ALGEBRAIZATION TECHNIQUES AND RIGID-ANALYTIC ARTIN–GROTHENDIECK VANISHING

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Abstract. First, we prove an algebraization result for rig-smooth algebras over a general noetherian ring; this positively answers the question raised in [Sta24, Tag 0GAX]. Then we prove a general partial algebraization result in non-archimedean geometry. The result says that we can always algebraize a geometrically reduced affinoid rigid-analytic space in "one direction" in an appropriate sense. As an application of this result, we show the remaining cases of the Artin–Grothendieck Vanishing for affinoid algebras, which were previously conjectured in [BM21, §7]. This allows us to deduce a stronger version of the rigid-analytic Artin–Grothendieck Vanishing Conjecture (see [Han20, Conj. 1.2]) over a field of characteristic 0. Using a completely different set of ideas, we also obtain a weaker version of this conjecture over a field of characteristic $p > 0$.

Contents

1. Introduction

1.1. Partial Algebraization.

In [Art69a], M. Artin pioneered the study of algebraization questions in algebraic geometry. Although algebraization problems may take different forms depending on the exact problem under consideration, usually they can be summarized by the following (imprecise) question:

Question 1.1. Let A be a ring with a finitely generated ideal I, and let $\overline{S}$ be an "algebraic structure" over $A^{\wedge}_{I}$. Can we "algebraize" (or at least "approximate") it to an "algebraic structure" S over A.

One positive answer to Question 1.1 is provided by [Art69a, Th. 1.10] and its later generalization due to Popescu (see [Sta24, Tag 0AH5]). It says that, for a noetherian G-ring A which is henselian along an ideal I, any solution $\widehat{y} \in (A^{\wedge}_I)^m$ of a (finite) system of polynomial equations $F_i \in A[X_1, \dots, X_m]$ can be approximated by a solution $y \in A^m$.

This result has become a standard (but very important) technical tool in many areas of algebraic geometry: various algebraization questions (see [Art69a, Cor. 2.6 and §3]), structure theory of algebraic stacks (see [AHR20]), homological and commutative algebra (see [HH90]), construction of moduli spaces (see [Art69b], [Ale02]), and étale cohomology (see [ILO14]).

Using the above approximation (and its version due to Elkik), we prove the following general algebraization result:

Theorem 1.2 (Noetherian rig-smooth algebraization; Theorem 5.13)**.** *Let A be a noetherian ring, and let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra such that B is rig-smooth over (A, I) (in the sense of* [Sta24, Tag 0GAI]*) and B/IB is a finite type A/I-algebra. Then there is a finite type A-algebra C such that C is smooth outside $\mathrm{V}(I)$ and there is an isomorphism $C^{\wedge}_I \simeq B$ of A-algebras.*

Theorem 1.2 positively answers the question raised in [Sta24, Tag 0GAX]. Of course, many special cases of Theorem 1.2 were known before. For instance, the case when A is a noetherian G-ring was settled in [Sta24, Tag 0GAT], and the case when I is a principal ideal was settled in [Elk73, Th. 7 on p. 582].

Theorem 1.2 provides a general tool for studying rig-smooth formal schemes. For instance, a version of Theorem 1.2 for rig-étale morphisms is used in the generalization of Artin's Theorem on dilatations (see [Sta24, Tag 0ARB] and [Sta24, Tag 0GDU], see also [Art70, Th. 3.2] for the original result), while *loc. cit.* is used in the proof of Artin's Theorem on contractions (see [Sta24, Tag 0GH7] and [Art70, Th. 3.1]).

However, the noetherian assumption in Theorem 1.2 is too limiting for the purposes of rigid-analytic geometry. Namely, one would like to apply this result to $A = \mathcal{O}_C$ for a rank-1 valuation ring in an algebraically closed non-archimedean field C. This ring is never noetherian, so Theorem 1.2 does not apply in this situation. Nevertheless, this issue has been overcome by R. Elkik, who studied an analogue of Artin approximation in certain non-noetherian situations. In particular, she proved the following remarkable fact:

Theorem 1.3 (Elkik's algebraization; special case of [Elk73, Th. 7 on p. 582 and Rmq. 2(c) on p.588])**.** *Let K be a non-archimedean field with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, and let A be a rig-smooth*[1]*, flat, topologically finite type $\mathcal{O}_K$-algebra. Then there is a flat, finite type $\mathcal{O}_K$-algebra B such that B_K is K-smooth and there exists an isomorphism $B^{\wedge}_{(\varpi)} \simeq A$ of $\mathcal{O}_K$-algebras.*

Theorem 1.3 provides a very general machinery to reduce certain (local) questions about smooth rigid-analytic spaces to analogous questions about smooth algebraic varieties. For example, this approach has been used in [Ber15] to study finiteness of étale cohomology of rigid-analytic spaces, in [Tem17] to study questions related to semi-stable reduction, and in [Zav21b, §2.5] to study properties of dualizing modules on rig-smooth admissible formal schemes.

However, Theorem 1.3 is still not strong enough to reduce questions about *singular* rigid-analytic spaces to analogous questions about singular algebraic varieties. Furthermore, it is well-known that a naive analogue of Theorem 1.3 is false if one does not impose any smoothness assumptions on B. One of the key results of this paper is that it is nevertheless always possible to algebraize a geometrically reduced affinoid rigid-analytic space in "one direction":

Theorem 1.4 (Partial Algebraization; Corollary 2.15)**.** *Let K be an non-archimedean field, let $\varpi \in \mathcal{O}_K$ be a pseudo-uniformizer, and let A_0 be a flat, topologically finite type $\mathcal{O}_K$-algebra such that $A \coloneqq A_0[\frac{1}{\varpi}]$ is a K-affinoid algebra of Krull dimension $d > 0$ and $\mathrm{Spa}\,(A, A^\circ)$ is geometrically reduced in the sense of* [Con99, §3.3]*. Put $R = \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$. Then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra B with an $\mathcal{O}_K$-linear isomorphism $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq A_0$.*

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[1] See [Tem08, Prop. 3.3.2] for the fact that the $\mathcal{O}_K \to A$ is formally smooth "outside $\mathrm{V}(\varpi)$" in the sense of [Elk73, p.581] if and only if $\mathcal{O}_K \to A$ is rig-smooth in the sense of [BLR95b, Def. 3.1]. Strictly speaking, [Tem08, Prop. 3.3.2] is written under the additional assumption that K is discretely valued, but the proof goes through for a general non-archimedean field K.

Corollary 1.5 (Partial Algebraization II; Corollary 2.16)**.** *Let K, ϖ, and A_0 be as in Theorem 1.4. Then there is a finitely presented, quasi-finite morphism $\mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d] \to B$ and an isomorphism $B^{\wedge}_{(\varpi)} \simeq A_0$ of $\mathcal{O}_K$-algebras.*

Unlike Theorem 1.3, Theorem 1.4 does not allow to directly reduce questions about geometrically reduced (affinoid) rigid-analytic spaces to similar questions about general (affine) algebraic varieties. However, it does give a quite robust tool of reducing problems about geometrically reduced (affinoid) rigid-analytic spaces of dimension d to algebro-geometric problems for (algebraic) curves over $\operatorname{Spec} K\langle T_1, \dots, T_{d-1}\rangle$. We implement this strategy to prove the remaining cases of Artin–Grothendieck Vanishing for affinoid algebras, we discuss this proof in more detail in the next section. We also expect this strategy to be useful for other problems in rigid-analytic geometry. For example, the first author knows how to use Corollary 1.5 to prove [Zav21a, Theorem 1.3.5] without using any input from [BS22] or [Sch17].

1.2. **Artin–Grothendieck Vanishing.** In this paper, we apply the partial algebraization techniques to study Artin–Grothendieck Vanishing for affinoid algebras. To put things into context, we recall that the theory of étale cohomology of schemes has been extensively studied in [SGA 4], [SGA $4\frac{1}{2}$], and [SGA 5]. Among the many important results obtained in *loc. cit.*, Artin and Grothendieck have proven the following celebrated vanishing result:

Theorem 1.6 (Artin–Grothendieck Vanishing; [SGA 4, Exp. XIV, Cor. 3.2])**.** *Let k be a separably closed field, and let A be a finite type k-algebra. Then, for any torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, we have*

$$\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} A, \mathcal{F}) = 0$$

for $i > \dim A$.

In this paper, we discuss two possible analogues of Theorem 1.6 in the rigid-analytic world. The first analogue is a version of Artin–Grothendieck Vanishing for affinoid algebras, which was conjectured in [BM21, Remark 7.4]. We use the partial algebraization techniques to prove this conjecture in full generality:

Theorem 1.7 (Artin–Grothendieck Vanishing for affinoid algebras; Theorem 3.1)**.** *Let C be an algebraically closed non-archimedean field, and let A be a C-affinoid algebra. Then, for any torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, we have*

$$\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} A, \mathcal{F}) = 0$$

for $i > \dim A$.

Remark 1.8. We recall that [BGR84, Prop. 3.4.1/6] and our convention that non-archimedean fields are complete imply that any separably closed non-archimedean field is algebraically closed. Therefore, there is no extra generality in considering separably closed non-archimedean fields in the formulation of Theorem 1.7.

Theorem 1.7 has been previously established under certain additional assumptions on $\mathcal{F}$:

1. For a non-archimedean field C of characteristic $p > 0$ and a sheaf $\mathcal{F}$ of $\mathbf{Z}/p\mathbf{Z}$-modules, the conclusion of Theorem 1.7 follows from [SGA 4, Exp. X, Th. 5.1];
2. Using the arc-topology, Bhatt and Mathew proved Theorem 1.7 for sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules when n is invertible in $\mathcal{O}_C$ (see [BM21, Th. 7.3]);
3. In general, [BM21, Th. 7.3] shows that $\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} A, \mathcal{F}) = 0$ for any torsion étale sheaf $\mathcal{F}$ and $i > \dim A + 1$.

Therefore, the main new contribution of our work is the case when C is a non-archimedean field of mixed characteristic $(0, p)$ and $\mathcal{F}$ is a sheaf of $\mathbf{Z}/p\mathbf{Z}$-modules. Nevertheless, our proof of Theorem 1.7 is independent of [BM21] and works uniformly for any ground field C and any sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules $\mathcal{F}$ (as long as n is invertible in C).

The other possible rigid-analytic analogue of Theorem 1.6 was conjectured by Hansen:

Conjecture 1.9 ([Han20, Conj. 1.2])**.** *Let C be an algebraically closed non-archimedean field, let A be an C-affinoid algebra, and let n be an integer invertible in $\mathcal{O}_C$. Then, for any Zariski-constructible sheaf $\mathcal{F}$ (see Definition 4.2) of $\mathbf{Z}/n\mathbf{Z}$-modules on $\operatorname{Spa}(A, A^\circ)_{\text{ét}}$, we have*

$$\mathrm{H}^i_{\text{ét}}(\operatorname{Spa}(A, A^\circ), \mathcal{F}) = 0$$

for $i > \dim A$.

Conjecture 1.9 is known when $\operatorname{char} C = 0$ due to [BM21, Th. 7.3] and [Han20, Th. 1.10]. In this paper, we prove a *stronger* version of this conjecture, allowing for more general n, when $\operatorname{char} C = 0$ or A is of dimension 1. We also establish some *weaker* statements when $\operatorname{char} C = p > 0$. We formulate the results of our paper more precisely below.

Theorem 1.10 (Rigid-analytic Artin–Grothendieck Vanishing; Theorem 4.4)**.** *Let C be an algebraically closed non-archimedean field, let A be an affinoid C-algebra, and let $\mathcal{F}$ be an algebraic torsion étale sheaf on* $\operatorname{Spa}(A, A^\circ)$ *(in the sense of Definition 4.1). Then*

$$\mathrm{H}^i_{\text{ét}}(\operatorname{Spa}(A, A^\circ), \mathcal{F}) = 0 \tag{1.11}$$

for $i > \dim A$. *In particular, (1.11) holds in either of the following situations:*

(1) $\mathcal{F}$ is a lisse sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer $n > 0$;

(2) $\operatorname{char} C = 0$ *and $\mathcal{F}$ is a Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer $n > 0$.*

We emphasize that Theorem 1.10 has no assumptions on the integer n. In particular, it applies to Zariski-constructible $\mathbf{F}_p$-sheaves on an affinoid space over a field C of mixed characteristic $(0, p)$. Unfortunately, Theorem 1.10 does not imply the full version of Conjecture 1.9 over a field of characteristic $p > 0$ due to the existence of non-algebraic constructible sheaves. Nevertheless, it recovers all previously-known versions of Artin–Grothendieck Vanishing in rigid-analytic geometry. We mention them below:

(1) In [Ber96, Th. 6.1], Berkovich treats the case when $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules, n is invertible in $\mathcal{O}_C$, and both A and $\mathcal{F}$ are "algebraizable" in a certain sense;

(2) In [Han20, Th. 1.4], Hansen treats the case when n is invertible in $\mathcal{O}_C$, $\mathcal{F} = \underline{\mathbf{Z}/n\mathbf{Z}}$ is the constant sheaf, and $\operatorname{Spa}(A, A^\circ)$ is defined over a discretely valued non-archimedean field $K \subset C$;

(3) In [Han20, Th. 1.3], Hansen also treats the case when $\operatorname{char} C = 0$, n is invertible in $\mathcal{O}_C$, $\mathcal{F}$ is a Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules, and both $\operatorname{Spa}(A, A^\circ)$ and $\mathcal{F}$ are defined over a discretely valued non-archimedean field $K \subset C$;

(4) Using [BM21, Th. 7.3], one can easily deduce Theorem 1.10 under the additional hypothesis that n is invertible in $\mathcal{O}_C$ and $\mathcal{F}$ is a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules.

We also establish a weaker version of Conjecture 1.9 that applies to all Zariski-constructible sheaves in all characteristics. Unlike Theorem 1.10, this result requires a completely new set of ideas and our proof does not use any partial algebraization techniques.

Theorem 1.12 (Rigid-analytic Artin–Grothendieck Vanishing in top degree; Theorem 4.10)**.** *Let C be an algebraically closed non-archimedean field, let A be a C-affinoid algebra of dimension $d \geq 1$, let n be an integer invertible in $\mathcal{O}_C$, and let $\mathcal{F}$ be a Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on* $\operatorname{Spa}(A, A^\circ)_{\text{ét}}$. *Then*

$$\mathrm{H}^{2d}_{\text{ét}}(\operatorname{Spa}(A, A^\circ), \mathcal{F}) = 0.$$

As far as we are aware, Theorem 1.12 is the first progress towards Conjecture 1.9 for non-algebraic Zariski-constructible sheaves.

We also note that Theorem 1.12 and Theorem 1.10 prove a stronger version of Conjecture 1.9, allowing for any n invertible in C, when $\operatorname{Spa}(A, A^\circ)$ is a curve; see Corollary 4.12 for more details.

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1.4. **Conventions.** Throughout this paper, we follow [Hub93] and use the multiplicative convention for valuations. A *non-archimedean field* is a complete topological field K whose topology is defined by a nontrivial rank-1 valuation $|.|\colon K \to \mathbf{R}_{\geq 0}$. We say that a K-algebra A is *K-affinoid* if there is surjection $K\langle X_1, \dots, X_d\rangle \twoheadrightarrow A$ for some $d \geq 0$.

For a commutative ring A and an ideal $I \subset A$ (resp. a finitely generated ideal I), we denote by A^{h}_I (resp. $A^{\wedge}_I$) the *I-adic henselization* (resp. *I-adic completion*) of A. We refer to [Sta24, Tag 0EM7] and [Sta24, Tag 00M9] for a detailed discussion of these notions.

We say that an A-module M is *I-adically complete* if the natural map $M \to \lim_n M/I^n M$ is an isomorphism, i.e., M is complete and separated in the I-adic topology.

In this paper, we will crucially use the notion of henselian pairs, we refer the reader to [Sta24, Tag 09XD] for the definition and detailed discussion of this notion. We also use the notion of rig-smoothness (over a pair (A, I) of a noetherian ring A and an ideal $I \subset A$) as defined in [Sta24, Tag 0GAI]; this definition is equivalent to the one from [BLR95a, §3].

2. PARTIAL ALGEBRAIZATION

2.1. **Weierstrass polynomials.** Throughout this section, we fix a non-archimedean field K with ring of integers $\mathcal{O}_K$ and a pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We denote the unique maximal ideal of $\mathcal{O}_K$ by $\mathfrak{m}_K$.

The main goal of this section is to recall the notion of a Weierstrass polynomial and study its properties. The main result of this section is Corollary 2.8 which will be crucial for our proof of the rigid-analytic Artin–Grothendieck Vanishing Theorem.

Notation 2.1. Throughout this paper, we denote the ring $\mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$ by R.

