MOD-p POINCARÉ DUALITY IN p-ADIC ANALYTIC GEOMETRY

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Abstract. We show Poincaré Duality for $\mathbf{F}_p$-étale cohomology of a smooth proper rigid space over a p-adic field K. It positively answers the question raised by P. Scholze in [Sch13a]. We also show versions of Poincaré Duality for $\mathbf{Z}/p^n\mathbf{Z}$, $\mathbf{Z}_p$, and $\mathbf{Q}_p$-cofficients. The main work is duality with $\mathbf{F}_p$-coefficients.

Contents

1. Introduction

1.1. **Historical Overview.** The classical theory of singular cohomology of a topological space has many useful properties. One of them is the Poincaré Duality Theorem that roughly says that, for a compact complex manifold X of pure dimension d, $\mathrm{H}^{2d-i}_{\mathrm{sing}}(X, K)$ is canonically dual to $\mathrm{H}^{i}_{\mathrm{sing}}(X, K)$ for any field K.

One can reformulate this result in a slightly more precise way. The duality says that, for any such X, there is a trace map

$$t_X \colon \mathrm{H}^{2d}_{\mathrm{sing}}(X, K) \to K$$

satisfying:

Theorem 1.1.1. (Classical Poincaré Duality) Let X be a compact complex manifold X of pure dimension d. Then the pairing

$$\mathrm{H}^{i}_{\mathrm{sing}}(X, K) \otimes_K \mathrm{H}^{2d-i}_{\mathrm{sing}}(X, K) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\mathrm{sing}}(X, K) \xrightarrow{t_X} K.$$

is perfect for any $i \geq 0$.

One can also try to show analogues of Theorem 1.1.1 for other cohomology theories. In general, it is expected that a version of Theorem 1.1.1 should hold for any reasonable cohomology theory. We mention some examples that will be relevant in this paper.

Suppose that X is a variety over a field k. Then Grothendieck et. al. defined étale cohomology groups of $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbf{F}_\ell)$ for any prime ℓ. This theory behaves especially well if the prime number ℓ is coprime to the characteristic of k. In this situation, one can show many analogues of the classical results in this algebraic set-up. In particular, one can show an analogue of Poincaré Duality:

Theorem 1.1.2. Let X be a smooth proper scheme of pure dimension d over a field k, and ℓ a prime number coprime with characteristic of k. Then there is a Galois-equivariant trace map

$$t_X \colon \mathrm{H}^{2d}_{\text{ét}}(X_{\overline{k}}, \mathbf{F}_\ell(d)) \to \mathbf{F}_\ell$$

such that the induced Galois-equivariant pairing

$$\mathrm{H}^{i}_{\text{ét}}(X_{\overline{k}}, \mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} \mathrm{H}^{2d-i}_{\text{ét}}(X_{\overline{k}}, \mathbf{F}_\ell(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X_{\overline{k}}, \mathbf{F}_\ell(d)) \xrightarrow{t_X} \mathbf{F}_\ell.$$

is perfect for any $i \geq 0$.

Between the world of complex-analytic manifolds considered in Theorem 1.1.1 and the world of algebraic varieties in Theorem 1.1.2, there is an intermediate world of non-archimedean rigid spaces. These spaces are roughly analytic manifolds over a non-archimedean field K. The theory of rigid-analytic varieties and their étale cohomology groups was extensively studied by Huber and Berkovich (separately) in [Hub96] and [Ber93]. In particular, they both were able to show a weak version of Theorem 1.1.2 in the rigid-analytic set-up:

Theorem 1.1.3. Let X be a smooth proper rigid-analytic variety of pure dimension d over a non-archimedean field K, and ℓ a prime number invertible in the ring of integers $\mathcal{O}_K$. Then there is a Galois-equivariant trace map

$$t_X\colon \mathrm{H}^{2d}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_\ell(d)) \to \mathbf{F}_\ell$$

such that the induced Galois-equivariant pairing

$$\mathrm{H}^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} \mathrm{H}^{2d-i}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_\ell(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_\ell(d)) \xrightarrow{t_X} \mathbf{F}_\ell.$$

is perfect for any $i \geq 0$.

Even though Theorem 1.1.3 looks similar to Theorem 1.1.2, it is weaker if K is of mixed characteristic $(0,p)$. Namely, in this case, $\mathbf{F}_p$-coefficients are not allowed in Theorem 1.1.3 even though K itself is a field of characteristic 0. So there is still an open question of what happens if we consider étale cohomology with $\mathbf{F}_p$-coefficients for rigid spaces over non-archimedean fields of mixed characteristic $(0,p)$.

The question of proving this version of Poincaré Duality was raised by Peter Scholze in [Sch13a]. We give a positive answer to this question proving the following theorem:

Theorem 1.1.4. (Theorem 5.4.2) Let X be a smooth proper rigid-analytic variety of pure dimension d over a non-archimedean field K of mixed characteristic $(0,p)$. Then there is a Galois-equivariant trace map

$$t_X\colon \mathrm{H}^{2d}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_p(d)) \to \mathbf{F}_p$$

such that the induced Galois-equivariant pairing

$$\mathrm{H}^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_p(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p.$$

is perfect for any $i \geq 0$.

We also give a version of Theorem 1.1.4 for $\mathbf{F}_p$-local systems (see Theorem 5.4.3), and also for $\mathbf{Z}/p^n\mathbf{Z}$, $\mathbf{Z}_p$, and $\mathbf{Q}_p$-coefficients (see Theorems 5.5.3, 5.5.5, and 5.5.7).

A proof of Theorem 1.1.4 was also announced by Ofer Gabber in 2015. But a written account of his proof has never appeared since then. Our proof was motivated by Gabber's Youtube lecture on the subject. In particular, both proofs use the perspective of formal models to eventually reduce Poincaré Duality on generic fiber to Grothendieck Duality on special fiber. However, it seems that our proof is quite different in many aspects. We use a refined version of the Temkin's local uniformization theorem from [Zav21b] and Berkovich's construction of the trace map from [Ber93] to simplify the proof. We also use some techniques (e.g. the theory of diamonds [Sch17] and a generalization of almost purity theorem from [BS19]) that were not available at the time.

In our future paper [Zav], we generalize Theorem 1.1.4 to the case of a smooth proper *morhism* $f\colon X \to Y$ and a Zariski-constructible complex $\mathcal{F}\colon \mathbf{D}_{zc}(X, \Lambda)$ for $\Lambda = \mathbf{Z}/p^n\mathbf{Z}$ for some $n \geq 1$. Namely, we show that the functor

$$f^! = f^* \otimes^L \Lambda(d)[2d] \colon \mathbf{D}_{zc}(Y) \to \mathbf{D}_{zc}(X)$$

is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}_{zc}(X) \to \mathbf{D}_{zc}(Y)$[1] for smooth and proper $f\colon X \to Y$.

[1]The fact that $\mathbf{R}f_*$ preserves Zariski-constructibility follows from [BH21, Theorem 3.10].

1.2. **Main Steps of Our Proof.** We now explain the plan of our proof that we follow in the paper. Roughly, we reduce Poincare Duality for $\mathbf{F}_p$-coefficients to almost duality for $\mathcal{O}_X^+/p$-cofficients by using the primitive comparison theorem [Sch13a, Theorem 5.1]. The complex $\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)$ can, in turn, be studied in two steps by writing it as

$$\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)\right)$$

for a choice of a formal model $\mathfrak{X}$, where $\nu\colon (X_{\text{proét}}, \mathcal{O}_X^+/p) \to (\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0})$[2] is the natural morphism of ringed sites. Then we can invoke (the almost version) of quasi-coherent Grothendieck duality on $\mathfrak{X}_0$ to construct the $\mathcal{O}_X^+/p$-version of the trace map, and show that the induced pairing is almost perfect (in the derived sense).

Now we explain the steps of this proof in more detail[3]:

(1) (Section 5.3) We adapt the construction of the trace map $t_X\colon \mathrm{H}^{2d}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{F}_p(d)) \to \mathbf{F}_p$ constructed in [Ber93, §7.2] to the world of rigid spaces. This allows us to assume that the base field $K = C$ is an algebraically closed p-adic non-archimedean field, and that X is (geometrically) connected.

(2) (Theorem 5.4.1) The next step is to show that the trace map $t_X\colon \mathrm{H}^{2d}_{\text{ét}}(X, \mathbf{F}_p(d)) \to \mathbf{F}_p$ is an isomorphism (under the assumption that X is connected). We note that Berkovich has already showed in [Ber93] that t_X is surjective. Thus, for the purpose of showing that t_X is an isomorphism, it suffices to show that $\mathrm{H}^{2d}_{\text{ét}}(X, \mathbf{F}_p(d))$ is 1-dimensional over $\mathbf{F}_p$.

