

FORMALITY FOR RIGID-ANALYTIC SPACES SATISFYING THE WEIGHT-MONODROMY CONJECTURE

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ABSTRACT. We prove that étale and de Rham cohomology algebras of a smooth proper rigid-analytic space over a finite extension of $\mathbf{Q}_p$ are formal if the rigid-analytic space satisfies the weight-monodromy conjecture. We give examples of smooth proper rigid-analytic surfaces whose cohomology algebras are not formal.

1. INTRODUCTION

In this paper, we study formality of cohomology of p -adic rigid-analytic spaces. Recall that complex Hodge theory implies the following two special properties of the cohomology of compact Kähler manifolds:

Theorem 1.1 ([DGMS75]). *Let X be a compact Kähler manifold.*

- (1) *Then the Hodge-to-de Rham spectral sequence $E_1^{i,j} = H^j(X, \Omega_X^i) \Rightarrow H^{i+j}(X, \mathbf{C})$ degenerates on the first page, and*
- (2) *the E_∞ -algebra $R\Gamma_{\mathrm{dR}}(X/\mathbf{C})$ is formal¹, i.e. it is quasi-isomorphic to the E_∞ -algebra $\bigoplus_{i \geq 0} H_{\mathrm{dR}}^i(X/\mathbf{C})[-i]$.*

We note that [Sul77, Theorem 12.1] and [HS79, Corollary 6.9] imply that the E_∞ -algebra $R\Gamma_{\mathrm{sing}}(X, \mathbf{Q})$ is also formal for a compact Kähler manifold X . The significance of this property is that for a simply connected Kähler manifold X its rational homotopy type is therefore recovered from the cohomology ring $H_{\mathrm{sing}}^*(X, \mathbf{Q})$ via Quillen’s rational homotopy theory.

For a prime number p , Hodge-to-de Rham degeneration holds for *all* smooth proper rigid-analytic spaces over $\mathbf{C}_p := \widehat{\overline{\mathbf{Q}}_p}$, cf. [Sch18, Theorem 2.1]. This is remarkable in that the degeneration fails for arbitrary compact complex manifolds, so in that respect arbitrary smooth proper rigid-analytic spaces behave as Kähler manifolds.

The main result of this paper is that formality for cohomology of a smooth proper rigid-analytic space defined over a finite extension of $\mathbf{Q}_p$ holds, provided that it satisfies the weight-monodromy conjecture:

Theorem 1.2 (Theorem 5.1 and Lemma 5.2). *Let p, ℓ be prime numbers, possibly equal, let K be a finite extension of $\mathbf{Q}_p$, and let X be a smooth proper rigid-analytic space over K . Assume that the weight-monodromy conjecture holds for $H_{\mathrm{ét}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ for all $i \geq 0$. Then the E_∞ -algebra $R\Gamma_{\mathrm{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ is formal. For $\ell = p$, the de Rham cohomology E_∞ -algebra $R\Gamma_{\mathrm{dR}}(X/K)$ is also formal.*

We refer to Definition 5.5 for the precise meaning of the phrase “the weight-monodromy conjecture holds for $H_{\mathrm{ét}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ ”. We also note that the example of a Hopf surface [Sch13b,

¹We note that the notion of E_∞ -algebras over a field of characteristic 0 is equivalent to the notion of commutative dg-algebras.

p. 16] shows that the weight-monodromy conjecture is false for a general smooth proper rigid-analytic space. Moreover, we find examples of smooth proper rigid-analytic spaces whose cohomology algebras are not formal:

Theorem 1.3 (Proposition 3.2). *There exists a smooth proper rigid-analytic surface X over $\mathbf{Q}_p$ such that the E_∞ -algebra $\mathrm{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ is not formal for every prime ℓ (including $\ell = p$). Furthermore, the E_∞ -algebra $\mathrm{R}\Gamma_{\text{dR}}(X/\mathbf{Q}_p)$ is also not formal.*

A simpler special case of Theorem 1.2 consists of smooth proper rigid-analytic spaces with good reduction. They satisfy the weight-monodromy conjecture by Weil conjectures, and their ℓ -adic cohomology algebras are formal as a consequence of the fact that Frobenius eigenvalues on different cohomology groups are distinct, cf. [Del80, Corollaire 5.3.7].

Theorem 1.2 applies to algebraic varieties as well, though it is only of academic interest in this case: even if the variety in question has bad reduction at p , we can spread it out and reduce modulo another prime of good reduction to deduce formality of cohomology from the Weil conjectures. This spreading out technique is notably unavailable for rigid-analytic spaces.

As an important technical step towards the proof of Theorem 1.2 for $\ell = p$, we give a description of the derived category of de Rham representations that mimics the definition of the abelian category of de Rham representations. We denote by $\mathrm{Rep}_{\mathbf{Q}_p} G_K$ the category of continuous finite-dimensional $\mathbf{Q}_p$ -linear representations of the absolute Galois group G_K , and by $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K$ its full subcategory of de Rham representations. We write $\mathcal{D}(\mathrm{Rep}_{\mathbf{Q}_p} G_K)$ and $\mathcal{D}(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$ for their respective derived ∞ -categories. Finally, $\mathcal{D}(G_K; \mathrm{B}_{\mathrm{dR}}^+)$ denotes an appropriate version of the derived ∞ -category of condensed $\mathrm{B}_{\mathrm{dR}}^+$ -semilinear G_K -representations (see Notation 4.5 for a precise definition).

Theorem 1.4 (Theorem 4.12). *Let p be a prime number, let K be a finite extension of $\mathbf{Q}_p$, and let G_K be its absolute Galois group. Then there is an equivalence of symmetric monoidal ∞ -categories*

$$\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K) \simeq \mathcal{D}_{\mathrm{fil}}(K) \times_{\mathcal{D}(G_K; \mathrm{B}_{\mathrm{dR}}^+)} \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K),$$

where $\mathcal{D}_{\mathrm{fil}}(K)$ is the filtered derived ∞ -category of K -vector spaces.

We refer to Theorem 4.12 for the precise formulation of this theorem, including the construction of all functors involved in the theorem. Theorem 1.4, combined with the de Rham comparison theorem, allows us to lift $\mathrm{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ to an E_∞ -algebra in $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$, cf. Proposition 4.17. We expect that one can alternatively do this by combining the results of [DN18] with the semi-stable comparison theorem of [CN20], see Remark 4.18.

Idea of the proof. To explain the idea of the proof of Theorem 1.2, recall the following point of view on the formality theorem over $\mathbf{C}$ provided by the theory of mixed Hodge complexes. For an algebraic variety X over $\mathbf{C}$, [Bei86, 4.1] naturally promotes $\mathrm{R}\Gamma(X(\mathbf{C}), \mathbf{Q})$ to an object of the derived category $\mathcal{D}^b(\mathrm{MHS}_{\mathbf{Q}})$ of the abelian category of mixed Hodge structures. Applying Deligne's fundamental observation that the weight filtration on an $\mathbf{R}$ -mixed Hodge structure comes with a natural splitting, we obtain a decomposition of the underlying complex $\mathrm{R}\Gamma(X(\mathbf{C}), \mathbf{R}) \in \mathcal{D}(\mathrm{Vect}_{\mathbf{R}})$ into weight pieces, which is compatible with the algebra structure. When X is smooth proper, $H^i(X(\mathbf{C}), \mathbf{R})$ is pure of weight i , hence this decomposition is exactly a formality isomorphism.

To prove Theorem 1.2, we show that for $\ell \neq p$ the ℓ -adic cohomology E_∞ -algebra $\mathrm{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ lifts to an algebra in the derived category $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_\ell}^{\mathrm{mix}} \mathrm{WD}_K)$ of the abelian

category of mixed Weil–Deligne representations, whenever X is a smooth proper rigid-analytic space satisfying the weight-monodromy conjecture. This is shown as a consequence of an abstract fact about the derived category of Weil–Deligne representations, provided by Theorem 5.13.

Formality for $\ell \neq p$ then follows as a consequence of the fact that the weight filtration on a mixed Weil–Deligne representation is naturally split.

To prove formality of $\mathbf{Q}_p$ -étale cohomology, we apply Fontaine’s functor D_{pst} to the complex $\text{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$, to be able to use the same general result about Weil–Deligne representations. We are in a position to apply the functor D_{pst} to the étale cohomology complex, because we have upgraded it to an object of the derived category $\mathcal{D}^b(\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K)$ of the abelian category of de Rham Galois representations, using Theorem 1.4, as described above.

Examples in Theorem 1.3 showing that formality does not always hold for rigid-analytic spaces are constructed by adapting a familiar construction from complex geometry. One of the best-known examples of a compact complex (necessarily non-Kähler) manifold whose cohomology is not formal is the Iwasawa manifold [DGMS75, p. 261], but its construction does not have a p -adic analog.

Instead, we use that primary Kodaira surfaces do have p -adic analogs. We show that their étale cohomology algebras are equivalent to those of their complex counterparts. In particular, they are not formal because the cohomology algebra contains the cohomology of the Heisenberg group as a tensor factor.

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2. RECOLLECTIONS ON FORMALITY

For a commutative ring R , the category of E_∞ -algebras over R is, by definition, the category of commutative algebra objects in the symmetric monoidal derived ∞ -category $D(R)$ of complexes of R -modules. Given any Grothendieck site $\mathcal{C}$ with a sheaf $\mathcal{O}$ of R -algebras, the cohomology complex $\text{R}\Gamma(\mathcal{C}, \mathcal{O}) \in D(R)$ can be naturally enhanced with an E_∞ -algebra structure, using the lax symmetric monoidal structure on the functor $\text{R}\Gamma(\mathcal{C}, -)$.

When R is a $\mathbf{Q}$ -algebra (as is always the case in this paper), the category of E_∞ -algebras over R is equivalent, via the Dold–Kan correspondence, to the ∞ -category of commutative differential graded R -algebras localized at quasi-isomorphisms; see [Lur17, Proposition 7.1.4.11]. For example, the fact that the singular cohomology complex $\text{R}\Gamma_{\text{sing}}(X, \mathbf{Q})$ of any topological space X can be naturally represented by a commutative differential graded $\mathbf{Q}$ -algebra goes back to Sullivan, cf. [DGMS75, §2].

We say that an E_∞ -algebra A is formal if it is equivalent to $\bigoplus_{i \in \mathbf{Z}} H^i(A)[-i]$ where the latter graded commutative algebra is viewed as an E_∞ -algebra. If R is a $\mathbf{Q}$ -algebra and A is represented by a commutative differential graded algebra $A^\bullet = \dots \xrightarrow{d} A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} \dots$ then formality is equivalent to the existence of another commutative differential graded algebra $B^\bullet$

admitting maps $B^\bullet \rightarrow A^\bullet$ and $B^\bullet \rightarrow (H^\bullet(A), 0)$ that induce isomorphisms on cohomology, as one shows using semi-free resolutions [Dri04, 13.4].

3. EXAMPLE

Let K be a finite extension of $\mathbf{Q}_p$, and denote by $\mathbf{C}_p = \widehat{K}$ the completion of its algebraic closure. We choose any elliptic curve E over K and a line bundle L of degree 1 on E . Consider its total space $Y := (\text{Tot}_E L) \setminus E$ with the zero section removed. Note that this is an example of a smooth non-proper algebraic variety whose cohomology is not formal, cf. [Mor78, Example on p. 203]. We will now turn it into a proper rigid-analytic space by taking an appropriate quotient. This smooth variety is equipped with an action of $\mathbf{G}_m$ given by dilation along the fibers. In particular, Y admits an action of $\mathbf{Z}$ via a homomorphism $\mathbf{Z} \rightarrow \mathbf{G}_m$ sending 1 to p .

Construction 3.1. We denote by $X := Y^{\text{an}}/p^{\mathbf{Z}}$ the rigid-analytic space over K obtained as the quotient of the analytification of Y by the action of $\mathbf{Z}$ described above.

By construction, $X \rightarrow E^{\text{an}}$ is a (non-trivial, as we will see) analytic torsor for the constant rigid-analytic group $\mathbf{G}_m^{\text{an}}/p^{\mathbf{Z}}$. The latter group is a Tate elliptic curve, see [BGR84, Theorem 9.7.3/3]². In particular, X is a smooth proper (geometrically connected) rigid-analytic surface over K . This construction goes back at least to [Uen87, §6 a)], see also [Wil11, 3.4] for a slightly different presentation of the same surface.

The main result of this section is that cohomology algebras of X are not formal:

Proposition 3.2. *For all primes ℓ (including $\ell = p$) the E_∞ -algebra $\text{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ is not equivalent to $\bigoplus_{i \geq 0} H_{\text{ét}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)[-i]$. Also, the de Rham cohomology E_∞ -algebra $\text{R}\Gamma_{\text{dR}}(X/K)$ is not equivalent to $\bigoplus_{i \geq 0} H_{\text{dR}}^i(X/K)[-i]$.*

We will reduce the computation of étale cohomology of this rigid-analytic surface to the computation of the cohomology of its classical complex-analytic analog.

Choose an embedding $K \hookrightarrow \mathbf{C}$ and denote by $Y_{\mathbf{C}}^{\text{an}}$ the complex-analytic space associated to $Y_{\mathbf{C}}$, and let $X_{\mathbf{C}}$ be the complex manifold obtained as the quotient of $Y_{\mathbf{C}}^{\text{an}}$ by the same action of $p^{\mathbf{Z}}$ as above. This compact complex surface is known as a *primary Kodaira surface*, cf. [BHPVdV04, V.5.B1b)].

Lemma 3.3. *There is an equivalence of E_∞ -algebras $\text{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell) \simeq \text{R}\Gamma_{\text{sing}}(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}_\ell)$.*

Proof. As mentioned above, the action of the group $\mathbf{Z}$ on the algebraic variety Y induces actions on the rigid-analytic space $Y_{\mathbf{C}_p}^{\text{an}}$ and the complex analytic space $Y_{\mathbf{C}}^{\text{an}}$. The spaces $X_{\mathbf{C}_p}$ and $X_{\mathbf{C}}$ are the quotients of $Y_{\mathbf{C}_p}^{\text{an}}$ and $Y_{\mathbf{C}}^{\text{an}}$ in the respective categories. In particular, we have equivalences of E_∞ -algebras

$$(3.1) \quad \begin{aligned} \text{R}\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell) &\simeq \text{R}\Gamma(\mathbf{Z}, \text{R}\Gamma_{\text{ét}}(Y_{\mathbf{C}_p}^{\text{an}}, \mathbf{Q}_\ell)) \\ &\quad \text{R}\Gamma_{\text{sing}}(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}_\ell) \simeq \text{R}\Gamma(\mathbf{Z}, \text{R}\Gamma_{\text{sing}}(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Q}_\ell)) \end{aligned}$$

where the outer $\text{R}\Gamma$ on the right-hand sides refers to the cohomology of the discrete group $\mathbf{Z}$.

Combining the comparison between rigid-analytic and algebraic étale cohomology (see [Hub96, Theorem 3.8.1]), the Artin comparison between algebraic étale and singular cohomology (see [AGV72, Exposé XVI, Théorème 4.1]), and the independence of étale cohomology

²Strictly speaking, [BGR84, Theorem 9.7.3/3] assumes that $p \neq 2$. However, this is not needed to justify that $\mathbf{G}_m^{\text{an}}/p^{\mathbf{Z}}$ is an elliptic curve.

of the algebraically closed ground field (see [AGV72, Expose XVI, Corollaire 1.6]), we get an equivalence of E_∞ -algebras in $\mathcal{D}(\text{Rep}_{\mathbf{Q}_\ell} \mathbf{Z})$

$$\text{R}\Gamma_{\text{ét}}(Y_{\mathbf{C}_p}^{\text{an}}, \mathbf{Q}_\ell) \simeq \text{R}\Gamma_{\text{ét}}(Y_{\overline{K}}, \mathbf{Q}_\ell) \simeq \text{R}\Gamma_{\text{sing}}(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Q}_\ell).$$

Applying the functor $\text{R}\Gamma(\mathbf{Z}, -)$ to this equivalence and using the equivalences (3.1) we get the result. $\square$

Denote by Γ the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbf{Z} \right\} \subset \text{GL}_3(\mathbf{Z}).$$

It can be characterized as the unique group that can be established as a central extension $1 \rightarrow \mathbf{Z} \rightarrow \Gamma \xrightarrow{\alpha} \mathbf{Z}^2 \rightarrow 1$ of $\mathbf{Z}^2$ by $\mathbf{Z}$ such that α induces an isomorphism between the abelianization Γ^{ab} and $\mathbf{Z}^2$.³

The following description of the topology of Kodaira surfaces is classical, see [Has05, (4) on p. 755] or [Bor84, p. 374], but let us include a proof for completeness.