Notation 2.2. We also denote by $\mathfrak{m}_K\langle X_1, \dots, X_d\rangle \subset \mathcal{O}_K\langle X_1, \dots, X_d\rangle$ the ideal defined as

$$\mathfrak{m}_K\langle X_1, \dots, X_d\rangle := \left\{ f = \sum_{i_1,\dots,i_d} a_{i_1,\dots,i_d} X_1^{i_1} \dots X_d^{i_d} \mid a_{i_1,\dots,i_d} \in \mathfrak{m}_K \, \forall \, i_1, \dots, i_d \right\}.$$

Definition 2.3. An element $f \in \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d] \subset \mathcal{O}_K\langle X_1, \dots, X_d\rangle$ is a *Weierstrass polynomial of degree s* if $f = X_d^s + \sum_{i=0}^{s-1} g_i(X_1, \dots, X_{d-1}) X_d^i$ for some $g_i \in \mathfrak{m}_K\langle X_1, \dots, X_{d-1}\rangle$.

For the next construction, we fix an element $f \in R$. We also note that [Sta24, Tag 0DYD] ensures that the ring $R^{\mathrm{h}}_{(\varpi)}$ is (ϖf)-adically henselian. Furthermore, [Sta24, Tag 05GG] and [Sta24, Tag 090T] imply that the rings $R^{\wedge}_{(\varpi f)}$ and $R^{\wedge}_{(\varpi)}$ are (ϖf)-adically complete. In particular, they are also (ϖf)-adically henselian by [Sta24, Tag 0ALJ].

Construction 2.4. In the notation as above, the universal property of henselization (see [Sta24, Tag 0A02]) and the universal property of completion imply that there are unique R-algebra homomorphisms

$$\alpha_f \colon R^{\mathrm{h}}_{(\varpi f)} \to R^{\mathrm{h}}_{(\varpi)},$$

$$\beta_f \colon R^{\wedge}_{(\varpi f)} \to R^{\wedge}_{(\varpi)}$$

such that the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] & R^{\mathrm{h}}_{(\varpi f)} \arrow[r] \arrow[d, "\alpha_f"] & R^{\wedge}_{(\varpi f)} \arrow[d, "\beta_f"] \\
& R^{\mathrm{h}}_{(\varpi)} \arrow[r] & R^{\wedge}_{(\varpi)}
\end{tikzcd}$$

commutes, where the vertical arrows are the natural ones.

The main goal of this section is to show that the morphisms α_f and β_f are isomorphisms when f is a *Weierstrass polynomial*. For this, we will need a number of preliminary lemmas:

Lemma 2.5. *Let $M \in D(\mathcal{O}_K)$ be an object of the derived category. Then $M = 0$ if and only if $M_K = 0$ and $M \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi) = 0$.*

Proof. The "only if" direction is clear, so we only need to show the "if" direction. We assume that $M_K = 0$ and $M \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi) = 0$. The latter assumption implies that the natural morphism $M \xrightarrow{\cdot \varpi} M$ is an isomorphism. Therefore, we conclude that $\mathrm{H}^i(M) \simeq \mathrm{H}^i(M_K) = 0$ for any integer i. Thus, $M = 0$. □

Lemma 2.6. *Let $f \in R$ be a Weierstrass polynomial of degree s. Then the natural morphism*

$$\gamma\colon R/(\varpi f) \to R^{\wedge}_{(\varpi)}/(\varpi f)$$

is an isomorphism.

Proof. First, we note that $R^{\wedge}_{(\varpi)}$ is simply the integral Tate algebra $\mathcal{O}_K\langle X_1, \dots, X_d\rangle$. Therefore, we need to show that the natural morphism

$$\gamma\colon \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]/(\varpi f) \to \mathcal{O}_K\langle X_1, \dots, X_d\rangle/(\varpi f)$$

is an isomorphism. Lemma 2.5 ensures that it suffices to show that γ_K and $\gamma \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi)$ are isomorphisms.

Step 1. *γ_K is an isomorphism.* Since ϖ is invertible in $K\langle X_1, \dots, X_{d-1}\rangle[X_d]$, it suffices to show that the natural morphism

$$K\langle X_1, \dots, X_{d-1}\rangle[X_d]/(f) \to K\langle X_1, \dots, X_d\rangle/(f)$$

is an isomorphism. This directly follows from [BGR84, Prop. 5.2.3/3(ii)].

Step 2. *$\gamma \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi)$ is an isomorphism.* We note that both R and $R^{\wedge}_{(\varpi)}$ are torsion-free (because they can be realized as subrings of $\mathcal{O}_K[\![X_1, \dots, X_d]\!]$), so

$$R/(\varpi) = R \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi) \quad \text{and} \quad R^{\wedge}_{(\varpi)}/(\varpi f) = R^{\wedge}_{(\varpi)} \otimes^L_R R/(\varpi f).$$

Therefore, $\gamma \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/(\varpi)$ can be identified with the morphism

$$R/(\varpi f) \otimes^L_R R/(\varpi) \to R^{\wedge}_{(\varpi)} \otimes^L_R R/(\varpi f) \otimes^L_R R/(\varpi).$$

But then it suffices to show that $R/(\varpi) \to R^{\wedge}_{(\varpi)} \otimes^L_R R/(\varpi)$ is an equivalence. Since $R^{\wedge}_{(\varpi)}$ is torsion-free, the question reduces to showing that the natural morphism

$$R/(\varpi) \to R^{\wedge}_{(\varpi)}/(\varpi)$$

is an isomorphism. This follows from directly [Sta24, Tag 05GG]. ☐

Corollary 2.7. *Let $f \in R$ be a Weierstrass polynomial. Then $R^{\mathrm{h}}_{(\varpi f)}$ is (ϖ)-adically henselian.*

Proof. First, we note that [Sta24, Tag 0DYD] and [Sta24, Tag 0AGU] imply that it suffices to show that

$$R^{\mathrm{h}}_{(\varpi f)}/(\varpi f) = R/(\varpi f)$$

is (ϖ)-adically henselian. Now Lemma 2.6 implies that

$$R/(\varpi f) \simeq R^{\wedge}_{(\varpi)}/(\varpi f).$$

Now note that $R^{\wedge}_{(\varpi)}$ is (ϖ)-adically henselian because it is (ϖ)-adically complete. Thus, its quotient $R^{\wedge}_{(\varpi)}/(\varpi f)$ is also (ϖ)-adically henselian due to [Sta24, Tag 09XK]. ☐

Corollary 2.8. *Let $f \in R$ be a Weierstrass polynomial. Then the natural morphisms*

$$\alpha_f\colon R^{\mathrm{h}}_{(\varpi f)} \to R^{\mathrm{h}}_{(\varpi)},$$
$$\beta_f\colon R^{\wedge}_{(\varpi f)} \to R^{\wedge}_{(\varpi)}$$

are isomorphisms.

Proof. We first show that α_f is an isomorphism. The universal property of $R^{\mathrm{h}}_{(\varpi)}$ and the fact that $R^{\mathrm{h}}_{(\varpi f)}$ is (ϖ)-adically henselian (see Corollary 2.7) allows us to define a canonical R-algebra homorphism $\alpha'_f\colon R^{\mathrm{h}}_{(\varpi)} \to R^{\mathrm{h}}_{(\varpi f)}$. Therefore, it suffices to show that $\alpha_f \circ \alpha'_f = \mathrm{id}$ and $\alpha'_f \circ \alpha_f = \mathrm{id}$, but this also easily follows from the universal properties of $R^{\mathrm{h}}_{(\varpi)}$ and $R^{\mathrm{h}}_{(\varpi f)}$ respectively.

Now we show that β_f is an isomorphism. Since both $R^{\wedge}_{(\varpi f)}$ and $R^{\wedge}_{(\varpi)}$ are (ϖf)-adically complete, it suffices to show that the natural morphism

$$R^{\wedge}_{(\varpi f)}/(\varpi f)^n = R/(\varpi^n f^n) \to R^{\wedge}_{(\varpi)}/(\varpi^n f^n)$$

is an isomorphism for any $n > 0$. This follows directly from Lemma 2.6 and the observation that f^n is a Weierstrass polynomial for any $n > 0$. ☐

2.2. Algebraization techniques.

Throughout this section, we fix the notation of the previous section. In particular, K is a non-archimedean field with ring of integers $\mathcal{O}_K$, and a pseudo-uniformizer $\varpi \in \mathcal{O}_K$. Recall that R denotes the ring $\mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$.

The key result of this section is Theorem 2.11. Combined with a generically étale version of Noetherian normalization (see Theorem 2.14), it will be the crucial ingredient in our proof of the rigid-analytic Artin–Grothendieck vanishing.

Besides this, we also recall the results of Elkik and Fujiwara–Gabber (see [Elk73] and [Fuj95]) in the non-noetherian context. These results are certainly well-known to the experts, but we prefer to spell them out explicitly since some care is nedeed in this non-noetherian situation.

We first discuss a version of Elkik's algebraization that will be relevant for our purposes.

Lemma 2.9. *(1) The ring $R^{\mathrm{h}}_{(\varpi)}$ is (ϖ)-adically universally adhesive (in the sense of* [FGK11, Def. 7.1.3]*);*

(2) The natural morphism $R^{\mathrm{h}}_{(\varpi)} \to R^{\wedge}_{(\varpi)}$ is faithfully flat;

(3) If $f \in R$ is a Weierstrass polynomial, then the natural morphism $R^{\mathrm{h}}_{(\varpi f)} \to R^{\wedge}_{(\varpi f)}$ is faithfully flat.

Proof. First, (1) follows directly from [FGK11, Th. 7.3.2] and [FK18, Prop. 0.8.5.10]. For the proof of (2), we note that the proof of [Sta24, Tag 0AGV] goes through if one uses [FGK11, Prop. 4.3.4] (and (1)) in place of [Sta24, Tag 00MB]. Now (3) follows directly from (2) and Corollary 2.8. □

Lemma 2.10 (Elkik's Algebraization)**.** *Let $f \in R$ be a Weierstrass polynomial and B a finite, finitely presented*[2] *$R^{\wedge}_{(\varpi f)}$-algebra such that $B[\frac{1}{\varpi f}]$ is étale over $R^{\wedge}_{(\varpi f)}[\frac{1}{\varpi f}]$. Then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi f)}$-algebra $\widetilde{B}$ such that $\widetilde{B} \otimes_{R^{\mathrm{h}}_{(\varpi f)}} R^{\wedge}_{(\varpi f)} \simeq B$.*

If R were a noetherian algebra, this would automatically follow from [Elk73, Th. 5].

Proof. First, we note that [GR03, Prop. 5.4.54] implies that there is a finite étale $R^{\mathrm{h}}_{(\varpi f)}[\frac{1}{\varpi f}]$-algebra $\widetilde{B}'$ such that

$$\widetilde{B}' \otimes_{R^{\mathrm{h}}_{(\varpi f)}[\frac{1}{\varpi f}]} R^{\wedge}_{(\varpi f)}\left[\frac{1}{\varpi f}\right] \simeq B\left[\frac{1}{\varpi f}\right].$$

Now Lemma 2.9(3) implies that $R^{\mathrm{h}}_{(\varpi f)} \to R^{\wedge}_{(\varpi f)}$ is faithfully flat. Thus formal gluing (see [Sta24, Tag 05ES] and [Sta24, Tag 05EU]) implies that we can find an $R^{\mathrm{h}}_{(\varpi f)}$-algebra $\widetilde{B}$ such that

$$\widetilde{B} \otimes_{R^{\mathrm{h}}_{(\varpi f)}} R^{\wedge}_{(\varpi f)} \simeq B, \, \widetilde{B}\left[\frac{1}{\varpi f}\right] \simeq \widetilde{B}'.$$

Since $R^{\mathrm{h}}_{(\varpi f)} \to R^{\wedge}_{(\varpi f)}$ is faithfully flat, faithfully flat descent (see [Sta24, Tag 03C4] and [Sta24, Tag 00QQ]) implies that $\widetilde{B}$ is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi f)}$-algebra. □

Theorem 2.11. *Let $f \in R$ be a Weierstrass polynomial and B a finite, finitely presented $R^{\wedge}_{(\varpi)} = \mathcal{O}_K\langle T_1, \dots, T_d\rangle$-algebra. If $B[\frac{1}{\varpi f}]$ is étale over $R^{\wedge}_{(\varpi)}[\frac{1}{\varpi f}] = K\langle T_1, \dots, T_d\rangle[\frac{1}{f}]$, then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra $\widetilde{B}$ such that*

$$\widetilde{B} \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq B.$$

Proof. Corollary 2.8 implies that $R^{\wedge}_{(\varpi)}$ (resp. $R^{\mathrm{h}}_{(\varpi)}$) is canonically isomorphic to $R^{\wedge}_{(\varpi f)}$ (resp. $R^{\mathrm{h}}_{(\varpi f)}$) as an R-algebra. Therefore, Lemma 2.10 provides us with a finite, finitely presented $R^{\mathrm{h}}_{(\varpi f)} \simeq R^{\mathrm{h}}_{(\varpi)}$-algebra $\widetilde{B}$ such that

$$\widetilde{B} \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq \widetilde{B} \otimes_{R^{\mathrm{h}}_{(\varpi f)}} R^{\wedge}_{(\varpi f)} \simeq B.$$ □

Now we discuss a version of the Fujiwara–Gabber comparison theorem that will be relevant for our purposes.

Lemma 2.12. *Let M be a finite $R^{\mathrm{h}}_{(\varpi)}$-module, and put $M' := M \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}$. Then there is an integer N such that $M[\varpi^\infty] = M[\varpi^N]$ and $M'[\varpi^\infty] = M'[\varpi^N]$.*

---

[2] Recall that an A-algebra A' is called finite, finitely presented if it is finitely generated as an A-module and finitely presented as an A-algebra.

Proof. First, Lemma 2.9(1) and the definition of adhesive rings imply that $M[\varpi^\infty] = M[\varpi^N]$ for some integer N. Then Lemma 2.9(2) and a standard flatness argument imply that $M[\varpi^m] \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} = M'[\varpi^m]$ for every integer $m \geq 0$. Therefore, $M'[\varpi^m] = M'[\varpi^N]$ for any $m \geq N$. By taking the union over all m, we get $M'[\varpi^\infty] = M'[\varpi^N]$. □

Lemma 2.13 (Fujiwara–Gabber)**.** *Let B be a finite $R^{\mathrm{h}}_{(\varpi)}$-algebra, $g\colon \operatorname{Spec}\left(B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}\left[\frac{1}{\varpi}\right]\right) \to \operatorname{Spec} B\left[\frac{1}{\varpi}\right]$ the natural morphism, and $\mathcal{F}$ a torsion étale sheaf on $\operatorname{Spec} B\left[\frac{1}{\varpi}\right]$. Then the natural morphism*

$$\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathcal{F}\right) \to \mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec}\left(B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}\left[\frac{1}{\varpi}\right]\right), g^*\mathcal{F}\right)$$

is an isomorphism.

If $R^{\mathrm{h}}_{(\varpi)}$ were a noetherian ring, this would almost immediately follow from [Fuj95, Cor. 6.6.4].

Proof. First, we note that [Sta24, Tag 09XK] ensures that B and $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}$ are (ϖ)-adically henselian. Furthermore, Lemma 2.12 ensures that both B and $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}$ have bounded ϖ^∞-torsion. Therefore, the result follows from [BČ22, Th. 2.3.4] with $A = B$, $A' = B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)}$, $t = \varpi$, $I = B$, and $U = \operatorname{Spec} B\left[\frac{1}{\varpi}\right]$ (alternatively, one can use [ILO14, Exp. XX, §4.4] or [BM21, Th. 6.11]). □

2.3. **Noether normalization and partial algebraization.** In this section, we show a generically étale version of the Noether Normalization Theorem. Then we combine it with the techniques of the previous section to get a "partial algebraization" result. This will be the key technical ingredient in our proof of Artin–Grothendieck vanishing for affinoid algebras.

Throughout the section, we fix a non-archimedean field K with ring of integers $\mathcal{O}_K$, residue field k, and a pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We recall that R denotes the ring $\mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$.

Theorem 2.14 (Generically Étale Noether Normalization)**.** *Let A_0 be a flat, topologically finite type $\mathcal{O}_K$-algebra such that $A \coloneqq A_0[\frac{1}{\varpi}]$ is a K-affinoid algebra of Krull dimension $d > 0$ and $\operatorname{Spa}(A, A^\circ)$ is geometrically reduced in the sense of [Con99, §3.3]. Then there is a finite, finitely presented morphism $h\colon \mathcal{O}_K\langle X_1, \dots, X_d\rangle \to A_0$ and a Weierstrass polynomial $f \in \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d]$ such that h is étale away from $\mathrm{V}(\varpi f)$, i.e., the induced map*

$$h_K\left[\frac{1}{f}\right]\colon K\langle X_1, \dots, X_d\rangle\left[\frac{1}{f}\right] \to A\left[\frac{1}{f}\right]$$

is finite étale.

Before we begin the proof of Theorem 2.14, we want to mention some immediate corollaries of this theorem and make some remarks about the needed generality of these results.

