_*(\mathrm{Tr}_i))} \operatorname{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big) \oplus \operatorname{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big),$$

where Tr_π and Tr_i denote the smooth-source trace and the closed trace, respectively. Thus, we only need to show that the images of the smooth trace of f and the proper trace of f under this injection agree with another. The first components in $\operatorname{Hom}\big(\mathrm{R}\pi_{Y,*}\omega_{X'}, \omega_Y\big)$ both agree with the smooth-source trace for $f \circ \pi\colon X' \to Y$ by Proposition 7.3.2. (v) and Theorem 7.4.1. (2) and (3). For the second components in $\operatorname{Hom}\big(\mathrm{R}i_{Y,*}\omega_Z, \omega_Y\big)$, we can use Lemma 7.2.14, Theorem 7.4.1. (1) and (3), and the fact that the smooth trace and the proper trace of $\bar{f}\colon Z \to A$ agree thanks to the induction hypothesis. This finishes the induction step. □

Remark 7.4.6. Proposition 7.4.5 implies a version of Theorem 7.2.19 when two maps are smooth and proper and the other two maps are proper. Unfortunately, we cannot establish an analogue of Theorem 7.2.19 for smooth and proper traces in general. The essential difficulty comes from the fact that we cannot prove an analogue of Corollary 7.3.16 for non-proper f (the key coconnectivity claim Theorem 7.3.14 used in the proof is unavailable in the non-proper case; see also Remark 7.3.15). It seems that the correct approach should be to construct trace maps for arbitrary taut separated maps compatible with compositions. The main obstacle to do this lies, again, in the fact that we cannot verify Corollary 7.3.16 beyond the proper case.

[38] In fact, as in Footnote 37, the U-modification π is U-admissible because f is smooth, hence open.

Next, we record the compatibility of proper traces under change of base field.

Definition 7.4.7. Let $f\colon X \to Y$ be a separated taut map of rigid-analytic spaces over K, which is either smooth or proper. Let $K \subset L$ be an extension of nonarchimedean fields, inducing the Cartesian diagram

$$\begin{tikzcd} X_L \arrow[r, "a_X"] \arrow[d, "f_L"] & X \arrow[d, "f"] \\ Y_L \arrow[r, "a_Y"] & Y. \end{tikzcd}$$

The smooth (resp. proper) trace map $\mathrm{Tr}_f\colon \mathrm{R}f_!\,\omega_X \to \omega_Y$ in the sense of Construction 7.2.7 (resp. Theorem 7.4.1) is *compatible with the base field extension* $K \subset L$ if the diagram

$$\begin{tikzcd} a_Y^*\left(\mathrm{R}f_!\,\omega_X\right) \arrow[r, "\mathrm{BC}_!"] \arrow[d, "a_Y^*(\mathrm{Tr}_f)"] & \mathrm{R}f_{L,!}\left(a_X^*\omega_X\right) \arrow[r, "\mathrm{R}f_{L,!}(\gamma_{X,L})", "\sim"'] & \mathrm{R}f_{L,!}\,\omega_{X_L} \arrow[d, "\mathrm{Tr}_{f_L}"] \\ a_Y^*\left(\omega_Y\right) \arrow[rr, "\gamma_{X,L}", "\sim"'] & & \omega_{Y_L}. \end{tikzcd}$$

commutes, where $\gamma_{X,L}$ is the base change isomorphism from [BH22, Th. 3.21.(6)] and $\mathrm{BC}_!$ is the base change map for compactly supported pushforward from [Hub96, Th. 5.4.6].

Proposition 7.4.8. *Let $f\colon X \to Y$ be a separated taut map of rigid-analytic spaces over K, which is either smooth or proper. Then Tr_f is compatible with every nonarchimedean base field extension $K \subset L$ in the sense of Definition 7.4.7.*

Recall that by Proposition 7.4.5, there is no ambiguity in the notation Tr_f when f is both smooth and proper.

Proof. *Step 1. Proof for smooth f.* It suffices to prove the statement when f is smooth of equidimension d. Using the definition of the smooth trace map from Construction 7.2.7, the diagram in Definition 7.4.7 can then be broken up as follows (using the canonical isomorphisms $\alpha_f\colon f^*\omega_Y(d)[2d] \xrightarrow{\sim} \omega_X$ provided by Corollary 7.1.2):

$$\begin{tikzcd} a_Y^*(\mathrm{R}f_!\omega_X) \arrow[d, "\mathrm{BC}_!"] & a_Y^*\mathrm{R}f_!f^*\omega_Y(d)[2d] \arrow[l, "a_Y^*\mathrm{R}f_!(\alpha_f)"', "\sim"] \arrow[d, "\mathrm{BC}_!"] & a_Y^*(\omega_Y \otimes^L \mathrm{R}f_!\underline{\Lambda}_X(d)[2d]) \arrow[l, "a_Y^*(\mathrm{PF}_f)"', "\sim"] \arrow[r, "a_Y^*(\mathrm{id}\otimes\mathrm{tr}_f)"] \arrow[d, equal] & a_Y^*(\omega_Y) \arrow[ddd, "\gamma_{Y,L}", "\wr"'] \\ \mathrm{R}f_{L,!}(a_X^*\omega_X) \arrow[dd, "\mathrm{R}f_{L,!}(\gamma_{X,L})", "\wr"'] & \mathrm{R}f_{L,!}a_X^*f^*\omega_Y(d)[2d] \arrow[l, "\mathrm{R}f_{L,!}a_X^*(\alpha_f)"', "\sim"] \arrow[d, equal] & a_Y^*\omega_Y \otimes^L a_Y^*\mathrm{R}f_!\underline{\Lambda}_X(d)[2d] \arrow[ur, "\mathrm{id}\otimes a_Y^*(\mathrm{tr}_f)"'] \arrow[d, "\mathrm{id}\otimes\mathrm{BC}_!"] & \\ & \mathrm{R}f_{L,!}f_L^*a_Y^*\omega_Y(d)[2d] \arrow[d, "\mathrm{R}f_{L,!}f_L^*(\gamma_{Y,L})(d)[2d]", "\wr"'] & a_Y^*\omega_Y \otimes^L \mathrm{R}f_{L,!}a_X^*\underline{\Lambda}_X(d)[2d] \arrow[l, "\mathrm{PF}_{f_L}"', "\sim"] \arrow[d, "\gamma_{Y,L}\otimes\mathrm{id}", "\wr"'] & \\ \mathrm{R}f_{L,!}\omega_{X_L} & \mathrm{R}f_{L,!}f_L^*\omega_{Y_L}(d)[2d] \arrow[l, "\mathrm{R}f_{L,!}(\alpha_{f_L})"', "\sim"] & \omega_{Y_L} \otimes^L \mathrm{R}f_{L,!}\underline{\Lambda}_{X_L}(d)[2d] \arrow[l, "\mathrm{PF}_{f_L}"', "\sim"] \arrow[r, "\mathrm{id}\otimes\mathrm{tr}_{f_L}"] & \omega_{Y_L} \end{tikzcd}$$

The upper left square commutes by the naturality of the base change map, the upper middle rectangle by the compatibility of the projection formula with base change (see e.g. [Sta22, Tag 0E48]), the lower middle square by the naturality of the projection formula, and the right trapezoid thanks to the compatibility of the smooth trace map for constant coefficients with pullbacks (Theorem 6.1.1. (2)).

To finish the argument for smooth f, it remains to see that the lower left rectangle commutes. We will show that the rectangle commutes even before applying $\mathrm{R}f_{L,!}$. In order to check this stronger claim, we may work locally on X and thus assume that f is of the form $f\colon \mathrm{Spa}\,(B, B^\circ) \to \mathrm{Spa}\,(A, A^\circ)$ with associated regular morphism $f^{\mathrm{alg}}\colon \mathrm{Spec}\,B \to \mathrm{Spec}\,A$. In this case, α_f and α_{f_L} (up to a twist) are defined as the analytifications of the isomorphisms from [ILO14, Exp. XVII, Prop. 4.1.1] applied to the regular morphisms f and f_L (see Remark 7.1.3), while $\gamma_{X,L}$ and $\gamma_{Y,L}$ are defined as the analytifications of the isomorphisms from [ILO14, Exp. XVII, Prop. 4.1.1] applied to regular morphisms $\mathrm{Spec}\left(B\widehat{\otimes}_K L\right) \to \mathrm{Spec}\,B$ and $\mathrm{Spec}\left(A\widehat{\otimes}_K L\right) \to \mathrm{Spec}\,A$ (see the last paragraph of the proof of [BH22, Prop. 3.24] for the fact that $A \to A\widehat{\otimes}_K L$ is regular and the proof of [BH22, Th. 3.21(6)] for the fact that the dimension function on Spec $\left(A\widehat{\otimes}_K L\right)$ introduced in [ILO14, Exp. XVII, Prop. 4.1.1] coincides with the dimension function introduced in [BH22, Prop. 3.18]). Therefore, the desired diagram commutes due to the compatibility of the isomorphisms from [ILO14, Exp. XVII, Prop. 4.1.1] with compositions (see [ILO14, Exp. XVII, Rmk. 4.3.1.3]).

Step 2. Proof for proper f: reduction to affinoid Y. Since the rest of the proof deals with proper f, we use $\mathrm{R}f_*$ instead of $\mathrm{R}f_!$. First, we show that the statement can be checked locally on Y; in particular, we may assume that Y is K-affinoid. To this end, it suffices to verify that $\mathrm{R}\mathscr{H}om\left(a_Y^*\mathrm{R}f_*\,\omega_X, \omega_{Y_L}\right) \in D^{\geq 0}(Y_{L,\text{ét}})$ thanks to the BBDG gluing lemma [BBDG18, Prop. 3.2.2]. Now we observe that Lemma 6.3.1 implies that the base change morphism $a_Y^*(\mathrm{R}f_*\omega_X) \to \mathrm{R}f_{L,*}\omega_{X_L}$ is an isomorphism. Hence, the desired result follows directly from Theorem 7.3.14.