Lemma 3.4. *The complex surface $X_{\mathbf{C}}(\mathbf{C})$ is homotopy equivalent to the classifying space of the group $\Gamma \times \mathbf{Z}$. In particular, the E_∞ -algebra $\text{R}\Gamma_{\text{sing}}(X_{\mathbf{C}}(\mathbf{C}), \mathbf{Q}_\ell)$ is equivalent to the group cohomology algebra $\text{R}\Gamma(\Gamma \times \mathbf{Z}, \mathbf{Q}_\ell)$.*

In the proof below, we will ignore the choice of a base point in the definition of homotopy groups π_i . Since all spaces considered are connected, this should not cause any confusion.

Proof. We will prove that $X_{\mathbf{C}}(\mathbf{C})$ is a $K(\pi, 1)$ -space whose fundamental group is isomorphic to $\mathbf{Z} \times \Gamma$, which is equivalent to the assertion of the lemma.

Step 1. $Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ is homotopy equivalent to $K(\Gamma, 1)$. First, we note that the natural map $f: Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}) \rightarrow E_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ is a fibration with fibers isomorphic to $\mathbf{C}^\times$. Since both $E_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ and $\mathbf{C}^\times$ are $K(\pi, 1)$ -spaces, we conclude that $Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ is also a $K(\pi, 1)$ -space. So we only need to show that $\pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}))$ is isomorphic to the Heisenberg group Γ .

For this, we consider the homotopy exact sequence for the fibration mentioned above (which is exact on the left because $\pi_2(E_{\mathbf{C}}^{\text{an}}(\mathbf{C})) = 0$):

$$1 \rightarrow \mathbf{Z} \rightarrow \pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \rightarrow \pi_1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \rightarrow 0.$$

We claim that this extension is central. To see this, it suffices to show that the action of $\pi_1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}))$ on $\mathbf{Z} \simeq \pi_1(\mathbf{C}^\times) \simeq H_1(\mathbf{C}^\times, \mathbf{Z}) \simeq H_c^1(\mathbf{C}^\times, \mathbf{Z})$ is trivial. This follows from the fact that the local system $R^1 f_! \mathbf{Z}$ is isomorphic to the constant local system $\mathbf{Z}$.⁴

Therefore, we conclude that $\pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}))$ fits into a central extension $1 \rightarrow \mathbf{Z} \rightarrow \pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \rightarrow \mathbf{Z}^2 \rightarrow 1$. We are only left to show that the induced map

$$\pi_1^{\text{ab}}(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \rightarrow \pi_1^{\text{ab}}(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}))$$

is an isomorphism. The Hurewicz theorem implies that this is equivalent to showing that the map $H_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \rightarrow H_1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z})$ is an isomorphism. Using the universal coefficient theorem, we see that it suffices to show that the natural map $H^1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \rightarrow H^1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z})$

³ $H^2(\mathbf{Z}^2, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$, under this isomorphism the class of a central extension $0 \rightarrow \mathbf{Z} \rightarrow G \xrightarrow{\beta} \mathbf{Z}^2 \rightarrow 0$ corresponds to the integer $[\tilde{x}, \tilde{y}]$ for any $\tilde{x}, \tilde{y} \in G$ lifting the chosen basis elements $x, y \in \mathbf{Z}^2$. The map β induces an isomorphism $G^{\text{ab}} \simeq \mathbf{Z}^2$ if and only if this integer is ± 1 , which is the case for $G = \Gamma$.

⁴To see this claim, note that f admits a compactification by a $\mathbf{P}^1$ -bundle $\bar{f}: \mathbf{P}_E^{\text{an}}(L \oplus \mathcal{O})(\mathbf{C}) \rightarrow E_{\mathbf{C}}^{\text{an}}(\mathbf{C})$. Using this, one sees that $R^1 f_! \mathbf{Z} \simeq \text{coker}(\bar{f}_* \mathbf{Z} = \mathbf{Z} \rightarrow \mathbf{Z}^2) \simeq \mathbf{Z}$.

is an isomorphism and both $H^2(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z})$ and $H^2(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z})$ are torsion-free. Clearly, $H^2(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \simeq \mathbf{Z}$ is torsion-free.

For other claims, we consider the Leray spectral sequence

$$E_2^{p,q} = H^p(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), R^q f_* \mathbf{Z}) \Rightarrow H_{\text{sing}}^{p+q}(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}).$$

Now we use that $f_* \mathbf{Z} \simeq \mathbf{Z}$, $R^1 f_* \mathbf{Z} \simeq \mathbf{Z}$, and $R^i f_* \mathbf{Z} = 0$ for $i \geq 2$.⁵ Hence, the only potentially non-trivial differential $H^0(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), R^1 f_* \mathbf{Z}) \rightarrow H^2(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), f_* \mathbf{Z})$ is given, up to a sign, by sending 1 to the first Chern class $c_1(L)$ of the line bundle L used to define Y . Since $\deg L = 1$, we conclude that this map is an isomorphism. This formally implies that $H^1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \rightarrow H^1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z})$ is an isomorphism and that

$$H^2(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \simeq H^1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), R^1 f_* \mathbf{Z}) \simeq H^1(E_{\mathbf{C}}^{\text{an}}(\mathbf{C}), \mathbf{Z}) \simeq \mathbf{Z}^2.$$

This finishes the proof that $Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ is homotopy equivalent to $K(\Gamma, 1)$.

Step 2. $X_{\mathbf{C}}(\mathbf{C})$ is homotopy equivalent to $K(\Gamma \times \mathbf{Z}, 1)$. Since $X_{\mathbf{C}}(\mathbf{C})$ is the quotient of $Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ by a free action of $\mathbf{Z}$, we conclude that $X_{\mathbf{C}}(\mathbf{C})$ is a $K(\pi, 1)$ -space and its fundamental group fits into the following short exact sequence:

$$1 \rightarrow \pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \rightarrow \pi_1(X_{\mathbf{C}}(\mathbf{C})) \xrightarrow{\pi} \mathbf{Z} \rightarrow 1.$$

This defines an outer action $\rho: \mathbf{Z} \rightarrow \text{Out}(\pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})))$. Since π obviously admits a group-theoretic section, it suffices to show that ρ is trivial to conclude that $\pi_1(X_{\mathbf{C}}(\mathbf{C})) \simeq \pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})) \times \mathbf{Z} \simeq \Gamma \times \mathbf{Z}$. This action is induced by the action of $\mathbf{Z}$ on $Y_{\mathbf{C}}^{\text{an}}(\mathbf{C})$ such that 1 acts by fiberwise multiplication by p . This automorphism is homotopic to the identity automorphism, therefore the induced outer action on $\pi_1(Y_{\mathbf{C}}^{\text{an}}(\mathbf{C}))$ is trivial. This finishes the proof. $\square$

Proof of Proposition 3.2. Combining Lemmas 3.3 and 3.4, we get that the étale cohomology $R\Gamma_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_{\ell})$ of our rigid-analytic surface is equivalent, even as an E_{∞} -algebra, to the group cohomology $R\Gamma(\Gamma \times \mathbf{Z}, \mathbf{Q}_{\ell})$. But the latter is non-formal even as an E_1 -algebra by the classical observation that there exists a non-trivial Massey product in the cohomology of the Heisenberg group Γ , cf. [DGMS75, p. 261]. Non-formality of de Rham cohomology now follows by Lemma 5.2 below. $\square$

Remark 3.5. One can also compute the cohomology of $Y_{\mathbf{C}_p}$ via purely p -adic methods. Assume for simplicity that E has split multiplicative reduction, so that its analytification is a Tate curve $\mathbf{G}_{m,K}^{\text{an}}/q^{\mathbf{Z}}$ for some $q \in \mathfrak{m}_K \subset \mathcal{O}_K$. Then $X_{\mathbf{C}_p}$ can be described as a quotient of $(\mathbf{G}_{m,\mathbf{C}_p}^{\text{an}})^2$ by the group $\mathbf{Z}^2$ which allows one to compute its cohomology via a Leray spectral sequence, see [Wil11, 3.4].

4. DERIVED CATEGORIES OF GALOIS REPRESENTATIONS

4.1. Representations. Throughout this subsection, we fix a finite extension K of $\mathbf{Q}_p$ with absolute Galois group G_K and a prime number ℓ (which could be equal to p). The main goal of this subsection is to define the correct derived category of continuous $\mathbf{Q}_{\ell}$ -linear representations of G_K .

⁵To see these isomorphisms, we also use the $\mathbf{P}^1$ -bundle compactification $\bar{f}$ of f and the associated distinguished triangle $\mathbf{Z}^{\oplus 2}[-2] \rightarrow R\bar{f}_* \mathbf{Z} \rightarrow Rf_* \mathbf{Z}$.

For this, we will use the proétale site $BG_{K,\text{proét}}$ as defined in [BS15, §4.3]. Its underlying category is the category⁶ $G_K\text{-pfsets} \simeq \text{Pro}(G_K\text{-fsets})$ of profinite sets with a continuous G_K -action with covers given by continuous surjections.

Any topological space X with a continuous G_K -action gives rise to a proétale sheaf $\underline{X} \in \text{Shv}(BG_{K,\text{proét}}; \text{Set})$ defined via the formula $\underline{X}(S) = \text{Map}_{G_K, \text{cont}}(S, X)$, see [BS15, Lemma 4.3.2]. In particular, topological ring $\mathbf{Q}_\ell$ equipped with a trivial action of G_K defines a sheaf of commutative algebras on $BG_{K,\text{proét}}$ that we denote by $\mathbf{Q}_\ell$ as well. From now on, we denote the abelian category of sheaves of $\mathbf{Q}_\ell$ -modules on $BG_{K,\text{proét}}$ by $\text{Mod}(G_K; \mathbf{Q}_\ell)$ and its derived ∞ -category by $\mathcal{D}(G_K; \mathbf{Q}_\ell)$.

Construction 4.1. There is a fully faithful, exact, symmetric monoidal functor

$$\text{Rep}_{\mathbf{Q}_\ell} G_K \rightarrow \text{Mod}(G_K; \mathbf{Q}_\ell)$$

which sends a continuous finite-dimensional $\mathbf{Q}_\ell$ -linear representation V to the associated sheaf $\underline{V}$ on $BG_{K,\text{proét}}$. In what follows, we will often simply write V in place of $\underline{V}$.

Since this functor is exact, it induces a symmetric monoidal functor $\mathcal{D}(\text{Rep}_{\mathbf{Q}_\ell} G_K) \rightarrow \mathcal{D}(G_K; \mathbf{Q}_\ell)$.

We will now prove that this functor is fully faithful, using the following lemma. Let $\varphi \in G_K$ be any lift of the arithmetic Frobenius element along the map $G_K \twoheadrightarrow G_K/I_K \simeq \hat{\mathbf{Z}}$.

Lemma 4.2. *Let $V, U \in \text{Rep}_{\mathbf{Q}_\ell} G_K$ and let $S = \mathbf{Q}_\ell \cdot e_1 \oplus \mathbf{Q}_\ell \cdot e_2$ be the two-dimensional continuous $\mathbf{Q}_\ell$ -linear G_K -representation such that the inertia I_K acts trivially on S , and φ acts via the $\mathbf{Q}_\ell$ -linear transformation such that $\varphi(e_1) = e_1$ and $\varphi(e_2) = e_1 + e_2$. Let $V \hookrightarrow V \otimes_{\mathbf{Q}_\ell} \text{Sym}^r S$ be a homomorphism of G_K -representations defined via the formula $v \mapsto v \otimes e_1^r$. Then the induced morphism*

$$\text{Hom}_{G_K}(V \otimes_{\mathbf{Q}_\ell} \text{Sym}^r S, U) \rightarrow \text{Hom}_{G_K}(V, U)$$

is zero for any $r \geq \dim_{\mathbf{Q}_\ell} V \cdot \dim_{\mathbf{Q}_\ell} U$.

Proof. Note that $\text{Hom}_{G_K}(V \otimes_{\mathbf{Q}_\ell} \text{Sym}^r S, U) \simeq \text{Hom}_{G_K}(\text{Sym}^r S, V^\vee \otimes_{\mathbf{Q}_\ell} U)$ and $\text{Hom}_{G_K}(V, U) \simeq \text{Hom}_{G_K}(\mathbf{Q}_\ell, V^\vee \otimes_{\mathbf{Q}_\ell} U)$. Therefore, it suffices to show that the inclusion $\mathbf{Q}_\ell \rightarrow \text{Sym}^r S$, defined via the formula $x \mapsto x e_1^r$, induces the zero morphism $\text{Hom}_{G_K}(\text{Sym}^r S, W) \rightarrow \text{Hom}_{G_K}(\mathbf{Q}_\ell, W)$ for any $W \in \text{Rep}_{\mathbf{Q}_\ell} G_K$ such that $\dim_{\mathbf{Q}_\ell} W \leq r$.

In other words, we need to show that for any homomorphism $T: \text{Sym}^r S \rightarrow W$ for W as above, we have $T(e_1^r) = 0$. For this, we note that $(\varphi - 1)(e_1^r) = 0$ and $(\varphi - 1)^r(e_2^r) = r! \cdot e_1^r$ in $\text{Sym}^r S$. Therefore, we conclude that for any homomorphism $T: \text{Sym}^r S \rightarrow W$, we have $T(e_1^r) \in \text{Im}((\varphi_W - 1)^r) \cap \text{Ker}(\varphi_W - 1)$. Now note that this intersection is trivial if $r \geq \dim_{\mathbf{Q}_\ell} W$. This finishes the proof. $\square$

Lemma 4.3. *Let K be a finite extension of $\mathbf{Q}_p$. Then the functor $\mathcal{D}^b(\text{Rep}_{\mathbf{Q}_\ell} G_K) \rightarrow \mathcal{D}(G_K; \mathbf{Q}_\ell)$ is fully faithful. Its essential image is characterized as the subcategory of objects $M \in \mathcal{D}(G_K; \mathbf{Q}_\ell)$ such that each $H^i(M) \in \text{Mod}(G_K; \mathbf{Q}_\ell)$ arises from an object of $\text{Rep}_{\mathbf{Q}_\ell} G_K$ and is zero for almost all $i \in \mathbf{Z}$.*

This was proven for $\ell = p$ in [EK, Proposition 1.3.2]. We give here a uniform argument for all ℓ .

Proof. An inductive argument reduces the question to showing that the natural morphism $\text{Ext}_{\text{Rep}_{\mathbf{Q}_\ell} G_K}^i(V, W) \rightarrow \text{Ext}_{\text{Shv}(BG_{K,\text{proét}}; \mathbf{Q}_\ell)}^i(V, W)$ is an isomorphism for any $V, W \in \text{Rep}_{\mathbf{Q}_\ell} G_K$

⁶This equivalence follows from [RZ10, Lemma 5.6.4].

and $i \geq 0$. Furthermore, we can replace V with $\mathbf{Q}_\ell$ and W with $V^\vee \otimes W$ and use [BS15, Lemma 4.3.9] to reduce the question to showing that $\mathrm{Ext}_{\mathrm{Rep}_{\mathbf{Q}_\ell} G_K}^i(\mathbf{Q}_\ell, W) \rightarrow H_{\mathrm{cont}}^i(G_K, W)$ is an isomorphism.