Corollary 2.15 (Partial Algebraization I)**.** *Keep the notation of Theorem 2.14. Then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra B with an $\mathcal{O}_K$-linear isomorphism $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq A_0$.*

Proof. This follows directly from Theorem 2.11 and Theorem 2.14. □

Corollary 2.16 (Partial Algebraization II)**.** *Keep the notation of Theorem 2.14. Then there is a finitely presented, quasi-finite morphism $R = \mathcal{O}_K\langle X_1, \dots, X_{d-1}\rangle[X_d] \to B$ and an isomorphism $B^{\wedge}_{(\varpi)} \simeq A_0$ of $\mathcal{O}_K$-algebras.*

Proof. We write $R^{\mathrm{h}}_{(\varpi)} = \operatorname{colim}_I R_i$ as a filtered colimit of étale R-algebras R_i such that the natural map $R/\varpi R \to R_i/\varpi R_i$ is an isomorphism. Then Corollary 2.15 and a standard approximation argument implies that we can find a finite, finitely presented morphism $R_i \to B_i$ such that $R^{\wedge}_{(\varpi)} \otimes_{R_i} B_i \simeq A_0$. Since $R \to R_i$ induces an isomorphism on ϖ-adic completions, we conclude that $A_0 \simeq (R_i)^{\wedge}_{(\varpi)} \otimes_{R_i} B_i$. Finally, [FGK11, Th. 7.3.2 and Prop. 4.3.4] imply that $(B_i)^{\wedge}_{(\varpi)} \simeq A_0$. Since étale morhpisms are quasi-finite, we conclude that $B \coloneqq B_i$ does the job. □

Corollary 2.17 (Partial Algebraization III)**.** *Let C be an algebraically closed non-archimedean field, and let A be a reduced C-affinoid algebra of dimension $d > 0$. Then there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra B with an $\mathcal{O}_C$-linear isomorphism $B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq A^\circ$.*

Proof. This follows directly from Corollary 2.15 and [BLR95b, Th. 1.2]. ☐

We note that if $\operatorname{char} K = 0$, then Theorem 2.14 is an easy consequence of the usual Noether Normalization [FK18, Th. 0.9.2.10] and Lemma 2.18 below. Furthermore, one can use (the proof of) [ALY23, Prop. A.2, Step 4] and Lemma 2.18 to prove Theorem 2.14 under the additional assumptions that K is algebraically closed, A is a domain, and $A_0 = A^\circ$. This generality suffices to conclude Corollary 2.17 when A is a domain. This is, in turn, the only result of this section that is used in the rest of the paper.

However, the strategy used in [ALY23] is inadequate for the purpose of proving Theorem 2.14 in full generality. The essential difficulty comes from the fact that the "special fiber" $A_0 \otimes_{\mathcal{O}_K} k$ does not need to be reduced in general. We believe Theorem 2.14 and Corollary 2.15 are itself of independent interest, so we decided to provide a complete proof of Theorem 2.14 in full generality.

The rest of this section is devoted to the proof of Theorem 2.14.

Lemma 2.18. *Let $d > 0$ be a positive integer, and let $f \in \mathcal{O}_K\langle X_1, \dots, X_d\rangle$ be an element of norm 1. Then there is a continuous $\mathcal{O}_K$-linear automorphism*

$$\sigma\colon \mathcal{O}_K\langle X_1, \dots, X_d\rangle \to \mathcal{O}_K\langle X_1, \dots, X_d\rangle,$$

an element $u \in \mathcal{O}_K\langle X_1, \dots, X_d\rangle^\times$, and a Weierstrass polynomial ω such that $\sigma(f) = u\omega$.

Proof. The proof of [Bos14, Lemma 2.2/7] implies that we can find an $\mathcal{O}_K$-linear continuous automorphism σ of $\mathcal{O}_K\langle X_1, \dots, X_d\rangle$ such that $\sigma(f)$ (considered as an object of $K\langle X_1, \dots, X_d\rangle$) is X_d-distinguished (in the sense of [Bos14, Def. 2.2/6]) and $|\sigma(f)| = |f| = 1$. Therefore, [Bos14, Cor. 2.2/9] implies that there is a unit $u \in K\langle T_1, \dots, X_d\rangle^\times$ and a Weierstrass polynomial ω such that $\sigma(f) = u\omega$. The only thing we are left to show is that $u \in \mathcal{O}_K\langle X_1, \dots, X_d\rangle$ and $u^{-1} \in \mathcal{O}_K\langle X_1, \dots, X_d\rangle$.

Since the Gauss-norm is multiplicative on $K\langle X_1, \dots, X_d\rangle$ (see [Bos14, p. 12]), it suffices to show that $|u| = 1$. But this follows from the equation $\sigma(f) = u\omega$ and the observation that $|\sigma(f)| = |\omega| = 1$. ☐

We recall that, for every K-affinoid algebra A, there is a functorial continuous map

$$r_A\colon |\operatorname{Spa}(A, A^\circ)| \to |\operatorname{Spec} A|$$

that sends a valuation $v \in \operatorname{Spa}(A, A^\circ)$ to a prime ideal $\operatorname{supp}(v) \coloneqq v^{-1}(\{0\}) \subset A$.

Lemma 2.19. *Let A be an affinoid K-algebra. Then the map $r_A\colon |\operatorname{Spa}(A, A^\circ)| \to |\operatorname{Spec} A|$ is surjective.*

Proof. For brevity, we denote by T_d the Tate algebra $K\langle X_1, \dots, X_d\rangle$. We pick a prime ideal $\mathfrak{p} \subset A$ and wish to find a valuation $v \in \operatorname{Spa}(A, A^\circ)$ such that $\operatorname{supp}(v) = \mathfrak{p}$. Since the construction of r is functorial in A, we may replace A by $A/\mathfrak{p}$ to achieve that A is a domain and $\mathfrak{p} = (0)$. Then we use [Bos14, Prop. 3.1/3] to get a finite monomorphism $f^\#\colon T_d \to A$. This induces morphisms $f^{\mathrm{alg}}\colon \operatorname{Spec} A \to \operatorname{Spec} T_d$ and $f^{\mathrm{ad}}\colon \operatorname{Spa}(A, A^\circ) \to \mathbf{D}^d$. One easily checks that f^{ad} is surjective and $f^{-1}(\eta_{T_d}) = \eta_A$, where η_{T_d} and η_A are the generic points of $\operatorname{Spec} T_d$ and $\operatorname{Spec} A$ respectively. Combining these facts with functoriality of r, we reduce the question to the case $A = T_d$ and $\mathfrak{p} = (0)$. In this case, we note that the supremum norm $v_\eta\colon T_d \to \mathbf{R}_{\geq 0}$ (see [Bos14, Prop. 2.2/3]) defines a point of $\operatorname{Spa}(T_d, T_d^\circ)$ with $\operatorname{supp}(v_\eta) = (0)$. This finishes the proof. ☐

Corollary 2.20. *Let A be an affinoid K-algebra, and let f be a non-zero element of A. Then $\left\{\operatorname{Spec} A\langle \frac{\varpi^n}{f}\rangle\right\}_{n\in\mathbf{N}} \to \operatorname{Spec} A\left[\frac{1}{f}\right]$ is a jointly surjective family of flat morphisms.*

Proof. First, we note that [Hub94, (II.1) (iv) on page 530] implies that $A \to A\langle\frac{\varpi^n}{f}\rangle$ is flat for any $n \in \mathbf{N}$. This formally implies that $A\left[\frac{1}{f}\right] \to A\langle\frac{\varpi^n}{f}\rangle$ is flat as well. Now, in order to see that the family $\left\{\operatorname{Spec} A\langle\frac{\varpi^n}{f}\rangle\right\}_{n\in\mathbf{N}} \to \operatorname{Spec} A\left[\frac{1}{f}\right]$ is jointly surjective, we invoke Lemma 2.19 to reduce the question to showing that the family $\left\{\operatorname{Spa}(A\langle\frac{\varpi^n}{f}\rangle, A\langle\frac{\varpi^n}{f}\rangle^\circ)\right\}_{n\in\mathbf{N}} \to \operatorname{Spa}(A, A^\circ)\big(f \neq 0\big)$ is jointly surjective. This follows from the standard observation that $\operatorname{Spa}(A, A^\circ)\big(f \neq 0\big) = \bigcup_{n\in\mathbf{N}} \operatorname{Spa}(A\langle\frac{\varpi^n}{f}\rangle, A\langle\frac{\varpi^n}{f}\rangle^\circ) \subset \operatorname{Spa}(A, A^\circ)$. ☐

We also recall the notion of completed differential forms. We fix a flat, topologically finite type $\mathcal{O}_K$-algebra A_0 and put $A = A_0\big[\frac{1}{\varpi}\big]$. We denote by $\widehat{\Omega}^1_{A_0/\mathcal{O}_K} := (\Omega^1_{A_0/\mathcal{O}_K})^{\wedge}_{(\varpi)}$ the (ϖ)-adic completion of the usual algebraic differentials $\Omega^1_{A_0/\mathcal{O}_K}$. Likewise, we denote by $\widehat{\Omega}^1_{A/K} := \widehat{\Omega}^1_{A_0/\mathcal{O}_K}\big[\frac{1}{\varpi}\big]$ the "generic fiber" of $\widehat{\Omega}^1_{A_0/\mathcal{O}_K}$; we note that [BLR95a, Prop. 1.5] implies that $\widehat{\Omega}^1_{A/K}$ is independent of the choice of $A_0 \subset A$ and, therefore, is well-defined for any K-affinoid algebra A. We refer to [FK18, §I.5.1], [BLR95a, §1-3], and [Hub96, §1.6] for an extensive discussion of these notions.

Lemma 2.21. *Let A be a K-affinoid algebra of Krull dimension d. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes of* Spec A *corresponding to the d-dimensional irreducible components. Then, for each $j = 1, \dots r$, the $k(\mathfrak{p}_j)$-vector space $\widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ is generated by the residue classes of the form $\overline{df}$ with $f \in \bigcap_{i\neq j} \mathfrak{p}_i$.*

Proof. First, we note that [FK18, Cor. I.5.1.11] implies that the A-module $\widehat{\Omega}^1_{A/K}$ is generated by the elements of the form df for $f \in A$. Now we note that [Sta24, Tag 00OK] ensures that, for each $j = 1, \dots, r$, we can pick an element $y \in \big(\cap_{i\neq j} \mathfrak{p}_i\big) \smallsetminus \mathfrak{p}_j$. Therefore, for any $f \in A$, we have an equality $d(fy) = fdy + ydf$ in $\widehat{\Omega}^1_{A/K}$. Now since y becomes invertible in $k(\mathfrak{p}_j)$, we conclude that we have an equation $\overline{df} = \frac{\overline{d(fy)}}{\overline{y}} - \frac{\overline{fdy}}{\overline{y}}$ in $\widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$. Now the result follows since fy and y lie in $\cap_{i\neq j}\mathfrak{p}_i$. ☐

Lemma 2.22. *Let A be a non-zero K-affinoid algebra of Krull dimension d such that* Spa (A, A°) *is geometrically reduced in the sense of* [Con99, §3.3]*, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subset A$ be the minimal prime ideals corresponding to the d-dimensional irreducible components of* Spec A*. Then*

(1) the $k(\mathfrak{p}_j)$-vector space $\widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ has dimension d;

(2) for any finite morphism $h\colon \mathrm{Spa}\,(A, A^\circ) \to \mathbf{D}^d$, there is a non-zero element $f \in K\langle X_1, \dots, X_d\rangle$ such that the Zariski-open subspace Spa $(A, A^\circ)\big(f \neq 0\big)$ *is K-smooth of pure dimension d.*

Proof. First, we note that [Con99, p. 512] implies that Spa (A, A°) is K-smooth away from a nowhere dense Zariski-closed subset $Z' \subset \mathrm{Spa}\,(A, A^\circ)$. In particular, $\dim Z' < d$. Put Z to be the union of Z' and all irreducible components of dimension less than d, and also put U to be the open complement of Z. By construction, $\dim Z < d$ and U is a smooth rigid-analytic space over K of pure dimension d.

Now we note that the map $r_A\colon |\mathrm{Spa}\,(A, A^\circ)| \to |\mathrm{Spec}\, A|$ induces a bijection between Zariski-open subspaces (see [Zav24, Cor. B.6.8]). Thus, the fact that $\dim Z < d$ and Lemma 2.19 imply that there are points $v_j \in U$ such that $r_A(v_j) = \mathfrak{p}_j$. Now since $\Omega^1_{\mathrm{Spa}\,(A,A^\circ)/K}$ is a coherent sheaf associated to a finite A-module $\widehat{\Omega}^1_{A/K}$ and since U is smooth of pure dimension d, we see that $\widehat{\Omega}^1_{A/K} \otimes_A k(v_j)$ is of dimension d. This formally implies that $\dim \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j) = d$.

(2): For this, we note that (the easier special case of) [BGR84, Prop. 9.6.3/3] implies that $h(Z) \subset \mathbf{D}^d$ is a Zariski-closed subspace of dimension less than d. In particular, $h(Z) \neq \mathbf{D}^d$, so there is a non-zero element $f \in K\langle X_1, \dots, X_d\rangle$ such that $h(Z) \subset \mathrm{V}(f)$. Then Spa $(A, A^\circ)\big(f \neq 0\big)$ is K-smooth of pure dimension d. ☐

Lemma 2.23. *Keep the notation of Lemma 2.22, let $f_1, \dots, f_d \in A$ be a d-tuple of elements in A, and let $U \subset A$ be an open neighborhood of 0. Then there is a d-tuple of elements $g_1, \dots, g_d \in U$ such that the residue classes $\overline{d(f_1 + g_1)}, \dots, \overline{d(f_d + g_d)} \in \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ form a basis for every $j = 1, \dots, r$.*

Proof. First, we note that there is a topologically finite type $\mathcal{O}_C$-subalgebra $A_0 \subset A$ and an integer $n \in \mathbf{N}$ such that $\varpi^n A_0 \subset U$. Therefore, we may and do assume that $U = \varpi^n A_0$ for A_0 and n as above.

Now Lemma 2.22(1) implies that $\widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ is a $k(\mathfrak{p}_j)$-vector space of dimension d for each $j = 1, \dots, r$. Now we fix $j \in [1, \dots, r]$ and choose a subset $I_j \subset [1, \dots, d]$ such that the set $\big\{\overline{df_i}\big\}_{i\in I_j} \subset \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ is a maximal linearly independent subset of $\{\overline{df_i}\}_{i=1}^d \subset \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ (of size $\#I_j$). We put $g_{i,j} = 0$ for each $i \in I_j$, and then Lemma 2.21 implies that we can find a $\#([1, \dots, d] \smallsetminus I_j)$-tuple of elements $g_{i,j} \in \bigcap_{k\neq j} \mathfrak{p}_k$ for $i \notin I_j$ such that the the residue classes $\overline{d(f_1 + g_{1,j})}, \dots, \overline{d(f_d + g_{d,j})}$ form a basis in $\widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$. Then we can replace each $g_{i,j}$ by $\varpi^N g_{i,j}$ for $N \gg 0$ to achieve that $g_{i,j} \in U$.

Now, for $i = 1, \dots, d$, we put $g_i = \sum_{j=1}^{r} g_{i,j} \in U$. By construction, $g_{i,j}$ lies in $\mathfrak{p}_{j'}$ for every i and every $j \neq j'$. Therefore, $\overline{g_i} = \overline{g_{i,j}} \in k(\mathfrak{p}_j)$ for every i and j. Thus, we conclude that, for each $j = 1, \dots, r$, the set

$$\left\{\overline{d(f_i + g_i)}\right\}_{i=1}^{d} = \left\{\overline{d(f_i + g_{i,j})}\right\}_{i=1}^{d} \subset \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$$

is a basis. □

Lemma 2.24. *Keep the notation of Lemma 2.22, and let $h\colon T_d = K\langle X_1, \dots, X_d\rangle \to A$ be a finite morphism of K-algebras such that $\left\{\overline{d\big(h(X_i)\big)}\right\} \subset \widehat{\Omega}^1_{A/K} \otimes_A k(\mathfrak{p}_j)$ is a basis for each $j = 1, \dots, r$. Then there is a non-zero element $f \in K\langle X_1, \dots, X_d\rangle$ such that*

$$h\Big[\frac{1}{f}\Big] \colon K\langle X_1, \dots, X_d\rangle\Big[\frac{1}{f}\Big] \to A\Big[\frac{1}{f}\Big]$$

is finite étale.