Step 3. Proof when f is a closed immersion. By Step 2, we may assume that $Y = \mathrm{Spa}\,(A, A^\circ)$. Since f is a closed immersion, X is a K-affinoid adic space of the form $X = \mathrm{Spa}\,(A/I, (A/I)^\circ)$ for some ideal $I \subset A$. We consider the following cartesian diagram:

$$\begin{tikzcd} X_L^{\mathrm{alg}} = \mathrm{Spec}\left((A/I)\widehat{\otimes}_K L\right) \arrow[r, "f_L^{\mathrm{alg}}"] \arrow[d, "a_{X^{\mathrm{alg}}}"] & Y_L^{\mathrm{alg}} = \mathrm{Spec}\left(A\widehat{\otimes}_K L\right) \arrow[d, "a_{Y^{\mathrm{alg}}}"] \\ X^{\mathrm{alg}} = \mathrm{Spec}\,A/I \arrow[r, "f^{\mathrm{alg}}"] & Y^{\mathrm{alg}} = \mathrm{Spec}\,A. \end{tikzcd}$$

After unraveling all the definitions and using the $(f_{L,*}, \mathrm{R}f_L^!)$-adjunction, we reduce the question to showing that the following diagram commutes:

$$\begin{tikzcd} a^*_{X^{\mathrm{alg}}}\omega_{X^{\mathrm{alg}}} \arrow[rr, "\gamma_{a_{X^{\mathrm{alg}}}}", "\sim"'] \arrow[d, "a^*_{X^{\mathrm{alg}}}(c_{f^{\mathrm{alg}}})", "\wr"'] & & \omega_{X_L^{\mathrm{alg}}} \arrow[d, "c_{f_L^{\mathrm{alg}}}", "\wr"'] \\ a^*_{X^{\mathrm{alg}}}\mathrm{R}f^{\mathrm{alg},!}\omega_{Y^{\mathrm{alg}}} \arrow[r, "\mathrm{BC}^{*,!}", "\sim"'] & \mathrm{R}f_L^{\mathrm{alg},!}a^*_{Y^{\mathrm{alg}}}\omega_{Y^{\mathrm{alg}}} \arrow[r, "\mathrm{R}f_L^{\mathrm{alg},!}(\gamma_{a_{Y^{\mathrm{alg}}}})", "\sim"'] & \mathrm{R}f_L^{\mathrm{alg},!}\omega_{Y_L^{\mathrm{alg}}}, \end{tikzcd}$$

where the γ-isomorphisms come from [ILO14, Exp. XVII, Prop. 4.1.1], the c-isomorphisms come from [ILO14, Exp. XVII, Prop. 4.1.2], and the isomorphism $\mathrm{BC}^{*,!}$ comes from [ILO14, Exp. XVII, Cor. 4.2.3]. Now we note that the composition

$$F := c^{-1}_{f_L^{\mathrm{alg}}} \circ \mathrm{R}f_L^{\mathrm{alg},!}(\gamma_{a_{Y^{\mathrm{alg}}}}) \circ \mathrm{BC}^{*,!} \circ a^*_{X^{\mathrm{alg}}}(c_{f^{\mathrm{alg}}}) \circ \gamma^{-1}_{a_{X^{\mathrm{alg}}}} : \omega_{X_L^{\mathrm{alg}}} \to \omega_{X_L^{\mathrm{alg}}}$$

is an automorphism of the potential dualizing morphism on X_L^{alg}. Therefore, [ILO14, Exp. XVII, Th. 5.1.1] implies, in order to show that $F = \mathrm{id}$, it suffices to show that F is compatible with pinnings. This, however, follows directly from [ILO14, Exp. XVII, Lemme 4.3.2.3] (whose proof does not use surjectivity of the map g).

Step 4. Proof for general proper f. Lastly, we show the statement for general proper morphisms by induction on $d := \dim X$. In the base case $d = -\infty$, there is nothing to prove. Now assume that $\dim X = d \geq 0$ and that the statement has been proven in dimensions $< d$. At first, we assume moreover that X is reduced. Then Proposition 7.3.9 allows us to choose a regular admissible modification $\pi\colon X' \to X$ as in the beginning of proof of Theorem 7.4.1, so we arrive again at the following commutative diagram from (7.4.2):

$$\begin{tikzcd} Z' \arrow[r, hook, "i'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \arrow[dr, bend left, "\pi_Y"] & \\ Z \arrow[r, hook, "i"] \arrow[rr, bend right, "i_Y"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

By the uniqueness from (7.4.3) in the proof of Theorem 7.4.1, it suffices to check that the traces Tr_i, Tr_{i_Y}, Tr_π, and Tr_{π_Y} are all compatible with change of base field. For Tr_i and Tr_{i_Y}, this follows from the induction hypothesis. Therefore, we may assume that X is smooth, in which case the proper trace agrees with the smooth-source trace in Construction 7.3.1, thanks to Theorem 7.4.1. (2). Our claim for reduced X is now a consequence of the compatibility of both smooth and closed traces with change of base field, as established in Step 1 and Step 3.

For general (not necessarily reduced) X, we consider the diagram

$$\begin{tikzcd}
X_{L,\mathrm{red}} \simeq X_{\mathrm{red},L} \arrow[r, "a_{X_{\mathrm{red}}}"] \arrow[d, "\iota_L"] \arrow[dd, bend right=60, "f_{L,\mathrm{red}}"'] & X_{\mathrm{red}} \arrow[d, "\iota"'] \arrow[dd, bend left=60, "f_{\mathrm{red}}"] \\
X_L \arrow[r, "a_X"] \arrow[d, "f_L"] & X \arrow[d, "f"'] \\
Y_L \arrow[r, "a_Y"] & Y,
\end{tikzcd}$$

where ι and ι_L denote the inclusions of the maximal reduced closed subspaces. Since the upper square and the outer rectangle are cartesian, the top right rectangle and the outer rectangle in the following diagram commute thanks to the case of reduced source treated before:

$$\begin{tikzcd}
a_Y^*\mathrm{R}f_{\mathrm{red},*}(\omega_{X_{\mathrm{red}}}) \arrow[r, "\mathrm{BC}"] \arrow[d, "a_Y^*\mathrm{R}f_*(\mathrm{Tr}_\iota)"', "\wr"] & \mathrm{R}f_{L,*}a_X^*(\iota_*\omega_{X_{\mathrm{red}}}) \arrow[r, "\mathrm{R}f_{L,*}(\mathrm{BC})"] \arrow[d, "\mathrm{R}f_{L,*}a_X^*(\mathrm{Tr}_\iota)"', "\wr"] & \mathrm{R}f_{L,\mathrm{red},*}a_{X_{\mathrm{red}}}^*(\omega_{X_{\mathrm{red}}}) \arrow[r, "\mathrm{R}f_{L,\mathrm{red},*}(\gamma_{X_{\mathrm{red}},L})", "\sim"'] & \mathrm{R}f_{L,\mathrm{red},*}(\omega_{X_{\mathrm{red},L}}) \arrow[d, "\mathrm{R}f_{L,*}(\mathrm{Tr}_{\iota_L})"', "\wr"] \\
a_Y^*\mathrm{R}f_*(\omega_X) \arrow[r, "\mathrm{BC}"] \arrow[d, "a_Y^*(\mathrm{Tr}_f)"'] & \mathrm{R}f_{L,*}a_X^*(\omega_X) \arrow[rr, "\mathrm{R}f_{L,*}(\gamma_{X,L})", "\sim"'] & & \mathrm{R}f_{L,*}(\omega_{X_L}) \arrow[d, "\mathrm{Tr}_{f_L}"'] \\
a_Y^*(\omega_Y) \arrow[rrr, "\gamma_{Y,L}", "\sim"'] & & & \omega_{Y_L}
\end{tikzcd}$$

On the other hand, the upper left rectangle in the diagram commutes by the naturality of the base change map and the left horizontal trace maps are isomorphisms (see Lemma 7.2.4). Thus, the right rectangle must also commute. This yields the statement for general X. $\square$

We recall that [BH22, Th. 3.21.(1)] provides us with a canonical isomorphism $c_f\colon \omega_X \xrightarrow{\sim} \mathrm{R}i^!\omega_Y$ for any finite morphism $f\colon X \to Y$ of rigid-analytic spaces over K. Therefore, for such f, we can give an alternative construction of a trace map $\mathrm{Tr}_f^{\mathrm{BH}}\colon f_*\omega_X \to \omega_Y$ as the composition

$$f_*\omega_X \xrightarrow[\sim]{f_*(c_f)} f_*\mathrm{R}f^!\omega_Y \xrightarrow{\epsilon_f} \omega_Y.$$

Proposition 7.4.9. *Let $f\colon X \to Y$ be a finite morphism of rigid-analytic spaces over K. Then*

$$\mathrm{Tr}_f^{\mathrm{BH}} = \mathrm{Tr}_f\colon f_*\omega_X \longrightarrow \omega_Y.$$

Proof. First, note that when f is a closed immersion, the statement follows from Theorem 7.4.1. (1). Now we can use Lemma 7.2.4 and the fact that both Tr_f and $\mathrm{Tr}_f^{\mathrm{BH}}$ respect compositions to reduce the question to the case when X and Y are reduced. Furthermore, we can assume that X is connected by arguing one clopen connected component of X at a time; cf. [Zav23a, Cor. 2.3]. Finally, $f(X) \subset Y$ is Zariski-closed by virtue of [BGR84, Prop. 9.6.3/3], so we can replace Y with $f(X)$ to assume that f is surjective.

Thanks to [BH22, Th. 3.21] and our assumption that X is connected,

$$\mathrm{R}\mathscr{H}om(f_*\omega_X, \omega_Y) = \mathrm{R}f_*\mathrm{R}\mathscr{H}om(\omega_X, \omega_X) = \mathrm{R}f_*\underline{\Lambda}_X \in D^{\geq 0}(Y_{\text{ét}}; \Lambda)$$

and $\mathscr{H}om(f_*\omega_X, \omega_Y)(Y) \to \mathscr{H}om(f_*\omega_X, \omega_Y)(U)$ is injective for any nonempty open subspace $U \subset Y$. Since both traces are étale local on Y, we may thus prove the statement after replacing Y with any such nonempty open $U \subset Y$ and X with $X_U \coloneqq X \times_Y U$ (after this procedure, X might be disconnected but it is not important for the rest of the proof). Recall that we assume that X and Y are reduced and that f is surjective, so there is a nonempty open $U \subset Y$ such that $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is finite étale. Hence, we can assume that f is finite étale.

Finally, both traces are étale local on the target and any finite étale morphism is étale locally a disjoint union of isomorphisms (or X is empty). Therefore, we reduce to the case when f is an isomorphism (or X is empty). In this case, the claim becomes trivial. $\square$

7.5. **Poincaré duality for Zariski constructible coefficients.** We recall that, throughout this section, we always assume that K is a nonarchimedean field of characteristic 0, n is a positive integer, and $\Lambda = \mathbf{Z}/n\mathbf{Z}$.

The goal of this subsection is to prove a version of Poincaré duality for general proper morphisms of rigid-analytic spaces and Zariski-constructible coefficients. More precisely, we will show that given a proper morphism $f\colon X \to Y$ of rigid-analytic spaces over K, the functor $\mathrm{R}f_*\colon D_{\mathrm{zc}}(X_{\text{ét}};\Lambda) \to D_{\mathrm{zc}}(Y_{\text{ét}};\Lambda)$ commutes with Verdier duality; this confirms an expectation of Bhatt–Hansen (see [BH22, Rmk. 3.23]). As an application of our main result, we deduce duality for intersection cohomology on certain non-smooth and non-proper rigid-analytic spaces. In particular, this confirms the expectation raised in the comment after [BH22, Th. 4.13].

We begin by setting up some notation:

Notation 7.5.1. Let $f\colon X \to Y$ be a morphism of rigid-analytic spaces over K.

(i) (see e.g. [Sta22, Tag 0B6D]) The *evaluation transformation*

$$\mathrm{Ev}_f\colon \mathrm{R}f_*\mathrm{R}\mathscr{H}om(-,-) \to \mathrm{R}\mathscr{H}om\big(\mathrm{R}f_*(-),\mathrm{R}f_*(-)\big)$$

is the natural transformation of functors given on objects $\mathcal{E},\mathcal{E}' \in D(X_{\text{ét}};\Lambda)$ as the tensor-hom adjoint to the composition

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \otimes^L \mathrm{R}f_*\mathcal{E} \xrightarrow{\cup} \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \otimes^L \mathcal{E}\big) \xrightarrow{\mathrm{R}f_*(\mathrm{eval})} \mathrm{R}f_*\mathcal{E}'$$

of the relative cup product from [Sta22, Tag 0B6C] (or Remark 6.3.6) and the derived pushforward of the evaluation map.