Since this map is an isomorphism for $i = 0$, [Buc60, Proposition 4.2] implies that it suffices to check that for any $V \in \mathrm{Rep}_{\mathbf{Q}_\ell} G_K$ and $u \in H_{\mathrm{cont}}^i(G_K, V)$, there is an injection $V \hookrightarrow V'$ in $\mathrm{Rep}_{\mathbf{Q}_\ell} G_K$ such that the image of u vanishes in $H^i(G_K, V')$. Since the ℓ -adic cohomological dimension of G_K is 2, we only need to deal with cases $i = 1$ and $i = 2$.

For $i = 1$, [HM03, Theorem A.1] implies that $\mathrm{Ext}^1(\mathbf{Q}_\ell, V) \rightarrow H_{\mathrm{cont}}^1(G_K, V)$ is an isomorphism. Thus, the class $u \in H_{\mathrm{cont}}^1(G_K, V)$ is represented by an extension $0 \rightarrow V \rightarrow W \rightarrow \mathbf{Q}_\ell \rightarrow 0$. Then the injection $V \hookrightarrow W$ annihilates the class $u \in H_{\mathrm{cont}}^1(G_K, V)$.

For $i = 2$, local Tate duality implies that $H_{\mathrm{cont}}^2(G_K, V)^\vee \simeq H_{\mathrm{cont}}^0(G_K, V^\vee(1)) \simeq \mathrm{Hom}_{G_K}(\mathbf{Q}_\ell, V^\vee(1)) \simeq \mathrm{Hom}_{G_K}(V, \mathbf{Q}_\ell(1))$. Therefore, it suffices to show that, for any $V \in \mathrm{Rep}_{\mathbf{Q}_\ell} G_K$, we can find an injection $V \hookrightarrow V'$ such that the natural morphism $\mathrm{Hom}_{G_K}(V', \mathbf{Q}_\ell(1)) \rightarrow \mathrm{Hom}_{G_K}(V, \mathbf{Q}_\ell(1))$ is zero. This is achieved by taking $V' := V \otimes_{\mathrm{Sym}^{\dim_{\mathbf{Q}_\ell} V} S} S$ from Lemma 4.2. $\square$

Remark 4.4. (1) Note that the analog of Lemma 4.3 is false for an arbitrary finitely generated profinite group Γ in place G_K . The category of continuous representations of the group $\Gamma = \mathrm{SL}_2(\mathbf{Z}_p)$ on finite-dimensional $\mathbf{Q}_p$ -vector spaces is semi-simple, but $H_{\mathrm{cont}}^3(\mathrm{SL}_2(\mathbf{Z}_p), \mathbf{Q}_p) \simeq \mathbf{Q}_p$ is non-zero by Lazard's theorem [Laz65, Théorème V.2.4.10].
 (2) We do not know if Lemma 4.3 holds for other fields in place of K . However, a consequence of Beilinson's basic lemma [Bha, Remark 21.5] is that for any finite type scheme X over a field F of characteristic coprime to ℓ there is a natural object of $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_\ell} G_F)$ lifting $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{F}}, \mathbf{Q}_\ell) \in \mathcal{D}(G_F; \mathbf{Q}_\ell)$. It seems natural to expect that the same applies to ℓ -adic cohomology of any smooth proper rigid-analytic space X over a non-archimedean field K of characteristic coprime to ℓ .

4.2. Derived category of de Rham representations. Recall that a finite-dimensional $\mathbf{Q}_p$ -representation V is called *de Rham* if there exists a K -vector space D together with a G_K -equivariant isomorphism

$$V \otimes_{\mathbf{Q}_p} \mathrm{B}_{\mathrm{dR}} \simeq D \otimes_K \mathrm{B}_{\mathrm{dR}}$$

of B_{dR} -vector spaces. Alternatively, we may rephrase this condition by saying that there exists a filtered K -vector space D together with an isomorphism

$$(4.1) \quad V \otimes_{\mathbf{Q}_p} \mathrm{B}_{\mathrm{dR}}^+ \simeq \mathrm{Fil}^0(D \otimes_K \mathrm{B}_{\mathrm{dR}})$$

where Fil^0 on the right-hand side is taken with respect to the tensor product of the given filtration on D with the filtration $\mathrm{Fil}^n \mathrm{B}_{\mathrm{dR}} := t^n \mathrm{B}_{\mathrm{dR}}^+$ on B_{dR} . The filtered vector space D is in fact naturally recovered from the de Rham representation V .

In this subsection, we prove that the verbatim derived analog of the condition (4.1) recovers the correct derived category of de Rham representations, see Theorem 4.12.

Notation 4.5. Topological $\mathbf{Q}_p$ -algebras $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR} equipped with a continuous action of G_K give rise to commutative algebras in the category $\mathrm{Mod}(G_K; \mathbf{Q}_p)$, and we denote by $\mathcal{D}(G_K; \mathrm{B}_{\mathrm{dR}}^+)$ the symmetric monoidal ∞ -category of modules over the corresponding commutative algebra object $\mathrm{B}_{\mathrm{dR}}^+ \in \mathcal{D}(G_K; \mathbf{Q}_p)$. We have a functor $V \mapsto V \otimes_{\mathbf{Q}_p} \mathrm{B}_{\mathrm{dR}}^+ : \mathcal{D}(G_K; \mathbf{Q}_p) \rightarrow \mathcal{D}(G_K; \mathrm{B}_{\mathrm{dR}}^+)$.

For an ∞ -category $\mathcal{C}$, we denote by

$$\mathcal{C}_{\mathrm{fil}} := \mathrm{Fun}((\mathbf{Z}, \leq)^{\mathrm{op}}, \mathcal{C})$$

the corresponding filtered ∞ -category, where our convention is to use decreasing filtrations. If $\mathcal{C}$ is equipped with a symmetric monoidal structure, $\mathcal{C}_{\text{fil}}$ can be endowed with the Day convolution symmetric monoidal structure (see [Lur17, Construction 2.2.6.7]). Explicitly, we have $\text{Fil}^n(X \otimes Y) = \text{colim}_{i+j \geq n} \text{Fil}^i X \otimes \text{Fil}^j Y$ for two objects $X, Y \in \mathcal{C}_{\text{fil}}$. The assignment

$\text{Fil}^\bullet C \mapsto \text{Fil}^0 C$ can be promoted to a lax symmetric monoidal functor $\text{Fil}^0: \mathcal{C}_{\text{fil}} \rightarrow \mathcal{C}$, see [BMS24, Corollary 3.8].

Notation 4.6. For $C \in \mathcal{C}$, we define C with the trivial filtration to be the filtered object

$$\text{Fil}^n C = \begin{cases} C & \text{if } n \leq 0 \\ 0 & \text{if } n > 0. \end{cases}$$

By a slight abuse of notation, we often denote this filtered object simply by C . This can be promoted to a symmetric monoidal functor $\text{triv}: \mathcal{C} \rightarrow \mathcal{C}_{\text{fil}}$ which is a left adjoint to Fil^0 (see [BMS24, Corollary 3.6]).

For brevity, we denote by $\mathcal{D}_{\text{fil}}(K) := \mathcal{D}(\text{Vect}_K)_{\text{fil}}$ the filtered derived ∞ -category of K -vector spaces.

Lemma 4.7. *For an object $D \in \mathcal{D}_{\text{fil}}(K)$ there is a natural equivalence $\text{RHom}_{\mathcal{D}_{\text{fil}}(K)}(K, D) \simeq \text{Fil}^0 D$, where $K \in \mathcal{D}_{\text{fil}}(K)$ is equipped with the trivial filtration from Notation 4.6.*

Proof. This follows immediately from the definition of $\text{triv}: \mathcal{D}(\text{Vect}_K) \rightarrow \mathcal{D}_{\text{fil}}(K)$ as the left adjoint to $\text{Fil}^0: \mathcal{D}_{\text{fil}}(K) \rightarrow \mathcal{D}(\text{Vect}_K)$. $\square$

Construction 4.8. We now formally define the symmetric monoidal functor

$$D \mapsto \text{Fil}^0(D \otimes_K B_{\text{dR}}): \mathcal{D}_{\text{fil}}(K) \rightarrow \mathcal{D}(G_K; B_{\text{dR}}^+)$$

which is a derived analog of the functor appearing on the RHS of (4.1).

The $\mathbf{Z}$ -indexed filtration $\text{Fil}^n B_{\text{dR}} := t^n B_{\text{dR}}^+$ on the commutative algebra object B_{dR} of the abelian symmetric monoidal category $\text{Mod}(G_K; \mathbf{Q}_p)$ makes B_{dR} into a commutative algebra object of the filtered derived ∞ -category $\mathcal{D}_{\text{fil}}(G_K; \mathbf{Q}_p) := \mathcal{D}(G_K; \mathbf{Q}_p)_{\text{fil}}$. We can then consider the composition

$$(4.2) \quad \mathcal{D}_{\text{fil}}(K) \rightarrow \text{Mod}_{K^\delta} \mathcal{D}_{\text{fil}}(G_K; \mathbf{Z}) \xrightarrow{D \mapsto D \otimes_{K^\delta} B_{\text{dR}}} \text{Mod}_{B_{\text{dR}}} \mathcal{D}_{\text{fil}}(G_K; \mathbf{Q}_p) \xrightarrow{\text{Fil}^0} \text{Mod}_{\text{Fil}^0 B_{\text{dR}}} \mathcal{D}(G_K; \mathbf{Q}_p) = \mathcal{D}(G_K; B_{\text{dR}}^+)$$

where the first functor sends an object $D \in \mathcal{D}_{\text{fil}}(K)$ to the corresponding object of $\text{Mod}_{K^\delta} \mathcal{D}_{\text{fil}}(G_K; \mathbf{Z})$, where K^δ denotes K viewed as a commutative algebra object of $\mathcal{D}_{\text{fil}}(G_K; \mathbf{Z})$ endowed with trivial filtration and Galois action, and with *discrete* topology.

The first two functors in (4.2) are symmetric monoidal, and the last functor Fil^0 has a structure of a lax symmetric monoidal functor. We reinterpret B_{dR}^+ -modules as filtered B_{dR} -modules satisfying an additional property, to prove that the composition (4.2) is strictly symmetric monoidal:

Lemma 4.9. *Consider the full ∞ -subcategory $\mathcal{L}$ of $\mathcal{D}_{\text{fil}}(G_K; B_{\text{dR}})$ consisting of filtered objects M such that the natural map $\text{Fil}^0 M \otimes_{\text{Fil}^0 B_{\text{dR}}} \text{Fil}^n B_{\text{dR}} \rightarrow \text{Fil}^n M$ is an equivalence, for every $n \in \mathbf{Z}$. Then the composition $\mathcal{L} \hookrightarrow \mathcal{D}_{\text{fil}}(G_K; B_{\text{dR}}) \xrightarrow{\text{Fil}^0} \mathcal{D}(G_K; B_{\text{dR}}^+)$ is an equivalence of symmetric monoidal ∞ -categories.*

Moreover, the functor $\mathcal{D}_{\text{fil}}(K) \xrightarrow{D \mapsto D \otimes_K B_{\text{dR}}} \text{Mod}_{B_{\text{dR}}} \mathcal{D}_{\text{fil}}(G_K; \mathbf{Q}_p)$ lands inside $\mathcal{L}$, and in particular the composition (4.2) is strictly symmetric monoidal.

Proof. The map of commutative algebras $B_{\text{dR}}^+ \rightarrow B_{\text{dR}}$ in $\text{Mod}(G_K; \mathbf{Q}_p)$ can be upgraded to a filtered map, where B_{dR}^+ is equipped with a trivial filtration, and B_{dR} is equipped with the standard t -adic filtration. Scalar extension along this map defines a symmetric monoidal functor $\mathcal{D}(G_K; B_{\text{dR}}^+) \rightarrow \mathcal{D}_{\text{fil}}(G_K; B_{\text{dR}})$. It lands inside the subcategory $\mathcal{L}$, and one readily sees that it is inverse to the functor $\text{Fil}^0: \mathcal{L} \rightarrow \mathcal{D}(G_K; B_{\text{dR}}^+)$.

Every object M of the form $D \otimes_K B_{\text{dR}}$ with $D \in \mathcal{D}_{\text{fil}}(K)$ is contained in $\mathcal{L}$, because the map $\text{Fil}^0 M \otimes_{B_{\text{dR}}^+} t^n B_{\text{dR}}^+ \rightarrow \text{Fil}^n M$ can be presented as a map of colimits

$$\text{colim}_{i+j \geq 0} (\text{Fil}^i D \otimes_K t^j B_{\text{dR}}^+ \otimes_{B_{\text{dR}}^+} t^n B_{\text{dR}}^+) \rightarrow \text{colim}_{i+j \geq 0} \text{Fil}^i D \otimes_K t^{n+j} B_{\text{dR}}^+$$

defining the Day convolution product, and this map is an equivalence already on each term of the colimit.

To check that the lax symmetric monoidal functor $D \mapsto \text{Fil}^0(D \otimes_K B_{\text{dR}}): \mathcal{D}_{\text{fil}}(K) \rightarrow \mathcal{D}(G_K; B_{\text{dR}}^+)$ is strictly symmetric monoidal we may compose it with the fully faithful symmetric monoidal embedding $\mathcal{D}(G_K; B_{\text{dR}}^+) \simeq \mathcal{L} \hookrightarrow \mathcal{D}_{\text{fil}}(G_K; B_{\text{dR}})$. This composition is simply the functor $D \mapsto D \otimes_K B_{\text{dR}}$ which is strictly symmetric monoidal by construction. $\square$

Before we discuss the description of $\mathcal{D}^b(\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K)$, we need the following two lemmas.

Lemma 4.10. *Let (V, D, α) be a triple of a finite-dimensional continuous $\mathbf{Q}_p$ -linear representation $V \in \text{Rep}_{\mathbf{Q}_p} G_K$, a filtered complex $D \in \mathcal{D}_{\text{fil}}(K)$, and an isomorphism $\text{Fil}^0(D \otimes_K B_{\text{dR}}) \simeq V \otimes_{\mathbf{Q}_p} B_{\text{dR}}^+$. Then, for any integer $i \in \mathbf{Z}$, the object $\text{Fil}^i D$ is concentrated in degree 0 and the transition map $\text{Fil}^{i+1} D \rightarrow \text{Fil}^i D$ is injective.*

Proof. We have that

$$V \otimes_{\mathbf{Q}_p} \mathbf{C}_p \simeq \text{Fil}^0(D \otimes_K B_{\text{dR}})/t \simeq \text{gr}^0(D \otimes_K B_{\text{dR}}) \simeq \bigoplus_{i \in \mathbf{Z}} \text{gr}^i D \otimes_K \mathbf{C}_p(-i)$$

is concentrated in degree 0, where the second isomorphism follows from Lemma 4.9. Therefore, each $\text{gr}^i D$ is concentrated in degree 0, and almost all of these graded quotients are zero. In particular, $\text{Fil}^i D$ stabilize as $i \rightarrow \infty$ and as $i \rightarrow -\infty$. Assume that $\text{Fil}^d D = \text{Fil}^{d+1} D = \dots$ and put $H := \text{Fil}^d D$ to be this stable value.

Define the object $\underline{H} \in \mathcal{D}_{\text{fil}}(K)$ such that $\text{Fil}^i \underline{H} = H$ for any $i \in \mathbf{Z}$ and all transition maps are the identity maps. Then there is a natural morphism $\underline{H} \rightarrow D$ and we set $D' := \text{Cone}(\underline{H} \rightarrow D)$.

Step 1. We show that each $\text{gr}^i D'$ is a finite-dimensional vector space concentrated in degree 0 and only finitely many of $\text{gr}^i D'$ are non-zero. We note that $\text{gr}^i \underline{H} \simeq 0$ for any $i \in \mathbf{Z}$. Thus, we conclude that $\text{gr}^i D \simeq \text{gr}^i D'$. Finally, the first paragraph of the proof showed that $\text{gr}^i D$ are finite-dimensional K -vector spaces and all but finitely many of them vanish.