Proof. First, we note that [Sta24, Tag 00OH] and reducedness of A imply that the natural morphism $A \otimes_{T_d} \mathrm{Frac}(T_d) \to \prod_{j=1}^{r} k(\mathfrak{p}_j)$ is an isomorphism. Therefore, [FK18, Prop. I.5.1.10] and our assumption on h imply that the natural morphism

$$\widehat{\Omega}^1_{T_d/K} \otimes_{T_d} A \otimes_{T_d} \mathrm{Frac}(T_d) \to \widehat{\Omega}^1_{A/K} \otimes_{T_d} \mathrm{Frac}(T_d)$$

is an isomorphism. Then a standard approximation argument implies that we can choose a non-zero element $f \in T_d$ such that $\big(\widehat{\Omega}^1_{T_d/K} \otimes_{T_d} A\big)\big[\frac{1}{f}\big] \to \widehat{\Omega}^1_{A/K}\big[\frac{1}{f}\big]$ is an isomorphism. In particular, the natural morphism

$$\widehat{\Omega}^1_{T_d/K} \otimes_{T_d} A\langle\frac{\varpi^n}{f}\rangle \to \widehat{\Omega}^1_{A/K} \otimes_A A\langle\frac{\varpi^n}{f}\rangle \tag{2.25}$$

is an isomorphism for any $n \in \mathbf{N}$. By Lemma 2.22(2) we can change f to ensure that $X(f \neq 0)$ is smooth over K. Then [BLR95a, Prop. 2.6] and (2.25) imply that the morphism $X(\frac{\varpi^n}{f}) \to \mathbf{D}^d(\frac{\varpi^n}{f})$ is finite étale for every $n \in \mathbf{N}$. In particular, $T_d\langle\frac{\varpi^n}{f}\rangle \to A\langle\frac{\varpi^n}{f}\rangle$ is finite étale for any $n \in \mathbf{N}$. Finally, we use Corollary 2.20 and faithfully flat descent [Sta24, Tag 02VN] (applied to the local rings of $T_d[\frac{1}{f}]$) to conclude that $T_d[\frac{1}{f}] \to A[\frac{1}{f}]$ is finite étale as well. □

Finally, we are ready to prove Theorem 2.14:

Proof of Theorem 2.14. First, we note that [FK18, Th. 0.9.2.10] and [Sta24, Tag 00OK] imply that we can find a finite morphism

$$h\colon \mathcal{O}_K\langle X_1, \dots, X_d\rangle \to A_0.$$

Now we explain how to modify h to assume that $h_K\big[\frac{1}{f}\big]$ is étale for some non-zero element $f \in K\langle X_1, \dots, X_d\rangle$. For this, we choose $g_1, \dots, g_d \in \varpi A_0$ as in Lemma 2.23 and consider the unique $\mathcal{O}_K$-linear continuous morphism

$$h'\colon \mathcal{O}_K\langle X_1, \dots, X_d\rangle \to A_0$$

which sends X_i to $h(X_i) + g_i$. Clearly, h and h' coincide modulo ϖ, so [FK18, Prop. I.4.2.1] implies that h' is finite. Furhermore, Lemma 2.24 implies that there is a non-zero element $f \in K\langle X_1, \dots, X_d\rangle$ such that $h'_K\big[\frac{1}{f}\big]$ is finite étale. Therefore, we can replace h by h' to achieve that $h_K[\frac{1}{f}]$ is finite étale for some non-zero $f \in K\langle X_1, \dots, X_d\rangle$.

Now we explain how to modify h to make f to be a Weierstrass polynomial. For this, we choose an element $c \in K^\times$ such that $|f| = |c|$ and replace f by f/c to achieve that $|f| = 1$. Therefore, Lemma 2.18 guarantees that, after a continuous $\mathcal{O}_K$-linear automorphism of $\mathcal{O}_K\langle X_1, \dots, X_d\rangle$, we can assume that $f = u\omega$ for a unit $u \in \mathcal{O}_K\langle X_1, \dots, X_d\rangle^\times$ and a Weierstrass polynomial ω. In this case, we can replace f by ω to achieve that the morphism

$$h\colon \mathcal{O}_K\langle X_1, \dots, X_d\rangle \to A_0$$

is étale away from $\mathrm{V}(\varpi f)$. Finally, we note that [Bos14, Th. 7.3/4] and [EGA IV_1, Prop. 1.4.7] imply that h is finitely presented. □

3. ARTIN–GROTHENDIECK VANISHING FOR AFFINOID ALGEBRAS

Throughout this section, we fix an algebraically closed non-archimedean field C with ring of integers $\mathcal{O}_C$ and a pseudo-uniformizer $\varpi \in \mathcal{O}_C$. The main goal of this section is to prove the following theorem:

Theorem 3.1 (Artin–Grothendieck Vanishing for affinoid algebras)**.** *Let A be a C-affinoid algebra and let $\mathcal{F}$ be a torsion étale sheaf on* $\operatorname{Spec} A$*. Then*

$$\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} A, \mathcal{F}\right) \in D^{\leq \dim A}(\mathbf{Z}).$$

For our notational convenience later on, we introduce the following notation:

Notation 3.2. For a C-affinoid algebra A and a torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, we say that $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds if $\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} A, \mathcal{F}\right) \in D^{\leq \dim A}(\mathbf{Z})$.

Remark 3.3. Even if one is only interested in cohomological bounds on $\mathrm{R}\Gamma\left(\operatorname{Spec} A, \mathbf{Z}/n\mathbf{Z}\right)$, it is crucial for the proof to consider more general n-torsion sheaves $\mathcal{F}$.

The last thing we discuss in this section is the easiest part of Theorem 3.1.

Lemma 3.4. *Let* $\operatorname{char} C = p > 0$*, let A be a C-affinoid algebra, and let $\mathcal{F}$ be a torsion étale sheaf on* $\operatorname{Spec} A$*. Then*

$$\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} A, \mathcal{F}\right) \in D^{\leq \dim A}(\mathbf{Z}).$$

Proof. If $\dim A = 0$, then the result is obvious. So we may assume that $\dim A \geq 1$. We put $X := \operatorname{Spec} A$, then [SGA 4, Exp. X, Th. 5.1] says that $\mathrm{cd}_p(X_{\text{ét}}) \leq 1 + \mathrm{cdqc}(X)$. Since X is affine, its quasi-coherent cohomological dimension is 0. So we conclude that $\mathrm{cd}_p(X_{\text{ét}}) \leq 1$. □

3.1. Preliminary reductions.

The main goal of this section is to show that, in order to prove Theorem 3.1, it suffices to show that

$$\mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec} A, \mathbf{Z}/p\mathbf{Z}) \in D^{\leq \dim A}(\mathbf{Z})$$

for any normal C-affinoid domain A and any prime number p.

In this section, we will freely use that any C-affinoid algebra A is excellent (in particular, noetherian). We refer to [Kie69, Th. 3] (and [Con99, §1.1]) for a proof of this fact.

Lemma 3.5. *Let A be an affinoid C-algebra. Suppose that $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds for any prime p and any constructible torsion étale sheaf $\mathcal{F}$ of $\mathbf{Z}/p\mathbf{Z}$-modules. Then $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds for any torsion étale sheaf $\mathcal{F}$.*

Proof. Pick a torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$. Then (the proof of) [Sta24, Tag 0F0N] implies that $\mathcal{F} = \operatorname{colim}_I \mathcal{F}_i$ is a filtered colimit of torsion *constructible* sheaves $\mathcal{F}_i$. Therefore, [Sta24, Tag 03Q5] implies that it suffices to prove $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ for torsion constructible $\mathcal{F}$.

The natural morphism

$$\bigoplus_{p \text{ prime}} \mathcal{F}[p^\infty] \to \mathcal{F}$$

is an isomorphism for any torsion sheaf $\mathcal{F}$. As $\operatorname{Spec} A$ is qcqs, it suffices to prove the result for p^∞-torsion constructible sheaves $\mathcal{F}$ for some prime number p. Furthermore, [Sta24, Tag 09YV] implies that any such $\mathcal{F}$ is actually p^N-torsion for some integer N. Therefore, it suffices to show vanishing for each $p^i\mathcal{F}/p^{i+1}\mathcal{F}$ for $i \leq N-1$. So it suffices to verify $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ for constructible sheaves of $\mathbf{Z}/p\mathbf{Z}$-modules for all primes p. □

In what follows, we are going to freely use the following fact:

Lemma 3.6. [BGR84, Prop. 6.1/6] *Let A be a C-affinoid algebra, and B a finite A-algebra. Then B is also a C-affinoid algebra.*

Lemma 3.7. *Let d be an integer. Suppose that $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds for*

(1) any C-affinoid algebra A with $\dim A < d$ and any torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$*;*

(2) any connected normal C-affinoid algebra A with $\dim A = d$, and $\mathcal{F} = \underline{\mathbf{Z}/p\mathbf{Z}}$ for any prime number p.

Then $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds for any C-affinoid algebra A of $\dim A \leq d$ and any torsion étale sheaf $\mathcal{F}$.

Proof. Lemma 3.5 implies that it suffices to prove $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ a constructible étale sheaf $\mathcal{F}$ of $\mathbf{Z}/p\mathbf{Z}$-modules for any prime number p. So we fix a prime number p, a C-affinoid algebra A of Krull dimension d, and a constructible sheaf of $\mathbf{Z}/p\mathbf{Z}$-modules $\mathcal{F}$. We wish to show that $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds under the assumptions of this lemma.

Step 1. *We reduce to the case of a normal domain A.* We use [Bos14, Prop. 3.1/3] and [Sta24, Tag 00OK] to find a finite morphism $f\colon \operatorname{Spec} A \to \operatorname{Spec} C\langle X_1, \dots, X_d\rangle$. Then [Sta24, Tag 095R] ensures that $f_*\mathcal{F}$ is constructible and [Sta24, Tag 03QP] implies that f_* is exact. Thus, $\mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec} A, \mathcal{F}) = \mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec} C\langle X_1, \dots, X_d\rangle, f_*\mathcal{F})$, and so we may replace the pair $(A, \mathcal{F})$ by $(C\langle X_1, \dots, X_d\rangle, f_*\mathcal{F})$ to achieve that A is a normal C-affinoid domain (see [Bos14, Prop. 2.2/15]).

Step 2. *We reduce to the case $\mathcal{F} = \underline{\mathbf{Z}/p\mathbf{Z}}$.* We note that constructibility of $\mathcal{F}$ implies that there is a qcqs dense open $j\colon V \hookrightarrow \operatorname{Spec} A$ such that $\mathcal{F}|_V = \mathcal{L}$ is a locally constant constructible sheaf of $\mathbf{Z}/p\mathbf{Z}$-modules. We denote its closed complement (with reduced scheme structure) by $i\colon Z = \operatorname{Spec} A' \hookrightarrow \operatorname{Spec} A$ and also note that $\dim A' < \dim A = d$. Therefore, the short exact sequence

$$0 \to j_!\mathcal{L} \to \mathcal{F} \to i_*\left(\mathcal{F}|_Z\right) \to 0$$

and the induction hypothesis imply that it suffices to show that $\left(\mathrm{AGV}_{A,j_!\mathcal{L}}\right)$ holds.

Now we note that V is irreducible because it is a dense open subscheme of an irreducible scheme $\operatorname{Spec} A$, so [Sta24, Tag 0A3R] implies that there is a finite étale covering

$$g\colon W \to V$$

of degree prime to p such that $g^*\mathcal{L}$ is a successive extension of the constant sheaves $\underline{\mathbf{Z}/p\mathbf{Z}}$. Using the $(g_! = \mathrm{R}g_*, g^* = \mathrm{R}g^!)$-adjunction, we get the trace map $g_*g^*\mathcal{L} \to \mathcal{L}$ such that the composition

$$\mathcal{L} \to g_*g^*\mathcal{L} \to \mathcal{L}$$

is equal to the multiplication by $\deg g$. Since $\deg g$ is coprime to p, we conclude that $\mathcal{L}$ is a direct summand of $g_*g^*\mathcal{L}$. Therefore, it suffices to prove that $\left(\mathrm{AGV}_{A,j_!g_*g^*\mathcal{L}}\right)$ holds. Since $g^*\mathcal{L}$ is a successive extension of constant sheaves, it actually suffices to show that $\left(\mathrm{AGV}_{A,j_!g_*\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ holds.

Now we define $g'\colon \operatorname{Spec} B \to \operatorname{Spec} A$ to be the relative normalization of $\operatorname{Spec} A$ in W (see [Sta24, Tag 035H]). We denote the closed complement (with its reduced scheme structure) of W in $\operatorname{Spec} B$ by $i'\colon Z' = \operatorname{Spec} B' \to \operatorname{Spec} B$. By construction, we have the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, "j'"] \arrow[d, "g"] & \operatorname{Spec} B \arrow[d, "g'"] & \operatorname{Spec} B' \arrow[l, "i'"'] \arrow[d] \\
V \arrow[r, "j"] & \operatorname{Spec} A & \operatorname{Spec} B \arrow[l, "i"']
\end{tikzcd}$$

with the left square being Cartesian. Now note that g' is finite due to [Sta24, Tag 07QV] and [Sta24, Tag 03GH] (thus, B is a C-affinoid algebra by Lemma 3.6). Thus, $g'_! = g'_*$, and so we conclude that

$$j_!g_*\underline{\mathbf{Z}/p\mathbf{Z}} = j_!g_!\underline{\mathbf{Z}/p\mathbf{Z}} = g'_!j'_!\underline{\mathbf{Z}/p\mathbf{Z}} = g'_*j'_!\underline{\mathbf{Z}/p\mathbf{Z}}.$$

Therefore, $\left(\mathrm{AGV}_{A,j_!g_*\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ is tautologically equivalent to $\left(\mathrm{AGV}_{A,g'_*j'_!\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$. Using that higher pushforwards along finite morphisms vanish, we also conclude that $\left(\mathrm{AGV}_{A,g'_*j'_!\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ is equivalent to $\left(\mathrm{AGV}_{B,j'_!\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$. In other words, we reduced the situation to showing that $\left(\mathrm{AGV}_{B,j'_!\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ holds.

The going-down lemma [Sta24, Tag 00H8] implies that W is dense in $\operatorname{Spec} B$, and so $\dim B' < \dim B = \dim A = d$. Therefore, the short exact sequence

$$0 \to j'_!\left(\underline{\mathbf{Z}/p\mathbf{Z}}\right) \to \underline{\mathbf{Z}/p\mathbf{Z}} \to i'_*\left(\underline{\mathbf{Z}/p\mathbf{Z}}\right) \to 0$$

and the induction hypothesis imply that it suffices to show that $\left(\mathrm{AGV}_{B,\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ holds. By construction, B is a normal C-affinoid ring. Therefore, passing to its finitely many clopen connected (see [Sta24, Tag 030C]) components B_i, we reduce the question to showing that $\left(\mathrm{AGV}_{B_i,\underline{\mathbf{Z}/p\mathbf{Z}}}\right)$ holds. This finishes the proof. ☐

3.2. **Crux of the argument.** In this section, we finally prove Theorem 3.1.

Proof of Theorem 3.1. We argue by induction on $d = \dim A$. The base case of $d = 0$ is evident. Thus we assume that $\left(\mathrm{AGV}_{A,\mathcal{F}}\right)$ holds for any C-affinoid algebra A of dimension $< d$ (and any $\mathcal{F}$) and prove the result for all A of dimension d.

First, we note Lemma 3.5 and Lemma 3.7 imply that it suffices to show that

$$\mathrm{R}\Gamma_{\text{ét}}(\operatorname{Spec} A, \mathbf{Z}/p\mathbf{Z}) \in D^{\leq d}(\mathbf{Z})$$

for any C-affinoid normal domain A of dimension d and any prime number p.

If $p = \operatorname{char} C$, then the result follows from Lemma 3.4. Therefore, we may and do assume that p is different from the characteristic of C.

Now we recall that R denotes the ring $\mathcal{O}_C\langle X_1, \dots, X_{d-1}\rangle[X_d]$, so Corollary 2.17 implies that there is a finite, finitely presented $R^{\mathrm{h}}_{(\varpi)}$-algebra B such that

$$B \otimes_{R^{\mathrm{h}}_{(\varpi)}} R^{\wedge}_{(\varpi)} \simeq A^{\circ}.$$

Therefore, Lemma 2.13 implies that the natural morphism

$$\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z}\right) \to \mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} A, \mathbf{Z}/p\mathbf{Z}\right)$$

is an isomorphism. Therefore, it suffices to show that $\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z}\right) \in D^{\leq d}(\mathbf{Z})$.