(ii) The adjunction between f^* and $\mathrm{R}f_*$ upgrades to an isomorphism

$$\mathrm{Adj}_f\colon \mathrm{R}f_*\mathrm{R}\mathscr{H}om(f^*\mathcal{E},\mathcal{E}') \xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*f^*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \xrightarrow{-\circ\eta_f} \mathrm{R}\mathscr{H}om(\mathcal{E},\mathrm{R}f_*\mathcal{E}')$$

for any $\mathcal{E} \in D(Y_{\text{ét}};\Lambda)$ and $\mathcal{E}' \in D(X_{\text{ét}};\Lambda)$, where η_f denotes the unit of the $(f^*,\mathrm{R}f_*)$-adjunction. This isomorphism is functorial in $\mathcal{E}$ and $\mathcal{E}'$ and hence gives rise to a natural equivalence of functors.

Remark 7.5.2. The evaluation and adjunction transformations are compatible with compositions. That is, given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of rigid-analytic spaces over K, we have $\mathrm{Ev}_{g\circ f} = \mathrm{Ev}_g \circ \mathrm{R}g_*(\mathrm{Ev}_f)$ and $\mathrm{Adj}_{g\circ f} = \mathrm{Adj}_g \circ \mathrm{R}g_*\,\mathrm{Adj}_f$. Unwinding definitions, the first identity concerning Ev amounts to the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{R}g_*\mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \otimes^L \mathrm{R}g_*\mathrm{R}f_*\mathcal{E}' \arrow[r, "\cup"] \arrow[d, "\mathrm{R}g_*(\mathrm{Ev}_f)\otimes^L\mathrm{id}"] \arrow[rr, bend left, "\cup"] & \mathrm{R}g_*(\mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \otimes^L \mathrm{R}f_*\mathcal{E}') \arrow[r, "\mathrm{R}g_*(\cup)"] \arrow[d, "\mathrm{R}g_*(\mathrm{Ev}_f \otimes^L \mathrm{id})"] & \mathrm{R}g_*\mathrm{R}f_*(\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \otimes^L \mathcal{E}') \arrow[d, "\mathrm{R}g_*\mathrm{R}f_*(\mathrm{eval})"] \\
\mathrm{R}g_*\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \otimes^L \mathrm{R}g_*\mathrm{R}f_*\mathcal{E}' \arrow[r, "\cup"] & \mathrm{R}g_*(\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \otimes^L \mathrm{R}f_*\mathcal{E}') \arrow[r, "\mathrm{R}g_*(\mathrm{eval})"] & \mathrm{R}g_*\mathrm{R}f_*\mathcal{E}';
\end{tikzcd}$$

note that the right square commutes already before applying the derived pushforward $\mathrm{R}g_*$ because the composition with the counit of the derived tensor-hom adjunction in the bottom horizontal map produces the adjoint of $\mathrm{Ev}_f \otimes^L \mathrm{id}$, which is by definition the composition in the clockwise direction. The case of Adj then quickly reduces to the case of Ev via the commutative diagram

$$\begin{tikzcd}
\mathrm{R}g_*\mathrm{R}f_*\mathrm{R}\mathscr{H}om(f^*g^*\mathcal{E},\mathcal{E}') \arrow[r, "\mathrm{Ev}_{g\circ f}"] \arrow[d, "\mathrm{R}g_*(\mathrm{Ev}_f)"] & \mathrm{R}\mathscr{H}om(\mathrm{R}g_*\mathrm{R}f_*f^*g^*\mathcal{E},\mathrm{R}g_*\mathrm{R}f_*\mathcal{E}') \arrow[r, "-\circ\eta_{g\circ f}"] \arrow[dr, "\mathrm{R}g_*(-\circ\eta_f)"'] & \mathrm{R}\mathscr{H}om(\mathcal{E},\mathrm{R}g_*\mathrm{R}f_*\mathcal{E}') \\
\mathrm{R}g_*\mathrm{R}\mathscr{H}om(\mathrm{R}f_*f^*g^*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \arrow[r, "\mathrm{R}g_*(-\circ\eta_f)"] \arrow[ur, "\mathrm{Ev}_g"] & \mathrm{R}g_*\mathrm{R}\mathscr{H}om(g^*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \arrow[r, "\mathrm{Ev}_g"] & \mathrm{R}\mathscr{H}om(\mathrm{R}g_*g^*\mathcal{E},\mathrm{R}g_*\mathrm{R}f_*\mathcal{E}'). \arrow[u, "-\circ\eta_g"']
\end{tikzcd}$$

The evaluation and adjunction isomorphism satisfy moreover the following compatibility:

Lemma 7.5.3. *Let $f\colon X \to Y$ be a morphism of rigid-analytic spaces over K. Let $\mathcal{E},\mathcal{E}' \in D(X_{\text{ét}};\Lambda)$ and denote the counit of the $(f^*,\mathrm{R}f_*)$-adjunction by $\epsilon_f\colon f^*\circ\mathrm{R}f_* \to \mathrm{id}$. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \arrow[r, "\mathrm{R}f_*\circ(-\circ\epsilon_f)"] \arrow[dr, "\mathrm{Ev}_f"'] & \mathrm{R}f_*\mathrm{R}\mathscr{H}om(f^*\mathrm{R}f_*\mathcal{E},\mathcal{E}') \arrow[d, "\mathrm{Adj}_f"] \\
& \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}')
\end{tikzcd}$$

Proof. Using Notation 7.5.1. (ii), we can expand the diagram in the statement as follows:

$$\begin{tikzcd}
\mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\mathcal{E}') \arrow[r, "\mathrm{R}f_*\circ(-\circ\epsilon_f)"] \arrow[d, "\mathrm{Ev}_f"'] & \mathrm{R}f_*\mathrm{R}\mathscr{H}om(f^*\mathrm{R}f_*\mathcal{E},\mathcal{E}') \arrow[d, "\mathrm{Ev}_f"] \\
\mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \arrow[r, "-\circ\mathrm{R}f_*(\epsilon_f)"] \arrow[dr, equal] & \mathrm{R}\mathscr{H}om(\mathrm{R}f_*f^*\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}') \arrow[d, "-\circ\eta_f"] \\
& \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\mathcal{E}');
\end{tikzcd}$$

here, η_f denotes the unit for the $(f^*,\mathrm{R}f_*)$-adjunction. The upper square commutes by the naturality of the evaluation transformation. The commutativity of the lower triangle is a standard exercise about the relationship of units and counits of adjunctions (see [Sta22, Tag 0GLL]). This yields the assertion. $\square$

Now we define the duality morphism.

Definition 7.5.4. (i) (*Duality functor*) Let X be a rigid-analytic space over K. The *Verdier duality functor* is given by

$$\mathbf{D}_X(-) := \mathrm{R}\mathscr{H}om(-,\omega_X)\colon D(X_{\text{ét}};\Lambda)^{\mathrm{op}} \to D(X_{\text{ét}};\Lambda).$$

(ii) (*Duality map*) Let $f\colon X\to Y$ be a proper morphism of rigid-analytic spaces over K. The *Poincaré duality transformation* is given on an object $\mathcal{E}\in D(X_{\text{ét}};\Lambda)$ by the composition

$$\begin{aligned}\mathrm{PD}_f(\mathcal{E})\colon \mathrm{R}f_*\mathbf{D}_X(\mathcal{E}) = \mathrm{R}f_*\mathrm{R}\mathscr{H}om(\mathcal{E},\omega_X) &\xrightarrow{\mathrm{Ev}_f} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\mathrm{R}f_*\omega_X)\\ &\xrightarrow{\mathrm{Tr}_f\,\circ-} \mathrm{R}\mathscr{H}om(\mathrm{R}f_*\mathcal{E},\omega_Y) = \mathbf{D}_Y(\mathrm{R}f_*\mathcal{E}),\end{aligned}$$

where the first morphism comes from Notation 7.5.1. (i) and the second morphism is given by postcomposition with the proper trace map Tr_f from Theorem 7.4.1.

This composition is functorial in $\mathcal{E}$ and hence defines a natural transformation of functors

$$\mathrm{PD}_f\colon \mathrm{R}f_*\circ\mathbf{D}_X \to \mathbf{D}_Y\circ\mathrm{R}f_*\colon D(X_{\text{ét}};\Lambda)^{\mathrm{op}} \to D(Y_{\text{ét}};\Lambda).$$

The main goal of this subsection is to show that $\mathrm{PD}_f(\mathcal{F})$ is an isomorphism for proper f and Zariski-constructible $\mathcal{F}\in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$. Before we embark on the proof, we need to establish some preliminary results. The first thing we discuss is the behavior of the Poincaré duality transformation with respect to the upper shriek functors.

Notation 7.5.5. Consider a cartesian diagram of rigid-analytic spaces over K

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "i"] & Y
\end{tikzcd}$$

such that f and f' are proper and i and i' are closed immersions. Then we have the following natural transformations:

(i) the *exchange transformation* $\mathrm{Ex}_i\colon \mathbf{D}_{Y'}\circ i^* \to \mathrm{R}i^!\circ\mathbf{D}_Y$, defined as the $(i_*,\mathrm{R}i^!)$-adjoint of the composition

$$i_*\circ\mathbf{D}_{Y'}\circ i^* = i_*\mathrm{R}\mathscr{H}om\big(i^*(-),\omega_{Y'}\big) \xrightarrow{\mathrm{Adj}_i} \mathrm{R}\mathscr{H}om(-,i_*\omega_{Y'}) \xrightarrow{\mathrm{Tr}_i\,\circ-} \mathrm{R}\mathscr{H}om(-,\omega_Y) = \mathbf{D}_Y,$$

where η_i is the unit of the (i^*,i_*)-adjunction;[39]

(ii) the *base change transformation* $\mathrm{BC}\colon i^*\circ\mathrm{R}f_* \to \mathrm{R}f'_*\circ i'^*$, defined as the $(f'^{,*},\mathrm{R}f'_*)$-adjoint of

$$f'^{,*}\circ i^*\circ\mathrm{R}f_* = i'^{,*}\circ f^*\circ\mathrm{R}f_* \xrightarrow{i'^{,*}(\epsilon_f)} i'^{,*};$$

[39]Note that the Verdier duality functor $\mathbf{D}_X$ is contravariant, so it reverses the direction of morphisms.