Step 2. We show that each $\text{Fil}^i D'$ is a vector space concentrated in degree 0 and the transition maps $\text{Fil}^{i+1} D' \rightarrow \text{Fil}^i D'$ are injective. Using Step 1, it suffices to show that $\lim_n \text{Fil}^n D' = 0$. This follows from the following sequence of equalities

$$\lim_n \text{Fil}^n D' \simeq \text{Cone}(\lim_n \text{Fil}^n \underline{H} \rightarrow \lim_n \text{Fil}^n D) \simeq \text{Cone}(H \rightarrow H) \simeq 0.$$

Step 3. We show that $H \simeq 0$. First, we note that Step 2 and the fact that there are no higher Ext groups in Vect_K imply that the filtration on D' splits non-canonically. This implies that we have a non-canonical isomorphism

$$\text{Fil}^0(D' \otimes_K B_{\text{dR}}) \cong \bigoplus_{i \in \mathbf{Z}} \text{gr}^i D' \otimes_K t^{-i} B_{\text{dR}}^+.$$

Furthermore, we observe that $\mathrm{Fil}^0(\underline{H} \otimes_K B_{\mathrm{dR}}) \simeq H \otimes_K B_{\mathrm{dR}}$ and we have $\mathrm{Fil}^0(D \otimes_K B_{\mathrm{dR}}) \simeq V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+$. Thus, after applying the functor $\mathrm{Fil}^0(- \otimes_K B_{\mathrm{dR}})$ to the exact triangle $\underline{H} \rightarrow D \rightarrow D'$, we get the following fiber sequence

$$H \otimes_K B_{\mathrm{dR}} \rightarrow V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+ \rightarrow \bigoplus_{i \in \mathbf{Z}} \mathrm{gr}^i D' \otimes_K t^{-i} B_{\mathrm{dR}}^+$$

in $\mathcal{D}(B_{\mathrm{dR}}^+)$. Now the assumption that V is finite-dimensional and Step 1 guarantee that the last two terms in the above exact triangle are concentrated in degree 0 and are finitely generated over B_{dR}^+ . This implies that $H \otimes_K B_{\mathrm{dR}}$ lies in $\mathcal{D}_{\mathrm{coh}}(B_{\mathrm{dR}}^+)$. However, this is possible only when $H = 0$.

Finally, the conclusions of Steps 2 and 3 finish the proof of the lemma. $\square$

Lemma 4.11. *Let $\mathcal{A}$ be an abelian category. Then the sequence $\{\mathrm{Ext}_{\mathcal{A}}^i(X, -) := \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, -[i]) : \mathcal{A} \rightarrow \mathrm{Ab}\}_{i \in \mathbf{Z}_{\geq 0}}$ is a universal δ -functor.*

We emphasize that we do not assume that $\mathcal{A}$ has enough injective or projective objects. This assumption would be inadequate for our applications.

Proof. First, [Sta26, Tag 06XP] ensures that $\{\mathrm{Ext}_{\mathcal{A}}^i(X, -)\}_{i \in \mathbf{Z}_{\geq 0}}$ can indeed be promoted to a δ -functor. Then the dual version of [Buc60, Proposition 4.2] implies that it suffices to show that, for all $i > 0$, for any $Y \in \mathcal{A}$ and $u \in \mathrm{Ext}_{\mathcal{A}}^i(X, Y)$, there is an injection $Y \hookrightarrow Y'$ such that the image of u in $\mathrm{Ext}_{\mathcal{A}}^i(X, Y')$ vanishes. Now [Sta26, Tag 06XU] identifies $\mathrm{Ext}_{\mathcal{A}}^i(X, Y)$ with the Yoneda Ext group. Therefore, the class u is represented by an exact sequence $0 \rightarrow Y \rightarrow Z_{i-1} \rightarrow \dots \rightarrow Z_0 \rightarrow X \rightarrow 0$. One readily checks that the image of u in $\mathrm{Ext}_{\mathcal{A}}^i(X, Z_{i-1})$ vanishes. $\square$

Now we are ready to prove the main result of this section.

Theorem 4.12. *The derived ∞ -category $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$ is equivalent, as a symmetric monoidal ∞ -category, to the fiber product of the diagram of ∞ -categories*

$$(4.3) \quad \begin{array}{ccc} \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K) & & \mathcal{D}_{\mathrm{fil}}(K) \\ & \searrow \scriptstyle V \mapsto V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+ & \swarrow \scriptstyle D \mapsto \mathrm{Fil}^0(D \otimes_K B_{\mathrm{dR}}) \\ & \mathcal{D}(G_K; B_{\mathrm{dR}}^+) & \end{array}$$

Here, the left functor is the composition of the fully faithful functor from Lemma 4.3 with the base change functor $\mathcal{D}(G_K; \mathbf{Q}_p) \rightarrow \mathcal{D}(G_K; B_{\mathrm{dR}}^+)$, and the right functor was introduced in Construction 4.8.

Remark 4.13. Our proof is specific to K being finite over $\mathbf{Q}_p$, as it uses Tate duality. We do not know if this equivalence holds for an arbitrary p -adic field K with a perfect residue field.

Proof. Denote by $\mathcal{D}_K$ the fiber product of (4.3), which is a symmetric monoidal stable ∞ -category. We denote objects of $\mathcal{D}_K$ by triples (V, D, α) with $V \in \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K)$, $D \in \mathcal{D}_{\mathrm{fil}}(K)$, and an equivalence $\alpha : \mathrm{Fil}^0(D \otimes_K B_{\mathrm{dR}}) \xrightarrow{\sim} V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+$. We have a functor $F : \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K \rightarrow \mathcal{D}_K$ sending a de Rham representation W to the triple $(W, D_{\mathrm{dR}}(W), \alpha)$, where $\alpha : \mathrm{Fil}^0(D_{\mathrm{dR}}(W) \otimes_K B_{\mathrm{dR}}) \xrightarrow{\sim} W \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+$ is the natural isomorphism. Since the functor D_{dR} is filtered exact and symmetric monoidal [Fon94b, Proposition 1.5.2], the functor F takes short exact sequences of de Rham representations to fiber sequences in $\mathcal{D}_K$ and induces a symmetric monoidal functor $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K) \rightarrow \mathcal{D}_K$.

Equivalence on the hearts. The three categories $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K)$, $\mathcal{D}(G_K; B_{\mathrm{dR}}^+)$, $\mathcal{D}_{\mathrm{fil}}(K)$ appearing in (4.3) are endowed with natural t -structures, using that they are the derived categories of the abelian categories $\mathrm{Rep}_{\mathbf{Q}_p} G_K$, $\mathrm{Mod}(G_K; B_{\mathrm{dR}}^+)$, $\mathrm{Fun}((\mathbf{Z}, \leq)^{\mathrm{op}}, \mathrm{Vect}_K)$, respectively. Both functors in (4.3) are t -exact, hence these t -structures induce a t -structure on the category $\mathcal{D}_K$. By definition, an object $(V, D, \alpha) \in \mathcal{D}_K$ lies in its heart if V and the objects $\mathrm{Fil}^n D$ for each $n \in \mathbf{Z}$ are concentrated in degree zero. Lemma 4.10 implies that this is equivalent to just requiring that V is concentrated in degree zero. Furthermore, in this case, all the transition maps $\mathrm{Fil}^{i+1} D \rightarrow \mathrm{Fil}^i D$ are injective. In other words, the heart of $\mathcal{D}_K$ consists of triples (V, D, α) where V is a Galois representation, D is a filtered K -vector space, and α is a Galois-equivariant isomorphism $V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+ \simeq \mathrm{Fil}^0(D \otimes B_{\mathrm{dR}})$. The existence of such an isomorphism implies that V is de Rham, and D together with α is uniquely recovered from V as $D_{\mathrm{dR}}(V)$. Thus, we conclude that the functor $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K) \rightarrow \mathcal{D}_K$ induces an equivalence of $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K$ with the heart of $\mathcal{D}_K$.

Full faithfulness of the functor $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K) \rightarrow \mathcal{D}_K$. It suffices to show that for any two de Rham representations $V_1, V_2 \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K$ the induced map

$$\mathrm{RHom}_{\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K}(V_1, V_2) \rightarrow \mathrm{RHom}_{\mathcal{D}_K}\left((V_1, D_{\mathrm{dR}}(V_1), \alpha_1), (V_2, D_{\mathrm{dR}}(V_2), \alpha_2)\right)$$

is an equivalence. Using the symmetric monoidal structure, it suffices to treat the case $V_1 = \mathbf{Q}_p$, and we denote V_2 by V for brevity.

By definition, $\mathrm{R}\Gamma(\mathcal{D}_K, V) := \mathrm{RHom}_{\mathcal{D}_K}\left((\mathbf{Q}_p, K, \alpha_{\mathbf{Q}_p}), (V, D_{\mathrm{dR}}(V), \alpha_V)\right)$ fits into the cartesian diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma(\mathcal{D}_K, V) & \xrightarrow{\quad} & \mathrm{RHom}_{\mathcal{D}_{\mathrm{fil}}(K)}(K, D_{\mathrm{dR}}(V)) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{RHom}_{\mathcal{D}(G_K; \mathbf{Q}_p)}(\mathbf{Q}_p, V) & \longrightarrow & \mathrm{RHom}_{\mathcal{D}(G_K; B_{\mathrm{dR}}^+)}(B_{\mathrm{dR}}^+, V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+), \end{array}$$

where we have appealed to the full faithfulness from Lemma 4.3 to be able to compute the bottom left object as RHom in $\mathcal{D}(G_K; \mathbf{Q}_p)$. More explicitly, the corners of this cartesian diagram can be rewritten as

$$(4.4) \quad \begin{array}{ccc} \mathrm{R}\Gamma(\mathcal{D}_K, V) & \xrightarrow{\quad} & \mathrm{Fil}^0 D_{\mathrm{dR}}(V) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{cont}}(G_K, V) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cont}}(G_K, V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+) \end{array}$$

where we used Lemma 4.7 for the top right corner, and [BS15, Lemma 4.3.9(1)] to identify the bottom terms with continuous group cohomology complexes. The right vertical map identifies $\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$ with $H_{\mathrm{cont}}^0(G_K, V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+)$, by definition of the filtration on $D_{\mathrm{dR}}(V)$. Hence, the cohomology groups $H^i(\mathcal{D}_K, V) := H^i(\mathrm{R}\Gamma(\mathcal{D}_K, V))$ fit into a long exact sequence

$$\begin{aligned}
 & 0 \longrightarrow H^0(\mathcal{D}_K, V) \longrightarrow H_{\text{cont}}^0(G_K, V) \longrightarrow 0 \\
 (4.5) \quad & \longrightarrow H^1(\mathcal{D}_K, V) \longrightarrow H_{\text{cont}}^1(G_K, V) \longrightarrow H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} B_{\text{dR}}^+) \longrightarrow \dots \\
 & \longrightarrow H^2(\mathcal{D}_K, V) \longrightarrow H_{\text{cont}}^2(G_K, V) \longrightarrow H_{\text{cont}}^2(G_K, V \otimes_{\mathbf{Q}_p} B_{\text{dR}}^+) \longrightarrow \dots
 \end{aligned}$$

Cohomology of G_K with coefficients in $V \otimes B_{\text{dR}}^+$ vanishes in degrees ≥ 2 by [Fon03, Proposition 3.3(iii)], and cohomology with coefficients in V vanishes in degrees ≥ 3 . Hence, the last displayed term, as well as all hidden terms in the sequence (4.5) are zero.

Since the map $\text{RHom}_{\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K}(\mathbf{Q}_p, V) \rightarrow \text{R}\Gamma(\mathcal{D}_K, V)$ induces an isomorphism on H^0 , [Buc60, Proposition 4.2] implies that it suffices to check that for any $V \in \text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K$ and $u \in H^i(\mathcal{D}_K, V)$, there is an injection $V \hookrightarrow V'$ in $\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K$ such that the image of u vanishes in $H^i(\mathcal{D}_K, V')$.

For $i = 1$, this amounts to the fact that

$$H^1(\mathcal{D}_K, V) = \ker(H_{\text{cont}}^1(G_K, V) \rightarrow H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} B_{\text{dR}}^+)) = \ker(H_{\text{cont}}^1(G_K, V) \rightarrow H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} B_{\text{dR}}))$$

classifies extensions $0 \rightarrow V \rightarrow W \rightarrow \mathbf{Q}_p \rightarrow 0$ in which W is de Rham. The class $[W] \in H_{\text{cont}}^1(G_K, V)$ then dies after the injection $V \hookrightarrow W$.

For $i = 2$, the observations that diagram (4.4) is cartesian and that $\text{Fil}^0 D_{\text{dR}}(V)$ is concentrated in degree 0 imply that $H^2(\mathcal{D}_K, V) \simeq H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} (B_{\text{dR}}^+/\mathbf{Q}_p))$. This functor is described as $(D_{\text{cris}}(V^\vee(1))^{\varphi=1})^\vee$ in Lemma 4.15 below. We will prove that already the functor $V \mapsto D_{\text{cris}}(V^\vee(1))^\vee$ is effaceable on $\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K$. Dualizing, this is equivalent to showing that for any de Rham representation V there exists a surjection of de Rham representations $W \rightarrow V$ such that the map $D_{\text{cris}}(W) \rightarrow D_{\text{cris}}(V)$ is zero. We denote by $K_0 \subset K$ the maximal subfield unramified over $\mathbf{Q}_p$.

Lemma 4.14. *If V_1, V_2 are two $\mathbf{Q}_p$ -representations of G_K such that V_2 is semi-stable, then the natural map $D_{\text{st}}(V_1) \otimes_{K_0} D_{\text{st}}(V_2) \rightarrow D_{\text{st}}(V_1 \otimes_{\mathbf{Q}_p} V_2)$ is an isomorphism.*

Proof. By definition of D_{st} , we have $D_{\text{st}}(V_1 \otimes_{\mathbf{Q}_p} V_2) = (V_1 \otimes_{\mathbf{Q}_p} V_2 \otimes_{\mathbf{Q}_p} B_{\text{st}})^{G_K}$, where the action of G_K is the diagonal one on all 3 tensor factors. Using that V_2 is semi-stable, we identify $V_2 \otimes_{\mathbf{Q}_p} B_{\text{st}}$ with $D_{\text{st}}(V_2) \otimes_{K_0} B_{\text{st}}$ where the Galois action is on the second factor only. Hence we have

$$\begin{aligned}
 (V_1 \otimes_{\mathbf{Q}_p} V_2 \otimes_{\mathbf{Q}_p} B_{\text{st}})^{G_K} &\simeq (V_1 \otimes_{\mathbf{Q}_p} D_{\text{st}}(V_2) \otimes_{K_0} B_{\text{st}})^{G_K} \\
 &\simeq (V_1 \otimes_{\mathbf{Q}_p} B_{\text{st}})^{G_K} \otimes_{K_0} D_{\text{st}}(V_2) \simeq D_{\text{st}}(V_1) \otimes_{K_0} D_{\text{st}}(V_2)
 \end{aligned}$$

In particular, the source and target of the natural map $D_{\text{st}}(V_1) \otimes_{K_0} D_{\text{st}}(V_2) \rightarrow D_{\text{st}}(V_1 \otimes_{\mathbf{Q}_p} V_2)$ have equal dimensions. But this map is injective for any pair of representations V_1, V_2 ([Fon94b, 1.5]), hence it is an isomorphism in our case. $\square$

Let T be the 2-dimensional representation fitting into a short exact sequence

$$0 \rightarrow \mathbf{Q}_p(1) \rightarrow T \rightarrow \mathbf{Q}_p \rightarrow 0$$

whose class in $H_{\text{cont}}^1(G_K, \mathbf{Q}_p(1))$ is equal to the image of $p \in K^\times$ under the Kummer map. The representation T is semi-stable with non-zero monodromy operator N on $D_{\text{st}}(T)$. Let r be an integer such that the r -th power N_V^r of the monodromy operator N_V on $D_{\text{st}}(V)$ vanishes. We claim that the surjection $W := \text{Sym}^r T \otimes_{\mathbf{Q}_p} V \rightarrow V$ induces the zero map on D_{cris} .