We fix an integer $i > d$ and a cohomology class $c \in \mathrm{H}^i_{\text{ét}}(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z})$. We wish to show that $c = 0$. For this, we note that a standard spreading out argument (see [Sta24, Tag 09YQ]) shows that we can find a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] \arrow[d, "\psi"] & \operatorname{Spec} B' \arrow[d, "\psi'"] \\
\operatorname{Spec} R^{\mathrm{h}}_{(\varpi)} \arrow[r] \arrow[d] & \operatorname{Spec} R' \arrow[dl] \\
\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_C\langle X_1, \dots, X_{d-1}\rangle[X_d] &
\end{tikzcd}$$

such that the top square is Cartesion, $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an étale morphism (and an isomorphism on the mod-ϖ fibers), B' is a finite, finitely presented R'-algebra, and the class $c \in \mathrm{H}^i_{\text{ét}}(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z})$ lies in the image of the natural morphism

$$\mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} B'\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z}\right) \to \mathrm{H}^i_{\text{ét}}\left(\operatorname{Spec} B\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z}\right).$$

Therefore, it suffices to show $\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} B'\left[\frac{1}{\varpi}\right], \mathbf{Z}/p\mathbf{Z}\right) \in D^{\leq d}(\mathbf{Z})$. We consider the composite morphism

$$\begin{tikzcd}
\operatorname{Spec} B'[\frac{1}{\varpi}] \arrow[r] \arrow[rr, bend left, "h"] & \operatorname{Spec} C\langle X_1, \dots, X_{d-1}\rangle[X_d] \arrow[r] & \operatorname{Spec} C\langle X_1, \dots, X_{d-1}\rangle.
\end{tikzcd}$$

This is an affine relative curve, i.e., all non-empty fibers of h are of dimension ≤ 1. Therefore, constructibility of higher pushforwards along h and the affine Lefschetz Theorem (see[3] [ILO14, Exp. XIII, Th. 1.1.1 and Exp. XV, Th. 1.1.2] respectively[4]) imply that $\mathrm{R}^{\beta}h_*\mathcal{F}$ is supported on a closed subscheme $\operatorname{Spec} D_{\beta} \subset \operatorname{Spec} C\langle X_1, \dots, X_{d-1}\rangle$ of dimension $\leq d - \beta$. The induction hypothesis implies that

$$\mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} C\langle X_1, \dots, X_{d-1}\rangle, \mathrm{R}^{\beta}h_*\underline{\mathbf{Z}/p\mathbf{Z}}\right) = \mathrm{R}\Gamma_{\text{ét}}\left(\operatorname{Spec} D_{\beta}, \mathrm{R}^{\beta}h_*\underline{\mathbf{Z}/p\mathbf{Z}}\right) \in D^{\leq d-\beta}(\mathbf{Z}).$$

---

[3] If $\operatorname{char} C = 0$, one can instead use [SGA 4, Exp. XIX, Th. 5.1 and Exp. XIX Th. 6.1] respectively.

[4] To apply [ILO14, Exp. XV, Th. 1.1.2], we use the dimension function constructed in Corollary A.4. We also note that the dimension function $f^*\delta$ in *loc.cit.* is exactly the dimension function from Corollary A.5.

Finally, an easy argument with spectral sequences ensures that

$$\mathrm{R}\Gamma_{\text{ét}}\Big(\operatorname{Spec} B'\Big[\frac{1}{\varpi}\Big], \mathbf{Z}/p\mathbf{Z}\Big) = \mathrm{R}\Gamma_{\text{ét}}\Big(\operatorname{Spec} C\langle X_1, \dots, X_{d-1}\rangle, \mathrm{R}h_*\underline{\mathbf{Z}/p\mathbf{Z}}\Big) \in D^{\leq d}(\mathbf{Z})$$

finishing the proof. ☐

4. Rigid-analytic Artin–Grothendieck vanishing

Throughout this section, we fix an algebraically closed non-archimedean field C with valuation $|.|\colon C \to \Gamma_C \cup \{0\}$, ring of integers $\mathcal{O}_C$, and a pseudo-uniformizer $\varpi \in \mathcal{O}_C$.

The main goal of this section is to prove Conjecture 1.9 in some particular situations. In particular, we show Conjecture 1.9 for affinoids over a field of characterstic 0 (in fact, a stronger version of it), and for affinoid curves over a field of arbitrary characteristic. We also provide a counter-example to some expectations from [BH22].

4.1. Artin–Grothendieck Vanishing for algebraic sheaves.

The main goal of this section is to show that Theorem 3.1 implies Conjecture 1.9 for a big class of sheaves.

In order to explicitly specify this class of sheaves, we need some preliminary discussion. We first recall that, for every C-affinoid algebra A, [Hub96, Cor. 1.7.3 and (3.2.8)] construct a morphism of topoi $c_A\colon \operatorname{Spa}(A, A^\circ)_{\text{ét}} \to \operatorname{Spec} A_{\text{ét}}$.

Definition 4.1. A torsion étale sheaf $\mathcal{F}$ on $\operatorname{Spa}(A, A^\circ)$ is *algebraic* if there is a torsion étale sheaf $\mathcal{G}$ on $\operatorname{Spec} A$ and an isomorphism $c_A^*\mathcal{G} \simeq \mathcal{F}$.

In order to get a good supply of algebraic sheaves, we recall the following definition:

Definition 4.2 ([Han20])**.** An étale sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules $\mathcal{F}$ on $\operatorname{Spa}(A, A^\circ)$ is *Zariski-constructible* if there is a locally finite stratification $\operatorname{Spa}(A, A^\circ) = \sqcup_{i\in I} X_i$ into Zariski locally closed subspaces[5] of X such that $\mathcal{F}|_{X_i}$ is lisse for every $i \in I$.

Lemma 4.3. *Let A be a C-affinoid algebra, and n an integer. Then*

(1) any lisse étale sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $\operatorname{Spa}(A, A^\circ)$ is algebraic;

(2) if $\operatorname{char} C = 0$, then any Zariski-constructible étale sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $\operatorname{Spa}(A, A^\circ)$ is algebraic.

Proof. In order to show (1), it suffices to show that the analytification functor induces an equivalence $(\operatorname{Spec} A)_{\text{fét}} \xrightarrow{\sim} \operatorname{Spa}(A, A^\circ)_{\text{fét}}$. This follows directly from the observation that both categories are canonically equivalent to the category $A_{\text{fét}}$ of finite étale A-algebras. (2) follows directly from [Han20, Th. 1.7]. ☐

Theorem 4.4 (Rigid-analytic Artin–Grothendieck Vanishing)**.** *Let A be an affinoid C-algebra, and let $\mathcal{F}$ be an algebraic torsion étale sheaf on $\operatorname{Spa}(A, A^\circ)$ (in the sense of Definition 4.1). Then*

$$\mathrm{R}\Gamma_{\text{ét}}\big(\operatorname{Spa}(A, A^\circ), \mathcal{F}\big) \in D^{\leq \dim A}(\mathbf{Z}). \tag{4.5}$$

In particular, (4.5) holds in either of the following situations:

(1) $\mathcal{F}$ is a lisse sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer n;

(2) $\operatorname{char} C = 0$ and $\mathcal{F}$ is a Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer n.

Proof. This follows directly from Theorem 3.1, Lemma 4.3, and [Han20, Th. 1.9]. ☐

We note that Theorem 4.4 proves a *stronger* version of Conjecture 1.9 if $\operatorname{char} C = 0$. But, if $\operatorname{char} C = p > 0$, then Theorem 4.4 does not solve Conjecture 1.9 in full generality due to the existence of Zariski-constructible sheaves that are not algebraic (see [Han20, p. 302]).

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[5] We recall that an immersion $X \overset{f}{\hookrightarrow} Y$ is called Zariski locally closed if f can be decomposed as a composition $X \overset{j}{\hookrightarrow} Z \overset{i}{\hookrightarrow} Y$ where j is a Zariski-open immersion and i is a closed immersion. We refer to [Zav24, Appendix B.6] for the basics on closed immersions in the rigid-analytic context.

4.2. **Artin–Grothendieck Vanishing in top degree.** In this section, we show vanishing of the top cohomology group of any Zariski-constructible sheaf on an affinoid space (without any assumptions on the characteristic of the ground field). In particular, this will be enough to conclude Conjecture 1.9 for affinoid curves over an algebraically closed field of *arbitrary* characteristic. We recall that the field C is assumed to be algebraically closed.

Definition 4.6. The *Gauss point* $\eta \in \mathbf{D}^d = \mathrm{Spa}\left(C\langle X_1, \dots, X_d\rangle, \mathcal{O}_C\langle X_1, \dots, X_d\rangle\right)$ is the point corresponding to the valuation

$$v_\eta \colon C\langle X_1, \dots, X_d\rangle \to \Gamma_C \cup \{0\}$$
$$v_\eta\left(\sum_{i_1, \dots i_d} a_{i_1, \dots, i_d} X_1^{i_1} \cdot \dots \cdot X_d^{i_d}\right) = \sup_{i_1, \dots, i_d} \left(|a_{i_1, \dots, i_d}|\right).$$

Lemma 4.7. *Let* $\mathbf{D}^d$ *be the d-dimension closed unit disc, and let* $\mathbf{D}^{d,c}$ *be its universal compactification (in the sense of* [Hub96, Def. 5.1.1]*). If* $d \geq 1$*, then there is a point* $x \in \mathbf{D}^{d,c} \smallsetminus \mathbf{D}^d$ *that generizes to the Gauss point* η*.*

Proof. For brevity, we denote by A the ring $C\langle X_1, \dots, X_d\rangle$ and by $A^{\min}$ the integral closure of $\mathcal{O}_C[A^{\circ\circ}]$ in A. Then [Hub01, Ex. 5.10(i)] ensures that $\mathbf{D}^{d,c} = \mathrm{Spa}\left(A, A^{\min}\right)$. Now we consider the map

$$v_x \colon C\langle X_1, \dots, X_d\rangle \to \Gamma_C \times \mathbf{Z} \cup \{0\}$$
$$v_x\left(\sum_{i_1, \dots i_d} a_{i_1, \dots, i_d} X_1^{i_1} \cdot \dots \cdot X_d^{i_d}\right) = \sup_{i_1, \dots, i_d \ \mid \ a_{i_1, \dots, i_d} \neq 0} \left(|a_{i_1, \dots, i_d}|, i_1\right).$$[6]

One easily checks that it is a valuation if $\Gamma_C \times \mathbf{Z}$ is endowed with the lexicographical order. We also see that $v_x(A^{\min}) \leq (1, 0)$ and v_x is continuous due to [Sem15, L. 9, Cor. 9.3.3], so v_x defines a point $x \in \mathbf{D}^{d,c}$. Furthermore, we note that $x \in \mathbf{D}^{d,c} \smallsetminus \mathbf{D}^d$ because $v_x(X_1) > (1, 0)$. Finally, we note that η is a generization of x since, for every $f, g \in A$, $v_x(f) \leq v_x(g)$ implies $v_\eta(f) \leq v_\eta(g)$. □

Lemma 4.8. *Let* $j \colon \mathbf{D}^d \hookrightarrow \mathbf{P}^{d,\mathrm{an}}$ *be the standard open immersion for some* $d \geq 1$*, let* $j' \colon U \hookrightarrow \mathbf{D}^d$ *be a Zariski-open immersion, let n be an integer invertible in* $\mathcal{O}_C$*, and let* $\mathcal{L}$ *be a lisse sheaf of* $\mathbf{Z}/n\mathbf{Z}$*-modules on* $U_{\mathrm{\acute{e}t}}$ *and* $\mathcal{F} = j'_! \mathcal{L}$*. Then any* $\mathbf{Z}/n\mathbf{Z}$*-linear homomorphism* $\varphi \colon j_*\mathcal{F} \to \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}$ *is the zero morphism.*

Proof. If U is empty, the claim is trivial. Therefore, we may and do assume that U is non-empty throughout the proof. We first note that a map of sheaves is uniquely determined by the induced morphisms on stalks. By [Hub96, Prop. 2.6.4] and its proof, every stalk of $j_*\mathcal{F}$ is either 0 or isomorphic to a stalk of $\mathcal{F}$ by a specialization map, and thus the restriction morphism

$$\begin{aligned} \mathrm{Hom}_{\mathbf{P}^{d,\mathrm{an}}}\left(j_*\mathcal{F}, \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}\right) &\to \mathrm{Hom}_{\mathbf{D}^d}\left(\mathcal{F}, \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{D}^d}\right) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^d}\left(j'_!\mathcal{L}, \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{D}^d}\right) \\ &\simeq \mathrm{Hom}_U\left(\mathcal{L}, \underline{\mathbf{Z}/n\mathbf{Z}}_U\right) \end{aligned}$$

is injective. Since U is connected (see [Han20, Cor. 2.7]) and $\mathcal{L}$ and $\underline{\mathbf{Z}/n\mathbf{Z}}$ are *lisse* sheaves, it suffices to show that the stalk morphism $\varphi_{\overline{y}} \colon (j_*\mathcal{F})_{\overline{y}} \to \left(\underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}\right)_{\overline{y}}$ vanishes at *one* single geometric point $\overline{y} \to U$.

Now we note that the Gauss point $\eta \in \mathbf{D}^d$ is a Shilov point in the sense of [BH22, Def. 2.4], so [BH22, Cor. 2.10] implies that $\eta \in U$. Therefore, it suffices to show that $\varphi_{\overline{\eta}} = 0$ for a geometric point $\overline{\eta}$ above η.

Fot this, we consider the closure $\overline{\mathbf{D}}^d \subset \mathbf{P}^{d,\mathrm{an}}$ of $\mathbf{D}^d$ inside $\mathbf{P}^{d,\mathrm{an}}$. Then [AGV22, Lemma 4.2.5 and Prop. 4.2.11] ensure[7] that $\overline{\mathbf{D}}^d$ is homeomorphic to the universal compactification $\mathbf{D}^{d,c}$. Therefore, Lemma 4.7

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[6] By our convention, supremum over the empty set is equal to 0.

[7] Strictly speaking, [AGV22, Prop. 4.2.11] assumes that S is universally uniform in the sense of [AGV22, Def. 4.2.7]. This hypothesis is essentially never satisfied due to the observation that $A\langle X\rangle/(X^2)$ is not uniform for a non-zero complete Tate ring A. However, the proof [AGV22, Prop. 4.2.11] does work for any locally strongly noetherian analytic space S.

implies that there is a point $x \in \overline{\mathbf{D}}^d \smallsetminus \mathbf{D}^d$ that generizes to η. We choose a geometric point $\overline{x}$ above x with a specialization morphism $\overline{\eta} \to \overline{x}$ (see [Hub96, Lemma 2.5.14 and (2.5.16)]).

Now we recall that $U \subset \mathbf{D}^d$ is Zariski-open, so it is closed under all specializations and generizations. Therefore, $\mathcal{F} = j'_! \mathcal{L}$ is overconvergent. Thus [Hub96, Prop. 8.2.3(ii)] ensures that $j_*\mathcal{F}$ is overconvergent as well. Since $\underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}$ is clearly overconvergent, we conclude that stalk-morphisms $\varphi_{\overline{\eta}}$ and $\varphi_{\overline{x}}$ are canonically identified. So we reduce the question to showing that $\varphi_{\overline{x}} = 0$ for any $\mathbf{Z}/n\mathbf{Z}$-linear morphism $\varphi\colon j_*\mathcal{F} \to \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}$.

For this, it suffices to show that, for any étale morphism $f_V\colon V \to \mathbf{P}^{d,\mathrm{an}}$ with a connected affinoid V and $x \in f_V(V)$, the morphism

$$\varphi(V)\colon \left(j_*\mathcal{F}\right)(V) \to \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}(V) = \mathbf{Z}/n\mathbf{Z}$$

is the zero morphism. Since $x \in f_V(V)$, we conclude that $f_V^{-1}\left(\mathbf{D}^d\right) \neq V$, so there is a classical point $z \in V \smallsetminus f_V^{-1}\left(\mathbf{D}^d\right)$. We choose a geometric point $\overline{z} \to V$ above z. Then we have the following commutative diagram

$$\begin{tikzcd}
\left(j_*\mathcal{F}\right)(V) \arrow[r, "\varphi(V)"] \arrow[d] & \left(\underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}\right)(V) \simeq \mathbf{Z}/n\mathbf{Z} \arrow[d, "\wr"] \\
\left(j_*\mathcal{F}\right)_{\overline{z}} \arrow[r, "\varphi_{\overline{z}}"] & \left(\underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}\right)_{\overline{z}} \simeq \mathbf{Z}/n\mathbf{Z}.
\end{tikzcd} \tag{4.9}$$

Since $z \in V$ is a classical point, it has no proper generizations. Therefore, [Sta24, Tag 0904] ensures that there is an open neighborhood $z \in W \subset V$ disjoint from $f_V^{-1}(\mathbf{D}^d)$. So we conclude that $(j_*\mathcal{F})_{\overline{z}} = 0$. Thus, Diagram (4.9) implies that $\varphi(V) = 0$ finishing the proof. ☐

Theorem 4.10 (Rigid-analytic Artin–Grothendieck Vanishing in top degree)**.** *Let $X = \mathrm{Spa}\,(A, A^\circ)$ be an affinoid rigid-analytic space over C with* $\dim A = d \geq 1$*, let n be an integer invertible in $\mathcal{O}_C$, and let $\mathcal{F}$ be a Zariski-constructible sheaf of* $\mathbf{Z}/n\mathbf{Z}$*-modules on $X_{\text{ét}}$. Then*

$$\mathrm{H}^{2d}_{\text{ét}}(X, \mathcal{F}) = 0.$$

Proof. First, [Bos14, Prop. 3.3/1] and [Sta24, Tag 00OK] imply that there is a finite morphism $f\colon X \to \mathbf{D}^d$. Then [Han20, Prop. 2.3] ensures that $f_*\mathcal{F}$ is Zariski-constructible, while [Hub96, Prop. 2.6.3] ensures that f_* is exact. In particular, this implies that $\mathrm{H}^{2d}_{\text{ét}}\left(X, \mathcal{F}\right) = \mathrm{H}^{2d}_{\text{ét}}\left(\mathbf{D}^d, f_*\mathcal{F}\right)$, so we can replace the pair $(X, \mathcal{F})$ by $\left(\mathbf{D}^d, f_*\mathcal{F}\right)$ to achieve that $X = \mathbf{D}^d$ is the closed unit disc.