(3) (Section 5.4) Now we invoke the Primitive Comparison Theorem that ensures that the natural morphism

$$\mathrm{H}^i_{\text{ét}}(X, \mathbf{F}_p(d)) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \to \mathrm{H}^i_{\text{ét}}(X, \mathcal{O}_X^+/p)$$

is an almost isomorphism. Therefore, in order to show that $\mathrm{H}^i_{\text{ét}}(X, \mathbf{F}_p(d))$ is one dimensional, it suffices to show that $\mathrm{H}^i_{\text{ét}}(X, \mathcal{O}_X^+/p)$ is almost isomorphic to $\mathcal{O}_C/p$. We will show it later.

(4) (Section 5.4) We note that once we establish that t_X is an isomorphism, perfectness of the pairing

$$\mathrm{H}^i_{\text{ét}}(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}_{\text{ét}}(X, \mathbf{F}_p(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p$$

is equivalent to perfectness of the pairing

$$\mathrm{H}^i_{\text{ét}}(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}_{\text{ét}}(X, \mathbf{F}_p(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X, \mathbf{F}_p(d)).$$

In particular, it does not depend on a choice of the trace map.

(5) (Section 5.4) Step (4) and the Primitive Comparison Theorem guarantee that it suffices to show almost duality for $\mathcal{O}_X^+/p$-coefficients on X. More precisely, it suffices to construct *some* map

$$\mathrm{Tr}^{2d}_X\colon \mathrm{H}^{2d}_{\text{ét}}(X, (\mathcal{O}_X^+/p)(d))^a \to \mathcal{O}_C^a/p$$

such that the induced pairing

$$\mathrm{H}^i_{\text{ét}}(X, \mathcal{O}_X^+/p)^a \otimes_{\mathcal{O}_C/p} \mathrm{H}^{2d-i}_{\text{ét}}(X, (\mathcal{O}_X^+/p)(d))^a \xrightarrow{-\cup-} \mathrm{H}^{2d}_{\text{ét}}(X, (\mathcal{O}_X^+/p)(d))^a \xrightarrow{\mathrm{Tr}_X} \mathcal{O}_C^a/p \tag{1.1}$$

is almost perfect (see Section 1.4 for the precise definition of an almost perfect pairing).

[2] $\mathfrak{X}_0$ is the mod-p fiber of $\mathfrak{X}$, not the special fiber.

[3] We warn the reader that the step order below does not coincide with the order of exposition in this paper.

Indeed, these results would firstly show that $\mathrm{H}^{2d}_{\text{ét}}(X, (\mathcal{O}^+_X/p)(d))^a$ is almost dual to $\mathrm{H}^0_{\text{ét}}(X, \mathcal{O}^+_X/p)^a$ and, therefore, almost isomorphic to $\mathcal{O}_C/p$. In particular, $\mathrm{H}^{2d}_{\text{ét}}(X, \mathbf{F}_p(d)) \simeq \mathbf{F}_p$. And so the argument from Step (4) would guarantee that almost perfectness of (1.1) implies Poincaré Duality.

(6) (Section 5.4) All in all, we showed that it suffices to define some map

$$\mathrm{H}^{2d}_{\text{ét}}(X, (\mathcal{O}^+_X/p)(d))^a \to \mathcal{O}^a_C/p \tag{1.2}$$

that induces an almost perfect pairing between $\mathrm{H}^i_{\text{ét}}(X, \mathcal{O}^+_X/p)^a$ and $\mathrm{H}^{2d-i}_{\text{ét}}(X, (\mathcal{O}^+_X/p)(d))^a$.

It will be more convenient to work with the full complex $\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p(d))$. In order to pass to this complex, we recall that [Zav21a, Theorem 6.13.5] guarantees that $\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)$ is almost concentrated in degrees $[0, 2d]$, so the map (1.2) is equivalent to a map

$$\mathrm{Tr}_X \colon \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to (\mathcal{O}^a_C/p)(-d)[-2d].$$

Moreover, almost perfectness of the pairing induced by (1.2) is equivalent to almost perfectness (in the derived sense) of the pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \xrightarrow{-\cup-} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \xrightarrow{\mathrm{Tr}_X} (\mathcal{O}^a_C/p)(-d)[-2d].$$

So now the problem of showing Poincaré Duality is reduced to two rather separate problems: constructing a trace map for $\mathcal{O}^+_X/p$-coefficients, and showing that this trace induces an almost perfect pairing.

(7) (Section 2) As a preliminary work to construct Tr_X, we need to develop part of the theory of dualizing sheaves and complexes on admissible formal $\mathcal{O}_C$-schemes. We prove only the bare minimum of the results we need for our purposes, and do not develop a general theory. We now only mention one result that is crucial for our construction of Tr_X. In particular, for any separated admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}$, we define its dualizing complex $\omega^\bullet_{\mathfrak{X}}$ and dualizing module $\omega_{\mathfrak{X}}$ (see Definition 2.1.2 and Definition 2.1.3 for the precise definitions)

Let $\mathfrak{X}$ a separated, admissible formal $\mathcal{O}_C$-model of X with reduced special fiber $\overline{\mathfrak{X}}$. Then the natural morphism $\omega_{\mathfrak{X}} \to j_{\mathfrak{X}^{\mathrm{sm}},*}(\omega_{\mathfrak{X}}|^{\mathrm{sm}}_{\mathfrak{X}}) \cap \omega_{\mathfrak{X}C}$ is an isomorphism (see Theorem 2.1.14 for the precise formulation of this result). If X is a sufficiently nice scheme over a discretely valued ring $\mathcal{O}_K$, the above property of the dualizing sheaf is well-known, and roughly follows from the fact that ω_X can be recovered from the codimension-1 points.

(8) (Section 3 and Section 5.2) Now we are ready to define Tr_X. We choose an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ of X. Then $\mathfrak{X}$ is proper by [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]), and the adjunction in the almost version of Grothendieck Duality [Zav21a, Theorem 6.13.5 and Theorem 6.13.6][4] guarantees that a map Tr_X that we want to construct corresponds to a map

$$\mathbf{R}\nu_*\left(\mathcal{O}^+_X/p\right)^a \to \omega^\bullet_{\mathfrak{X}_0}(-d)[-2d] \coloneqq \mathfrak{f}^!_0(\mathcal{O}_C/p)(-d)[-2d],$$

where $\mathfrak{f} \colon \mathfrak{X} \to \operatorname{Spf} \mathcal{O}_C$ is the structure map. Now [Zav21a, Theorem 6.13.5 and Theorem 6.13.6] ensures that both $\mathbf{R}\nu_*\left(\mathcal{O}^+_X/p\right)^a$, and $\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a$ are (almost) concentrated in degrees $[0, d]$ and $\omega^\bullet_{\mathfrak{X}}(-d)[-2d]$ is concentrated in degrees $[d, 2d]$ by Theorem 2.1.6. So, it is

[4]We note that ν is denoted by ν' in [Zav21a].

sufficient to define a map

$$\mathrm{Tr}^{d,+}_{F,\mathfrak{X}} \colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a \to \omega^a_{\mathfrak{X}}(-d)$$

that provides certain perfectness properties.

By the Reduced Fiber Theorem, we can focus our attention on the case of an admissible formal scheme $\mathfrak{X}$ with reduced special fiber. In this case, the isomorphism

$$\omega_{\mathfrak{X}} \to j_{\mathfrak{X}^{\mathrm{sm}},*}(\omega_{\mathfrak{X}}|^{\mathrm{sm}}_{\mathfrak{X}}) \cap \omega_{\mathfrak{X}C}$$

ensures that it suffices to define $\mathrm{Tr}^d_{F,\mathfrak{X}}$ on the generic fiber and on the smooth locus in a compatible manner. We use the map from [Sch13a, Proposition 3.23] on the generic fiber, and we adapt the map from [BMS18, §8.2] on the smooth locus. This finishes the construction of Tr_X. A priori, this construction depends on a choice of an admissible model $\mathfrak{X}$, but we show that it is canonically independent of it in Lemma 5.2.1.

We want to emphasize that even though we are mostly interested in the $\mathcal{O}^+_X/p$ cohomology groups, it was crucial here to work integrally here to be able to have access to the generic fiber of $\mathfrak{X}$.