Lemma 4.14 implies that we have a sequence of isomorphisms

$$D_{\text{st}}(W) = D_{\text{st}}(\text{Sym}^r T \otimes_{\mathbf{Q}_p} V) \simeq D_{\text{st}}(\text{Sym}^r T) \otimes_{K_0} D_{\text{st}}(V) \simeq K_0[x]/x^{r+1} \otimes_{K_0} D_{\text{st}}(V),$$

where the monodromy operator on $D_{\text{st}}(W) \simeq K_0[x]/x^{r+1} \otimes_{K_0} D_{\text{st}}(V)$ is given by $N_W = x \otimes 1 + 1 \otimes N_V$ and the map $D_{\text{st}}(W) \simeq K_0[x]/x^{r+1} \otimes_{K_0} D_{\text{st}}(V) \rightarrow D_{\text{st}}(V)$ is equal to the evaluation at $x = 0$. The space $D_{\text{cris}}(W)$ is, by definition, the kernel of N_W on $D_{\text{st}}(W)$ which can be identified with the space of elements of the form $\sum_{i=0}^r x^i \otimes v_i$ for $v_i \in D_{\text{st}}(V)$ such that $v_i = -N_V(v_{i+1})$ for all $i = 0, \dots, r-1$ and $N_V(v_0) = 0$. In particular, the vector $v_0 = (-1)^r N_V^r(v_r)$ vanishes by our assumption on the order of nilpotence of N_V . That is, the map $D_{\text{st}}(W)^{N_W=0} \rightarrow D_{\text{st}}(V)^{N_V=0}$ is zero, as desired.

This finishes the proof of effaceability of $H^2(\mathcal{D}_K, V)$ and concludes the proof of full faithfulness. Together with the equivalence on the hearts, this proves the theorem, because every object of $\mathcal{D}_K$ is a successive extension of shifts of objects from the heart. $\square$

Lemma 4.15. *For any finite-dimensional representation $V \in \text{Rep}_{\mathbf{Q}_p} G_K$ there is a natural isomorphism*

$$H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} (B_{\text{dR}}^+/\mathbf{Q}_p)) \simeq (D_{\text{cris}}(V^\vee(1))^{\varphi=1})^\vee.$$

Proof. This is a special case of Tate duality for Galois cohomology with coefficients in Banach–Colmez spaces, proved in [Fon03, Théorème 6.1].

First, [Fon03, Théorème 2.14(iii)] ensures that $H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} t^n B_{\text{dR}}^+) = 0$ for any sufficiently large n (it suffices to take n to be larger than all integral generalized Hodge–Tate weights of V). The natural map $H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} (B_{\text{dR}}^+/\mathbf{Q}_p)) \rightarrow H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} (B_{\text{dR},n}^+/\mathbf{Q}_p))$ is then an isomorphism, where $B_{\text{dR},n}^+$ denotes B_{dR}^+/t^n .

Then [Fon03, Proposition 6.8(ii)] implies that we have a perfect pairing of finite-dimensional $\mathbf{Q}_p$ -vector spaces

$$H_{\text{cont}}^1(G_K, V \otimes_{\mathbf{Q}_p} (B_{\text{dR},n}^+/\mathbf{Q}_p)) \otimes_{\mathbf{Q}_p} \text{Ext}_{\mathcal{C}(G_K)}^1(B_{\text{dR},n}^+/\mathbf{Q}_p, V^\vee(1)) \rightarrow H_{\text{cont}}^2(G_K, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p$$

where Ext^1 is taken in Fontaine’s category of almost $\mathbf{C}_p$ -representations of G_K (see [Fon20, §1, 1.2]). Therefore, the result follows from Lemma 4.16 applied to $W = V^\vee(1)$ and a sufficiently large n . $\square$

Lemma 4.16. *Let $W \in \text{Rep}_{\mathbf{Q}_p} G_K$. Then there is an integer n_0 such that, for every $n \geq n_0$, there is an isomorphism $\text{Ext}_{\mathcal{C}(G_K)}^1(B_{\text{dR},n}^+/\mathbf{Q}_p, W) \simeq D_{\text{cris}}(W)^{\varphi=1}$.*

The proof below involves formulas containing both Tate twists by $\mathbf{Q}_p(1)$ and twists by the line bundle $\mathcal{O}(1)$ on the Fargues–Fontaine curve. In order to avoid any confusion, we write $(n)_T$ for Tate twists and $(n)_{FF}$ for twists by $\mathcal{O}(n)$.

Proof. To calculate this Ext^1 group, we use the equivalence

$$(4.6) \quad D^b(\mathcal{C}(G_K)) \simeq D^b(\text{Coh}^{G_K}(\text{FF}_{\mathbf{C}_p}))$$

of the derived category of $\mathcal{C}(G_K)$ with the derived category of (continuously) Galois-equivariant coherent sheaves on the Fargues–Fontaine curve provided by [Fon20, Theorem 6.4]. Recall that for any integer n we have a G_K -equivariant line bundle $\mathcal{O}(n)_{FF}$ on $\text{FF}_{\mathbf{C}_p}$

associated to the φ -module $(K_0, p^{-n} \cdot \varphi_{K_0})$ over K_0 . We have Galois-equivariant isomorphisms $H^0(\mathrm{FF}_{\mathbf{C}_p}, \mathcal{O}(n)_{FF}) \simeq B_{\mathrm{cris}}^{+, \varphi=p^n}$ and $H^1(\mathrm{FF}_{\mathbf{C}_p}, \mathcal{O}(-n)_{FF}) \simeq (B_{\mathrm{dR}, n}^+/\mathbf{Q}_p)(-n)_T$ for $n > 0$.

As explained in the proof of [Fon20, Proposition 6.8] (see also [Li24, Theorem 4.23]), the equivalence (4.6) carries the object $\mathcal{O}(-n)_{FF} \in \mathrm{Coh}^{G_K}(\mathrm{FF}_{\mathbf{C}_p})$ to $H^1(\mathrm{FF}_{\mathbf{C}_p}, \mathcal{O}(-n)_{FF})[-1]$ for all $n > 0$, hence we obtain an isomorphism

$$\begin{aligned} \mathrm{Ext}_{\mathcal{C}(G_K)}^1(B_{\mathrm{dR}, n}^+/\mathbf{Q}_p, W) &\simeq \mathrm{Ext}_{\mathcal{C}(G_K)}^1(B_{\mathrm{dR}, n}^+/\mathbf{Q}_p(-n)_T, W(-n)_T) \\ &\simeq \mathrm{Ext}_{D^b(\mathrm{Coh}^{G_K}(\mathrm{FF}_{\mathbf{C}_p}))}^1(\mathcal{O}(-n)_{FF}[1], W(-n)_T \otimes_{\mathbf{Q}_p} \mathcal{O}) \end{aligned}$$

The final Ext^1 is simply the space $\mathrm{Hom}_{\mathrm{FF}_{\mathbf{C}_p}}(\mathcal{O}(-n)_{FF}, W(-n)_T \otimes_{\mathbf{Q}_p} \mathcal{O})^{G_K}$ of Galois-equivariant homomorphisms of vector bundles on $\mathrm{FF}_{\mathbf{C}_p}$, which equals to the space of Galois invariant global sections

$$H^0(\mathrm{FF}_{\mathbf{C}_p}, W(-n)_T \otimes_{\mathbf{Q}_p} \mathcal{O}(n)_{FF})^{G_K} \simeq (B_{\mathrm{cris}}^{+, \varphi=p^n} \otimes_{\mathbf{Q}_p} W(-n)_T)^{G_K}.$$

Since $(W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{\varphi=1})^{G_K}$ is finite-dimensional and $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t]$, one sees that the map $(W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{+, \varphi=p^n}(-n)_T)^{G_K} \xrightarrow{\mathrm{id} \otimes t^{-n}} (W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{\varphi=1})^{G_K}$ is an isomorphism for large enough n . Combining the results, we conclude that

$$\mathrm{Ext}_{\mathcal{C}(G_K)}^1(B_{\mathrm{dR}, n}^+/\mathbf{Q}_p, W) \simeq (W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{+, \varphi=p^n}(-n)_T)^{G_K} \simeq (W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}^{\varphi=1})^{G_K} \simeq D_{\mathrm{cris}}(W)^{\varphi=1}$$

for sufficiently large n . $\square$

4.3. p -adic étale cohomology as an object of $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$. In this subsection we apply the description of $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$ provided by Theorem 4.12 to upgrade p -adic étale cohomology $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ of any smooth proper rigid-analytic space over K to an object of this category.

Proposition 4.17. *Let X be any smooth proper rigid-analytic space over K . Then its p -adic étale cohomology algebra $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \in \mathcal{D}(G_K; \mathbf{Q}_p)$ naturally lifts to an E_∞ -algebra in the category $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$.*

Remark 4.18. We expect that Proposition 4.17 can also be deduced from the results of [DN18]. When X arises as the analytification of an algebraic variety this result follows from [DN18, Lemma 2.22], which produces the requisite object of $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$ by applying the functor V_{pst} to an object of the derived category of admissible filtered (φ, N, G_K) -modules constructed out of Hyodo–Kato cohomology. For a general X , the same strategy should give a proof, using the construction of Hyodo–Kato cohomology provided in [CN20, Theorem 1.1].

First, we recall how the complex of $\mathbf{Q}_p$ -vector spaces $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ is endowed with a structure of an object of $\mathcal{D}(G_K; \mathbf{Q}_p)$. Denote by $f: X_{\mathrm{qproét}} \rightarrow (\mathrm{Spa} K)_{\mathrm{qproét}}$ the structure map of quasi-proétale sites, as defined in [Sch17, Definition 14.1]. We have an object $\mathrm{R}f_* \mathbf{Q}_p \in \mathcal{D}((\mathrm{Spa} K)_{\mathrm{qproét}}; \mathbf{Q}_p)$ whose value on $\mathrm{Spa} C$ is the étale cohomology complex $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$, by the comparison between quasi-proétale and étale cohomology for quasi-compact rigid-analytic spaces. Moreover, the complex $\mathrm{R}f_* \mathbf{Q}_p$ has a natural structure of an E_∞ -algebra in the symmetric monoidal category $\mathcal{D}((\mathrm{Spa} K)_{\mathrm{qproét}}; \mathbf{Q}_p)$ obtained by applying the lax symmetric monoidal functor $\mathrm{R}f_*$ to the commutative algebra $\mathbf{Q}_p \in \mathcal{D}(X_{\mathrm{qproét}}, \mathbf{Q}_p)$.

To obtain an object of $\mathcal{D}(G_K; \mathbf{Q}_p)$ from $\mathrm{R}f_* \mathbf{Q}_p$, we identify the quasi-proétale topos of $\mathrm{Spa} K$ with the topos of G_K -profinite sets $BG_{K, \mathrm{proét}}$ (see [BS15, Definition 4.3.1]). As before, we denote its underlying category by $G_K\text{-pfsets} \simeq \mathrm{Pro}(G_K\text{-fsets})$.

Construction 4.19. There is a natural functor $G_K\text{-pfsets} \rightarrow (\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}$ which sends a profinite set S with a continuous action of G_K to the quasi-pro\acute{e}tale morphism $(\mathrm{Spa} \mathbf{C}_p \times S)/\underline{G_K} \rightarrow \mathrm{Spa} K$, where the action of $\underline{G_K}$ on $\mathrm{Spa} \mathbf{C}_p \times S$ is diagonal. This functor sends covers to covers and commutes with pullbacks, so [Sta26, Tag 00X6] ensures that this defines a morphism of sites

$$\pi: (\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}} \rightarrow BG_{K,\mathrm{pro\acute{e}t}}.$$

Lemma 4.20. *The morphism of sites $\pi: (\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}} \rightarrow BG_{K,\mathrm{pro\acute{e}t}}$ induces an equivalence of topoi $\pi_*: \mathrm{Shv}((\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}; \mathrm{Set}) \rightarrow \mathrm{Shv}(BG_{K,\mathrm{pro\acute{e}t}}; \mathrm{Set})$.*

Proof. We note $(\mathrm{Spa} K)_{\mathrm{f\acute{e}t}} \simeq (\mathrm{Spa} K)_{\mathrm{\acute{e}t}, \mathrm{qc}, \mathrm{sep}}$. Therefore, [Sch17, Proposition 11.23 and 11.24] imply that the natural functor

$$\mathrm{Pro}((\mathrm{Spa} K)_{\mathrm{f\acute{e}t}}) \simeq \mathrm{Pro}((\mathrm{Spa} K)_{\mathrm{\acute{e}t}, \mathrm{qc}, \mathrm{sep}}) \hookrightarrow (\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}$$

is fully faithful and defines a basis for the quasi-pro\acute{e}tale topology on $\mathrm{Spa} K$. Therefore it induces an equivalence of topoi $\mathrm{Shv}(\mathrm{Pro}(\mathrm{Spa} K)_{\mathrm{f\acute{e}t}}; \mathrm{Set}) \rightarrow \mathrm{Shv}((\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}; \mathrm{Set})$.

By Galois theory, the category $(\mathrm{Spa} K)_{\mathrm{f\acute{e}t}}$ is equivalent to the category $G_K\text{-fsets}$, hence the site $\mathrm{Pro}((\mathrm{Spa} K)_{\mathrm{f\acute{e}t}})$ is equivalent to $\mathrm{Pro}(G_K\text{-fsets}) \simeq BG_{K,\mathrm{pro\acute{e}t}}$. Hence we obtain an equivalence of topoi $\mathrm{Shv}(BG_{K,\mathrm{pro\acute{e}t}}; \mathrm{Set}) \simeq \mathrm{Shv}((\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}; \mathrm{Set})$ which is readily seen to be induced by the morphism π . $\square$

Applying the equivalence of topoi from Lemma 4.20, the E_∞ -algebra $\mathrm{R}f_* \mathbf{Q}_p \in \mathcal{D}((\mathrm{Spa} K)_{\mathrm{qpro\acute{e}t}}, \mathbf{Q}_p)$ is carried to an E_∞ -algebra in $\mathcal{D}(G_K; \mathbf{Q}_p)$ that we denote by $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ as well. We now turn to proving that this object can be presented by a complex of de Rham representations:

Proof of Proposition 4.17. First, the structure of a condensed G_K -module on each cohomology group $H_{\mathrm{\acute{e}t}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ arises from a continuous finite-dimensional $\mathbf{Q}_p$ -representation (see [Bos24, Proposition 4.9]), hence $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ lies in the full subcategory $\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K) \subset \mathcal{D}(G_K; \mathbf{Q}_p)$ by Lemma 4.3.

By Theorem 4.12, we need to lift $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \in \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K)$ to an E_∞ -algebra object of the fiber product

$$\begin{array}{ccc} \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p} G_K) & & \mathcal{D}_{\mathrm{fil}}(K) \\ & \searrow V \mapsto V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+ & \swarrow D \mapsto \mathrm{Fil}^0(D \otimes_K B_{\mathrm{dR}}) \\ & \mathcal{D}(G_K; B_{\mathrm{dR}}^+) & \end{array}$$

We will take the corresponding object $D \in \mathcal{D}_{\mathrm{fil}}(K)$ to be the de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$ equipped with its Hodge filtration. It is made into a commutative algebra object of $\mathcal{D}_{\mathrm{fil}}(K)$ by applying the lax symmetric monoidal functor $\mathrm{R}\Gamma(X, -)$ to the filtered commutative differential graded algebra $\mathcal{O}_X \xrightarrow{d} \Omega_{X/K}^1 \xrightarrow{d} \dots$ in sheaves of K -vector spaces on X .