Now the definition of Zariski-constructible sheaves implies that there is a non-empty Zariski-open subspace $j'\colon U \hookrightarrow \mathbf{D}^d$ such that $\mathcal{L} := \mathcal{F}|_U$ is lisse. We denote the closed complement (with the reduced adic space structure) by $i\colon Z \to \mathbf{D}^d$. Then we have a short exact sequence

$$0 \to j'_!\,\mathcal{L} \to \mathcal{F} \to i_*\left(\mathcal{F}|_Z\right) \to 0.$$

Now [Hub96, Lemma 1.8.6(ii)] guarantees that $\dim Z = \dim.\mathrm{tr}\, Z \leq d-1$, so [Hub96, Cor. 2.8.3] implies that $\mathrm{R}\Gamma_{\text{ét}}\left(X, i_*\left(\mathcal{F}|_Z\right)\right) = \mathrm{R}\Gamma_{\text{ét}}\left(Z, \mathcal{F}|_Z\right) \in D^{\leq 2d-2}(\mathbf{Z})$. Therefore, it suffices to prove the claim for $\mathcal{F} = j'_!\mathcal{L}$.

In this situation, we consider the compactification $j\colon \mathbf{D}^d \hookrightarrow \mathbf{P}^{d,\mathrm{an}}$. Then [Hub96, Prop. 2.6.4] implies that j_* is exact, and so $\mathrm{H}^{2d}_{\text{ét}}\left(\mathbf{D}^d, \mathcal{F}\right) = \mathrm{H}^{2d}_{\text{ét}}\left(\mathbf{P}^{d,\mathrm{an}}, j_*\mathcal{F}\right)$. In particular, it suffices to show that *the dual* $\mathbf{Z}/n\mathbf{Z}$-module $\mathrm{H}^{2d}_{\text{ét}}\left(\mathbf{P}^{d,\mathrm{an}}, j_*\mathcal{F}\right)^\vee$ vanishes. Now, after choosing a trivialization of the Tate twist $\underline{\mathbf{Z}/n\mathbf{Z}}(d) \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, Poincaré duality (see [Hub96, Cor. 7.5.6], [Ber93, Th. 7.3.4], or [Zav23, Th. 1.3.2]) implies that

$$\mathrm{H}^{2d}_{\text{ét}}\left(\mathbf{P}^{d,\mathrm{an}}, j_*\mathcal{F}\right)^\vee = \mathrm{Hom}_{\mathbf{P}^{d,\mathrm{an}}}\left(j_*\mathcal{F}, \underline{\mathbf{Z}/n\mathbf{Z}}_{\mathbf{P}^{d,\mathrm{an}}}\right).$$

Thus, the desired vanishing follows directly from Lemma 4.8. ☐

Remark 4.11. We want to mention that it is possible to prove Theorem 4.10 in greater generality. Namely, in the formulation of Theorem 4.10, it suffices to assume that

(1) the space X is a quasi-compact separated rigid-analytic C-space of dimension d such that none of its irreducible components is a proper rigid-analytic C-space of dimension d;

(2) $\mathcal{F}$ is a Zariski-constructibe sheaf of Λ-modules for a $\mathbf{Z}/n\mathbf{Z}$-algebra Λ (and an integer n invertible in $\mathcal{O}_C$).

Since we do not know any interesting application of this extra generality and the argument becomes significantly longer; we prefer not to spell it out in this paper.

Corollary 4.12. *Let A be an C-affinoid algebra of dimension 1, let n be an integer invertible in C, and let $\mathcal{F}$ be a Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $\operatorname{Spa}(A, A^\circ)_{\text{ét}}$. Then*

$$\mathrm{R}\Gamma_{\text{ét}}\big(\operatorname{Spa}(A, A^\circ), \mathcal{F}\big) \in D^{\leq 1}(\mathbf{Z}).$$

Proof. If $\operatorname{char} C = 0$, the result follows from Theorem 4.4. If $\operatorname{char} C = p > 0$, then n is invertible in $\mathcal{O}_C$. Therefore, the result follows from Theorem 4.10 and [Hub96, Cor. 2.8.3]. □

4.3. **Counterexample to perverse exactness of nearby cycles.** Throughout this subsection, we assume that C is an algebraically closed non-archimedean field of mixed characteristic $(0, p)$.

In [BH22, Th. 4.2], it is claimed that Theorem 4.4 should imply that, for any admissible formal $\mathcal{O}_C$-scheme $\mathcal{X}$, the nearby cycles functor $\mathrm{R}\lambda_* \colon D^b_{zc}(\mathcal{X}_\eta; \mathbf{F}_p) \to D^+(\mathcal{X}_s; \mathbf{F}_p)$ is perverse exact (using the perverse t-structure on the source constructed in [BH22, Def. 4.1] and the perverse t-structure on the target constructed[8] in [Gab04]). However, this claim is false: the next lemma shows that this functor is not perverse right t-exact even for the (formal) affine line.

Lemma 4.13. *Let $\mathcal{X} = \widehat{\mathbf{A}}^1_{\mathcal{O}_C} = \operatorname{Spf} \mathcal{O}_C\langle T\rangle$ be the formal affine line over $\mathcal{O}_C$, let $\zeta \in \mathcal{X}_s = \mathbf{A}^1_s$ be the generic point in the special fiber of $\mathcal{X}$, and let $\mathrm{R}\lambda_* \colon D^b_{zc}(\mathcal{X}_\eta; \mathbf{F}_p) \to D^+(\mathcal{X}_s; \mathbf{F}_p)$ be the nearby cycles functor. Then*

$$\mathcal{H}^0\left(\mathrm{R}\lambda_* \mu_p[1]\right)_{\overline{\zeta}} = \left(\mathrm{R}^1\lambda_* \mu_p\right)_{\overline{\zeta}} \neq 0.$$

In particular, $\mathrm{R}\lambda_*$ is not perverse right t-exact.

Proof. Since $\mathcal{X}_\eta$ is smooth, we conclude that $\mu_p[1]$ is a perverse sheaf on $\mathcal{X}_\eta$. So it suffices to show that $\mathrm{R}\lambda_*\mu_p[1] \notin {}^pD^{\leq 0}(\mathcal{X}_s; \mathbf{F}_p)$. Using the definition of ${}^pD^{\leq 0}(\mathcal{X}_s; \mathbf{F}_p)$ (see [Cas22, p.7]), we see that it is enough to show that $\mathcal{H}^0\left(\mathrm{R}\lambda_*\mu_p[1]\right)_{\overline{\zeta}} \neq 0$, where $\zeta \in \mathcal{X}_s$ is the generic point and $\overline{\zeta}$ is the geometric point of $\mathcal{X}_s$ above ζ.

Now we put $X \coloneqq \mathbf{A}^1_{\mathcal{O}_C} = \operatorname{Spec} \mathcal{O}_C[T]$ to be the schematic affine line over $\mathcal{O}_C$, $j \colon X_\eta \to X$ to be the open immersion of the generic fiber X_η into X, and $i \colon X_s \to X$ to be the closed immersion of the special fiber X_s into X. Since $|\mathcal{X}_s| = |X_s|$, we can consider ζ as a point of X corresponding to the generic point of the special fiber X_s. We denote by $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ the strict henselization of the local ring $\mathcal{O}_{X,\zeta}$ (corresponding to $\overline{\zeta}$). Then combining [Hub96, Th. 3.5.13] with [Sta24, Tag 03Q9], we conclude that

$$\left(\mathrm{R}^1\lambda_*\mu_p\right)_{\overline{\zeta}} \simeq \left(i^*\mathrm{R}^1 j_*\mu_p\right)_{\overline{\zeta}} \simeq \left(\mathrm{R}^1 j_*\mu_p\right)_{\overline{\zeta}} \simeq \mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{X,\zeta}\left[\frac{1}{p}\right], \mu_p\right).$$

For brevity, we denote $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}[\frac{1}{p}]$ by K_ζ. Then the Kummer exact sequence implies that there is an injection $K_\zeta^\times/K_\zeta^{\times,p} \subset \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K_\zeta, \mu_p)$. Therefore, for the purpose of proving that $\left(\mathrm{R}^1\lambda_*\mu_p\right)_{\overline{\zeta}} \neq 0$, it suffices to show $K_\zeta^\times/K_\zeta^{\times,p} \neq 0$.

Now [GR18, Prop. 9.1.32] implies that $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ is a rank-1 valuation ring with the maximal ideal $\mathfrak{m}_C\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$. Thus $p \in \mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ is a pseudo-uniformizer, so K_ζ is the fraction field of $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$. In particular, $T \in K_\zeta^\times$. Therefore, in order to show that $K_\zeta^\times/K_\zeta^{\times,p} \neq 0$, it suffices to show that there is no element $S \in K_\zeta$ such that $S^p = T$. Suppose that there was such an element S, then $S \in \mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ since $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ is a valuation ring (so it is integrally closed in K_ζ). On the other hand, since $\mathfrak{m}_C\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$ is the maximal ideal of $\mathcal{O}^{\mathrm{sh}}_{X,\zeta}$, we conclude that

$$\mathcal{O}^{\mathrm{sh}}_{X,\zeta}/\mathfrak{m}_C\mathcal{O}^{\mathrm{sh}}_{X,\zeta} \simeq k(T)^{\mathrm{sep}},$$

where $k = \mathcal{O}_C/\mathfrak{m}_C$ is the residue field of $\mathcal{O}_C$ and $k(T)^{\mathrm{sep}}$ is a separable closure of $k(T)$. Therefore, we conclude that the element $T \in k(T)^{\mathrm{sep}}$ must also admit a p-th root, which is clearly false by separability of the extension $k(T) \subset k(T)^{\mathrm{sep}}$. This implies that no such $S \in K_\zeta$ exists, and thus finishes the proof that $K_\zeta^\times/K_\zeta^{\times,p} \neq 0$. □

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[8] See also [Cas22, §2, p.7] for the discussion of the perverse t-structure on $D^+(\mathcal{X}_s; \mathbf{F}_p)$.

Remark 4.14. The nearby cycles functor is expected to be left t-exact with respect to the perverse t-structures. In the case of an "algebraizable" admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}$, this is going to be proven in [BL].

5. Noetherian rig-smooth algebraization

In this section, we give a proof of Theorem 1.2 for any noetherian ring A and any ideal $I \subset A$. The strategy of the argument is to first establish some "weak" uniqueness of an algebraization; this occupies the first three lemmas of this section. Then we use this weak uniqueness and some deformation-theoretic arguments to run induction on the number of generators of the ideal I. This occupies the rest of the section.

Lemma 5.1. *Let (A, I) be a noetherian henselian pair, let B be a finite type A-algebra, and let $\varphi\colon A \to B^{\mathrm{h}}_I$ be a morphism of A-algebras. If the induced morphism $\varphi^\wedge_I\colon A^\wedge_I \to B^\wedge_I$ is an isomorphism, then so is φ.*

Proof. Our assumption on φ implies that the natural morphism $A/I \to B/IB$ is an isomorphism. Then [MB23, Prop. 2.3.2] ensures that there is a clopen subscheme $\operatorname{Spec} B' \subset \operatorname{Spec} B$ such that $A \to B'$ is a finite morphism and $B/IB \to B'/IB'$ is an isomorphism. Therefore, we conclude that $B^{\mathrm{h}}_I \xrightarrow{\sim} (B')^{\mathrm{h}}_I$. Thus we can replace B by B' to achieve that $A \to B$ is a finite morphism. In this case, [Sta24, Tag 0DYE] implies that $B = B^{\mathrm{h}}$, so $A \to B^{\mathrm{h}}_I$ is a finite morphism. Then we conclude that $B^\wedge_I \simeq B^{\mathrm{h}}_I \otimes_A A^\wedge_I$, so $\varphi^\wedge_I = \varphi \otimes_A A^\wedge_I$. We conclude that φ is an isomorphism since $\varphi \otimes_A A^\wedge_I$ is an isomorphism and $A \to A^\wedge_I$ is faithfully flat (see [Sta24, Tag 0AGV]). $\square$

Lemma 5.2. *Let A be a noetherian ring, let $I \subset A$ be an ideal, let B and C be finite type A-algebras, let $\psi\colon B^{\mathrm{h}}_I \to C^{\mathrm{h}}_I$ and $\varphi\colon B^\wedge_I \to C^\wedge_I$ be homomorphisms of A-algebras. If φ is an isomorphism and $\psi \bmod I = \varphi \bmod I$, then ψ is an isomorphism.*

Proof. We denote by $\psi'\colon B \to C^{\mathrm{h}}_I$ the composition of ψ with the the morphism $B \to B^{\mathrm{h}}_I$. Then a standard approximation argument ensures that there is an étale morphism $C \to C'$ such that $C/IC \to C'/IC'$ is an isomorphism and ψ' factors as $B \xrightarrow{\psi''} C' \to (C')^{\mathrm{h}}_I \simeq C^{\mathrm{h}}_I$. Since $C^{\mathrm{h}}_I \simeq (B^{\mathrm{h}}_I \otimes_B C')^{\mathrm{h}}_I$, we conclude that C^{h}_I (with the B^{h}_I-algebra structure coming from ψ) is obtained as the I-adic henselization of a finite type B^{h}_I-algebra. Thus Lemma 5.1 implies that it suffices to show that $\psi^\wedge_I\colon B^\wedge_I \to C^\wedge_I$ is an isomorphism. This follows from [Bou98, Ch. III, §2, n. 8, Cor. 3] since $\operatorname{gr}_I(\psi^\wedge_I) = \operatorname{gr}_I(\varphi)$ by our assumption. $\square$

For the next definition, we fix a ring A with an ideal $I \subset A$.

Definition 5.3. A finitely presented morphism $A \to B$ is *smooth outside* $\mathrm{V}(I)$ (or *smooth outside* I) if $\operatorname{Spec} B \smallsetminus \mathrm{V}(IB) \to \operatorname{Spec} A \smallsetminus \mathrm{V}(I)$ is a smooth morphism.

Now we are ready to show the "weak" uniqueness result promised at the beginning of the section.

Lemma 5.4. *Let A be a noetherian ring, let $I \subset A$ be an ideal, let B be a finite type A-algebra that is smooth outside $\mathrm{V}(I)$, let C be a finite type A-algebra, and let $\varphi\colon B^\wedge_I \to C^\wedge_I$ be a morphism of A-algebras. Then, for every integer $n > 0$, there is a morphism of A-algebras $\psi\colon B^{\mathrm{h}}_I \to C^{\mathrm{h}}_I$ such that $\psi \bmod I^n = \varphi \bmod I^n$. Furthermore, if φ is an isomorphism, then ψ is an isomorphism as well.*

Proof. We fix a presentation $B = A[X_1, \dots, X_m]/(f_1, \dots, f_s)$. Then the morphism φ is uniquely defined by the set of elements $\widehat{c}_1 \coloneqq \varphi(X_1), \dots, \widehat{c}_m \coloneqq \varphi(X_m) \in C^\wedge_I$ such that $f_i(\underline{\widehat{c}}) = 0$ for $i = 1, \dots, s$. Now [Elk73, Th. 2bis on p.560] implies that there are elements $c_1, \dots, c_m \in C^{\mathrm{h}}_I$ such that $f_i(\underline{c}) = 0$ for $i = 1, \dots, s$ and $c_j - \widehat{c}_j \in I^n C^{\mathrm{h}}_I$ for $j = 1, \dots, m$. Then we define $\psi'\colon B \to C^{\mathrm{h}}_I$ to be the unique A-linear morphism that sends X_j to c_j for $j = 1, \dots, m$. By construction, we have $\psi' \bmod I^n = \varphi \bmod I^n$. Now the universal property of henselizations (see [Sta24, Tag 0A02]) implies that ψ' uniquely extends to a morphism $\psi\colon B^{\mathrm{h}}_I \to C^{\mathrm{h}}_I$. We conclude that $\psi \bmod I^n = \varphi \bmod I^n$ since the same was true for ψ'. Finally, if φ is an isomorphism, then Lemma 5.2 guarantees that ψ must be an isomorphism as well. $\square$

Before we prove the main algebraization result, we need to establish a certain result in deformation theory. It is proven in the two lemmas below. For a ring A and an A-module M, we denote by $\widetilde{M}$ the quasi-coherent sheaf on $\operatorname{Spec} A$ associated to M.