(9) (Section 4 and Section 5.2) Finally, we need to show that the trace map Tr_X constructed in (8) induces an almost perfect pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \xrightarrow{-\cup-} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \xrightarrow{\mathrm{Tr}_X} (\mathcal{O}^a_C/p)(-d)[-2d].$$

Again, using the almost version of Grothendieck Duality and properness of any admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ of X, we reduce the question to showing almost perfectness of the pairing

$$\mathbf{R}\nu_*(\mathcal{O}^+_X/p)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*(\mathcal{O}^+_X/p)^a \xrightarrow{-\cup-} \mathbf{R}\nu_*(\mathcal{O}^+_X/p)^a \xrightarrow{\mathrm{Tr}_{F,\mathfrak{X}}} (\omega^{\bullet,a}_{\mathfrak{X}_0})(-d)[-2d]. \tag{1.3}$$

This is now a *local* question on $\mathfrak{X}$. We use the local uniformization theorem [Zav21b, Theorem 1.4] that roughly says that any admissible formal $\mathcal{O}_C$-model of smooth X locally looks like a "nice" quotient by a finite group of a polystable formal $\mathcal{O}_C$-scheme up to some rig-isomorphisms. So it suffices to show almost perfectness for polystable formal $\mathcal{O}_C$-models and then show that it descends through rig-isomorphisms and "nice" quotients by finite groups.

We verify almost perfectness of pairing (1.3) in the polystable case by an explicit computation that eventually reduces the claim to almost duality in the continuous group cohomology of the profinite group $\mathbf{Z}_p(1)^d$. And then we show almost perfectness of pairing (1.3) by using the almost version of Grothendieck Duality and duality between homotopy invariants and coinvariants for an action of a finite group G. This finishes the proof.

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1.4. **Conventions.** Let K be a complete rank-1 valued field. We denote its ring of integers by $\mathcal{O}_K$, its maximal ideal by $\mathfrak{m}_K$, and residue field by $k \coloneqq \mathcal{O}_K/\mathfrak{m}_K$.

An *admissible* formal $\mathcal{O}_K$-scheme is a quasi-compact, quasi-separated, flat, topologically finitely presented formal $\mathcal{O}_K$-scheme $\mathfrak{X}$. If $\varpi \in \mathcal{O}_K$ is a fixed pseudo-uniformizer, we define

$$\mathfrak{X}_i \coloneqq \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$$

to be the mod-ϖ^{i+1} fiber of $\mathfrak{X}$. We denote also the *special fiber* by

$$\overline{X} \coloneqq \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K.$$

In this paper, a *rigid K-space* will always mean an adic space locally finite type over $\operatorname{Spa}(K, K^\circ)$. We note that there is a fully faithful functor r_K[5] from the category of (classical) Tate rigid K-spaces to the category of rigid K-spaces in our definition. This functor induces an equivalence between quasi-separated Tate rigid K-spaces and quasi-separated rigid K-spaces by [Hub94, Proposition 4.5].

We recall the notion of the *cotangent complex* $L_f = L_{X/S} = L_{\mathcal{O}_X/\mathcal{O}_S}$ for a morphism of ringed sites $f\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$. We refer to [Sta19, Tag 08UT] for the definition and a self-contained systematic development of this theory. We follow Stacks Project and use cohomological notations for the cotangent complex, in particular, $L_f \in \mathbf{D}^{\leq 0}(\mathcal{O}_X)$ for a morphism f. If $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of topologically finitely presented formal $\mathcal{O}_C$-schemes, we define $\widehat{L}_{\mathfrak{X}/\mathfrak{Y}}$ as the derived p-adic completion of the complex $L_{\mathfrak{X}/\mathfrak{Y}}$. This definition coincides with the definition of the *analytic cotangent complex* $L^{\mathrm{an}}_{\mathfrak{X}/\mathfrak{Y}}$ from [GR03, Definition 7.2.3]. In particular, [GR03, Proposition 7.2.10] implies that $\widehat{L}_{\mathfrak{X}/\mathfrak{Y}} \in \mathbf{D}^{\leq 0}_{coh}(\mathfrak{X})$ and the natural morphism $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to \mathcal{H}^0(\widehat{L}_{\mathfrak{X}/\mathfrak{Y}})$ is an isomorphism for any $\mathfrak{f}$.

We use the formalism of *almost coherent* sheaves on (formal) $\mathcal{O}_C$-schemes as developed in [Zav21a]. In what follows, we always do almost mathematics with respect to the ideal $\mathfrak{m} \subset \mathcal{O}_C$. It is straightforward to see that $\mathfrak{m}$ is $\mathcal{O}_C$-flat and that $\mathfrak{m} = \mathfrak{m}^2$, so this does satisfy the condition for the ideal of almost mathematics.

We define *the pro-étale site* of a rigid-space X similarly to [Sch13a] and [Sch16], but we fix a cardinal κ and consider only κ-small pro-systems and coverings by κ-small sets of objects. It is easily seen that this does not change the all the results of this paper. We also consider the morphisms of ringed topoi $\lambda\colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (X_{\text{ét}}, \mathcal{O}^+_X)$ and $t\colon (X_{\text{ét}}, \mathcal{O}^+_X) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ for any rigid-space X with an admissible formal model $\mathfrak{X}$. So, there is a commutative triangle

$$\begin{tikzcd}
\left(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X\right) \arrow[d, "\lambda"] \arrow[dr, "\nu"] & \\
\left(X_{\text{ét}}, \mathcal{O}^+_X\right) \arrow[r, "t"] & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})
\end{tikzcd}$$

that also induces a commutative diagram

$$\begin{tikzcd}
\left(X_{\text{proét}}, \mathcal{O}^+_X/p\right) \arrow[d, "\lambda"] \arrow[dr, "\nu"] & \\
\left(X_{\text{ét}}, \mathcal{O}^+_X/p\right) \arrow[r, "t"] & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/p) = (\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}).
\end{tikzcd}$$

[5]We refer to [Hub94, §4] for the details on the construction of this functor.

Whenever we talk about cohomology of $\mathbf{Z}_p$ or $\mathbf{Q}_p$ local systems on a rigid space X, we always mean pro-étale cohomology groups. In particular, if X is a (quasi-compact, quasi-separated) rigid space over a non-archimedean field K with $\widehat{\overline{K}} = C$,

$$\mathbf{R}\Gamma(X_C, \mathbf{Z}_p) \coloneqq \mathbf{R}\Gamma_{\text{proét}}(X_C, \widehat{\mathbf{Z}}_p) \text{ and}$$

$$\mathbf{R}\Gamma(X_C, \mathbf{Q}_p) \coloneqq \mathbf{R}\Gamma_{\text{proét}}(X_C, \widehat{\mathbf{Q}}_p),$$

where $\widehat{\mathbf{Z}}_p \coloneqq \lim_n \underline{\mathbf{Z}/p^n\mathbf{Z}}$ in $X_{\text{proét}}$ and $\widehat{\mathbf{Q}}_p = \widehat{\mathbf{Z}}_p[1/p]$.

If $(\mathcal{C}, \otimes, \mathbf{1})$ is a closed symmetric monoidal category with the inner Hom-functor $\underline{\mathrm{Hom}}$, we say that a pairing

$$A \otimes B \to C$$

is *perfect* if both duality morphisms

$$B \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, C)$$
$$A \to \underline{\mathrm{Hom}}_{\mathcal{C}}(B, C)$$

are isomorphisms. Throughout the paper, we are mostly interested in the cases $\mathcal{C}$ is equal to one of the following symmetric monoidal categories

$$(\mathbf{Mod}_R, \otimes_R)\,,\ \left(\mathbf{D}(R), \otimes^L_R\right),\ (\mathbf{Mod}^a_R, \otimes_R)\,,\ \left(\mathbf{D}(R)^a, \otimes^L_R\right)$$

for some $\mathcal{O}_C$-algebra R, or

$$(\mathbf{Mod}_X, \otimes_{\mathcal{O}_X})\,,\ \left(\mathbf{D}(X), \otimes^L_{\mathcal{O}_X}\right),\ (\mathbf{Mod}^a_X, \otimes_{\mathcal{O}_X})\,,\ \left(\mathbf{D}(X)^a, \otimes^L_{\mathcal{O}_X}\right)$$

for an $\mathcal{O}_C$-ringed space $(X, \mathcal{O}_X)$. If the monoidal category $(\mathcal{C}, \otimes)$ is one of $(\mathbf{Mod}^a_R, \otimes_R)$, $\left(\mathbf{D}(R)^a, \otimes^L_R\right)$, $(\mathbf{Mod}^a_X, \otimes_{\mathcal{O}_X})$, or $\left(\mathbf{D}(X)^a, \otimes^L_{\mathcal{O}_X}\right)$, we say that a pairing is *almost perfect* instead of just perfect.