(iii) the *shriek base change transformation* $\mathrm{BC}^!\colon \mathrm{R}f'_* \circ \mathrm{R}i'^! \to \mathrm{R}i^! \circ \mathrm{R}f_*$, defined as the $(i_*, \mathrm{R}i^!)$-adjoint of

$$i_* \circ \mathrm{R}f'_* \circ \mathrm{R}i'^! = \mathrm{R}f_* \circ i'_* \circ \mathrm{R}i'^! \xrightarrow{\mathrm{R}f_*(\epsilon_{i'})} \mathrm{R}f_*.$$

These natural transformations interact with the Poincaré duality transformation from Definition 7.5.4. (ii) in the following manner:

Proposition 7.5.6. *With Notation 7.5.5, the following diagram of natural transformations between contravariant functors from $D(X)$ to $D(Y')$ commutes:*

$$\begin{tikzcd}
\mathrm{R}f'_* \circ \mathbf{D}_{X'} \circ i'^{,*} \arrow[r, "\mathrm{R}f'_*\circ\mathrm{Ex}_{i'}"] \arrow[d, "\mathrm{PD}_{f'}\circ i'^{,*}"] & \mathrm{R}f'_* \circ \mathrm{R}i'^{,!} \circ \mathbf{D}_X \arrow[r, "\mathrm{BC}^!\circ\mathbf{D}_X"] & \mathrm{R}i^! \circ \mathrm{R}f_* \circ \mathbf{D}_X \arrow[d, "\mathrm{R}i^!\circ\mathrm{PD}_f"] \\
\mathbf{D}_{Y'} \circ \mathrm{R}f'_* \circ i'^{,*} \arrow[r, "\mathbf{D}_{Y'}\circ\mathrm{BC}"] & \mathbf{D}_{Y'} \circ i^* \circ \mathrm{R}f_* \arrow[r, "\mathrm{Ex}_i\circ\mathrm{R}f_*"] & \mathrm{R}i^! \circ \mathbf{D}_Y \circ \mathrm{R}f_*.
\end{tikzcd} \tag{7.5.7}$$

Furthermore, all horizontal transformations become isomorphisms when evaluated on $\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$.

Here, the first bottom horizontal arrow has its direction seemingly reversed, which is due to the fact that the Verdier duality functor $\mathbf{D}_{Y'}$ is contravariant.

Proof. To lighten notation, we drop all the derived notation for pushforwards and exceptional inverse images for the duration of this proof; for example, we write f_* instead of $\mathrm{R}f_*$ and $i^!$ instead of $\mathrm{R}i^!$. Moreover, we omit all the "$\circ$"-symbols between functors and natural transformations. As a further simplification, we denote the functor $\mathrm{R}\mathscr{H}om(-, f_*\omega_X)\colon D(Y_{\text{ét}};\Lambda)^{\mathrm{op}} \to D(Y_{\text{ét}};\Lambda)$ by $\mathbf{D}_{X\to Y}(-)$, and similarly for other morphisms. With these shorthands, we have for example

$$\mathrm{PD}_f\colon f_*\mathbf{D}_X \xrightarrow{\mathrm{Ev}_f} \mathbf{D}_{X\to Y} f_* \xrightarrow{\mathrm{Tr}_f\,\circ -} \mathbf{D}_Y f_* \quad \text{and} \quad \mathrm{Adj}_f\colon f_*\mathbf{D}_X f^* \xrightarrow{\mathrm{Ev}_f} \mathbf{D}_{X\to Y} f_* f^* \xrightarrow{-\circ\eta_f} \mathbf{D}_{X\to Y}.$$

We begin with the commutativity of the diagram (7.5.7). By the $(i_*, i^!)$-adjunction, it suffices to show the commutativity of the corresponding diagram where we add i_* to the left four terms and drop the $i^!$ from the right two terms. Plugging in the definitions of all the transformations, this adjoint diagram becomes the outer rim of the following diagram:

$$\begin{tikzcd}[column sep=small]
i_*f'_*\mathbf{D}_{X'}i'^{,*} \arrow[r, "i_*f'_*\eta_{i'}"] \arrow[dr, equal] \arrow[ddr, "i_*f'_*\mathbf{D}_{X'}i'^{,*}\epsilon_f"'] \arrow[dddd, "i_*\,\mathrm{Ev}_{f'}"] & i_*f'_*i'^{,!}i'_*\mathbf{D}_{X'}i'^{,*} \arrow[r, "i_*f'_*i'^{,!}\,\mathrm{Adj}_{i'}"] \arrow[dr, equal] & i_*f'_*i'^{,!}\mathbf{D}_{X'\to X} \arrow[r, "i_*f'_*i'^{,!}(\mathrm{Tr}_{i'}\,\circ-)"] \arrow[dr, equal] & i_*f'_*i'^{,!}\mathbf{D}_X \arrow[r, equal] & f_*i'_*i'^{,!}\mathbf{D}_X \arrow[r, "f_*\epsilon_{i'}"] & f_*\mathbf{D}_X \arrow[dddd, "\mathrm{Ev}_f"] \\
 & f_*i'_*\mathbf{D}_{X'}i'^{,*} \arrow[r, "f_*i'_*\eta_{i'}"] \arrow[dr, equal] & f_*i'_*i'^{,!}i'_*\mathbf{D}_{X'}i'^{,*} \arrow[r, "f_*i'_*i'^{,!}\,\mathrm{Adj}_{i'}"] \arrow[d, "f_*\epsilon_{i'}"] \arrow[dl, phantom, "(7.5.8)"] & f_*i'_*i'^{,!}\mathbf{D}_{X'\to X} \arrow[r, "f_*\epsilon_{i'}"] \arrow[ur, "f_*i'_*i'^{!}(\mathrm{Tr}_{i'}\,\circ-)"] & f_*\mathbf{D}_{X'\to X} \arrow[ur, "f_*(\mathrm{Tr}_{i'}\,\circ-)"'] \arrow[dl, "f_*\mathbf{D}_{X'\to X}\epsilon_f"] \arrow[ddd, "\mathrm{Ev}_f"] & \\
 & i_*f'_*\mathbf{D}_{X'}i'^{,*}f^*f_* \arrow[dd, "i_*\,\mathrm{Ev}_{f'}"] & f_*i'_*\mathbf{D}_{X'}i'^{,*} \arrow[urr, "f_*\,\mathrm{Adj}_{i'}"] \arrow[dr, "f_*i'_*\mathbf{D}_{X'}i'^{,*}\epsilon_f"'] & f_*\mathbf{D}_{X'\to X}f^*f_* \arrow[ddr, bend left, "\mathrm{Adj}_f"] \arrow[r, phantom, "(7.5.9)"] & & \\
 & & i_*f'_*\mathbf{D}_{X'}f'^{,*}i^*f_* \arrow[r, equal] \arrow[d, "i_*\,\mathrm{Ev}_{f'}"] \arrow[dr, "i_*\,\mathrm{Adj}_{f'}"] \arrow[drr, "\mathrm{Adj}_{i\circ f'}"] & f_*i'^{,*}\mathbf{D}_{X'}i'^{,*}f^*f_* \arrow[u, "f_*\,\mathrm{Adj}_{i'}"] \arrow[dr, "\mathrm{Adj}_{f\circ i'}"] \arrow[r, phantom, "(7.5.10)"] & & \\
i_*\mathbf{D}_{X'\to Y'}f'_*i'^{,*} \arrow[r, "i_*\mathbf{D}_{X'\to Y'}f'_*i'^{,*}\epsilon_f"] \arrow[d, "i_*(\mathrm{Tr}_{f'}\,\circ-)"] & i_*\mathbf{D}_{X'\to Y'}f'_*i'^{,*}f^*f_* \arrow[r, equal] \arrow[d, "i_*(\mathrm{Tr}_{f'}\,\circ-)"] & i_*\mathbf{D}_{X'\to Y'}f'_*f'^{,*}i^*f_* \arrow[r, "i_*\mathbf{D}_{X'\to Y'}\eta_{f'}"'] \arrow[d, "i_*(\mathrm{Tr}_{f'}\,\circ-)"] \arrow[u, phantom, "(7.5.11)"] & i_*\mathbf{D}_{X'\to Y'}i^*f_* \arrow[r, "\mathrm{Adj}_i"'] \arrow[d, "i_*(\mathrm{Tr}_{f'}\,\circ-)"] \arrow[u, phantom, "(7.5.12)"] & \mathbf{D}_{X'\to Y}f_* \arrow[r, "(f_*\,\mathrm{Tr}_{i'}\,\circ-)"] \arrow[d, "(i_*\,\mathrm{Tr}_{f'}\,\circ-)"'] \arrow[dr, phantom, "(7.5.13)"] & \mathbf{D}_{X\to Y}f_* \arrow[d, "(\mathrm{Tr}_f\,\circ-)"] \\
i_*\mathbf{D}_{Y'}f'_*i'^{,*} \arrow[r, "i_*\mathbf{D}_{Y'}f'_*i'^{,*}\epsilon_f"] & i_*\mathbf{D}_{Y'}f'_*i'^{,*}f^*f_* \arrow[r, equal] & i_*\mathbf{D}_{Y'}f'_*f'^{,*}i^*f_* \arrow[r, "i_*\mathbf{D}_{Y'}\eta_{f'}"] & i_*\mathbf{D}_{Y'}i^*f_* \arrow[r, "\mathrm{Adj}_i"] & \mathbf{D}_{Y'\to Y}f_* \arrow[r, "(\mathrm{Tr}_i\,\circ-)"] & \mathbf{D}_Y f_*
\end{tikzcd}$$

For the most part, the various cells in this diagram commute thanks to the naturality of the evaluation transformations Ev, the adjunction transformations Adj, the counit $\epsilon_{i'}$, and the transformations given by precomposition with traces, with the following exceptions:

- the triangle (7.5.8) commutes by the unit-counit identity for adjunctions (see [Sta22, Tag 0GLL]);
- the triangle (7.5.9) commutes by Lemma 7.5.3;

- the triangles (7.5.10) and (7.5.12) commute by Remark 7.5.2;
- the triangle (7.5.11) commutes due to the definition of Adj_f (see Notation 7.5.1. (ii)); and
- the bottom right square (7.5.13) commutes by Theorem 7.4.1. (3).

This finishes the proof of the first assertion. To prove the second assertion, we first note that [BH22, Th. 3.10, Cor. 3.12, and Cor. 3.14] imply that all functors involved in (7.5.7) preserve $D^{(b)}_{\mathrm{zc}}$. Therefore, a combination of Lemma 6.3.1, Lemma 7.3.4, and [BH22, Th. 3.21.(4)] shows that all horizontal transformations become isomorphisms when evaluated on $\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$. □

In order to be able to use Proposition 7.5.6 effectively, we need the following general lemma:

Lemma 7.5.14. *Let X be a rigid-analytic space over K and let $\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$ be a locally bounded complex with Zariski-constructible cohomology sheaves. Assume that $\mathrm{R}i_x^!\mathcal{F} = 0$ for every classical point $i_x\colon x \hookrightarrow X$. Then $\mathcal{F} = 0$.*

Proof. We pick a classical point $i_x\colon x \hookrightarrow X$. Then [BH22, Cor. 3.12] ensures that $\mathrm{R}i_x^!$ carries $D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$ to $D^b_{\mathrm{zc}}(x_{\text{ét}};\Lambda)$. Furthermore, (the proof of) [BH22, Th. 3.21(4)] implies that $\mathbf{D}_x\big(\mathrm{R}i_x^!\mathcal{F}\big) \simeq i_x^*\mathbf{D}_X\mathcal{F}$. Since $\mathbf{D}_x$ induces an anti-equivalence of $D^b_{\mathrm{zc}}(x_{\text{ét}};\Lambda)$ (see [BH22, Th. 3.21(3)]), we conclude that $i_x^*\mathbf{D}_X(\mathcal{F}) = 0$. Moreover, *loc. cit.* implies that $\mathbf{D}_X$ induces an anti-equivalence of $D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$. In particular, $\mathbf{D}_X(\mathcal{F}) \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$. Since $x \in X$ was an arbitrary classical point, we conclude that $i_x^*\mathbf{D}_X(\mathcal{F}) = 0$ for any classical point $x \in X$. This implies that $\mathbf{D}_X(\mathcal{F}) = 0$ because $\mathbf{D}_X(\mathcal{F})$ has Zariski-constructible cohomology. Finally, we use that $\mathbf{D}_X$ induces an anti-equivalence of $D^{(b)}_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$ once again to conclude that $\mathcal{F} = 0$. □

We also discuss a particularly nice set of generators in $D^b_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$ for a quasi-compact rigid-analytic space X over a non-archimedean field K of characteristic 0.