Lemma 5.5. *Let A be a ring, let $I \subset A$ be a finitely generated ideal, let $U := \operatorname{Spec} A \smallsetminus \mathrm{V}(I)$ be the open complement of $\mathrm{V}(I)$, let M be an A-module, and let $A \to B$ be a flat morphism such that $A/I \to B/IB$ is an isomorphism. We put $M_B := M \otimes_A B$ and $U_B := U \times_{\operatorname{Spec} A} \operatorname{Spec} B$. Then the natural morphism*

$$\mathrm{H}^i(U, \widetilde{M}|_U) \to \mathrm{H}^i(U_B, \widetilde{M}_B|_{U_B})$$

is an isomorphism for any $i \geq 1$.

Proof. We put $Z := \mathrm{V}(I) \subset \operatorname{Spec} A$ and $Z_B := \mathrm{V}(IB) \subset \operatorname{Spec} B$. Then [Sta24, Tag 0DWR] implies that it suffices to show that the natural morphism $\mathrm{H}^i_Z(M) \to \mathrm{H}^i_{Z_B}(M_B)$ is an isomorphism for any $i \geq 2$. We show that this actually holds for any $i \geq 0$. For this, we note that [Sta24, Tag 0ALZ] implies that it suffices to show that the natural morphism

$$\mathrm{H}^i_Z(M) \to \mathrm{H}^i_Z(M) \otimes_A B$$

is an isomorphism for any $i \geq 0$. This follows directly from [Sta24, Tag 05E9]. □

Lemma 5.6. *Let A be a ring, let $I \subset A$ be a finitely generated ideal, let $J \subset A$ be a square-zero ideal, let C be a finitely presented A-algebra that is smooth outside $\mathrm{V}(I)$. We put $A_0 := A/J$ and consider a diagram*

(5.7)
$$\begin{tikzcd}
Z_0 = \operatorname{Spec} C_0 \arrow[r, hook, "i_Z"] \arrow[d, "f_0"] & Z = \operatorname{Spec} C \arrow[dd, "h"] \\
Y_0 = \operatorname{Spec} B_0 \arrow[d, "g_0"] & \\
X_0 = \operatorname{Spec} A_0 \arrow[r, hook, "i_X"] & X = \operatorname{Spec} A
\end{tikzcd}$$

such that i_X is the natural closed immersion, the ambient square is Cartesian[9]*, $Y_0 \to X_0$ is a finitely presented morphism that is smooth outside $\mathrm{V}(IA_0)$, and $Z_0 \to Y_0$ is an étale morphism that induces an isomorphism $B_0/IB_0 \xrightarrow{\sim} C_0/IC_0$. Then we can fill in Diagram 5.7 to a commutative diagram*

$$\begin{tikzcd}
Z_0 = \operatorname{Spec} C_0 \arrow[r, hook, "i_Z"] \arrow[d, "f_0"] & Z = \operatorname{Spec} C \arrow[d, "f"] \arrow[dd, bend left=60, "h"] \\
Y_0 = \operatorname{Spec} B_0 \arrow[r, hook] \arrow[d, "g_0"] & Y = \operatorname{Spec} B \arrow[d, "g"] \\
X_0 = \operatorname{Spec} A_0 \arrow[r, hook, "i_X"] & X = \operatorname{Spec} A,
\end{tikzcd}$$

such that each square is a Cartesian, $Y \to X$ is a finitely presented morphism that is smooth outside $\mathrm{V}(I)$, and $Z \to Y$ is an étale morphism that induces an isomorphism $B/IB \xrightarrow{\sim} C/IC$.

Proof. First, we denote by $U \subset X$, $U_0 \subset X_0$, $V_0 \subset Y_0$, $W \subset Z$, and $W_0 \subset Z_0$ the open complement of the vanishing locus of the ideal I. We denote by $\mathrm{Def}_{X_0 \hookrightarrow X}(W_0)$ (resp. $\mathrm{Def}_{X_0 \hookrightarrow X}(V_0)$) the set of isomorphism classes of flat X-lifts of W_0 (resp. of V_0); see [FGI+05, (8.5.7)] for more detail. For a smooth morphism $S \to S'$, we denote by $\mathrm{T}_{S/S'}$ the relative tangent bundle.

Now the classical deformation theory (see [FGI+05, Th. 8.5.9(b)]) implies that the obstruction to lifting W_0 (resp. V_0) to a flat X-scheme is given by a class in $\mathrm{H}^2(W_0, \mathrm{T}_{W_0/X_0} \otimes_{A_0} J)$ (resp. $\mathrm{H}^2(V_0, \mathrm{T}_{V_0/X_0} \otimes_{A_0} J)$) and, if this class vanishes, then the set of isomorphism classes of flat X-lifts is a torsor under the group $\mathrm{H}^1(W_0, \mathrm{T}_{W_0/X_0} \otimes_{A_0} J)$ (resp. $\mathrm{H}^1(V_0, \mathrm{T}_{V_0/X_0} \otimes_{A_0} J)$). Since f_0 is étale, we conclude $\mathrm{T}_{Y_0/X_0} \otimes_{B_0} C_0 \simeq \mathrm{T}_{Z_0/X_0}$. Therefore, Lemma 5.5 implies that the natural morphism

$$\mathrm{H}^i(V_0, \mathrm{T}_{V_0/X_0} \otimes_{A_0} J) \to \mathrm{H}^i(W_0, \mathrm{T}_{W_0/X_0} \otimes_{A_0} J) \tag{5.8}$$

is an isomorphism for $i \geq 1$. Therefore, (5.8) applied to $i = 2$ and the functoriality of the obstruction class (see [FGI+05, Rmk. 8.5.10(a)]) imply that the obstruction to the existence of a flat X-lift of V_0 vanishes (since W_0 admits a flat X-lift).

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[9] In particular, this implies that $C_0 \simeq C/JC$ as an A_0-algebra.

Topological invariance of the small étale site (see [Sta24, Tag 04DZ]) implies that any flat X-lift of V_0 uniquely defines a flat X-lift of W_0. This induces a map $\alpha\colon \mathrm{Def}_{X_0 \hookrightarrow X}(V_0) \to \mathrm{Def}_{X_0\hookrightarrow X}(W_0)$. Now (5.8) applied to $i = 1$ implies that both of these sets are (compatibly) torsors under the same group. Since both torsors are non-empty, we conclude that α is a bijection. In other words, we can find a flat X-lifting V of V_0 which fits in a commuting diagram

$$\begin{tikzcd}
W_0 \arrow[r, hook, "(i_Z)|_{W_0}"] \arrow[d, "(f_0)|_{W_0}"'] & W \arrow[d, "f'"] \\
V_0 \arrow[r, hook] \arrow[d, "(g_0)|_{V_0}"'] & V \arrow[d, "g'"] \\
X_0 \arrow[r, hook, "i_X"] & X
\end{tikzcd}$$

such that all squares are Cartesian.

We wish to lift the whole X_0-scheme Y_0 to an X-scheme Y. This is quite subtle because Y_0 is not a flat X_0-scheme, so the usual deformation theory does not apply in this case. Instead, we use the theory of algebraic spaces. Namely, we note that $W \to V$ is a separated étale morphism due to [Sta24, Tag 06AG]. Therefore, [Sta24, Tag 0DVJ] implies that we can construct an algebraic space Y over X as a pushout of the diagram

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "f'"] & Z \arrow[d, "f"] \\
V \arrow[r, hook] & Y.
\end{tikzcd} \tag{5.9}$$

Furthermore, *loc.cit.* implies that (5.9) is an elementary distinguished square in the sense of [Sta24, Tag 08GM]. In particular, the morphism f is étale. Then [Sta24, Tag 08GN] and [Sta24, Tag 0DVI] imply that $Y \times_X X_0 \simeq Y_0$ as X_0-schemes. Since $X_0 \to X$ is a nilpotent thickening, [Sta24, Tag 07VT] implies that $Y = \operatorname{Spec} B$ is an affine X-scheme. By construction, the affine X-scheme Y fits into the following diagram

$$\begin{tikzcd}
Z_0 = \operatorname{Spec} C_0 \arrow[r, hook, "i_Z"] \arrow[d, "f_0"] & Z = \operatorname{Spec} C \arrow[d, "f"] \arrow[dd, bend left=60, "h"] \\
Y_0 = \operatorname{Spec} B_0 \arrow[r, hook] \arrow[d, "g_0"] & Y = \operatorname{Spec} B \arrow[d, "g"] \\
X_0 = \operatorname{Spec} A_0 \arrow[r, hook, "i_X"] & X = \operatorname{Spec} A
\end{tikzcd}$$

such that each square is Cartesian and f is étale. Therefore, [Sta24, Tag 036N] implies g is finitely presented. By construction $g' = g|_W \colon W \to X$ is flat, so g is smooth due to [Sta24, Tag 01V8]. Equivalently, $A \to B$ is smooth outside $\mathrm{V}(I)$. Finally, [Sta24, Tag 051H] implies that the natural morphism $B/IB \to C/IC$ is an isomorphism since so is $B/(I+J)B = B_0/IB_0 \to C_0/IC_0 = C/(I+J)C$. This finishes the proof. ☐

Before we start the proof of the main result of this section, we need to verify the following basic lemma:

Lemma 5.10. *Let A be a noetherian ring, let $J \subset I$ be two ideals, and let B be a J-adically complete A-algebra such that B/J is a finite type A/J-algebra. Put $A_n = A/J^n$ and $B_n = B/J^nB$. If $B^\wedge_I$ is rig-smooth*[10] *over (A, I) and B_n is smooth outside $\mathrm{V}(IA_n)$ for every $n \geq 1$, then B is rig-smooth over (A, J).*

Proof. We refer to [Sta24, Tag 0AJL] for the construction of the naive (completed) cotangent complexes $(\mathrm{NL}_{B/A})^\wedge_J$ and $(\mathrm{NL}_{B^\wedge_I/A})^\wedge_I$. We choose some compatible generators $J = (f_1, \dots, f_s)$ and $I = (f_1, \dots, f_s, \dots, f_r)$.

For brevity, we denote by N^{-1} and N^0 the cohomology modules of $(\mathrm{NL}_{B/A})^\wedge_J$. Then [Sta24, Tag 0GAJ] implies that it suffices to show that N^{-1} is J-power torsion and $\mathrm{N}^0[\frac{1}{f_i}]$ is projective for $i = 1, \dots, s$. We will

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[10] Since $B^\wedge_I/IB^\wedge_I \simeq B/IB$ is a finite type A/I-algebra, the notion of rig-smoothness over (A, I) is well-defined for $B^\wedge_I$.

prove a stronger claim that N^{-1} is I-power torsion and $\mathrm{N}^0[\frac{1}{f_i}]$ is projective for $i = 1, \dots, r$. For this, we consider the morphism

$$\alpha\colon B \to B_I^\wedge \times \prod_{j=1}^{r} B[\frac{1}{f_j}]_J^\wedge.$$

Using that B is J-adically complete, one easily checks that α is faithfully flat. Therefore, faithfully flat descent implies that it suffices to show that $\mathrm{N}^{-1} \otimes_B B_I^\wedge$ and $\mathrm{N}^{-1} \otimes_B B[\frac{1}{f_j}]_J^\wedge$ are I-power torsion for any $j = 1, \dots, r$, and that $(\mathrm{N}^0 \otimes_B B_I^\wedge)[\frac{1}{f_i}]$ and $(\mathrm{N}^0 \otimes_B B[\frac{1}{f_j}]_J^\wedge)[\frac{1}{f_i}]$ are projective for any $i, j = 1, \dots, r$.

Using the explicit description of $\mathrm{NL}^\wedge$ and the basic properties of completions, we see that $(\mathrm{NL}_{B/A})_J^\wedge$ lies in $D_{coh}^{[-1,0]}(B)$ and there are isomorphisms

$$(\mathrm{NL}_{B/A})_J^\wedge \otimes_B B_I^\wedge \simeq (\mathrm{NL}_{B_I^\wedge/A})_I^\wedge, \tag{5.11}$$

$$(\mathrm{NL}_{B/A})_J^\wedge \otimes_B \left(B[\frac{1}{f_j}]\right)_J^\wedge \simeq (\mathrm{NL}_{(B[\frac{1}{f_j}])_J^\wedge/A})_J^\wedge \tag{5.12}$$

for $j = 1, \dots, r$.

We first deal with $\mathrm{N}^{-1} \otimes_B B_I^\wedge$ and $\mathrm{N}^0 \otimes_B B_I^\wedge$. For this, we observe that (the proof of) [Sta24, Tag 0GAJ] and (5.11) directly imply $\mathrm{N}^{-1} \otimes_B B_I^\wedge$ is I-power torsion and $(\mathrm{N}^0 \otimes_B B_I^\wedge)[\frac{1}{f_i}]$ is projective for any $i = 1, \dots, r$. Then we are only left to deal with $\mathrm{N}^{-1} \otimes_B B[\frac{1}{f_j}]_J^\wedge$ and $\mathrm{N}^0 \otimes_B B[\frac{1}{f_j}]_J^\wedge$ for $j = 1, \dots, r$. We will actually show a stronger claim that $\mathrm{N}^{-1} \otimes_B B[\frac{1}{f_j}]_J^\wedge = 0$ and $\mathrm{N}^0 \otimes_B B[\frac{1}{f_j}]_J^\wedge$ is projective for any $j = 1, \dots, r$. First, [Sta24, Tag 0AJS] implies that $(\mathrm{NL}_{A[\frac{1}{f_j}]_J^\wedge/A})_J^\wedge \simeq 0$ for every $j = 1, \dots, r$. Therefore, [Sta24, Tag 0ALM] implies that $(\mathrm{NL}_{B[\frac{1}{f_j}]_J^\wedge/A})_J^\wedge \simeq (\mathrm{NL}_{B[\frac{1}{f_j}]_J^\wedge/A[\frac{1}{f_j}]_J^\wedge})_J^\wedge$. Thus the assumption that each B_n is smooth outside $\mathrm{V}(IA_n)$, [Sta24, Tag 00T2], [Sta24, Tag 0AJS], and basic properties of completions imply that $\mathrm{H}^{-1}\big((\mathrm{NL}_{B[\frac{1}{f_j}]_J^\wedge/A})_J^\wedge\big) = 0$ and $\mathrm{H}^0\big((\mathrm{NL}_{B[\frac{1}{f_j}]_J^\wedge/A})_J^\wedge\big)$ is a projective for any $j = 1, \dots, r$. Combining it with (5.12), we conclude that $\mathrm{N}^{-1} \otimes_B B[\frac{1}{f_j}]_J^\wedge = 0$ and $\mathrm{N}^0 \otimes_B B[\frac{1}{f_j}]_J^\wedge$ is projective for any $j = 1, \dots, r$. This finishes the proof. □

Theorem 5.13 (Noetherian rig-smooth algebraization)**.** *Let A be a noetherian ring, and let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra such that B/IB is a finite type A/I-algebra and B is rig-smooth over (A, I) (in the sense of* [Sta24, Tag 0GAI]*). Then there is a finite type A-algebra C such that C is smooth outside $\mathrm{V}(I)$ and there is an isomorphism $C_I^\wedge \simeq B$ of A-algebras.*

Proof. First, we note that the natural morphism $A \to B$ uniquely factors as $A \to A_I^{\mathrm{h}} \to B$. Then we note that B is rig-smooth over $(A_I^{\mathrm{h}}, IA_I^{\mathrm{h}})$ (this follows directly from [Sta24, Tag 0GAI]). Now we choose some generators $I = (f_1, \dots, f_r)$ and argue inductively on the number of generators r. If $r = 1$, then [Elk73, Th. 7 on p. 582] implies[11] that there is a finite type A_I^{h}-algebra C' such that it is smooth outside $\mathrm{V}(I)$ and there is an isomorphism $(C')_I^\wedge \simeq B$ of A_I^{h}-algebras. Then a standard approximation argument (similar to [Sta24, Tag 0GAS]) implies that there is a finite type A-algebra C such that it is smooth outside $\mathrm{V}(I)$ and there is an isomorphism $C_I^\wedge \simeq B$ of A-algebras.