We say that a diagram of the form

$$\begin{array}{ccc} A \otimes A' & \xrightarrow{\alpha} & A'' \\ f\uparrow \quad \downarrow g & & \downarrow h \\ B \otimes B' & \xrightarrow{\beta} & B'' \end{array}$$

commutes if two natural morphisms

$$B \otimes A' \xrightarrow{\mathrm{id}\otimes f} A \otimes A' \xrightarrow{\alpha} A'' \xrightarrow{h} B,$$

$$B \otimes A' \xrightarrow{g\otimes \mathrm{id}} B \otimes B' \xrightarrow{\beta} B$$

are equal.

We also recall the definitions of the Tate and Breuil-Kisin twists. We fix a complete rank-1 valuation field K of mixed characteristic $(0, p)$ with $\widehat{\overline{K}} = C$. The absolute Galois group $G_K \coloneqq \mathrm{Gal}(\overline{K}/K)$ acts continuously on C and $\mathcal{O}_C$.

The *Tate twist* $\mathbf{Z}_p(1)$ is defined as

$$\mathbf{Z}_p(1) \coloneqq \mathrm{T}_p(\mathbf{G}_{m,K}) = \lim \mu_{p^n}(\overline{K}),$$

where $\mathrm{T}_p(\mathbf{G}_{m,K})$ stands for the Tate module of $\mathbf{G}_{m,K}$. This is a rank-1 free $\mathbf{Z}_p$-module with a continuous action of G_K. However, this module does not have any canonical trivialization.

Now $\mathcal{O}_C(1)$ is defined as an $\mathcal{O}_C$-module as $\mathcal{O}_C \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1) = \mathcal{O}_C \otimes_{\mathbf{Z}_p} (\lim \mu_{p^n}(\overline{K}))$. We define $\mathcal{O}_C(n)$, for $n \geq 1$, as tensor products $\mathcal{O}_C(n) \coloneqq \mathcal{O}_C(1)^{\otimes n}$. Finally, we define $\mathcal{O}_C(-n)$ as $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C(n), \mathcal{O}_C)$.

If $\mathcal{C}$ is a site, and $\mathcal{F} \in \mathbf{Shv}(\mathcal{C})$ is a sheaf of $\mathcal{O}_C$-modules, we define its Tate twist $\mathcal{F}(n)$ as $\mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_C(n)$. It is straightforward to see that there is a canonical isomorphism of G_K-modules $\mathrm{H}^i(U, \mathcal{F}(n)) \cong \mathrm{H}^i(U, \mathcal{F})(n)$ for $U \in \mathcal{C}$ and $i \geq 0$. We will freely use this isomorphism in this paper.

We also recall the definition of the *Breuil-Kisin twist* $\mathcal{O}_C\{1\}$ from [BMS18, Definition 8.2]. We set

$$\mathcal{O}_C\{1\} \coloneqq \mathrm{T}_p(\Omega^1_{\mathcal{O}_C/\mathbf{Z}_p}) = \widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p}[-1],$$

where $\widehat{L}$ stands for the derived p-adically complete cotangent complex. We define the dlog map $\mathrm{dlog}\colon \mathbf{Z}_p(1) \to \mathcal{O}_C\{1\}$ as the morphism induced by the map

$$\mu_{p^\infty}(\bar{K}) \to \Omega^1_{\mathcal{O}_C/\mathbf{Z}_p}$$

that sends $f \in \mu_{p^\infty}(\bar{K})$ to $\frac{df}{f}$. The theorem of Fontaine [Fon82, Theorem 1], [Bei12, §1.3], [SZ18, Theorem 3.1] says $\mathcal{O}_C\{1\}$ is a free rank-1 $\mathcal{O}_C$-module, and the dlog map, after an $\mathcal{O}_C$-linearization,

$$\mathrm{dlog}_{\mathcal{O}_C}\colon \mathcal{O}_C(1) \to \mathcal{O}_C\{1\}$$

is injective with image being equal to $(\zeta_p - 1)\mathcal{O}_C\{1\}$ for any choice of a primitive p-th root of unity ζ_p. We define the Brieul-Kisin twists $\mathcal{O}_C\{n\}$ and $\mathcal{F}\{n\}$ similarly to the case of analogous Tate twists.

2. Poor Man's Version of Duality on Formal Schemes

2.1. **Dualizing Modules on Separated Admissible Formal Schemes.** We define "naive dualizing complex" on separated admissible formal $\mathcal{O}_K$-schemes. This object will not be quite functorial because its construction will involve certain derived limits that are not functorial. In order to deal with this problem, we mainly restrict our attention to the study of a *dualizing module* that is defined as the bottom cohomology sheaves of naive dualizing complex. This turns out to be a more functorial object, and this functoriality is sufficient for all our purposes.

For the rest of the section, we fix a complete rank-1 valuation ring $\mathcal{O}_K$ with a pseudo-uniformizer ϖ and *algebraically closed*[6] fraction field K. We do not assume here that K is of characteristic 0.

We recall that if $f_i\colon Y_i \to Z_i$ is a separated morphism of finitely presented $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$-schemes. We can define the twisted inverse image functor[7]

$$f_i^!\colon \mathbf{D}^+_{qcoh}(Z_i) \to \mathbf{D}^+_{qcoh}(Y_i).$$

If $f_i\colon Y_i \to \operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$ a separated finitely presented morphism, we can define the *dualizing complex*

$$\omega^\bullet_{Y_i} \coloneqq f_i^!\left(\mathcal{O}_{\operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K}\right)$$

Now [Sta19, Tag 0E9W] guarantees that $\omega^\bullet_{Y_i}$ is $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$-perfect and comes with the additional "rigidity structure", i.e. an isomorphism

$$\xi\colon \mathbf{R}\Delta_*\mathcal{O}_{Y_i} \to \mathbf{L}p_1^*\omega^\bullet_{Y_i}$$

that induces an isomorphism

$$\Delta_*\mathcal{O}_X \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\Delta_*\mathcal{O}_X, \mathbf{L}p_1^*\omega^\bullet_{Y_i}),$$

[6] This will be used only in Definition 2.4.6. It is possible to develop a more general theory without this assumption. But we have decided not to do this as we will never need more general valuation rings in this paper.

[7] We refer to [Sta19, Tag 0A9Y] and [Sta19, Tag 0ATZ] for the construction and properties of these functors. We note that there is an extra noetherian assumption. This assumption can be weakened to the universally coherent assumption by using the suitable version of the Proper Mapping Theorem [FK18].

where $\Delta\colon Y_i \to Y_i \times_{\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K} Y_i$ is the diagonal map, and $p_1\colon Y_i \times_{\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K} Y_i \to Y_i$ is the first projection map. In other words, the pair $(\omega^\bullet_{Y_i}, \xi)$ is the relative dualizing complex in the sense of [Sta19, Tag 0E2T]. Therefore, [Sta19, Tag 0E2Y] and [Sta19, Tag 0E2W] provide us with the canonical isomorphisms

$$\mathrm{BC}_{\omega^\bullet_{Y_i}}\colon \omega^\bullet_{Y_i} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y_i}} \mathcal{O}_{Y_{i-1}} \to \omega^\bullet_{Y_{i-1}}$$

and

$$\overline{\mathrm{BC}}_{\omega^\bullet_{Y_i}}\colon \omega^\bullet_{Y_i} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y_i}} \mathcal{O}_{\overline{Y_i}} \to \omega^\bullet_{\overline{Y_i}},$$

where $Y_{i-1} \coloneqq Y_i \times_{\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\varpi^i\mathcal{O}_K$ and $\overline{Y_i} \coloneqq Y_i \otimes_{\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$. In particular, these morphisms induce $\mathcal{O}_{Y_i}$-linear morphisms

$$\mathrm{BC}_{\omega^\bullet_{Y_i}}\colon \omega^\bullet_{Y_i} \to \omega^\bullet_{Y_{i-1}}.$$

Lemma 2.1.1. Let Y_i be a separated, flat, finitely presented $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$-scheme. Then $\omega^\bullet_{Y_i} \in \mathbf{D}^{[-d,0]}_{coh}(Y_i)$, where d is the Krull dimension of the special fiber $\overline{Y} \coloneqq Y_i \times_{\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$.

Proof. We firstly note that $\omega^\bullet_{Y_i} \in \mathbf{D}_{coh}(Y_i)$ by [Sta19, Tag 0AU1]. So, we only need to show that $\omega^\bullet_{Y_i} \in \mathbf{D}^{[-d,0]}(Y_i)$.