Lemma 7.5.15. *Let X be a quasi-compact rigid-analytic space over K. Then $D^b_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$ is the smallest thick triangulated subcategory of $D(X_{\text{ét}};\Lambda)$ containing all objects of the form $\mathrm{R}f_*\underline{M}_{X'}$ for a finitely generated Λ-module M and a proper morphism $f\colon X' \to X$ such that X' is smooth and $\dim f^{-1}(x) < \max(\dim X, 1)$ for any classical point $x \in X$.*

Proof. We denote by $D' \subset D(X_{\text{ét}};\Lambda)$ the smallest thick triangulated subcategory which contains $\mathrm{R}f_*\underline{M}_{X'}$ for f and M as in the formulation of the lemma. Using [BH22, Th. 3.10], we conclude that $D' \subset D^b_{\mathrm{zc}}(X_{\text{ét}};\Lambda)$. Therefore, it suffices to show that $D^b_{\mathrm{zc}}(X_{\text{ét}};\Lambda) \subset D'$. We prove this by induction on $d = \dim X$ (note that $\dim X$ is finite since X is quasi-compact).

If $d \le 0$, then the claim is essentially obvious because either X is empty or $X_{\mathrm{red}} = \sqcup_{i=1}^m \mathrm{Spa}\,(L_i, L_i^\circ)$ for some finite extensions $K \subset L_i$. Now we assume that $\dim X = d > 0$ and the result is known for all spaces of dimension strictly less then d. Then [BH22, Prop. 3.6] implies that it suffices to show that sheaves of the form $g_*\underline{M}_{X'}$ lie in D', where $g\colon X' \to X$ is a finite morphism and M is a finitely generated Λ-module. By the topological invariance of the étale site (see [Hub96, Prop. 2.3.7]), we may assume that both X and X' are reduced. Now denote by U' the smooth locus of X' and by Z' its Zariski-closed complement (with reduced adic space structure).

Then Proposition 7.3.9 implies that there is a regular U'-admissible modification $h\colon X'' \to X'$. We denote by $f\colon X'' \to X$ the composed morphism to X' and by $Z'' \coloneqq X'' \times_X Z'$ the pre-image of Z' in X''.

First, we show that, for every classical point $x \in X$, we have $\dim f^{-1}(x) < \max(\dim X, 1)$. To see this, we first note that $\dim X' \le \dim X$ since g is finite and, thus, Lemma 7.3.10 implies that $\dim h^{-1}(x') < \max(\dim X', 1) \le \max(\dim X, 1)$. Then we observe that, for every classical point $x \in X$, the fiber $|f^{-1}(x)|$ is (topologically) equal to $\bigsqcup_{x' \in X'_{\mathrm{cl}}\colon f(x') = x} |h^{-1}(x')|$. Therefore, we also have $\dim f^{-1}(x) < \max(\dim X, 1)$.

Now we consider the following exact triangle:

$$\underline{M}_{X'} \to \mathrm{R}h_*\underline{M}_{X''} \to C. \tag{7.5.16}$$

By construction, $\mathrm{supp}(C) \subset Z'$. After applying g_* to (7.5.16), we get the following exact triangle

$$g_*\underline{M}_{X'} \to \mathrm{R}f_*\underline{M}_{X''} \to g_*(C). \tag{7.5.17}$$

By construction, $g_*(C)$ is supported on $Z \coloneqq g(Z')$, which is a Zariski-closed subset of X due to [BGR84, Th. 9.6.3/3]. Since $Z' \to Z$ is surjective and Z' is nowhere dense in X', we conclude that $\dim Z \le \dim Z' <$

$\dim X' \le \dim X = d$. Therefore, the induction hypothesis implies that $g_*(C) \in D'$. We also have $\mathrm{R}f_*\underline{M}_{X''}$ by the very definition of D'. Thus, (7.5.17) ensures that $g_*\underline{M}_{X'} \in D'$ finishing the proof. □

Now we are ready to prove the general Poincaré duality result as expected by Bhatt and Hansen (see [BH22, Rmk. 3.23]).

Theorem 7.5.18. *Let $f\colon X \to Y$ be a proper morphism of rigid-analytic spaces over K, and let $\mathcal{F} \in D_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. Then the Poincaré duality transformation*

$$\mathrm{PD}_f(\mathcal{F})\colon \mathrm{R}f_*\big(\mathbf{D}_X(\mathcal{F})\big) \to \mathbf{D}_Y\big(\mathrm{R}f_*\mathcal{F}\big)$$

is an isomorphism.

Proof. *Step 0. We reduce to the case when X and Y are qcqs.* First, Theorem 7.4.1 (4) implies that the question is local on Y. Therefore, we can assume that Y is qcqs. In this case, X is automatically qcqs as well.

Step 1. We reduce to the case when $\mathcal{F}$ lies in $D^b_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. First, we note that $\mathrm{R}f_*$ commutes with sequential homotopy colimits (e.g., as defined in [Sta22, Tag 0A5K]) due to [Zav23a, Lem. 9.1]. This implies that both the source and target of PD_f (viewed as functors in $\mathcal{E}$) transform sequential homotopy colimits into sequential homotopy limits (e.g., as defined in [Sta22, Tag 08TB]). Since the natural morphism $\mathrm{hocolim}_n\, \tau^{\le n}\mathcal{F} \to \mathcal{F}$ is an isomorphism (this can be deduced from [Sta22, Tag 0CRK]), we reduce to the case when $\mathcal{F} \in D^-_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. In this case, we consider the exact triangle

$$\tau^{\le -N}\mathcal{F} \to \mathcal{F} \to \tau^{> -N}\mathcal{F}.$$

Recall that $\mathrm{R}f_*$ has cohomological dimension $2\dim(f)$ by virtue of [Hub96, Prop. 5.3.11]. Furthermore, $\omega_X \in D^{[-2\dim X, 0]}(X_{\text{ét}}; \Lambda)$ and $\omega_Y \in D^{[-2\dim Y, 0]}(Y_{\text{ét}}; \Lambda)$ by virtue of [BH22, Lem. 3.30]. Therefore, we conclude that

$$\mathrm{R}f_*\big(\mathbf{D}_X(\tau^{\le -N}\mathcal{F})\big) = \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om_\Lambda(\tau^{\le -N}\mathcal{F}, \omega_X)\big) \in D^{\ge N - 2\dim X}(X_{\text{ét}}; \Lambda),$$
$$\mathbf{D}_Y\big(\mathrm{R}f_*\tau^{\le -N}\mathcal{F}\big) = \mathrm{R}\mathscr{H}om_\Lambda\big(\mathrm{R}f_*(\tau^{\le -N}\mathcal{F}), \omega_Y\big) \in D^{\ge N - 2\dim f - 2\dim Y}(Y_{\text{ét}}; \Lambda).$$

Given an integer q, the map on cohomology sheaves $\mathcal{H}^q\big(\mathrm{PD}_f(\mathcal{F})\big)$ is therefore an isomorphism if and only if $\mathcal{H}^q\big(\mathrm{PD}_f(\tau^{> -M}\mathcal{F})\big)$ is an isomorphism for any large $M \gg 0$. In particular, if $\mathrm{PD}_f(\tau^{> -N}\mathcal{F})$ is an isomorphism for all N, then $\mathrm{PD}_f(\mathcal{F})$ is an isomorphism as well. Thus, we reduce to the case when $\mathcal{F}$ is bounded.

Step 2. We reduce to the case when $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$. Pick a classical point $i_y\colon y \hookrightarrow Y$ and consider the fiber sequence

$$\begin{tikzcd} X_y \arrow[r, "i'_y"] \arrow[d, "f_y"] & X \arrow[d, "f"] \\ y \arrow[r, "i_y"] & Y. \end{tikzcd}$$

Then [BH22, Th. 3.10 and Th. 3.21(3)] imply that both $\mathrm{R}f_*\mathbf{D}_X(\mathcal{F})$ and $\mathbf{D}_Y(\mathrm{R}f_*\mathcal{F})$ lie in $D^b_{\mathrm{zc}}(Y_{\text{ét}}; \Lambda)$. Therefore, Lemma 7.5.14 (applied to $\mathrm{cone}(\mathrm{PD}_f(\mathcal{F}))$) implies that it suffices to show that $\mathrm{R}i^!_y\big(\mathrm{PD}_f(\mathcal{F})\big)$ is an isomorphism for any classical point $y \in Y$. Furthermore, Proposition 7.5.6 then ensures that it suffices to show that $\mathrm{PD}_{f_y}(i^*_y\mathcal{F})$ is an isomorphism for any classical point $y \in Y$. In other words, we can assume that $Y = \mathrm{Spa}\,(L, \mathcal{O}_L)$ for some finite extension $K \subset L$ (and $\mathcal{F}$ still lies in $D^b_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$). After replacing K by L, we can even assume that $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$.

Step 3. End of proof. Finally, we complete the argument under the extra assumptions that $Y = \mathrm{Spa}\,(K, \mathcal{O}_K)$ and $\mathcal{F} \in D^b_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. In this case, we argue by induction on $d = \dim X$. If $d \le 0$, then the claim is essentially obvious because either X is empty or $X_{\mathrm{red}} = \sqcup_{i=1}^m \mathrm{Spa}\,(L_i, L_i^\circ)$ for some finite extensions $K \subset L_i$. So we assume that $\dim X = d > 0$ and that the result is known for all spaces of dimension strictly less then d.