To lift the pair $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p), \mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$ to an object of the above fiber product, it remains to construct an equivalence of commutative algebra objects

$$\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^+ \simeq \mathrm{Fil}^0(\mathrm{R}\Gamma_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}})$$

in $\mathcal{D}(G_K; B_{\mathrm{dR}}^+)$. It is obtained by passing to Fil^0 in the de Rham comparison isomorphism for rigid-analytic spaces [Sch13a, Theorem 8.4] providing an equivalence of filtered commutative

algebra objects in $\mathcal{D}(G_K; \mathbf{B}_{\mathrm{dR}})$:

$$(4.7) \quad \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X/K) \otimes_K \mathbf{B}_{\mathrm{dR}}.$$

In loc. cit. this equivalence is stated on the level of individual cohomology groups, but it is constructed by proving that the natural maps

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} &\rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}) \\ \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{B}_{\mathrm{dR}}) &\rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_{\mathbf{C}_p}, \mathcal{O}\mathbf{B}_{\mathrm{dR}} \xrightarrow{\nabla} \dots) \\ \mathrm{R}\Gamma_{\mathrm{dR}}(X/K) \otimes_K \mathbf{B}_{\mathrm{dR}} &\rightarrow \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathcal{O}\mathbf{B}_{\mathrm{dR}} \xrightarrow{\nabla} \dots) \end{aligned}$$

of filtered commutative algebras are equivalences, which induces the equivalence (4.7). $\square$

5. FORMALITY FROM THE WEIGHT-MONODROMY CONJECTURE

5.1. Formulation of the main theorem. Let K be a finite extension of $\mathbf{Q}_p$, and let ℓ be a prime number (possibly equal to p). In this subsection we formulate our main theorem, which will be proven in Subsection 5.4.

Theorem 5.1. *Let X be a smooth proper rigid-analytic space over K . Assume that the weight-monodromy conjecture holds for $\mathrm{H}_{\mathrm{\acute{e}t}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ for all $i \geq 0$. Then the E_∞ -algebra $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ is formal.*

Definition 5.5 explains what it means for “the weight-monodromy conjecture to hold for $\mathrm{H}_{\mathrm{\acute{e}t}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$ ” and Theorem 5.1 is proven in Subsection 5.4. Together with the following lemma relating formality of $\mathbf{Q}_p$ -étale cohomology with that of de Rham cohomology, Theorem 5.1 gives the main Theorem 1.2 from the introduction.

Lemma 5.2. *Let X be a smooth proper rigid-analytic space over K . Then the E_∞ -algebra $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ is formal if and only if the de Rham cohomology E_∞ -algebra $\mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$ is formal.*

Proof. As discussed in the proof of Proposition 4.17, we have an equivalence of E_∞ -algebras over the field $\mathbf{B}_{\mathrm{dR}}$

$$(5.1) \quad \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X/K) \otimes_K \mathbf{B}_{\mathrm{dR}}.$$

Formality of an E_∞ -algebra can be checked after an arbitrary extension of characteristic zero fields [HS79, Corollary 6.9], hence the identification (5.1) proves the lemma. $\square$

5.2. Weil–Deligne representations. In this subsection, we fix a finite extension $\mathbf{Q}_p \subset K$ with Galois group $G_K := \mathrm{Aut}(\overline{K}/K)$, a prime number ℓ , and a field E of characteristic 0 that will serve as the field of coefficients. One should feel free to assume that $E = \mathbf{Q}_\ell$ when $\ell \neq p$ and that $E = \mathbf{Q}_p^{\mathrm{nr}}$ when $\ell = p$. We denote by W_K the Weil group of K and by q the size of the residue field $k = \mathbf{F}_q$ of K . The main goal of this subsection is to recall the notion of Weil–Deligne representations and prove its basic properties.

An E -linear Weil–Deligne representation is a finite-dimensional E -vector space equipped with an action of the Weil group W_K , on which the inertia acts through a finite quotient, and an E -linear map $N: V \rightarrow V(-1)$ of W_K -representations. We denote by $\mathrm{Rep}_E WD_K$ the category of E -linear Weil–Deligne representations.

The category $\mathrm{Rep}_E WD_K$ has a natural symmetric monoidal structure, where $V_1 \otimes_E V_2$ is defined to be the usual tensor product of W_K -representations and the monodromy operator $N_{V_1 \otimes_E V_2}$ is defined as $N_{V_1} \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes N_{V_2}$. This makes $\mathrm{Rep}_E WD_K$ into a symmetric

monoidal abelian category. We denote by $\mathcal{D}(\text{Rep}_E WD_K)$ the associated symmetric monoidal derived ∞ -category (see [NS18, Proposition A.5]).

We now discuss some examples of Weil–Deligne representations attached to (some) Galois representations.

Examples 5.3. (1) Let $\ell \neq p$. Then there is a fully faithful exact symmetric monoidal functor

$$\mathbf{D}: \text{Rep}_{\mathbf{Q}_\ell} G_K \rightarrow \text{Rep}_{\mathbf{Q}_\ell} WD_K.$$

We refer the reader to [Fon94a, §2.2] and [BL23, Example 2.3] for the construction of this functor. Choosing a compatible system of ℓ -power roots of unity in $\overline{K}$ and a Frobenius element, we may identify the underlying vector space of $\mathbf{D}(V)$ with V for every $V \in \text{Rep}_{\mathbf{Q}_\ell} G_K$, compatibly with tensor product. The exactness of this functor and [NS18, Proposition A.5] allow us to derive it to a t -exact symmetric monoidal functor $\mathbf{D}: \mathcal{D}(\text{Rep}_{\mathbf{Q}_\ell} G_K) \rightarrow \mathcal{D}(\text{Rep}_{\mathbf{Q}_\ell} WD_K)$.

(2) Let $\ell = p$. Then there is an exact symmetric monoidal functor

$$\mathbf{D}: \text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K \rightarrow \text{Rep}_{\mathbf{Q}_p}^{\text{nr}} WD_K,$$

sending a de Rham representation V to the Weil–Deligne representation constructed out of the (φ, N, G_K) -module $D_{\text{pst}}(V)$. Here, we use that every de Rham representation is potentially semi-stable [Ber02, Théorème 0.7], so that $\mathbf{D}$ is faithful and monoidal. We refer to [Fon94b, §4.2] and [BL23, Example 2.4] for the details of the definition of this functor. For any $V \in \text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K$ there is a natural isomorphism

$$V \otimes_{\mathbf{Q}_p} B_{\text{st}, \overline{\mathbf{Q}}_p} \simeq \mathbf{D}(V) \otimes_{\mathbf{Q}_{\text{nr}}} B_{\text{st}, \overline{\mathbf{Q}}_p}$$

of $B_{\text{st}, \overline{\mathbf{Q}}_p} := B_{\text{st}} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ -modules compatible with tensor products. Similarly to the ℓ -adic situation, we can derive to a functor $\mathbf{D}: \mathcal{D}(\text{Rep}_{\mathbf{Q}_p}^{\text{dR}} G_K) \rightarrow \mathcal{D}(\text{Rep}_{\mathbf{Q}_p}^{\text{nr}} WD_K)$.

Our next goal is to define what it means for cohomology of smooth rigid-analytic spaces to satisfy the weight-monodromy conjecture. For this, we need the following definition.

Definition 5.4. An E -linear Weil–Deligne representation V is *monodromy-pure* of weight i if, for any integer j , a (fixed) Frobenius lift $\varphi \in W_K$ acts on $\text{gr}_j^M V$ of the monodromy filtration with eigenvalues that are q -Weil numbers of weight $i + j$.

We refer to [Del80, Proposition 1.6.1] for the definition of the monodromy filtration. The definition of monodromy-pure representation is independent of the choice of a Frobenius lift because, for any two Frobenius lifts $\varphi, \varphi' \in W_K$, there is an integer $n \geq 1$ such that the actions of φ^n and φ'^n on V (and, therefore, on each $\text{gr}_i^M V$) coincide. Furthermore, [Del80, Proposition 1.6.9(i)] implies that a tensor product of monodromy-pure Weil–Deligne representations of weights i and j is monodromy-pure of weight $i + j$.

Definition 5.5. Let X be a smooth proper rigid-analytic space over K . We say that *the weight-monodromy conjecture holds for $H_{\text{ét}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell)$* if the associated Weil–Deligne representation $\mathbf{D}(H_{\text{ét}}^i(X_{\mathbf{C}_p}, \mathbf{Q}_\ell))$ is monodromy-pure of weight i .

Remark 5.6. According to the ℓ -independence conjecture [Sch13b, Conjecture 3.9], if the weight-monodromy conjecture for a given X and i holds for one prime ℓ then it should hold for every other prime as well.

Now we consider the functor $\mathrm{R}\Gamma(WD_K, -) := \mathrm{R}\mathrm{Hom}_{WD_K}(E, -) : \mathcal{D}(\mathrm{Rep}_E WD_K) \rightarrow \mathcal{D}(\mathrm{Vect}_E)$. We also set $H^i(WD_K, M) := H^i(\mathrm{R}\Gamma(WD_K, M))$ for any $M \in \mathrm{Rep}_E WD_K$. Our last main goal of this subsection is to present a complex computing $\mathrm{R}\Gamma(WD_K, M)$ for any $M \in \mathrm{Rep}_E WD_K$.

The following lemma will be used to identify the Ext functors in the category $\mathrm{Rep}_E WD_K$ and its mixed version:

Lemma 5.7. *Let $V, U \in \mathrm{Rep}_E WD_K$ and let $S = E \cdot e_1 \oplus E \cdot e_2$ be a two-dimensional Weil–Deligne representation such that $N_S = 0$, the inertia I_K acts trivially on S , and any Frobenius lift φ acts via the unique E -linear transformation such that $\varphi(e_1) = e_1$ and $\varphi(e_2) = e_1 + e_2$. Let $V \hookrightarrow V \otimes_E \mathrm{Sym}^r S$ be a homomorphism of Weil–Deligne representations defined via the formula $v \mapsto v \otimes e_1^r$. Then the induced morphism*

$$\mathrm{Hom}_{WD_K}(V \otimes_E \mathrm{Sym}^r S, U) \rightarrow \mathrm{Hom}_{WD_K}(V, U)$$

is zero for any $r \geq \dim_E V \cdot \dim_E U$.

Proof. Completely analogous to the proof of Lemma 4.2. $\square$

Now we show that $\mathrm{R}\mathrm{Hom}$ in the derived category of finite-dimensional representations of WD_K is computed by a Koszul-type complex:

Lemma 5.8. *Let $V \in \mathrm{Rep}_E WD_K$ and let $\varphi \in W_K$ be a lift of Frobenius. Then the complex*

$$C^\bullet(V) := [V^{I_K} \xrightarrow{(\varphi_V - \mathrm{id}, N)} V^{I_K} \oplus V(-1)^{I_K} \xrightarrow{(N, \mathrm{id} - \varphi_{V(-1)})} V(-1)^{I_K}]$$

is quasi-isomorphic to $\mathrm{R}\Gamma(WD_K, V)$.

Note that, after identifying $V^{I_K} \simeq V(-1)^{I_K}$ by trivializing $E(-1)$, the Frobenius operator $\varphi_{V(-1)}$ becomes equal to $q \cdot \varphi_V$.

Proof. First, we note that the functor of inertia invariants $(-)^{I_K}$ is exact on $\mathrm{Rep}_E WD_K$ since the action of I_K factors through a finite quotient and $\mathrm{char} E = 0$. This implies that, for any short exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$, the associated sequence of complexes $0 \rightarrow C^\bullet(V') \rightarrow C^\bullet(V) \rightarrow C^\bullet(V'') \rightarrow 0$ is exact. Thus, the sequence $\{V \mapsto H^i(C^\bullet(V))\}_{i \in \mathbb{Z}_{\geq 0}}$ is naturally a δ -functor. Lemma 4.11 implies that it suffices to show that there is a functorial isomorphism $\mathrm{Hom}_{WD_K}(E, V) \simeq H^0(C^\bullet(V))$ for any V and that $\{V \mapsto H^i(C^\bullet(V))\}_{i \in \mathbb{Z}_{\geq 0}}$ is a universal δ -functor.

The first claim is clear since $H^0(C^\bullet(V)) \simeq V^{N=0, W_K} \simeq \mathrm{Hom}_{WD_K}(E, V)$ naturally in V . Therefore, it suffices to show that $\{V \mapsto H^i(C^\bullet(V))\}_{i \in \mathbb{Z}_{\geq 0}}$ is a universal δ -functor. Now [Buc60, Proposition 4.2] implies that it suffices to show that, for any $V \in \mathrm{Rep}_E WD_K$ and a class $u \in H^i(C^\bullet(V))$ for $i = 1$ or 2 , there is an injection $V \hookrightarrow V'$ such that the image of u in $H^i(C^\bullet(V'))$ is zero.

For $i = 1$, we suppose that u is represented by an element $(x, y) \in V^{I_K} \oplus V^{I_K}(-1)$. Then we take $V' := V \oplus E \cdot e$ and endow it with the unique structure of a Weil–Deligne representation such that $V \subset V'$ is a sub-representation, and e is invariant under all of I_K and we have $N_{V'}(e) = y, \varphi_{V'}(e) = x \oplus e$. This Weil–Deligne representation is well-defined by the assumption that (x, y) is a cocycle in $C^1(V)$. One readily checks that the map $H^1(C^\bullet(V)) \rightarrow H^1(C^\bullet(V'))$ induced by natural morphism $V \hookrightarrow V'$ kills the class of u .

For $i = 2$, we note that there is an evident isomorphism of complexes $C^\bullet(V)^\vee[-2] \simeq C^\bullet(V^\vee(1))$. By passing to cohomology, we conclude that $H^2(C^\bullet(V))^\vee \simeq H^0(C^\bullet(V^\vee(1))) \simeq \mathrm{Hom}_{WD_K}(E, V^\vee(1)) \simeq \mathrm{Hom}_{WD_K}(V, E(1))$. Therefore, it suffices to show that, for any $V \in \mathrm{Rep}_E WD_K$, we can find an injection $V \hookrightarrow V'$ such that the natural morphism

$\mathrm{Hom}_{WD_K}(V', E(1)) \rightarrow \mathrm{Hom}_{WD_K}(V, E(1))$ is zero. This is achieved by taking $V' \simeq V \otimes_E \mathrm{Sym}^{\dim_E V} S$ from Lemma 5.7. $\square$

Corollary 5.9. *Let $V \in \mathrm{Rep}_E WD_K$. Then $\mathrm{R}\Gamma(WD_K, V)$ is concentrated in cohomological degrees $[0, 2]$ and each of its cohomology groups is finite-dimensional over E . Also, there is a natural equivalence $\mathrm{R}\Gamma(WD_K, V)^\vee \simeq \mathrm{R}\Gamma(WD_K, V^\vee(1)[2])$.*

Proof. The first claim follows directly from the isomorphism $\mathrm{R}\Gamma(WD_K, V) \simeq C^\bullet(V)$ from Lemma 5.8. The second claim follows immediately from the isomorphism of complexes $C^\bullet(V)^\vee \simeq C^\bullet(V^\vee(1))[2]$. $\square$

5.3. Mixed Weil–Deligne representations. In this subsection, we fix a finite extension $\mathbf{Q}_p \subset K$ with Galois group G_K , a prime number ℓ , and a field E of characteristic 0. The main goal of this subsection is to recall the notion of mixed and monodromy-pure mixed Weil–Deligne representations and prove that two versions of the derived ∞ -categories of mixed Weil–Deligne representations coincide.

Definition 5.10. A *mixed Weil–Deligne representation* is a pair $(V, W_\bullet V)$ of an object $V \in \mathrm{Rep}_E WD_K$ and a complete exhaustive increasing $\mathbf{Z}$ -indexed filtration $W_i V \subset V$ in $\mathrm{Rep}_E WD_K$ such that each $\mathrm{gr}_i^W V := W_i V / W_{i-1} V$ is monodromy-pure of weight i .