Now we use that $\omega^\bullet_{Y_i} \otimes^L_{\mathcal{O}_K/\varpi^{i+1}} \mathcal{O}_K/\mathfrak{m}_K \simeq \omega^\bullet_{\overline{Y}} \in \mathbf{D}^{[-d,0]}_{coh}(\overline{Y})$ by [Sta19, Tag 0AWN]. Recall that $\omega^\bullet_{Y_i}$ is $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$-perfect, so it is, in particular, bounded above. Therefore, an easy application of Nakayama's lemma implies that $\omega^\bullet_{Y_i} \in \mathbf{D}^{\leq 0}_{coh}(Y_i)$.

The last thing we need to check is that $\omega^\bullet_{Y_i} \in \mathbf{D}^{\geq -d}_{coh}(Y_i)$. We will prove this without the flatness assumption on Y_i. We note that Y_i admits a finitely type $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$ compactification by [Con07, Theorem 4.1] and standard approximation techniques to ensure that a compactification can be chosen to be finitely presented. Then it suffices to show that $\omega^\bullet_{Y_i}$ lies in $\mathbf{D}^{\geq -d}(Y_i)$ for a proper, finitely presented $\mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K$-scheme Y_i.

In this case, we know that $f^!$ is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{qcoh}(Y_i) \to \mathbf{D}_{qcoh}(\operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K)$. We also note that $|Y_i| = |\overline{Y_i}|$, so the Krull dimension of Y_i is equal to d. So the cohomological dimension of f_* is at most d. Therefore, we conclude that

$$\operatorname{Hom}_{\mathbf{D}(Y_i)}\left(\tau^{<-d}\omega^\bullet_{Y_i}, \omega^\bullet_{Y_i}\right) \simeq \operatorname{Hom}_{\mathbf{D}(\operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K)}\left(\mathbf{R}f_*\tau^{<-d}\omega^\bullet_{Y_i}, \mathcal{O}_{\operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K}\right).$$

However,

$$\mathbf{R}f_*\tau^{<-d}\omega^\bullet_{Y_i} \in \mathbf{D}^{<0}(\operatorname{Spec} \mathcal{O}_K/\varpi^{i+1}\mathcal{O}_K),$$

as the cohomological dimension of f_* is at most d. Therefore, we conclude that

$$\operatorname{Hom}_{\mathbf{D}(Y_i)}(\tau^{<-d}\omega^\bullet_{Y_i}, \omega^\bullet_{Y_i}) \simeq 0.$$

In other words, $\omega^\bullet_{Y_i} \in \mathbf{D}^{\geq -d}(Y_i)$ finishing the proof. $\square$

Now we define a dualizing complex on a separated admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}$. Lemma 2.1.1 guarantees that $\omega^\bullet_{\mathfrak{X}_i} \in \mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X}_i)$, where $d = \dim \overline{X}$. And the canonical isomorphisms $\mathrm{BC}_{\omega^\bullet_{\mathfrak{X}_i}}\colon \omega^\bullet_{\mathfrak{X}_i} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_i}} \mathcal{O}_{\mathfrak{X}_{i-1}} \to \omega^\bullet_{\mathfrak{X}_{i-1}}$ induce morphisms

$$\mathrm{BC}_{\omega^\bullet_{\mathfrak{X}_i}}\colon \omega^\bullet_{\mathfrak{X}_i} \to \omega^\bullet_{\mathfrak{X}_{i-1}}.$$

In what follows, We slightly abuse the notation and denote the base change morphisms simply as $\mathrm{BC}_{\mathfrak{X}_i}$.

Definition 2.1.2. We define *a naive dualizing complex* $\omega^\bullet_{\mathfrak{X}}$ of a separated admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ as the derived limit $\omega^\bullet_{\mathfrak{X}} \coloneqq \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n}$ with the transition map $\mathrm{BC}_{\mathfrak{X}_n}$. This object is well-defined up to an isomorphism (but not up to a unique isomorphism).

The naive dualizing complex will play the role of an intermediate device in our paper. So the lack of functoriality of $\omega^\bullet_{\mathfrak{X}}$ will not be a major problem for our purposes. The actual thing that will be important for us the dualizing sheaf $\omega_{\mathfrak{X}} \coloneqq \lim_n \omega_{\mathfrak{X}_n}$ that is, indeed, functorial in $\mathfrak{X}$.

Definition 2.1.3. We define the *dualizing sheaf* (also called *dualizing module*) $\omega_{\mathfrak{X}}$ on a separated admissible formal $\mathcal{O}_K$-scheme with the special fiber of pure dimension d as the limit $\omega_{\mathfrak{X}} \coloneqq \lim_n \omega_{\mathfrak{X}_n}$, with the transition maps induced by $\mathrm{BC}_{\mathfrak{X}_n}$. This object is defined up to a unique isomorphism.

Remark 2.1.4. We note that [FK18, Corollary II.10.1.11] implies that $\overline{\mathfrak{X}}$ is of pure dimension d, if so is its adic generic fiber $\mathfrak{X}_K$.

Remark 2.1.5. Theorem 2.1.6 below implies that there is an isomorphism $\mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}}) \cong \omega_{\mathfrak{X}}$.

We note that there is a canonical isomorphism $\omega_{\mathfrak{U}} \to (\omega_{\mathfrak{X}})_{\mathfrak{U}}$ for any open $\mathfrak{U} \subset \mathfrak{X}$, and a non-canonical isomorphism $\omega^\bullet_{\mathfrak{U}} \to (\omega^\bullet_{\mathfrak{X}})_{\mathfrak{U}}$. The dualizing module is the object we are more interested in, but it is more convenient to study some of its properties by relating it to a less canonical dualizing complex $\omega^\bullet_{\mathfrak{X}}$. The main issue with studying $\omega_{\mathfrak{X}}$ on its own right is that it does not commute with the base change, while the formation $\omega^\bullet_{\mathfrak{X}}$ does commute with the derived base change.

Theorem 2.1.6. Let $\mathfrak{X}$ be a separated admissible formal $\mathcal{O}_K$-scheme with the adic generic fiber $X = \mathfrak{X}_K$ of pure dimension d. Then the dualizing complex $\omega^\bullet_{\mathfrak{X}}$ lies in $\mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X})$ and the canonical morphism $\omega^\bullet_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_n} \to \omega^\bullet_{\mathfrak{X}_n}$ is an isomorphism. Moreover, there is an isomorphism $\mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}}) \cong \omega_{\mathfrak{X}}$, so the dualizing module $\omega_{\mathfrak{X}}$ is a coherent sheaf on $\mathfrak{X}$. There is also the canonical isomorphism $r_{\mathfrak{X}^{\mathrm{sm}}} \colon \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}[d] \to \omega^\bullet_{\mathfrak{X}^{\mathrm{sm}}}$ on the smooth locus of $\mathfrak{X}$. This induces a canonical isomorphism $r_{\mathfrak{X}^{\mathrm{sm}}} \colon \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}} \to \omega_{\mathfrak{X}}|_{\mathfrak{X}^{\mathrm{sm}}}$.

Proof. Step 1. The dualizing complex $\omega^\bullet_{\mathfrak{X}}$ is in $\mathbf{D}^{\le 0}_{coh}(\mathfrak{X})$ and commutes with base change: The question is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \operatorname{Spf} A$ for an admissible $\mathcal{O}_K$-algebra A. As $\mathfrak{X}_0$ is a coherent scheme, we note that any object of $\mathbf{D}^-_{coh}(\mathfrak{X}_0)$ is pseudo-coherent. Since $\omega^\bullet_{\mathfrak{X}_0} \in \mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X}_0)$ by Lemma 2.1.1, we conclude that $\omega^\bullet_{\mathfrak{X}_0}$ is a pseudo-coherent object on $\mathfrak{X}_0$. So [Sta19, Tag 08D9] and [Sta19, Tag 08E7] imply that there is a resolution of $\omega^\bullet_{\mathfrak{X}_0}$ by finite free $\mathcal{O}_{\mathfrak{X}_0}$-module $\mathcal{E}^\bullet_0 \to \omega^\bullet_{\mathfrak{X}_0}$. Now we recall that there are canonical isomorphisms $\mathrm{BC}_{\omega^\bullet_{\mathfrak{X}_n}} \colon \omega^\bullet_{\mathfrak{X}_n} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_n}} \mathcal{O}_{\mathfrak{X}_{n-1}} \to \omega^\bullet_{\mathfrak{X}_{n-1}}$ as $\mathfrak{X}_n$ is flat over $\mathcal{O}_K/\varpi^{n+1}\mathcal{O}_K$. Thus, we use [Sta19, Tag 0BCB] to find resolutions $\mathcal{E}^\bullet_n \to \omega^\bullet_{\mathfrak{X}_n}$ by finite free modules such that $\mathcal{E}^\bullet_n \otimes_{\mathcal{O}_{\mathfrak{X}_n}} \mathcal{O}_{\mathfrak{X}_{n-1}} \simeq \mathcal{E}^\bullet_{n-1}$. Then we see that