Let $D' \subset D(X_{\text{ét}}; \Lambda)$ be the full subcategory consisting of objects $\mathcal{F}$ such that $\mathrm{PD}_f(\mathcal{F})$ is an isomorphism. We wish to show that D' contains $D^b_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. Now note that D' is a thick triangulated subcategory of $D(X_{\text{ét}}; \Lambda)$. Therefore, Lemma 7.5.15 implies that it suffices to show that $\mathrm{R}g_*\underline{M}_{X'} \in D'$ for a finitely generated Λ-module M and a proper morphism $g\colon X' \to X$ such that X' is smooth and $\dim f^{-1}(x) < \max(\dim X, 1)$

for any classical point $x \in X$. Since proper trace is compatible with compositions (see Theorem 7.4.1. (3)), we see that the composition

$$\mathrm{R}f_* \circ \mathrm{R}g_* \circ \mathbf{D}_{X'}(\underline{M}_{X'}) \xrightarrow{\mathrm{R}f_*(\mathrm{PD}_g(\underline{M}_{X'}))} \mathrm{R}f_* \circ \mathbf{D}_X \circ \mathrm{R}g_*(\underline{M}_{X'}) \xrightarrow{\mathrm{PD}_f(\mathrm{R}g_*\underline{M}_{X'})} \mathbf{D}_Y \circ \mathrm{R}f_* \circ \mathrm{R}g_*(\underline{M}_{X'})$$

is given by $\mathrm{PD}_{f\circ g}(\underline{M}_{X'})$. Hence, we are reduced to showing that both $\mathrm{PD}_g(\underline{M}_{X'})$ and $\mathrm{PD}_{f\circ g}(\underline{M}_{X'})$ are isomorphisms. The latter map is an isomorphism due to Theorem 6.4.10 and the combination of Theorem 7.4.1 (2) and Proposition 7.3.2 (i) (see also Construction 7.2.7).

Therefore, we reduce the question to showing that $\mathrm{PD}_g(\underline{M}_{X'})$ is an isomorphism. For any classical point $x \in X$, we denote by $X'_x \coloneqq X' \times_X x$ the fiber over x and by $g_x \colon X'_x \to x$ the restriction of g. Arguing as in Step 2, we reduce this question to showing that $\mathrm{PD}_{g_x}(\underline{M}_{X'_x})$ is an isomorphism for any classical point $x \in X$. Since we chose g such that $\dim X'_x < \dim X$, the induction hypothesis implies that $\mathrm{PD}_{g_x}(\underline{M}_{X'_x})$ is indeed an isomorphism for any classical point $x \in X$. This finishes the proof. □

As the main application of the general form of Poincaré duality, we show that a version of Poincaré duality holds for some non-smooth and non-proper spaces. For this, we need to recall some definitions.

Definition 7.5.19. A rigid-analytic space X over K is *Zariski-compactifiable* if there is a Zariski-open immersion $j \colon X \hookrightarrow \overline{X}$ such that $\overline{X}$ is proper over K.

In order to formulate this version of duality on non-smooth spaces, we also need to recall the definition of intersection cohomology of rigid-analytic varieties due to Bhatt and Hansen. For this, we fix a rigid-analytic space X over K of equidimension d, a smooth Zariski-open subspace $j \colon U \hookrightarrow X$, and a lisse sheaf of Λ-modules $\mathbf{L}$ on $U_{\text{ét}}$.

Definition 7.5.20. The *IC sheaf* $\mathrm{IC}_X(\mathbf{L})$ *associated to* $\mathbf{L}$ is the intermediate extension $\mathrm{IC}_X(\mathbf{L}) \coloneqq j_{!*}\big(\mathbf{L}[d]\big)$ (see [BH22, Th. 4.2(5)]).

The *intersection cohomology complex* $\mathrm{IH}(X, \mathbf{L})$ *with coefficients in* $\mathbf{L}$ is the complex $\mathrm{IH}(X, \mathbf{L}) \coloneqq \mathrm{R}\Gamma\big(X, \mathrm{IC}_X(\mathbf{L})\big)$.

The *compactly supported intersection cohomology complex* $\mathrm{IH}_c(X, \mathbf{L})$ *with coefficients in* $\mathbf{L}$ is the complex $\mathrm{IH}_c(X, \mathbf{L}) \coloneqq \mathrm{R}\Gamma_c\big(X, \mathrm{IC}_X(\mathbf{L})\big)$.

The *i-th intersection cohomology* $\mathrm{IH}^i(X, \mathbf{L})$ *with coefficients in* $\mathbf{L}$ is the Λ-module $\mathrm{IH}^i(X, \mathbf{L}) \coloneqq \mathrm{H}^i\left(\mathrm{IH}(X, \mathbf{L})\right)$.

The *i-th compactly supported intersection cohomology* $\mathrm{IH}^i_c(X, \mathbf{L})$ *with coefficients in* $\mathbf{L}$ is the Λ-module $\mathrm{IH}^i_c(X, \mathbf{L}) \coloneqq \mathrm{H}^i_c\left(\mathrm{IH}(X, \mathbf{L})\right)$.

In order to prove the desired above version of Poincaré duality, we need the following preliminary lemma:

Lemma 7.5.21. *Let $j \colon U \to X$ be a Zariski-open immersion of rigid-analytic spaces over K, let $\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(U_{\text{ét}}; \Lambda)$. If there is $\overline{\mathcal{F}} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$ such that $j^*\overline{\mathcal{F}} = \mathcal{F}$, then $\mathrm{R}j_*\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$ and $j_!\mathcal{F} \in D^{(b)}_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$.*

Proof. It clearly suffices to show that $\mathrm{R}j_*j^*\overline{\mathcal{F}}$ and $j_!j^*\overline{\mathcal{F}}$ lie in $D^{(b)}_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$. We denote by $i \colon Z \to X$ the closed complement of U (with the reduced adic space structure on it). Then the exact triangles

$$i_*\mathrm{R}i^!\overline{\mathcal{F}} \to \overline{\mathcal{F}} \to \mathrm{R}j_*j^*\overline{\mathcal{F}},$$

$$j_!j^*\overline{\mathcal{F}} \to \overline{\mathcal{F}} \to i_*i^*\overline{\mathcal{F}}$$

imply that it suffices to show that the functors i_*, i^*, and $\mathrm{R}i^!$ preserve locally bounded Zariski-constructible complexes. The claim is evident for the first two functors, and [BH22, Cor. 3.12] implies the claim for the third functor above. □

Theorem 7.5.22. *Let $\overline{X}$ be a proper rigid-analytic space over K, let $U \subset X \subset \overline{X}$ be two rigid-analytic subspaces which are both Zariski-open in $\overline{X}$,*[40] *let $\mathbf{L}$ be a local system of finite free Λ-modules on $U_{\text{ét}}$, and let*

[40] We note that this is stronger than requiring the inclusions $U \subset X$ and $X \subset \overline{X}$ to be Zariski-open. For example, $\left(\mathbf{A}^{1,\mathrm{an}}_K \smallsetminus \{(1/p)^{\mathbf{N}}\}\right) \subset \mathbf{A}^{1,\mathrm{an}}_K$ and $\mathbf{A}^{1,\mathrm{an}}_K \subset \mathbf{P}^{1,\mathrm{an}}_K$ are Zariski-open, but $\left(\mathbf{A}^{1,\mathrm{an}}_K \smallsetminus \{(1/p)^{\mathbf{N}}\}\right) \subset \mathbf{P}^{1,\mathrm{an}}_K$ is not because any Zariski-closed subset of $\mathbf{P}^{1,\mathrm{an}}_K$ is either finite or $\mathbf{P}^{1,\mathrm{an}}_K$ by rigid GAGA.

$\mathbf{L}^\vee$ *be its* Λ*-linear dual. Assume that* U *is smooth of equidimension* d *and set* $C \coloneqq \widehat{\overline{K}}$*. Then* $\mathrm{IH}_c(X_C, \mathbf{L})$ *and* $\mathrm{IH}(X_C, \mathbf{L}^\vee(d))$ *lie in* $D^b_{\mathrm{coh}}(\Lambda)$ *and there is a Galois-equivariant isomorphism*

$$\mathrm{RHom}_\Lambda\big(\mathrm{IH}_c(X_C, \mathbf{L}), \Lambda\big) \simeq \mathrm{IH}\big(X_C, \mathbf{L}^\vee(d)\big) \tag{7.5.23}$$

which is functorial in $\mathbf{L}$*. In particular, there is a Galois-equivariant isomorphism* $\mathrm{IH}_c^{-i}(X_C, \mathbf{L})^\vee \simeq \mathrm{IH}^i(X_C, \mathbf{L}^\vee)(d)$ *for any integer* i*.*

Proof. First, we denote by $j\colon X \to \overline{X}$ the natural open immersion and by $f\colon X \to \mathrm{Spa}\,(K, \mathcal{O}_K)$ and $\overline{f}\colon \overline{X} \to \mathrm{Spa}\,(K, \mathcal{O}_K)$ the structure morphisms. Then we note that [BH22, Th. 4.2(5)] implies that $\mathrm{IC}_X(\mathbf{L})$ lies in $D^b_{\mathrm{zc}}(X_{\text{ét}}; \Lambda)$, $\mathrm{IC}_{\overline{X}}(\mathbf{L})$ lies in $D^b_{\mathrm{zc}}(\overline{X}_{\text{ét}}; \Lambda)$, and $\overline{j}^*_X\mathrm{IC}_{\overline{X}}(\mathbf{L}) \simeq \mathrm{IC}_X(\mathbf{L})$. Furthermore, Lemma 7.5.21 guarantees that $j_!\mathrm{IC}_X(\mathbf{L})$ lies in $D^b_{\mathrm{zc}}(\overline{X}_{\text{ét}}; \Lambda)$, so [BH22, Th. 3.10] implies that $\mathrm{IH}_c(X, \mathbf{L}) \in D^b_{\mathrm{coh}}(\Lambda)$. Then we have the following sequence of isomorphisms:

$$\begin{aligned} \mathbf{D}_{\mathrm{Spa}\,(K,\mathcal{O}_K)}\big(\mathrm{R}f_!\mathrm{IC}_X(\mathbf{L})\big) &\simeq \mathbf{D}_{\mathrm{Spa}\,(K,\mathcal{O}_K)}\big(\mathrm{R}\overline{f}_*j_!\mathrm{IC}_X(\mathbf{L})\big) \\ &\simeq \mathrm{R}\overline{f}_*\mathbf{D}_{\overline{X}}\big(j_!\mathrm{IC}_X(\mathbf{L})\big) \simeq \mathrm{R}\overline{f}_*\mathrm{R}j_*\mathbf{D}_X\big(\mathrm{IC}_X(\mathbf{L})\big) \simeq \mathrm{R}f_*\mathrm{IC}_X\big(\mathbf{L}^\vee(d)\big), \end{aligned} \tag{7.5.24}$$

where the first isomorphism follows from the formula $\mathrm{R}f_! \simeq \mathrm{R}\overline{f}_* \circ j_!$, the second isomorphism follows from Theorem 7.5.18 and the observation that $j_!\mathrm{IC}_X(\mathbf{L}) \in D^b_{\mathrm{zc}}(\overline{X}; \Lambda)$, the third isomorphism follows from [BH22, Th. 3.21(5)], and the last isomorphism follows from the formula $\mathrm{R}f_* \simeq \mathrm{R}\overline{f}_* \circ \mathrm{R}j_*$, [BH22, Th. 4.2(5)] and the assumption that U is smooth over K. Then the isomorphism (7.5.23) follows directly (7.5.24) by passing to (derived) global section over $\mathrm{Spa}\,(C, \mathcal{O}_C)$. Likewise, we immediately conclude that $\mathrm{IH}(X_C, \mathbf{L}^\vee(d))$ lies in $D^b_{\mathrm{coh}}(\Lambda)$. The last assertion follows directly from (7.5.23) and the observation that $\Lambda = \mathbf{Z}/n\mathbf{Z}$ is an injective Λ-module. □

Remark 7.5.25. The condition that $U \subset \overline{X}$ is Zariski-open is automatically satisfied if $U = X^{\mathrm{sm}}$ is the smooth locus of X. To justify this, we first observe that $\overline{X}^{\mathrm{sm}} \subset \overline{X}$ is Zariski-open because the smooth locus is always Zariski-open. Therefore, $X^{\mathrm{sm}} = \overline{X}^{\mathrm{sm}} \cap X \subset \overline{X}$ is Zariski-open as an intersection of two Zariski-open subspaces. In particular, Theorem 7.5.22 proves Poincaré duality for intersection cohomology (with constant coefficients) for any Zariski-compactifiable X.