The *weights* of a mixed Weil–Deligne representation $(V, W_\bullet V)$ are the integers $i \in \mathbf{Z}$ such that $\mathrm{gr}_i^W V \neq 0$. We denote the *category of mixed Weil–Deligne representations* by $\mathrm{Rep}_E^{\mathrm{mix}} WD_K$.

Since the tensor product of monodromy-pure Weil–Deligne representations of weights i and j is monodromy-pure of weight $i + j$, one readily sees that the tensor product of two mixed Weil–Deligne representations has a natural structure of a mixed Weil–Deligne representation. So $\mathrm{Rep}_E^{\mathrm{mix}} WD_K$ is a symmetric monoidal category. Remarkably, the category $\mathrm{Rep}_E^{\mathrm{mix}} WD_K$ happens to be abelian [Vol03, Proposition 20] (see also [BL23, Theorem 2.10]) because any morphism of mixed representations is strictly compatible with weight filtrations. Moreover, the forgetful functor $\mathrm{Rep}_E^{\mathrm{mix}} WD_K \rightarrow \mathrm{Rep}_E WD_K$ is symmetric monoidal and exact.

In particular, the tensor product on $\mathrm{Rep}_E^{\mathrm{mix}} WD_K$ is exact in each variable. Thus, [NS18, Proposition A.5] ensures that $\mathcal{D}(\mathrm{Rep}_E^{\mathrm{mix}} WD_K)$ is a symmetric monoidal ∞ -category and the forgetful functor $\mathcal{D}(\mathrm{Rep}_E^{\mathrm{mix}} WD_K) \rightarrow \mathcal{D}(\mathrm{Rep}_E WD_K)$ is symmetric monoidal and t -exact.

Since the functor of inner Hom is exact on both $\mathrm{Rep}_E^{\mathrm{mix}} WD_K$ and $\mathrm{Rep}_E WD_K$, we can extend them to t -exact functors

$$\underline{\mathrm{Hom}}(-, -): \mathcal{D}(\mathrm{Rep}_E^{\mathrm{mix}} WD_K)^{\mathrm{op}} \times \mathcal{D}(\mathrm{Rep}_E^{\mathrm{mix}} WD_K) \rightarrow \mathcal{D}(\mathrm{Rep}_E^{\mathrm{mix}} WD_K),$$

$$\underline{\mathrm{Hom}}(-, -): \mathcal{D}(\mathrm{Rep}_E WD_K)^{\mathrm{op}} \times \mathcal{D}(\mathrm{Rep}_E WD_K) \rightarrow \mathcal{D}(\mathrm{Rep}_E WD_K),$$

which are compatible with respect to the forgetful functors.

Definition 5.11. A mixed Weil–Deligne representation $(V, W_\bullet V) \in \mathrm{Rep}_E^{\mathrm{mix}} WD_K$ is *monodromy-pure of weight i* if $W_{i-1} V = 0$ and $W_i V = V$.

This definition clearly implies that the underlying Weil–Deligne representation is monodromy-pure of weight i . However, the property introduced in this definition is stronger. For instance, one can take V to be a non-trivial extension $0 \rightarrow \mathbf{Q}_\ell(1) \rightarrow V \rightarrow \mathbf{Q}_\ell \rightarrow 0$ equipped with the filtration $W_{-3} V = 0$, $W_{-2} V = W_{-1} V = \mathbf{Q}_\ell(1)$, and $W_0 V = V$. Then V is monodromy-pure of weight -1 , but $(V, W_\bullet V) \in \mathrm{Rep}_{\mathbf{Q}_\ell}^{\mathrm{mix}} WD_K$ is not monodromy-pure of any weight.

Definition 5.12. We define $\mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K) \subset \mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)$ to be the full ∞ -subcategory consisting of objects M such that $H^i(M)$ is a monodromy-pure mixed Weil–Deligne representation of weight i for any $i \in \mathbf{Z}$ (see Definition 5.11).

Likewise, we define $\mathcal{D}_{\text{pure}}(\text{Rep}_E WD_K) \subset \mathcal{D}(\text{Rep}_E WD_K)$ to be the full ∞ -subcategory consisting of objects M such that $H^i(M)$ is a monodromy-pure Weil–Deligne representation of weight i for any $i \in \mathbf{Z}$ (see Definition 5.4).

Since the tensor products on $\mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)$ and $\mathcal{D}(\text{Rep}_E WD_K)$ are t -exact in each variable and the tensor product of (mixed) monodromy-pure Weil–Deligne representations is (mixed) monodromy-pure of the correct weight, we conclude that $\mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K)$ and $\mathcal{D}_{\text{pure}}(\text{Rep}_E WD_K)$ are symmetric monoidal ∞ -subcategories of $\mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)$ and $\mathcal{D}(\text{Rep}_E WD_K)$, respectively.

Therefore, the symmetric monoidal forgetful functor derives to a symmetric monoidal functor $F: \mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}_{\text{pure}}(\text{Rep}_E WD_K)$. We wish to show that this functor is an equivalence on bounded objects.

Theorem 5.13. *The forgetful functor $F: \mathcal{D}_{\text{pure}}^b(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}_{\text{pure}}^b(\text{Rep}_E WD_K)$ is a symmetric monoidal equivalence.*

We start with the following lemma.

Lemma 5.14. *Let $(V', W_\bullet V') \in \text{Rep}_E^{\text{mix}} WD_K$ be a mixed Weil–Deligne representation of non-positive weights and let $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ be an extension in $\text{Rep}_E WD_K$ such that V'' is monodromy-pure of weight 0. Set $W_i V = W_i V'$ if $i < 0$ and $W_i V = V$ if $i \geq 0$. Then $(V, W_\bullet V)$ is a mixed Weil–Deligne representation of non-positive weights.*

Proof. The only property we need to check is that $\text{gr}_0^W V$ is monodromy-pure of weight 0. By construction, it fits into the short exact sequence $0 \rightarrow \text{gr}_0^W V' \rightarrow \text{gr}_0^W V \rightarrow V'' \rightarrow 0$, where both $\text{gr}_0^W V'$ and V'' are monodromy-pure of weight 0. This follows from (the proof of) [BL23, Proposition 2.11]. $\square$

Now we begin the proof of the fact that the two versions of $\mathcal{D}_{\text{pure}}$ are equivalent. As the first step, we need a formula for the Ext groups in $\mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)$. For this, we recall that [BL23, Theorem 2.10] ensures that the functor $W_0: \text{Rep}_E^{\text{mix}} WD_K \rightarrow \text{Rep}_E WD_K$ is exact, therefore, it extends to a t -exact functor

$$W_0: \mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}(\text{Rep}_E WD_K).$$

We show that Ext's in the derived category $\mathcal{D}^b(\text{Rep}_E^{\text{mix}} WD_K)$ coincide with Exts in the filtered derived category of Weil–Deligne representations:

Lemma 5.15. *Let $V_1, V_2 \in \mathcal{D}^b(\text{Rep}_E^{\text{mix}} WD_K)$. Then there is a natural isomorphism $\text{RHom}_{WD_K^{\text{mix}}}(V_1, V_2) \simeq \text{R}\Gamma(WD_K, W_0 \underline{\text{Hom}}(V_1, V_2))$.*

Proof. First, we note that $\text{RHom}_{WD_K^{\text{mix}}}(V_1, V_2) \simeq \text{RHom}_{WD_K^{\text{mix}}}(E, \underline{\text{Hom}}(V_1, V_2))$. Since $W_0 E = E$, the functor $W_0: \mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}(\text{Rep}_E WD_K)$ induces a natural morphism

$$\text{RHom}_{WD_K^{\text{mix}}}(E, \underline{\text{Hom}}(V_1, V_2)) \rightarrow \text{RHom}_{WD_K}(E, W_0 \underline{\text{Hom}}(V_1, V_2))$$

which can be rewritten as a morphism $\text{RHom}_{WD_K^{\text{mix}}}(V_1, V_2) \rightarrow \text{R}\Gamma(WD_K, W_0 \underline{\text{Hom}}(V_1, V_2))$. Therefore, we reduce the question to showing that the natural morphism

$$(5.2) \quad \text{RHom}_{WD_K^{\text{mix}}}(E, V) \rightarrow \text{RHom}_{WD_K}(E, W_0 V)$$

is an isomorphism for any $V \in \mathcal{D}^b(\text{Rep}_E^{\text{mix}} WD_K)$. Since W_0 is t -exact, we reduce the question to showing that (5.2) is an isomorphism for $V \in \text{Rep}_E^{\text{mix}} WD_K$. In this case, Lemma 4.11 reduces the question to showing that the natural map $\text{Hom}_{WD_K^{\text{mix}}}(E, V) \rightarrow \text{Hom}_{WD_K}(E, W_0V)$ is an isomorphism for any $V \in \text{Rep}_E^{\text{mix}} WD_K$ and that $\{\text{Ext}_{WD_K}^i(E, W_0(-))\}_{i \in \mathbf{Z}_{\geq 0}} = \{H^i(WD_K, W_0(-))\}_{i \in \mathbf{Z}_{\geq 0}}$ is a universal δ -functor on $\text{Rep}_E^{\text{mix}} WD_K$.

The claim that the map $\text{Hom}_{WD_K^{\text{mix}}}(E, V) \rightarrow \text{Hom}_{WD_K}(E, W_0V)$ is an isomorphism is clear. So we only need to show the latter claim. For this, we first note that $\{H^i(WD_K, W_0(-))\}_{i \in \mathbf{Z}_{\geq 0}}$ is indeed a δ -functor since $W_0: \text{Rep}_E^{\text{mix}} WD_K \rightarrow \text{Rep}_E WD_K$ is an exact functor.

Now [Buc60, Proposition 4.2] ensures that it suffices to show that, for any $V \in \text{Rep}_E^{\text{mix}} WD_K$, $i \in \mathbf{Z}_{>0}$, and $u \in H^i(WD_K, W_0V)$, there is an injection $V \hookrightarrow V'$ in $\text{Rep}_E^{\text{mix}} WD_K$ such that the image of u in $H^i(WD_K, W_0V')$ vanishes. Furthermore, by Corollary 5.9 we only need to treat the cases of $i = 1$ and $i = 2$.

Case of $i = 1$. Fix a class $u \in H^1(WD_K, W_0V) = \text{Ext}_{WD_K}^1(E, W_0V)$. By [Sta26, Tag 06XU], it is represented by an extension $0 \rightarrow W_0V \rightarrow U \rightarrow E \rightarrow 0$ in $\text{Rep}_E WD_K$. Lemma 5.14 ensures that U can be naturally promoted to an object $(U, W_\bullet U) \in \text{Rep}_E^{\text{mix}} WD_K$ such that $W_0U = U$ and such that $0 \rightarrow W_0V \rightarrow U \rightarrow E \rightarrow 0$ is exact in $\text{Rep}_E^{\text{mix}} WD_K$. The image of u in $H^1(WD_K, U) = \text{Ext}_{WD_K}^1(E, U)$ is 0, because the extension U becomes split upon being pushed out to U itself.

Now we define $V' \in \text{Rep}_E^{\text{mix}} WD_K$ as the pushout $V \oplus_{W_0V} U := \text{coker}(W_0V \rightarrow V \oplus U)$ in $\text{Rep}_E^{\text{mix}} WD_K$. The natural map $V \rightarrow V'$ is injective since so is $W_0(V) \rightarrow U$. Furthermore, the exactness of W_0 implies that $W_0V' \simeq U$. Therefore, the image of u in $H^1(WD_K, W_0V') = H^1(WD_K, U)$ is zero as shown in the previous paragraph. This finishes the proof when $i = 1$.

Case of $i = 2$. We need to show that, for any $V \in \text{Rep}_E^{\text{mix}} WD_K$, there is an injection $V \hookrightarrow V'$ in $\text{Rep}_E^{\text{mix}} WD_K$ such that $H^2(WD_K, W_0V) \rightarrow H^2(WD_K, W_0V')$ is zero. By Tate's duality for Weil–Deligne cohomology (see Corollary 5.9), this is equivalent to asking that

$$\begin{aligned} \text{Hom}_{WD_K}(E, (W_0V')^\vee(1)) &\simeq \text{Hom}_{WD_K}(W_0V', E(1)) \\ &\longrightarrow \text{Hom}_{WD_K}(W_0V, E(1)) \simeq \text{Hom}_{WD_K}(E, (W_0V)^\vee(1)) \end{aligned}$$

is zero. Let S be the two-dimensional E -linear Weil–Deligne representation from Lemma 5.7. Note that it is monodromy-pure of weight 0, so $(S, W_\bullet S)$ lies in $\text{Rep}_E^{\text{mix}} WD_K$ with the filtration defined by $W_i S = 0$ if $i < 0$ and $W_i S = S$ if $i \geq 0$. For any $r \geq 0$, the object $W_0V \otimes_E \text{Sym}^r S \in \text{Rep}_E^{\text{mix}} WD_K$ has only non-positive weights. Now set $d := \dim_E W_0V$ and $U := W_0V \otimes_E \text{Sym}^d S$. It comes with the natural inclusion of Weil–Deligne representations $W_0V \hookrightarrow U$ and Lemma 5.7 implies that the natural morphism

$$\text{Hom}_{WD_K}(U, E(1)) \rightarrow \text{Hom}_{WD_K}(W_0V, E(1))$$

is zero.

As in the case $i = 1$, we define $V' \in \text{Rep}_E^{\text{mix}} WD_K$ as the pushout $V \oplus_{W_0V} U := \text{coker}(W_0(V) \rightarrow V \oplus U)$ in $\text{Rep}_E^{\text{mix}} WD_K$. The natural map $V \rightarrow V'$ is injective and passing to W_0 it induces the map $W_0V \rightarrow U$. Hence it induces the zero map on $H^2(WD_K, -)$, as we just established. This finishes the proof. $\square$

Before demonstrating the equivalence of the two versions of pure complexes, we only need one additional preliminary lemma.

Lemma 5.16. *Let $V_1, V_2 \in \mathcal{D}_{\text{pure}}^b(\text{Rep}_E^{\text{mix}} WD_K)$ such that $V_1 \in \mathcal{D}_{\text{pure}}^{\geq n}(\text{Rep}_E^{\text{mix}} WD_K)$ is concentrated in cohomological degrees $\geq n$ and $V_2 \in \mathcal{D}_{\text{pure}}^{\leq n}(\text{Rep}_E^{\text{mix}} WD_K)$ is concentrated in degrees $\leq n$. Then the natural morphism $W_0 \underline{\text{Hom}}(V_1, V_2) \rightarrow \underline{\text{Hom}}(V_1, V_2)$ is an isomorphism.*

Proof. First, we have $\underline{\text{Hom}}(V_1, V_2) \simeq V_1^\vee \otimes_E V_2$. Furthermore, the dual object $V_1^\vee$ belongs to $\mathcal{D}_{\text{pure}}^{\leq -n}(\text{Rep}_E^{\text{mix}} WD_K)$, hence the tensor product $V_1^\vee \otimes_E V_2$ lies in $\mathcal{D}_{\text{pure}}^{\leq 0}(\text{Rep}_E^{\text{mix}} WD_K)$. Thus, it suffices to show that the natural morphism $W_0 V \rightarrow V$ is an isomorphism for any $V \in \mathcal{D}_{\text{pure}}^{\leq 0}(\text{Rep}_E^{\text{mix}} WD_K)$. This can be checked on individual cohomology groups, where the claim is clear. $\square$

Recall that, for an ∞ -category $\mathcal{C}$, we denote by $\text{Map}_{\mathcal{C}}(X, Y)$ the mapping anima between objects X and Y . If $\mathcal{C} = \mathcal{D}(\mathcal{A})$ for an abelian category $\mathcal{A}$, this mapping anima can be identified with $\tau^{\leq 0} \text{RHom}_{\mathcal{A}}(X, Y)$.