$$\omega^\bullet_{\mathfrak{X}} \cong \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n} \cong \mathbf{R}\lim_n \mathcal{E}^\bullet_n \cong \lim_n \mathcal{E}^\bullet_n$$

where the last equality comes from [Sta19, Tag 0D60] as cohomology coherent sheaves vanish on affine schemes. This already shows that $\omega^\bullet_{\mathfrak{X}} \in \mathbf{D}^{\le 0}(\mathfrak{X})$. In order to show that $\omega^\bullet_{\mathfrak{X}} \in \mathbf{D}^{\le 0}_{coh}(\mathfrak{X})$, it suffices to show that $\lim_n \mathcal{E}^i_n \in \mathbf{Coh}(\mathfrak{X})$ for any $i \le 0$, but this is easily seen to be finite free $\mathcal{O}_{\mathfrak{X}}$-module of the same rank as $\mathcal{E}^i_0$. So, $\omega^\bullet_{\mathfrak{X}} \in \mathbf{D}^{\le 0}_{coh}(\mathfrak{X})$. And, moreover, we established that the map $\omega^\bullet_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathcal{O}_{\mathfrak{X}_n}} \to \omega^\bullet_{\mathfrak{X}_n}$ has a representative $(\lim_n \mathcal{E}^\bullet_n) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_n} \to \mathcal{E}^\bullet_n$ that is an isomorphism by the construction.

Step 2. The dualizing complex $\omega^\bullet_{\mathfrak{X}}$ is in $\mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X})$ and there is an isomorphism $\mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}}) \cong \omega_{\mathfrak{X}}$: Now we show that $\mathcal{H}^m(\omega^\bullet_{\mathfrak{X}}) \cong 0$ for $m < -d$. As $\mathcal{H}^m(\omega^\bullet_{\mathfrak{X}})$ is coherent on $\mathfrak{X}$, it is sufficient to show that $\Gamma(\mathfrak{X}, \mathcal{H}^m(\omega^\bullet_{\mathfrak{X}})) \cong 0$ by [FK18, Proposition I.3.2.1]. We use [Zav21a, Lemma 4.8.13] to conclude that

$$\mathrm{H}^m(\mathfrak{X}, \omega^\bullet_{\mathfrak{X}}) := \mathrm{H}^m\mathbf{R}\Gamma(\mathfrak{X}, \omega^\bullet_{\mathfrak{X}}) \simeq \Gamma(\mathfrak{X}, \mathcal{H}^m(\omega^\bullet_{\mathfrak{X}})),$$

so it is actually sufficient to show that $\mathrm{H}^m(\mathfrak{X}, \omega^\bullet_{\mathfrak{X}})$ for $m < -d$. Now we apply the exact sequence [Sta19, Tag 0D60] to $\omega^\bullet_{\mathfrak{X}} = \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n}$ to get the exact sequence:

$$0 \to \mathrm{R}^1 \lim_n \mathrm{H}^{m-1}(\mathfrak{X}_n, \omega^\bullet_{\mathfrak{X}_n}) \to \mathrm{H}^m(\mathfrak{X}, \omega^\bullet_{\mathfrak{X}}) \to \lim_n \mathrm{H}^m(\mathfrak{X}_n, \omega^\bullet_{\mathfrak{X}_n}) \to 0 .$$

This easily implies that $\mathrm{H}^m(\mathfrak{X}, \omega^\bullet_{\mathfrak{X}}) \cong 0$ for $m < -d$ by using 2.1.1 as all $\omega^\bullet_{\mathfrak{X}_i}$ and $\omega^\bullet_{\mathfrak{X}}$ are concentrated in degrees $[-d, 0]$. We also conclude that the natural morphism $\lim_n \mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}_n}) \to \mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}})$ is an isomorphism.

Step 3. Construction of the canonical isomorphism $r_{\mathfrak{X}^{sm}} \colon \widehat{\Omega}^d_{\mathfrak{X}^{sm}}[d] \to \omega_{\mathfrak{X}^{sm}}$: As $\mathfrak{X}$ is flat over $\mathcal{O}_K$, we conclude that $(\mathfrak{X}^{\mathrm{sm}})_i = (\mathfrak{X}_i)^{\mathrm{sm}}$. So we can use [Sta19, Tag 0BRT][8] to get a canonical isomorphism $r_{\mathfrak{X}_n^{\mathrm{sm}}} \colon \Omega^d_{\mathfrak{X}_n^{\mathrm{sm}}}[d] \to \omega^\bullet_{\mathfrak{X}_n^{\mathrm{sm}}}$ that commutes with base change. In particular, each dualizing complex $\omega^\bullet_{\mathfrak{X}_n^{\mathrm{sm}}}$ is concentrated in degree $-d$ and locally free. The fact that $r_{\mathfrak{X}_n^{\mathrm{sm}}}$ commutes with base change means that the diagram

$$\begin{tikzcd}
\Omega^d_{\mathfrak{X}_n^{\mathrm{sm}}}[d] \otimes_{\mathcal{O}_{\mathfrak{X}_n^{\mathrm{sm}}}} \mathcal{O}_{\mathfrak{X}_{n-1}^{\mathrm{sm}}} \arrow[r, "r_{\mathfrak{X}_n^{\mathrm{sm}}} \otimes_{\mathcal{O}_{\mathfrak{X}_n^{\mathrm{sm}}}} \mathcal{O}_{\mathfrak{X}_{n-1}^{\mathrm{sm}}}"] \arrow[d] & \omega^\bullet_{\mathfrak{X}_n^{\mathrm{sm}}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}_n^{\mathrm{sm}}}} \mathcal{O}_{\mathfrak{X}_{n-1}^{\mathrm{sm}}} \arrow[d] \\
\Omega^d_{\mathfrak{X}_{n-1}^{\mathrm{sm}}}[d] \arrow[r, "r_{\mathfrak{X}_{n-1}^{\mathrm{sm}}}"] & \omega^\bullet_{\mathfrak{X}_{n-1}^{\mathrm{sm}}} ,
\end{tikzcd}$$

where the vertical maps are canonical base change isomorphisms, is commutative. In particular, we see that the canonical morphisms $\lim_n \Omega^d_{\mathfrak{X}_n^{\mathrm{sm}}}[d] \to \mathbf{R}\lim_n \Omega^d_{\mathfrak{X}_n^{\mathrm{sm}}}[d] \to \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n^{\mathrm{sm}}} \cong \omega^\bullet_{\mathfrak{X}^{\mathrm{sm}}}$ are isomorphisms[9]. Now we apply $\mathcal{H}^{-d}(-)$ to get a canonical isomorphism $r_{\mathfrak{X}^{\mathrm{sm}}} \colon \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}} \to \omega_{\mathfrak{X}}|_{\mathfrak{X}^{\mathrm{sm}}}$ □

We say that an admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ is *algebraizable*, if there is a flat, locally finite type $\mathcal{O}_K$-scheme X and an isomorphism $\widehat{X} \to \mathfrak{X}$, where $\widehat{X}$ is the ϖ-adic completion of X. In what follows, we denote by $c \colon \widehat{X} \to X$ the completion morphism, and we denote by $i'_n \colon X_n \to X$ the closed immersion $X \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/\varpi^{n+1}\mathcal{O}_K \to X$.

Lemma 2.1.7. Let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be an étale morphism of flat, finitely presented formal $\mathcal{O}_K$-schemes. Then there is an isomorphism $\mathbf{L}\mathfrak{f}^*\omega^\bullet_{\mathfrak{Y}} \cong \omega^\bullet_{\mathfrak{X}}$ that induces a canonical isomorphism $\mathfrak{f}^*\omega_{\mathfrak{Y}} \to \omega_{\mathfrak{X}}$.