Finally, we deduce a usual version of Poincaré duality for local systems on smooth Zariski-compactifiable rigid-analytic spaces:

Corollary 7.5.26. *Let* U *be a smooth Zariski-compactifiable rigid-analytic space over* K *of equidimension* d*. Let* $\mathbf{L}$ *be a local system of finite free* Λ*-modules on* $U_{\text{ét}}$*, let* $\mathbf{L}^\vee$ *be its* Λ*-linear dual, and let* $\mathrm{ev}\colon \mathbf{L} \otimes_\Lambda \mathbf{L}^\vee \to \underline{\Lambda}_U$ *be the natural evaluation map. Set* $C \coloneqq \widehat{\overline{K}}$*. Then* $\mathrm{R}\Gamma_c(U_C, \mathbf{L})$ *and* $\mathrm{R}\Gamma(U_C, \mathbf{L}^\vee(d)[2d])$ *lies in* $D^b_{\mathrm{coh}}(\Lambda)$ *and the Galois-equivariant pairing*

$$\mathrm{R}\Gamma_c(U_C, \mathbf{L}) \otimes^L_\Lambda \mathrm{R}\Gamma(U_C, \mathbf{L}^\vee(d)[2d]) \xrightarrow{\cup} \mathrm{R}\Gamma_c(U_C, \mathbf{L} \otimes \mathbf{L}^\vee(d)[2d]) \xrightarrow{\mathrm{R}\Gamma_c(U_C, \mathrm{ev}(d)[2d])} \mathrm{R}\Gamma_c(U_C, \Lambda(d)[2d]) \xrightarrow{\mathrm{tr}_f} \Lambda$$

is perfect (in the derived sense), where tr_f *is the smooth trace from Theorem 6.1.1.*

Note that for $\mathbf{Z}_p$-local systems and discretely valued K, a rational duality statement was obtained in [LLZ23, Th. 1.3].

Proof. First, we note that the first conclusion of Theorem 7.5.22 implies that both $\mathrm{R}\Gamma_c(U_C, \mathbf{L})$ and $\mathrm{R}\Gamma(U_C, \mathbf{L}^\vee(d)[2d])$ lie in $D^b_{\mathrm{coh}}(\Lambda)$. Therefore, it suffices to show that the natural morphism

$$\mathrm{R}\Gamma(U_C, \mathbf{L}^\vee(d)[2d]) \to \mathrm{RHom}_\Lambda(\mathrm{R}\Gamma_c(U_C, \mathbf{L}), \Lambda) \tag{7.5.27}$$

is an isomorphism. This follows directly from Theorem 7.5.22. □

A. Universal compactifications

Theorem A.0.1 ([Hub96, Th. 5.1.5 & Cor. 5.1.6]). *Let X be a separated, $+$-weakly finite type adic space over $\mathrm{Spa}\,(C, \mathcal{O}_C)$. Then there exists a proper adic space X^c over $\mathrm{Spa}\,(C, \mathcal{O}_C)$ and an open embedding $j\colon X \hookrightarrow X^c$ with the following universal property: if Y is a proper adic space over $\mathrm{Spa}\,(C, \mathcal{O}_C)$ and $j'\colon X \hookrightarrow Y$ an open embedding, then there exists a unique morphism $f\colon X^c \to Y$ such that $j' = f \circ j$. Moreover every point of X^c is a specialization of a point of $j(X)$, and $\mathcal{O}_{X^c} \to j_*(\mathcal{O}_X)$ is an isomorphism of sheaves of topological rings.*

Definition A.0.2. We call an embedding $X \hookrightarrow X^c$ satisfying the conclusion of Theorem A.0.1 the *universal compactification* of X.

In the affinoid case, Theorem A.0.1 specializes to the following lemma. While the statement is implicit in the proof of [Hub96, Th. 5.1.5], we recall the argument here for the convenience of the reader.

Lemma A.0.3. *Let C be an algebraically closed nonarchimedean field, and $f\colon \mathrm{Spa}\,(A, A^+) \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ be a finite type morphism. Then its universal compactification is equal to*

$$\begin{tikzcd} \mathrm{Spa}\,(A, A^+) \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathrm{Spa}\,(A, \mathcal{O}_C[A^{\circ\circ}]^+) \arrow[dl, "f^c"] \\ & \mathrm{Spa}\,(C, \mathcal{O}_C), & \end{tikzcd}$$

where $\mathcal{O}_C[A^{\circ\circ}]^+$ denotes the integral closure of the smallest subring of A that contains $\mathcal{O}_C$ and $A^{\circ\circ}$.

Proof. Set $X := \mathrm{Spa}\,(A, A^+)$ and $X^c := \mathrm{Spa}\,(A, \mathcal{O}_C[A^{\circ\circ}]^+)$. Let $j\colon X \to X^c$ be the map induced by the identity map on A. Since (A, A^+) is of topologically finite type, hence $+$-weakly finite type (in the sense of [Hub96, Def. 1.2.1]) over $(C, \mathcal{O}_C)$, there exists a finite set $E \subset A^+$ such that A^+ is the smallest integrally closed subring of A which contains $\mathcal{O}_C[A^{\circ\circ}]^+$ and E. Therefore, j can be identified with the inclusion of the rational subset $\{|E| \leq 1\}$ and is in particular an open embedding.

To show that $f^c\colon X^c \to \mathrm{Spa}\,(C, \mathcal{O}_C)$ is proper, we first note that f^c is of $+$-weakly finite type and X^c is spectral, hence quasicompact and quasiseparated. Thus, we can invoke the valuative criterion for properness [Hub96, Lem. 1.3.10]: for any nonarchimedean field K over C and any open and bounded valuation subring $\mathcal{O}_C \subset K^+ \subset K$, every diagram

$$\begin{tikzcd} \mathrm{Spa}\,(K, \mathcal{O}_K) \arrow[r] \arrow[d] & X^c = \mathrm{Spa}\,(A, \mathcal{O}_C[A^{\circ\circ}]^+) \arrow[d, "f^c"] \\ \mathrm{Spa}\,(K, K^+) \arrow[r] \arrow[ur, dashed] & \mathrm{Spa}\,(C, \mathcal{O}_C) \end{tikzcd}$$

admits a lift as indicated by the dashed arrow because the map $A \to K$ which determines the top row is continuous and thus sends $A^{\circ\circ}$ into $K^{\circ\circ} \subset K^+$.

Lastly, every point of $x \in X^c$ has a generization $y \in X^c$ corresponding to a valuation of rank 1 [Hub96, Lem. 1.1.10.ii)]. Then necessarily $y(a) \leq 1$ for all $a \in A^\circ$, so $y \in X$. Since the rational structure sheaf does not depend on the ring of integral elements, the natural map $\mathcal{O}_{X^c} \to j_*\mathcal{O}_X$ is an isomorphism of sheaves of topological rings. Therefore, X^c is a universal compactification of X by [Hub96, Lem. 5.1.7]. $\square$

Lemma A.0.4. *Let X be a separated taut C-rigid space, then any $x \in |X^c| \smallsetminus |X|$ has rank > 1.*

Proof. According to Theorem A.0.1, every point in X^c is a specialization of a point in X, in particular it admits generalization, hence of rank > 1. $\square$

B. Pseudo-adic spaces

One of the major subtleties while working with adic spaces is that a (locally) closed subset of an analytic adic spaces is rarely an adic space itself. In fact, a higher rank closed point $x \in X$ of an analytic adic space X *never* admits a structure of an adic space.

In order to circumvent this issue, Huber has defined the notion of a pseudo-adic space and its étale topos. This theory has not been widely used beyond his book [Hub96], where this notion does play a crucial role to make many arguments work. This theory is also quite crucial for the results of this paper[41], so we have decided to recall the main definitions and constructions from this theory in the Appendix.

B.1. **Basic definitions.**

Definition B.1.1. A *(strongly) pseudo-adic space* X is a pair $X = (\underline{X}, |X|)$ consisting of an adic space $\underline{X}$ and a closed subset $|X| \subset \underline{X}$. A morphism of (strongly) pseudo-adic spaces $f\colon X = (\underline{X}, |X|) \to Y = (\underline{Y}, |Y|)$ is a morphism of adic spaces $f\colon \underline{X} \to \underline{Y}$ such that $f(|X|) \subset |Y|$.

Remark B.1.2. R. Huber defines a more general notion of a pseudo-adic space in [Hub96, Def. 1.10.3]. This level generality is convenient to set up foundations. However, we will never need this level of generality in this paper, so we do not discuss it. We include the word "strongly" to emphasize that our definition of pseudo-adic spaces is stronger than the definition given by Huber.

Lemma B.1.3. *Let $X = (\underline{X}, |X|)$ be a (strongly) pseudo-adic space. Then X is a pseudo-adic space in the sense of* [Hub96, Def. 1.10.3].

Proof. We need to show that $|X| \subset \underline{X}$ is locally pro-constructible and convex (see [Hub96, (1.1.3)]). We note that closed subspaces of locally spectral spaces are closed under specialization, so convexity of $|X|$ inside $\underline{X}$ is clear. Now we show a stronger claim that any closed subset of locally spectral space is pro-constructible. This is a local statement, so we can assume that X is spectral, then this is [Wed19, Prop. 3.23(i)]. ☐

We give two examples of pseudo-adic spaces that will be important for this paper:

Example B.1.4.
(1) (Closed points) Let $x \in X$ be a closed point of an adic space X. Then $(X, \{x\})$ is a (strongly) pseudo-adic space. We will usually denote it simply by $\{x\}$.
(2) (Closed pro-special subsets) Let $X = \mathrm{Spa}\,(A, A^+)$ be an affinoid adic space (with possibly non-complete (A, A^+)) and $\{f_i\}_{i\in I}$ a set of functions in A^+. Then
$$X\,(|f_i| < 1) = \{x \in X \mid |f_i(x)| < 1 \text{ for } i \in I\}$$
is a closed subspace of X. So $(X, X\,(|f_i| < 1))$ is a (strongly) pseudo-adic space.

Remark B.1.5. Let (k, k^+) be an affinoid field and $s \in \mathrm{Spa}\,(k, k^+)$ a closed point. Then $\{s\} \in \mathrm{Spa}\,(k, k^+)$ is an example of a closed pro-special subset with the set of function $\{f_i\}_{i\in I}$ equal to the set of elements of the maximal ideal $\mathfrak{m} \subset k^+$.