Now we do the induction step. We assume that $r > 1$ and the result is known for all ideals generated by less than r elements. Then we put $I_2 \coloneqq (f_2, \dots, f_r)$, and we also put $A_n \coloneqq A/(f_1^n)$ and $B_n = B/(f_1^n)$ for any integer $n \geq 1$. We note that B_n is I-adically complete due to [Sta24, Tag 0AJQ] and that B_n is rig-smooth over (A_n, IA_n) due to [Sta24, Tag 0GAM]. Since the assertion depends only on $\mathrm{rad}(I)$ and $\mathrm{rad}(IA_n) = \mathrm{rad}(I_2A_n)$, we can apply the inductive hypothesis to $A_n \to B_n$ to find finite type A_n-algebras C_n smooth outside $\mathrm{V}(IA_n)$ together with isomorphisms $\alpha_n\colon (C_n)_I^\wedge \xrightarrow{\sim} B_n$ of A_n-algebras (equivalently, of A-algebras). Since $B_n/(f_1^{n-1}) = B_{n-1}$, we use the isomorphisms α_n to get isomorphisms

$$\overline{\varphi}_n\colon (C_n)_I^\wedge/(f_1^{n-1}) \xrightarrow{\sim} (C_{n-1})_I^\wedge$$

of A-algebras. We denote by $\varphi_n\colon (C_n)_I^\wedge \to (C_{n-1})_I^\wedge$ the induced morphisms. We also denote by

$$\pi_n\colon B_n \to B_{n-1}$$

---

[11] We use [Sta24, Tag 0GAJ] to justify that B is formally smooth outside $\mathrm{V}(I)$ in the sense of [Elk73, p. 581].

the "reduction by f_1^{n-1}" morphism. Now we wish to modify α_n and C_n such that all φ_n can be "decompleted" to morphisms $C_n \to C_{n-1}$.

Step 1. *We partially "decomplete" φ_n to achieve that they come from maps $(C_n)^{\mathrm{h}}_I \to (C_{n-1})^{\mathrm{h}}_I$.* To do this, we note that Lemma 5.4 ensures that, for each integer $n > 1$, we can find an A-linear isomorphism

$$\overline{\psi}_n \colon (C_n)^{\mathrm{h}}_I/(f_1^{n-1}) \simeq \big(C_n/(f_1^{n-1})\big)^{\mathrm{h}}_I \xrightarrow{\sim} (C_{n-1})^{\mathrm{h}}_I$$

such that $(\overline{\psi}_n)^{\wedge}_I \bmod I^n = \overline{\varphi}_n \bmod I^n$. We denote by $\psi_n \colon (C_n)^{\mathrm{h}}_I \to (C_{n-1})^{\mathrm{h}}_I$ the induced morphism. For each $n > m \geq 1$, we denote by

$$\psi_{n,m} \colon (C_n)^{\mathrm{h}}_I \to (C_m)^{\mathrm{h}}_I,$$
$$\varphi_{n,m} \colon (C_n)^{\wedge}_I \to (C_m)^{\wedge}_I,$$
$$\pi_{n,m} \colon B_n \to B_m$$

the evident composition of ψ_i, φ_i, and π_i for $i = n, \dots, m+1$ respectively. These morphisms induce isomorphisms $\overline{\psi}_{n,m} \colon (C_n)^{\mathrm{h}}_I/(f_1^m) \simeq \big(C_n/(f_1^m)\big)^{\mathrm{h}}_I \xrightarrow{\sim} (C_m)^{\mathrm{h}}_I$ and $\overline{\varphi}_{n,m} \colon (C_n)^{\wedge}_I/(f_1^m) \simeq \big(C_n/(f_1^m)\big)^{\wedge}_I \xrightarrow{\sim} (C_m)^{\wedge}_I$. Then we define

$$\theta_{n,m} \coloneqq \overline{\varphi}_{n,m} \circ \big((\overline{\psi}_{n,m})^{\wedge}_I\big)^{-1} \colon (C_m)^{\wedge}_I \xrightarrow{\sim} (C_m)^{\wedge}_I$$

for any $n > m \geq 1$. By construction, the diagram

$$\begin{tikzcd}
(C_n)^{\wedge}_I \arrow[r, "\alpha_n", "\sim"'] \arrow[d, "(\psi_{n,m})^{\wedge}_I"] & B_n \arrow[d, "\pi_n"] \\
(C_m)^{\wedge}_I \arrow[r, "\alpha_{n-1}\circ\theta_{n,m}", "\sim"'] & B_{n-1}
\end{tikzcd} \tag{5.14}$$

commutes for any $n > m \geq 1$. Now, for any $n \geq 1$ and an element $x \in (C_n)^{\wedge}_I$, the sequence

$$x, \theta_{n+1,n}(x), \theta_{n+2,n}(x), \theta_{n+3,n}(x), \dots$$

is Cauchy with respect to the I-adic uniform structure because $(\psi_{n+i})^{\wedge}_I \bmod I^{n+i} = \varphi_{n+i} \bmod I^{n+i}$ for any integer $i \geq 0$. Since each $(C_n)^{\wedge}_I$ is I-adically complete, we can define

$$\theta_n(x) = \lim_{m\to\infty} \theta_{n+m,n}(x) \in (C_n)^{\wedge}_I$$

for any $x \in (C_n)^{\wedge}_I$. One easily checks that $\theta_n \colon (C_n)^{\wedge}_I \to (C_n)^{\wedge}_I$ is an isomorphism of A-algebras. Using (5.14) and a standard limit argument, we conclude that the diagram

$$\begin{tikzcd}
(C_n)^{\wedge}_I \arrow[r, "\alpha_n\circ\theta_n", "\sim"'] \arrow[d, "(\psi_n)^{\wedge}_I"] & B_n \arrow[d, "\pi_n"] \\
(C_{n-1})^{\wedge}_I \arrow[r, "\alpha_{n-1}\circ\theta_{n-1}", "\sim"'] & B_{n-1}
\end{tikzcd}$$

commutes for any $n \geq 2$. Therefore, we can replace each α_n by $\alpha_n \circ \theta_n$ to achieve that the diagram

$$\begin{tikzcd}
(C_n)^{\mathrm{h}}_I \arrow[r] \arrow[d, "\psi_n"] & (C_n)^{\wedge}_I \arrow[r, "\alpha_n", "\sim"'] \arrow[d, "(\psi_n)^{\wedge}_I"] & B_n \arrow[d, "\pi_n"] \\
(C_{n-1})^{\mathrm{h}}_I \arrow[r] & (C_{n-1})^{\wedge}_I \arrow[r, "\alpha_{n-1}", "\sim"'] & B_{n-1}
\end{tikzcd}$$

commutes for any $n \geq 2$.

Step 2. *We "dehenselize" ψ_n to achieve that they come from maps $C_n \to C_{n-1}$ (possibly after changing C_n).* More precisely, we show that there exists finite type A_n-algebras C'_n smooth outside $\mathrm{V}(IA_n)$, isomorphisms $\beta_n \colon (C_n)^{\mathrm{h}}_I \xrightarrow{\sim} (C'_n)^{\mathrm{h}}_I$ of A_n-algebras, and morphisms of A_n-algebras $\rho_n \colon C'_n \to C'_{n-1}$ such that $(\rho_n)^{\mathrm{h}}_I = \psi_n$ for every[12] $n \geq 1$ (using the identifications β_n).

We construct C'_n, β_n, and ρ_n inductively on n. For $n = 1$, we put $C'_1 = C_1$ and $\beta_1 = \mathrm{id}$. Now we suppose that $n > 1$ and that we have constructed C'_i, β_i, and ρ_i for $i \leq n-1$, and try to construct it for n. For the purpose of this construction, we can safely replace C_i by C'_i to achieve that $C_i = C'_i$ and $\beta_i = \mathrm{id}$ for $i \leq n-1$.

---

[12] For $n = 1$ there is no datum of ρ_1.

First, Lemma 5.1 ensures that ψ_n induces an isomorphism $\overline{\psi}_n \colon \big(C_n/(f_1^{n-1})\big)^{\mathrm{h}}_I \xrightarrow{\sim} (C_{n-1})^{\mathrm{h}}_I$. Then a simple approximation argument implies that there is a finite type A_{n-1}-algebra D_{n-1} and A-linear étale morphisms $a_{n-1} \colon C_{n-1} \to D_{n-1}$ and $b_{n-1} \colon C_n/(f_1^{n-1}) \to D_{n-1}$ that induce isomorphisms $C_{n-1}/IC_{n-1} \xrightarrow{\sim} D_{n-1}/ID_{n-1}$ and $C_n/(f_1^{n-1}, IC_n) \xrightarrow{\sim} D_{n-1}/ID_{n-1}$ and such that[13]

$$\big((a_{n-1})^{\mathrm{h}}_I\big)^{-1} \circ (b_{n-1})^{\mathrm{h}}_I = \overline{\psi}_n \colon \big(C_n/(f_1^{n-1})\big)^{\mathrm{h}}_I \to (C_{n-1})^{\mathrm{h}}_I$$

The topological invariance of the small étale site (see [Sta24, Tag 039R]) imply that we can find lift the morphism $b_{n-1} \colon C_n/(f_1^{n-1}) \to D_{n-1}$ to an étale A-linear morphism $b_n \colon C_n \to D_n$. Then [Sta24, Tag 051H] ensures that $C_n/IC_n \to D_n/ID_n$ is an isomorphism, thus we have

$$\big((a_{n-1})^{\mathrm{h}}_I\big)^{-1} \circ (r_n)^{\mathrm{h}}_I \circ (b_n)^{\mathrm{h}}_I = \psi_n \colon (C_n)^{\mathrm{h}}_I \to (C_{n-1})^{\mathrm{h}}_I,$$

where $r_n \colon D_n \to D_{n-1}$ is the natural reduction morphism. Using Lemma 5.6, we can find a commutative diagram

$$\begin{tikzcd}
D_n \arrow[r, "r_n"] & D_{n-1} \\
C'_n \arrow[u, "a_n"] \arrow[r, "\rho_n"] & C_{n-1} \arrow[u, "a_{n-1}"'] \\
A_n \arrow[u] \arrow[r] & A_{n-1} \arrow[u]
\end{tikzcd}$$

such that each square is a push-out square, C'_n is a finite type A_n-algebra smooth away from $\mathrm{V}(IA_n)$, and a_n is an étale morphism inducing an isomorphism $C'_n/IC'_n \to D_n/ID_n$. Then we put $\beta_n \coloneqq \big((a_n)^{\mathrm{h}}_I\big)^{-1} \circ (b_n)^{\mathrm{h}}_I \colon (C_n)^{\mathrm{h}}_I \xrightarrow{\sim} (C'_n)^{\mathrm{h}}_I$; this is evidently an isomorphism of A-algebras. Then we have

$$(\rho_n)^{\mathrm{h}}_I \circ \beta_n = (\rho_n)^{\mathrm{h}}_I \circ \big((a_n)^{\mathrm{h}}_I\big)^{-1} \circ (b_n)^{\mathrm{h}}_I = \big((a_{n-1})^{\mathrm{h}}_I\big)^{-1} \circ (r_n)^{\mathrm{h}}_I \circ (b_n)^{\mathrm{h}}_I = \psi_n.$$

Therefore, C'_n, ρ_n, and β_n as defined above do the job. Thus induction finishes the proof of Step 2.

Now we can replace each C_n by C'_n to achieve that we are in the situation where we have A-linear morphisms $\rho_n \colon C_n \to C_{n-1}$ fitting into a commutative diagram

$$\begin{tikzcd}
C_n \arrow[r] \arrow[d, "\rho_n"] & (C_n)^{\wedge}_I \arrow[r, "\alpha_n", "\sim"'] \arrow[d, "(g_n)^{\wedge}_I"] & B_n \arrow[d, "\pi_n"] \\
C_{n-1} \arrow[r] & (C_{n-1})^{\wedge}_I \arrow[r, "\alpha_{n-1}", "\sim"'] & B_{n-1}.
\end{tikzcd}$$

End of proof. Now we note that $\widetilde{C} \coloneqq \lim_n C_n$ is an (f_1)-adically complete A-algebra due to [Sta24, Tag 0AJP]. Also, *loc. cit.* and our assumption on C_n imply that, for any $n \geq 1$, the finite type A_n-algebra $\widetilde{C}/(f_1^n) \simeq C_n$ is smooth outside $\mathrm{V}(IA_n)$. Furthermore, there is an isomorphism[14] $\widetilde{C}^{\wedge}_I \simeq B$ of A-algebras, so $\widetilde{C}^{\wedge}_I$ is rig-smooth over (A, I). Therefore, Lemma 5.10 ensures that $\widetilde{C}$ is rig-smooth over $(A, (f_1))$. Thus, we can apply the base of induction to find a finite type A-algebra C such that it is smooth outside $\mathrm{V}(f_1A)$ and there is an isomorphism $C^{\wedge}_{(f_1)} \simeq \widetilde{C}$ of A-algebras. Since $C/(f_1^n) \simeq \widetilde{C}/(f_1^n)$ is smooth outside $\mathrm{V}(IA_n)$, [Sta24, Tag 0523] and [Sta24, Tag 00TF] imply that $\operatorname{Spec} C \to \operatorname{Spec} A$ is smooth at all points of $\mathrm{V}(f_1C) \smallsetminus \mathrm{V}(IC)$. Since it is also assumed to be smooth outside $\mathrm{V}(f_1A)$, we conclude that C is smooth outside $\mathrm{V}(IA)$. Finally, we note that

$$C^{\wedge}_I \simeq \big(C^{\wedge}_{(f_1)}\big)^{\wedge}_I \simeq (\widetilde{C})^{\wedge}_I \simeq B.$$

Thus, C does the job. □

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[13] In the formula below, we implicitly use that $(a_{n-1})^{\mathrm{h}}_I$ is an isomorphism since it is étale and induces an isomorphism on mod-I fibers.

[14] This follows from the following sequence of isomorphism $\widetilde{C}^{\wedge}_I = \lim_n \widetilde{C}/I^n\widetilde{C} = \lim_n \widetilde{C}/(f_1^n, I^n\widetilde{C}) = \lim_{n,m} \widetilde{C}/(f_1^n, I^m\widetilde{C}) = \lim_{n,m} C_n/I^mC_n = \lim_n(\lim_m C_n/I^mC_n) = \lim_n B_n = B$.

APPENDIX A. DIMENSION FUNCTION ON AFFINOIDS

Throughout this section, we fix a non-archimedean field K and a K-affinoid algebra A. The main goal of this section is to construct a canonical dimension function on $\operatorname{Spec} A$.

Definition A.1. Let X be a spectral space. A point $y \in X$ is a *specialization of* a point x if $y \in \overline{\{x\}}$. We denote the specialization relation by the symbol $x \rightsquigarrow y$.

A point $y \in X$ is an *immediate specialization of* a point x if $x \rightsquigarrow y$, and there is no $z \in X \setminus \{x, y\}$ such that $x \rightsquigarrow z$ and $z \rightsquigarrow y$.

Definition A.2. [ILO14, Exp. XIV, Déf. 2.1.8, Prop. 2.1.4] Let X be a noetherian, universally catenary scheme. A *dimension function* on X is a map

$$\delta\colon |X| \to \mathbf{Z}$$

such that $\delta(y) = \delta(x) - 1$ for any immediate specialization $x \rightsquigarrow y$.

Lemma A.3. [Con99, Lemma 2.1.5] *Let A be a K-affinoid domain, and $\mathfrak{m} \subset A$ a(ny) maximal ideal. Then* $\dim A = \dim A_{\mathfrak{m}}$.

Corollary A.4. *Let A be a K-affinoid algebra. Then the function*

$$\delta_A\colon |\operatorname{Spec} A| \to \mathbf{Z}$$

defined by $\delta_A(x) = \dim A/\mathfrak{p}_x$ is a dimension function on $\operatorname{Spec} A$.

We note that A is a quotient of a regular algebra due to [Bos14, §2.2, Prop. 17]. Therefore, [Sta24, Tag 00NQ and Tag 00NM] imply that A is universally catenary. In particular, Definition A.2 applies to $\operatorname{Spec} A$.

Proof. The only thing we have to check is that, for any prime ideals $\mathfrak{p}' \subset \mathfrak{p}$ with no proper prime ideals between them, we have

$$\dim A/\mathfrak{p} = \dim A/\mathfrak{p}' - 1.$$

For this, we can assume that $\mathfrak{p}' = (0)$ (so A is a domain) and so $\mathfrak{p} \subset A$ is an ideal of height-1. In this case, we pick a maximal ideal $\mathfrak{m} \subset A$. Then Lemma A.3 reduces the question to showing that

$$\dim A_{\mathfrak{m}}/\mathfrak{p} = \dim A_{\mathfrak{m}} - 1 = \dim A_{\mathfrak{m}} - \operatorname{ht}(\mathfrak{p}).$$

This follows from [Sta24, Tag 0ECF]. □

Corollary A.5. [ILO14, Exp. XIV, Cor. 2.5.2] *Let A be a K-affinoid algebra, and let $f\colon X \to \operatorname{Spec} A$ be a finite type morphism. Then*

$$\delta_X(x) = \delta_A(f(x)) + \operatorname{trdeg}\left(k(x)/k(f(x))\right)$$

is a dimension function of X.

Remark A.6. In the notation of Corollary A.5, we have $\delta_X(x) \leq \dim A + d'$, where d' is the relative dimension of f.

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