Proof. We note that $\mathbf{L}\mathfrak{f}^*(\omega^\bullet_{\mathfrak{Y}}) \in \mathbf{D}^b_{coh}(\mathfrak{X})$ because $\omega^\bullet_{\mathfrak{Y}} \in \mathbf{D}^b_{coh}(\mathfrak{Y})$ and $\mathfrak{f}$ is étale. So the complex $\mathbf{L}\mathfrak{f}^*(\omega^\bullet_{\mathfrak{Y}})$ is derived ϖ-adically complete. Therefore, we use [Sta19, Tag 0A0E] to see that

$$\mathbf{L}\mathfrak{f}^*(\omega^\bullet_{\mathfrak{Y}}) \cong \mathbf{R}\lim_n \left(\mathbf{L}\mathfrak{f}^*\left(\omega^\bullet_{\mathfrak{Y}}\right) \otimes^L_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n}\right) \simeq \mathbf{R}\lim_n \mathbf{L}\mathfrak{f}^*(\omega^\bullet_{\mathfrak{Y}}) \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_n}.$$

Here we used that $\mathfrak{X}$ is flat over $\mathcal{O}_K$ to conclude that ϖ is a regular element of $\mathcal{O}_{\mathfrak{X}}$, so $(\mathcal{O}_{\mathfrak{X}} \xrightarrow{\varpi^{n+1}} \mathcal{O}_{\mathfrak{X}}) \cong \mathcal{O}_{\mathfrak{X}_n}[0]$. Now

$$\mathbf{L}\mathfrak{f}^*\left(\omega^\bullet_{\mathfrak{Y}}\right) \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_n} \simeq \mathbf{L}\mathfrak{f}_n^*\left(\omega^\bullet_{\mathfrak{Y}} \otimes^L_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n}\right) \simeq \mathbf{L}\mathfrak{f}_n^*\left(\omega^\bullet_{\mathfrak{Y}_n}\right)$$

[8] This proof does not really require properness of f.

[9] A morphism $\mathbf{R}\lim_n \Omega^d_{\mathfrak{X}_n^{\mathrm{sm}}}[d] \to \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n^{\mathrm{sm}}}$ is not quite canonical as cone is not functorial in the derived category. However, any such choice induces the same morphism on cohomology sheaves. And as all complexes a posteriori are concentrated in one degree, we get that the constructed morphism is canonical.

as $\omega^\bullet_{\mathfrak{Y}} \otimes^L_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n} \simeq \omega^\bullet_{\mathfrak{Y}_n}$ by Theorem 2.1.6. Finally, [Sta19, Tag 0FWI] guarantees that dualizing complex commutes with étale base change in the algebraic world to construct the isomorphism

$$\mathbf{L}\mathfrak{f}^*(\omega^\bullet_{\mathfrak{Y}}) \cong \mathbf{R}\lim_n \mathbf{L}\mathfrak{f}_n^*(\omega^\bullet_{\mathfrak{Y}_n}) \cong \mathbf{R}\lim_n \omega^\bullet_{\mathfrak{X}_n} \cong \omega^\bullet_{\mathfrak{X}} .$$

To see canonicity of the induced map on dualizing modules, we just note that it coincides with the morphism

$$\mathfrak{f}^*\omega_{\mathfrak{Y}} \simeq \mathfrak{f}^* \lim_n \omega_{\mathfrak{Y}_n} \simeq \lim_n \mathfrak{f}_n^*(\omega_{\mathfrak{Y}_n}) \xrightarrow{\sim} \lim_n \omega_{\mathfrak{X}_n} \simeq \omega_{\mathfrak{X}}$$

that is functorial due to functoriality of $\lim_n$ (as opposed to $\mathbf{R}\lim_n$). $\square$

Lemma 2.1.8. Let X be a flat, finitely presented $\mathcal{O}_K$-scheme with the adic generic fiber of pure dimension d, and let $c\colon \widehat{X} \to X$ be the ϖ-adic completion morphism. Then, for any $\mathcal{F} \in \mathbf{D}^b_{coh}(X)$, there is a functorial (in $\mathcal{F}$) isomorphism $\mathbf{L}c^*\mathcal{F} \to \mathbf{R}\lim_n \mathbf{L}i_n'^*\mathcal{F}$. In particular, if X is separated then there is an isomorphism $\mathbf{L}c^*\omega^\bullet_X \to \omega^\bullet_{\widehat{X}}$ that induces a canonical isomorphism $c^*(\omega_X) \to \omega_{\widehat{X}}$.

Proof. We observe that we have a commutative square

$$\begin{array}{ccc} \widehat{X}_n & \xrightarrow{=} & X_n \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_n'} \\ \widehat{X} & \xrightarrow{c} & X \end{array},$$

so we have a canonical isomorphism $\mathbf{L}i_n^*\mathbf{L}c^*\mathcal{F} \cong \mathbf{L}i_n'^*\mathcal{F}$. Moreover, we note that $\widehat{X}$ is $\mathcal{O}_K$-flat, thus the Koszul complex $(\mathcal{O}_{\mathfrak{X}} \xrightarrow{\varpi^{n+1}} \mathcal{O}_{\mathfrak{X}})$ is naturally quasi-isomorphism to $\mathcal{O}_{\mathfrak{X}_n}[0]$. So we can identify $\mathbf{R}\lim_n \mathbf{L}i_n'^*\mathcal{F}$ with the derived ϖ-adic completion of $\mathbf{L}c^*\mathcal{F}$ by [Sta19, Tag 0A0E]. As the derived ϖ-adic completion is functorial, we get a functorial morphism $\mathbf{L}c^*\mathcal{F} \to \mathbf{R}\lim_n \mathbf{L}i_n'^*\mathcal{F}$. In order, to check that this is an isomorphism, it suffices to show that $\mathbf{L}c^*\mathcal{F}$ is derived ϖ-adically complete.

As derived complete $\mathcal{O}_{\mathfrak{X}}$-modules are closed under taking cones, it suffices to show that $\mathbf{L}c^*\mathcal{F}$ is bounded and has derived ϖ-adically complete cohomology sheaves. As the morphism $\widehat{X} \to X$ is flat by [FK18, Proposition I.1.4.7], we see that $\mathbf{L}c^*\mathcal{F} \in \mathbf{D}^b_{coh}(\widehat{X})$ and $\mathcal{H}^m := \mathcal{H}^m(\mathbf{L}c^*\mathcal{F}) \simeq c^*\mathcal{H}^m(\mathcal{F})$ that is coherent by [FK18, Proposition I.1.4.7]. Thus we conclude that there are isomorphisms

$$\mathcal{H}^m \simeq \lim_n \mathcal{H}^m_n \simeq \mathbf{R}\lim_n \mathcal{H}^m_n$$

by [FK18, Definition I.3.1.3, Theorem I.7.1.1] and [Sta19, Tag 0BKS]. Each $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{H}^m_n$ is derived ϖ-adically complete as it is ϖ^{n+1}-torsion, so the derived limit is also derived ϖ-adically complete.

This discussion provides us with an isomorphism $\mathbf{L}c^*\omega^\bullet_X \to \omega^\bullet_{\widehat{X}}$ of complexes concentrated in degrees $[-d, 0]$. Using that the map c is flat, we easily see that this morphism is canonical in the bottom degree $-d$ and induces an isomorphism $c^*(\omega_X) \to \omega_{\widehat{X}}$ that comes as a limit of the base change morphisms $c^*(\omega_X) \to \omega_{X_n} \cong \omega_{\widehat{X}_n}$. $\square$

Lemma 2.1.9. Let $\mathfrak{X}$ is a separated admissible formal $\mathcal{O}_K$-scheme with the adic generic fiber $\mathfrak{X}_K$ of pure dimension d. Then the dualizing module $\omega_{\mathfrak{X}}$ is flat over $\mathcal{O}_K$.

Proof. Theorem 2.1.6 implies that we have a distinguished triangle

$$\omega^\bullet_{\mathfrak{X}} \xrightarrow{\varpi} \omega^\bullet_{\mathfrak{X}} \to \omega^\bullet_{\mathfrak{X}_0}$$

in $\mathbf{D}^b_{coh}(\mathfrak{X})$. We note that Theorem 2.1.6 and Lemma 2.1.1 imply that both complexes $\omega^\bullet_{\mathfrak{X}}$ and $\omega^\bullet_{\mathfrak{X}_0}$ are concentrated in degrees $[-d, 0]$. This implies that multiplication by ϖ is injective on $\mathcal{H}^{-d}(\omega^\bullet_{\mathfrak{X}})$. Now we recall that $\mathcal{O}_K$-flat modules are exactly ϖ-torsionfree modules. This finishes the proof. $\square$

Definition 2.1.10. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$, and let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{X}$. We define the $\mathcal{O}_{\mathfrak{X}}$-module $j_{\mathfrak{U},*}\mathcal{F}_{\mathfrak{U}} \cap \mathcal{F}_K$ the kernel of the map

$$j_{\mathfrak{X}_K,*}\mathcal{F}_K \oplus j_{\mathfrak{U},*}\mathcal{F}|_{\mathfrak{U}} \xrightarrow{f-g} j_{\mathfrak{U}_K,*}(\mathcal{F}|_{\mathfrak{U}})_K,$$

where $j_{\mathfrak{U}}\colon \mathfrak{U} \to \mathfrak{X}$ (resp. $j_{\mathfrak{X}_K,*}\colon \mathfrak{X}_K \to \mathfrak{X}$, resp. $j_{\mathfrak{U}_K}\colon \mathfrak{U}_K \to \mathfrak{X}$) is the natural open immersion (resp. generic fiber map) and f, g are the natural morphisms that come from the adjunction.