Now we wish to discuss the notion of an étale topos of a (strongly) pseudo-adic space (X, Z). Let $U \coloneqq X \setminus Z$ be the open complement of Z in X. Then the étale topos $U_{\text{ét}}$ is identified with the slice topos $(X_{\text{ét}})_{/h_U}$ and the natural morphism $U_{\text{ét}} \to X_{\text{ét}}$ is fully faithful. Therefore, $U_{\text{ét}}$ is an open subtopos of $X_{\text{ét}}$ in the sense of [Sta22, Tag 08LX].

Definition B.1.6. The *étale topos* $(X, Z)_{\text{ét}}$ of a (strongly) pseudo-adic space (X, Z) is the closed subtopos of $X_{\text{ét}}$ obtained as the closed complement of an open subtopos $U_{\text{ét}} \subset X_{\text{ét}}$ (see [Sta22, Tag 08LZ] and [Sta22, Tag 08LZ]).

Remark B.1.7. Explicitly, $(X, Z)_{\text{ét}}$ is a full subcategory of $X_{\text{ét}}$ that consists of sheaves $\mathcal{F} \in X_{\text{ét}}$ such that the natural morphism $\mathcal{F} \times h_U \to h_U$ is an isomorphism.

Remark B.1.8. The étale topos of a (strongly) pseudo-adic space is functorial with respect to morphism of (strongly) pseudo-adic spaces.

Remark B.1.9. Let $j\colon U \to X$ be an open immersion with the complement $i\colon Z \to X$. The étale topos $(X, Z)_{\text{ét}}$ comes with a morphism of topoi $i\colon (X, Z)_{\text{ét}} \to X_{\text{ét}}$ by construction. It is essentially formal[42] that the sequence
$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$
is exact for any sheaf of abelian groups $\mathcal{F} \in X_{\text{ét}}$.

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[41] Though, we need only a mild part of it.

[42] For example, it can be deduced from [SGA72, Exp. 4, Prop. 9.4.1].

Lemma B.1.10. *Let (X, Z) be a (strongly) pseudo-adic space. Then $(X, Z)_{\text{ét}}$ defined in Definition B.1.6 coincides with the definition of the étale topos of a pseudo-adic space from* [Hub96, Def. 2.3.1].

Proof. For the purpose of this proof, we denote by $(X, Z)^{\mathrm{H}}_{\text{ét}}$ the topos defined in [Hub96, Def. 2.3.1]. Then [Hub96, Rmk. 2.3.4(i)] gives a particularly nice site $(X, Z)_{\text{ét.w}}$ defining $(X, Z)^{\mathrm{H}}_{\text{ét}}$:

(1) the underlying category of $(X, Z)^{\mathrm{H}}_{\text{ét}}$ is the category $\text{Ét}/X$ of adic spaces étale over X;
(2) a family $\{f_i : Y_i \to Y\}_{i \in I}$ of morphisms in $\text{Ét}/X$ is a covering if $h^{-1}(S) \subset \cup_{i \in I} f_i(Y_i)$ where $h: Y \to X$ is the structure morphism.

Now the proof of [Sta22, Tag 08LY] implies that the same site defines the closed subtopos $(X, Z)_{\text{ét}}$. This finishes the proof. □

Remark B.1.11. The definition of an étale topos of a (strongly) pseudo-adic space (X, Z) depends on the ambient space X. However, [Hub96, Cor. 2.3.8] shows that it is independent of X in some precise sense.

Lemma B.1.12. *Let $X = (\underline{X}, |X|)$ be a (strongly) pseudo-adic space, and let $|X| = |X_1| \sqcup |X_2|$ is a disjoint union of two closed subsets X_1 and X_2. Then there is a canonical equivalence*

$$\left(\underline{X}, |X|\right)_{\text{ét}} \simeq \left(\underline{X} \sqcup \underline{X}, |X_1| \sqcup |X_2|\right)_{\text{ét}} \simeq \left(\underline{X}, |X_1|\right)_{\text{ét}} \times \left(\underline{X}, |X_2|\right)_{\text{ét}}.$$

Proof. For the first isomorphism, it suffices to show that the natural morphism of pseudo-adic spaces $\left(\underline{X} \sqcup \underline{X}, |X_1| \sqcup |X_2|\right) \xrightarrow{\mathrm{id} \sqcup \mathrm{id}} \left(\underline{X}, |X|\right)$ induces an equivalence on the associated étale topoi. This follows from [Hub96, Prop. 2.3.7]. The second isomorphism follows from the following sequence of isomorphisms

$$\left(\underline{X} \sqcup \underline{X}, |X_1| \sqcup |X_2|\right)_{\text{ét}} \simeq \Big(\left(\underline{X}, |X_1|\right) \sqcup \left(\underline{X}, |X_2|\right)\Big)_{\text{ét}} \simeq \left(\underline{X}, |X_1|\right)_{\text{ét}} \times \left(\underline{X}, |X_2|\right)_{\text{ét}}.$$ □

B.2. **Étale topos of a closed point.** The main goal of this subsection is to give an explicit characterization of the étale topos of a pseudo-adic space (X, x) for $x \in X$ a closed point. For the rest of the subsection, we fix a locally noetherian analytic adic space X.

Let $x \in X$ be a closed point of an analytic locally noetherian adic space X; we wish to understand the cohomology groups of the pseudo-adic space $\{x\} = (X, x)$. For this, we define (K_x, K_x^+) to be either $\left(k(x)^{\mathrm{h}}, k(x)^{+,\mathrm{h}}\right)$ or $\left(\widehat{k(x)}^{\mathrm{h}}, \widehat{k(x)}^{+,\mathrm{h}}\right)$ (see Definition 2.1.8), and s to be the unique closed point of $\operatorname{Spa}(K_x, K_x^+)$. Then the morphism of pseudo-adic space $(\operatorname{Spa}(K_x, K_x^+), s) \to (X, x)$ induces a morphism of topoi

$$b\colon \left(\operatorname{Spa}(K_x, K_x^+), \{s\}\right)_{\text{ét}} \to (X, x)_{\text{ét}}.$$

The universal property of affine scheme (see [Sta22, Tag 01I1]) gives us a canonical morphism $\operatorname{Spa}(K_x, K_x^+) \to \operatorname{Spec} K_x^+$ that can be easily extended to a morphism of étale topoi

$$a\colon \left(\operatorname{Spa}(K_x, K_x^+), \{s\}\right)_{\text{ét}} \to (\operatorname{Spec} K_x^+)_{\text{ét}}$$

Theorem B.2.1. *In the notation as above, both a and b are equivalences of topoi. In particular,*

$$\gamma = a \circ b^{-1}\colon (X, x)_{\text{ét}} \to (\operatorname{Spec} K_x)_{\text{ét}}$$

is an equivalence of topoi. In particular, there are canonical isomorphisms

$$\mathrm{R}\Gamma\left(\{x\}, \mu_n\right) \simeq \mathrm{R}\Gamma\left(\operatorname{Spec} k(x)^{\mathrm{h}}, \mu_n\right) \simeq \mathrm{R}\Gamma\left(\operatorname{Spec} \widehat{k(x)}^{\mathrm{h}}, \mu_n\right).$$

for every integer n invertible in $\mathcal{O}_X$.

Proof. The first part follows from the proof of [Hub96, Prop. 2.3.10]. The second part is a formal consequence of the first part. □

Warning B.2.2. The result of Theorem B.2.1 is *false* if we put $K_x = k(x)$ or $K_x = \widehat{k(x)}$. The (implicit) henselian assumption on K_x is essential for the proof.

B.3. **Cohomology of closed pro-special subsets.** The main goal of this subsection is to understand cohomology groups of closed pro-special pseudo-adic spaces (see Example B.1.4). Unlike the case of a closed point, we will not be able to describe the whole étale topos of this pseudo-adic space.

In what follows, we fix a (possibly non-complete) Tate–Huber pair (A, A^+) with a pseudo-uniformizer $\varpi \in A^+$ and a set of elements $\{f_i \in A^+\}_{i \in I}$. We define $X \coloneqq \mathrm{Spa}\,(A, A^+)$ and a closed subspace $Z = X\,(|f_i| < 1) \subset X$. Then (X, Z) is a pseudo-adic space.

Definition B.3.1. We define the *henselization of A along Z* to be the ring

$$A(Z) \coloneqq (A^+)^{\mathrm{h}}_I \left[\frac{1}{\varpi}\right],$$

where the henselization is taken with respect to the ideal $I = (f_i, \varpi)_{i \in I} \subset A^+$.

Remark B.3.2. The ring $A(Z)$ is easily seen to be independent of the choice of a pseudo-uniformizer ϖ. A much harder result is that the ring $A(Z)$ is also independent of the choice of generators $\{f_i\}$ and is intrinsic to the pro-special set Z. We refer to [Hub96, Prop and Def. 3.1.12] for a proof of this result.

Example B.3.3. Let (k, k^+) be an affinoid field, and $s \in X = \mathrm{Spa}\,(k, k^+)$ the closed point considered as a closed pro-special subset (see Remark B.1.5). Then $k(\{s\})$ from Definition B.3.1 coincides with the henselized residue field k^{h} in the sense of Definition 2.1.6.

Example B.3.4. If $Z = \varnothing$, we denote $A(\varnothing)$ by A^{h}.

The main result of [Hub96, § 3] says that the *algebraic* cohomology of $\mathrm{Spec}\, A(Z)$ coincide with the *analytic* cohomology of the pseudo-adic space (X, Z).

Theorem B.3.5. *Let (A, A^+) be a strongly noetherian (possibly not complete) Tate–Huber pair, and $Z \subset X = \mathrm{Spa}\,(A, A^+)$ a closed pro-special subset (see Example B.1.4). Then there is a morphism of topoi*

$$\gamma \colon (X, Z)_{\mathrm{\acute{e}t}} \to (\mathrm{Spec}\, A(Z))_{\mathrm{\acute{e}t}}$$

such that

(1) for each n invertible in A, the natural morphism

$$\mathrm{R}\Gamma(\mathrm{Spec}\, A(Z), \mu_n) \to \mathrm{R}\Gamma((X, Z)_{\mathrm{\acute{e}t}}, \mu_n)$$

is an isomorphism;

(2) the morphism γ is functorial in (X, Z);

(3) if $(A, A^+) = (k, k^+)$ is an affinoid field and $Z \subset X$ is a closed point, then γ coincides with the morphism c constructed in Theorem B.2.1;

In particular, one gets a functorial isomorphism $\mathrm{R}\Gamma(\mathrm{Spa}\,(A, A^+), \mu_n) \simeq \mathrm{R}\Gamma(\mathrm{Spec}\, A^{\mathrm{h}}, \mu_n)$ by putting $Z = \varnothing$.

Proof. This is essentially [Hub96, Th. 3.2.9]. Unfortunately, these properties (and the existence of a topos-theoretic morphism γ) is not explicitly stated in [Hub96, Th. 3.2.9], but it does follow from the proof. The reader willing to verify these properties should read [Hub96, Rmk. 3.2.10 and § 3.3 and 3.4]. We especially refer to the proof of [Hub96, Th. 3.3.3] and the discussion on [Hub96, p. 194] for the construction of the morphism γ. □

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