Proof of Theorem 5.13. The forgetful functor is symmetric monoidal, so we only need to show that it is an equivalence of underlying ∞ -categories.

First, we show that F is fully faithful. For this, we need to show that the natural morphism $\text{Map}_{\mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)}(V_1, V_2) \rightarrow \text{Map}_{\mathcal{D}(\text{Rep}_E WD_K)}(V_1, V_2)$ is an equivalence for any $V_1, V_2 \in \mathcal{D}_{\text{pure}}^b(\text{Rep}_E^{\text{mix}} WD_K)$. In other words, we wish to show that the natural morphism

$$\tau^{\leq 0} \text{RHom}_{WD_K^{\text{mix}}}(V_1, V_2) \rightarrow \tau^{\leq 0} \text{RHom}_{WD_K}(V_1, V_2)$$

is an isomorphism for any $V_1, V_2 \in \mathcal{D}_{\text{pure}}^b(\text{Rep}_E^{\text{mix}} WD_K)$. Since $\text{RHom}_{WD_K^{\text{mix}}}(V_1, V_2) \simeq \text{RHom}_{WD_K^{\text{mix}}}(E, V_1^\vee \otimes_E V_2)$ and the same formula holds in $\mathcal{D}(\text{Rep}_E WD_K)$, we can assume that $V_1 = E$. In this case, we denote V_2 simply by V .

In this situation, we consider the following commutative diagram

$$\begin{array}{ccccc} \text{RHom}_{WD_K^{\text{mix}}}(E, \tau^{\leq 0} V) & \longrightarrow & \text{RHom}_{WD_K^{\text{mix}}}(E, V) & \longrightarrow & \text{RHom}_{WD_K^{\text{mix}}}(E, \tau^{> 0} V) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ \text{RHom}_{WD_K}(E, \tau^{\leq 0} V) & \longrightarrow & \text{RHom}_{WD_K}(E, V) & \longrightarrow & \text{RHom}_{WD_K}(E, \tau^{> 0} V), \end{array}$$

where the rows are fiber sequences. We wish to show that $\tau^{\leq 0} \beta$ is an isomorphism. Since both $\text{RHom}_{WD_K^{\text{mix}}}(E, \tau^{> 0} V)$ and $\text{RHom}_{WD_K}(E, \tau^{> 0} V)$ lie in $\mathcal{D}^{\geq 1}(\mathbf{Z})$, it suffices to show that α is an isomorphism. This follows directly from Lemma 5.15 and Lemma 5.16.

Now we show that F is essentially surjective. First, we note that, by the very definition, every object of the form $V[-d]$ for a monodromy-pure V of weight d lies in the essential image of F . Now we pick $V \in \mathcal{D}_{\text{pure}}^{[a, b]}(\text{Rep}_E WD_K)$ and wish to show that it lies in the essential image. We argue by induction on $b - a \geq 0$. If $b - a = 0$, this was proved above. So we assume that $n := b - a > 0$ and the claim is known for any $b - a < n$. Consider the exact triangle $H^a(V)[-a] \rightarrow V \rightarrow \tau^{> a} V$. By induction, $H^a(V)[-a]$ and $\tau^{> a} V$ lie in the essential image of F . Say $H^a(V)[-a] \simeq F(U_1)$ and $\tau^{> a} V \simeq F(U_2)$. Since F is t -exact, we conclude that $U_1 \in \mathcal{D}_{\text{pure}}^{[a, a]}(\text{Rep}_E^{\text{mix}} WD_K)$ and $U_2 \in \mathcal{D}_{\text{pure}}^{[a+1, b]}(\text{Rep}_E^{\text{mix}} WD_K)$. Now it suffices to show that the natural map

$$\text{Ext}_{WD_K^{\text{mix}}}^1(U_2, U_1) \rightarrow \text{Ext}_{WD_K}^1(F(U_2), F(U_1))$$

is an isomorphism. This follows immediately from Lemma 5.15 and Lemma 5.16. $\square$

5.4. Consequences for formality. The main goal of this subsection is to prove Theorem 5.1. The proof is a formal consequence of Theorem 5.13 combined with the decomposition result given in Lemma 5.17.

Lemma 5.17. *There is an equivalence of symmetric monoidal functors between the forgetful functor $\mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}(\text{Vect}_E)$ and its composition with the functor $M \mapsto \bigoplus_{i \in \mathbb{Z}} H^i(M)[-i]: \mathcal{D}(\text{Vect}_E) \rightarrow \mathcal{D}(\text{Vect}_E)$.*

Proof. This lemma follows formally from the fact that the weight filtration on any mixed Weil–Deligne representation is naturally split, on the level of underlying vector spaces, as we now explain.

We denote by $\text{obl}: \text{Rep}_E^{\text{mix}} WD_K \rightarrow \text{Vect}_E$ the forgetful functor sending a Weil–Deligne representation to its underlying vector space. By [BL23, Definition 2.13], this functor is isomorphic as a symmetric monoidal functor to the functor $V \mapsto \bigoplus_i \text{gr}_i^W V$. In particular, obl factors through the category $\text{Vect}_E^{\mathbb{Z}}$ of $\mathbb{Z}$ -graded vector spaces.

Passing to derived functors, we obtain a factorization of the symmetric monoidal functor $\mathcal{D}(\text{obl})$:

$$\mathcal{D}(\text{obl}): \mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K) \xrightarrow{\text{gr}_W^\bullet} \mathcal{D}(\text{Vect}_E^{\mathbb{Z}}) \xrightarrow{\oplus} \mathcal{D}(\text{Vect}_E)$$

Restricted to the full subcategory $\mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K) \subset \mathcal{D}(\text{Rep}_E^{\text{mix}} WD_K)$, the functor $\text{gr}_W^\bullet$ lands in the full subcategory $\mathcal{D}_{\text{pure}}(\text{Vect}_E^{\mathbb{Z}})$ consisting of graded complexes M such that $\text{gr}^i M$ is concentrated in the cohomological degree i . This is a symmetric monoidal subcategory because tensor product over E is exact.

However, the endofunctor $M \mapsto \bigoplus H^i(M)[-i]: \mathcal{D}_{\text{pure}}(\text{Vect}_E^{\mathbb{Z}}) \rightarrow \mathcal{D}_{\text{pure}}(\text{Vect}_E^{\mathbb{Z}})$ is equivalent to the identity functor, as symmetric monoidal functors. Indeed, $\mathcal{D}_{\text{pure}}(\text{Vect}_E^{\mathbb{Z}})$ is equivalent to the nerve of a symmetric monoidal 1-category because all the mapping spaces in this category are discrete. Thus constructing an equivalence between two endofunctors amounts to doing so on the homotopy category of this ∞ -category, where the equivalence is evident.

Since $\mathcal{D}_{\text{pure}}(\text{obl}): \mathcal{D}_{\text{pure}}(\text{Rep}_E^{\text{mix}} WD_K) \rightarrow \mathcal{D}(\text{Vect}_E)$ factors through $\mathcal{D}_{\text{pure}}(\text{Vect}_E^{\mathbb{Z}})$, its composition with $M \mapsto \bigoplus H^i(M)[-i]$ is equivalent to $\mathcal{D}_{\text{pure}}(\text{obl})$ as a symmetric monoidal functor. $\square$

Combining Lemma 5.17 with Theorem 5.13, we get a general formality result:

Corollary 5.18. *For any E_∞ -algebra $A \in \mathcal{D}_{\text{pure}}^b(\text{Rep}_E WD_K)$ there exists an equivalence $A \simeq \bigoplus H^i(A)[-i]$ of E_∞ -algebras in $\mathcal{D}(\text{Vect}_E)$.*

It remains to apply Corollary 5.18 to the cohomology of rigid-analytic spaces satisfying the weight-monodromy conjecture:

Proof of Theorem 5.1. We have $\text{R}\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{Q}_\ell)$ as a commutative algebra object of $\mathcal{D}(G_K; \mathbb{Q}_\ell)$. Each cohomology group $H_{\text{ét}}^i(X_{\mathbb{C}_p}, \mathbb{Q}_\ell)$ is a condensed representation of G_K arising from a continuous representation on a finite-dimensional $\mathbb{Q}_\ell$ -vector space (cf. [Bos24, Proposition 4.9]). Hence, Lemma 4.3 makes $\text{R}\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{Q}_\ell)$ into a commutative algebra object of the symmetric monoidal category $\mathcal{D}^b(\text{Rep}_{\mathbb{Q}_\ell} G_K)$. Moreover, for $\ell = p$ the p -adic cohomology $\text{R}\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{Q}_p)$ further lifts to an object of $\mathcal{D}^b(\text{Rep}_{\mathbb{Q}_p}^{\text{dR}} G_K)$ by Proposition 4.17.

The desired formality will now be a consequence of Corollary 5.18. Let us first treat the case $\ell \neq p$. Applying the symmetric monoidal functor $\mathbf{D}: \mathcal{D}^b(\text{Rep}_{\mathbb{Q}_\ell} G_K) \rightarrow \mathcal{D}^b(\text{Rep}_{\mathbb{Q}_\ell} WD_K)$ from Example 5.3(1), we obtain a commutative algebra object $\mathbf{D}(\text{R}\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{Q}_\ell))$ of

$\mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_\ell} WD_K)$ whose i -th cohomology is monodromy-pure of weight i , by the assumption of the theorem. That is, it lies in the full subcategory $\mathcal{D}_{\mathrm{pure}}^b(\mathrm{Rep}_{\mathbf{Q}_\ell} WD_K)$ and Corollary 5.18 implies that $\mathbf{D}(\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_\ell))$ is a formal algebra. Since the underlying commutative algebra object of $\mathcal{D}(\mathrm{Vect}_{\mathbf{Q}_\ell})$ of $\mathbf{D}(\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_\ell))$ is equivalent to $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_\ell)$, we obtain the formality of ℓ -adic cohomology for $\ell \neq p$.

Similarly, for $\ell = p$ we apply the functor from Example 5.3(2) to the commutative algebra object $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p) \in \mathcal{D}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}} G_K)$ to obtain an object $\mathbf{D}(\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p)) \in \mathcal{D}_{\mathrm{pure}}^b(\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{nr}} WD_K)$. By de Rham comparison isomorphism, we have an equivalence of commutative algebras in $\mathcal{D}(\mathrm{Vect}_{B_{\mathrm{dR}}})$

$$(5.3) \quad \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq \mathbf{D}(\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p)) \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} B_{\mathrm{dR}}.$$

Since $\mathbf{D}(\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p))$ is formal by Corollary 5.18 and formality can be checked after an arbitrary field extension (see [HS79, Corollary 6.9]), we conclude that the p -adic étale cohomology algebra $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_{C_p}, \mathbf{Q}_p)$ is formal as well. $\square$

REFERENCES

- [AGV72] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, editors. *Théorie des topos et cohomologie étale des schémas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). With the collaboration of Nicolas Bourbaki, Pierre Deligne et Bernard Saint-Donat.
- [Bei86] A. A. Beilinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.
- [Ber02] Laurent Berger. Représentations p -adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.
- [BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- [Bha] Bhargav Bhatt. Perverse sheaves, lecture notes for a Fall 2015 course. <https://www.math.purdue.edu/~murayama/MATH731.pdf>.
- [BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.
- [BL23] L. Alexander Betts and Daniel Litt. Semisimplicity of the Frobenius action on π_1 . In *p -adic Hodge theory, singular varieties, and non-abelian aspects*, Simons Symp., pages 17–64. Springer, Cham, [2023] ©2023.
- [BMS24] Shay Ben-Moshe and Tomer M. Schlank. Higher semiadditive algebraic K-theory and redshift. *Compos. Math.*, 160(2):237–287, 2024.
- [Bor84] Ciprian Borcea. Moduli for Kodaira surfaces. *Compositio Math.*, 52(3):373–380, 1984.
- [Bos24] Guido Bosco. Torsion in p -adic étale cohomology: remarks and conjectures. *arXiv preprint arXiv:2411.07355* <https://web.math.princeton.edu/~gb5927/docs/Torsion.pdf>, 2024.
- [BS15] B. Bhatt and P. Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.
- [Buc60] David A. Buchsbaum. Satellites and universal functors. *Ann. of Math. (2)*, 71:199–209, 1960.
- [CN20] Pierre Colmez and Wiesław Nizioł. On p -adic comparison theorems for rigid analytic varieties, I. *Münster J. Math.*, 13(2):445–507, 2020.
- [Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.
- [DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.
- [DN18] Frédéric Déglise and Wiesław Nizioł. On p -adic absolute Hodge cohomology and syntomic coefficients. I. *Comment. Math. Helv.*, 93(1):71–131, 2018.

- [Dri04] Vladimir Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.
- [EK] Matthew Emerton and Mark Kisin. Extensions of crystalline representations. <https://www.math.uchicago.edu/~emerton/pdffiles/ext.pdf> accessed on June 30, 2026.
- [Fon94a] Jean-Marc Fontaine. Représentations l -adiques potentiellement semi-stables. *Astérisque*, (223):321–347, 1994. Périodes p -adiques (Bures-sur-Yvette, 1988).
- [Fon94b] Jean-Marc Fontaine. Représentations p -adiques semi-stables. *Astérisque*, (223):113–184, 1994. With an appendix by Pierre Colmez, Périodes p -adiques (Bures-sur-Yvette, 1988).
- [Fon03] Jean-Marc Fontaine. Presque C_p -représentations. *Doc. Math.*, pages 285–385, 2003. Kazuya Kato’s fiftieth birthday.
- [Fon20] Jean-Marc Fontaine. Almost C_p Galois representations and vector bundles. *Tunis. J. Math.*, 2(3):667–732, 2020.
- [Has05] Keizo Hasegawa. Complex and Kähler structures on compact solvmanifolds. volume 3, pages 749–767. 2005. Conference on Symplectic Topology.
- [HM03] Richard Hain and Makoto Matsumoto. Tannakian fundamental groups associated to Galois groups. In *Galois groups and fundamental groups*, volume 41 of *Math. Sci. Res. Inst. Publ.*, pages 183–216. Cambridge Univ. Press, Cambridge, 2003.
- [HS79] Stephen Halperin and James Stasheff. Obstructions to homotopy equivalences. *Adv. in Math.*, 32(3):233–279, 1979.
- [Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.
- [Laz65] Michel Lazard. Groupes analytiques p -adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.
- [Li24] Zhenghui Li. Duality for Arithmetic p -adic Pro-étale Cohomology of Analytic Spaces. *arXiv preprint arXiv:2412.11786*, 2024.
- [Lur17] Jacob Lurie. Higher algebra. <https://www.math.ias.edu/~lurie/papers/HA.pdf>, 2017.
- [Mor78] John W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.
- [NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.
- [RZ10] Luis Ribes and Pavel Zalesskii. *Profinite Groups*. A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 2010.
- [Sch13a] Peter Scholze. p -adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi*, 1:e1, 77, 2013.
- [Sch13b] Peter Scholze. Perfectoid spaces: a survey. In *Current developments in mathematics 2012*, pages 193–227. Int. Press, Somerville, MA, 2013.
- [Sch17] Peter Scholze. Étale cohomology of diamonds. Preprint, available at <https://arxiv.org/abs/1709.07343>, 2017.
- [Sch18] Peter Scholze. p -adic geometry. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 899–933. World Sci. Publ., Hackensack, NJ, 2018.
- [Sta26] The Stacks Project authors. The Stacks Project. <https://stacks.math.columbia.edu>, 2026.
- [Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331, 1977.
- [Uen87] Kenji Ueno. Compact rigid analytic spaces with special regard to surfaces. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 765–794. North-Holland, Amsterdam, 1987.
- [Vol03] Vadim Vologodsky. Hodge structure on the fundamental group and its application to p -adic integration. *Mosc. Math. J.*, 3(1):205–247, 260, 2003.
- [Wil11] Rolf Stefan Wilke. Line bundles on totally degenerated formal schemes. *Int. Math. Res. Not. IMRN*, (22):5019–5044, 2011.

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