We recall that a coherent module $\mathcal{F}$ on $\mathfrak{X}$ is called *reflexive* if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ to its double-dual is an isomorphism.

We now recall some terminology from [GR18] that we will need in the proof of Theorem 2.1.12. Let X be a scheme, $Z \subset X$ a closed subset, $x \in X$ a point, and $\mathcal{F} \in \mathbf{D}(X)$, then

$$\operatorname{depth}_X(Z, \mathcal{F}) \coloneqq \sup\{n \in \mathbf{Z} \mid \mathcal{H}^i_Z(\mathcal{F}) = 0 \text{ for all } i < n\} \in \mathbf{Z} \cup \infty,$$

$$\delta_X(x, \mathcal{F}) \coloneqq \operatorname{depth}_{\operatorname{Spec} \mathcal{O}_{X,x}}(x, \mathcal{F}|_{\operatorname{Spec} \mathcal{O}_{X,x}})$$

where $\mathcal{H}^i_Z(\mathcal{F})$ is understood as in [Sta19, Tag 0G6Y] (if $\mathcal{F}$ is zero then $\delta_X(x, \mathcal{F}) = \infty$).

Lemma 2.1.11. Let X be a scheme with noetherian underlying topological space $|X|$, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, and $x \in X$. Then $x \in \operatorname{Spec} \mathcal{O}_{X,x}$ is a weakly associated prime of $M_x \coloneqq \mathcal{F}_x$ if and only if $\delta_X(x, \mathcal{F}) = 0$.

Proof. We note that $|\operatorname{Spec} \mathcal{O}_{X,x}|$ is noetherian since it is a subspace of noetherian $|X|$. Then $\{x\} \in \operatorname{Spec} \mathcal{O}_{X,x}$. So we can replace X with $\operatorname{Spec} \mathcal{O}_{X,x}$ to assume that X is a spectrum of a local ring, and x is its (unique) closed point.

Now we use noetherianness of $|X|$ and [Sta19, Tag 0G72] to conclude that sheaves $\mathcal{H}^i_x(X, \mathcal{F})$ are quasi-coherent. So, we see that

$$\delta_X(x, \mathcal{F}) = \sup\{n \in \mathbf{Z} \mid \mathrm{H}^i_x(X, \mathcal{F}) = 0 \text{ for all } i < n\},$$

where $\mathrm{H}^i_x(X, \mathcal{F}) = \Gamma(X, \mathcal{H}^i_x(X, \mathcal{F})$. Now

$$\mathrm{H}^0_x(X, \mathcal{F}) = \{s \in \mathcal{F}_x \mid \operatorname{Supp}(s) \subset x\} = \{s \in \mathcal{F}_x \mid \operatorname{rad}(\operatorname{Ann}_{\mathcal{O}_{X,x}}(s)) = \mathfrak{m}_x\}.$$

Therefore, [Sta19, Tag 0566] implies that if $\mathrm{H}^0_x(X, \mathcal{F}) \neq 0$ if and only if x is a weakly associated prime of $M_x = \mathcal{F}_x$. $\square$

Theorem 2.1.12. Let X be a separated flat finitely presented $\mathcal{O}_K$-scheme with smooth generic fiber X_K of pure dimension d, and with geometrically reduced special fiber $\overline{X}$. Then the relative dualizing sheaf $\omega_X \coloneqq \mathcal{H}^{-d}\left(f^!\left(\mathcal{O}_{\operatorname{Spec} \mathcal{O}_K}\right)\right)$ is a reflexive $\mathcal{O}_X$-module.

Proof. Step 1*: The relative dualizing complex $\omega^\bullet_X \coloneqq f^!\left(\mathcal{O}_{Spec\, \mathcal{O}_K}\right) \in \mathbf{D}^{[-d,0]}_{coh}(X)$ and $\omega_{\mathfrak{X}} \simeq \mathcal{H}^{-d}(\omega^\bullet_X)$ is $\mathcal{O}_K$-flat.* The first claim can be proven similarly to Lemma 2.1.1. Then an easy modification of the proof of Lemma 2.1.9 shows that ω_X is $\mathcal{O}_K$-flat.

We now denote the special fiber of X by $\overline{X}$, its relative dualizing complex over $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m}_K$ by $\omega^\bullet_{\overline{X}} \in \mathbf{D}^{[-d,0]}_{coh}(\overline{X})$, and the associated dualzing sheaf by $\omega_{\overline{X}} \coloneqq \mathcal{H}^{-d}(\omega^\bullet_{\overline{X}})$.

Step 2*: $\omega_X/\mathfrak{m}_K\omega_X \subset \omega_{\overline{X}}$.* We consider the short exact sequence:

$$0 \to \mathfrak{m}_K \to \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{m}_K \to 0$$

and take its derived tensor product against $\omega^\bullet_X$ to get the distinguished triangle:

$$\mathfrak{m}_K \otimes^L_{\mathcal{O}_K} \omega^\bullet_X \to \omega^\bullet_X \to \omega^\bullet_{\overline{X}}.$$

Using that $\mathfrak{m}_K$ is $\mathcal{O}_K$-flat and that both $\omega^\bullet_X$ and $\omega^\bullet_{\overline{X}}$ are concentrated in degrees $[-d, 0]$, we conclude that there is a short exact sequence

$$0 \to \mathfrak{m}_K \otimes_{\mathcal{O}_K} \omega_X \to \omega_X \to \omega_{\overline{X}}.$$

Now we use that ω_X is $\mathcal{O}_K$-flat to conclude that $\mathfrak{m}_K \otimes_{\mathcal{O}_K} \omega_X \simeq \mathfrak{m}_K\omega_X$ to see that $\omega_X/\mathfrak{m}_K\omega_X \subset \omega_{\overline{X}}$.

Step 3*: The* $\mathcal{O}_{\overline{X}}$*-module* $\omega_X/\mathfrak{m}_K\omega_X$ *is torsion-free.* Since $\omega_X/\mathfrak{m}_K\omega_X \subset \omega_{\overline{X}}$, it is sufficient to show $\omega_{\overline{X}}$ is torsion-free, i.e. it suffices to show that, for any reduced separated finite type $\mathcal{O}_K/\mathfrak{m}_K$-scheme Y, ω_Y is torsion-free. This follows from [Sta19, Tag 0AWN] as (S_2) sheaf on a reduced scheme is torsion-free.

Step 4*: The* $\mathcal{O}_{X_K}$*-module* $(\omega_X)_K$ *is line bundle.* This follows from the assumption that X_K is K-smooth and a sequence of isomorphisms:

$$(\omega_X)_K \simeq \omega_{X_K} \simeq \Omega^d_{X_K}.$$

Step 5*: Some cohomological considerations.* We choose $x \in X$ a non-generic point in the special fiber. And we roughly want to show that ω_X is at least (S_2) at this point and $\omega_X^{\vee\vee}$ is at least (S_1). However, it turns out that it requires extra care as all rings involved must not be noetherian, so we need to use some extra input from from [GR18].

Namely, we apply [GR18, Corollary 10.4.46] to the finitely presented morphism $X \to \operatorname{Spec} \mathcal{O}_K$, $\mathcal{O}_K$-flat coherent $\mathcal{O}_X$-module ω_X, and a quasi-coherent $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_K}$-module $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_K}$ to conclude that

$$\delta_X(x, \omega_X) = \delta_{\overline{X}}(x, \omega_X/\mathfrak{m}_K\omega_X) + \delta_{\operatorname{Spec} \mathcal{O}_K}(\{\mathfrak{m}_K\}, \mathcal{O}_{\operatorname{Spec} \mathcal{O}_K}). \tag{2.1}$$

Now we note that $\omega_X/\mathfrak{m}_K\omega_X$ is torsion-free by Step 3. Therefore, [Sta19, Tag 0566] guarantees that its only weakly associated primes are generic points of $\overline{X}$. Therefore, Lemma 2.1.11 guarantees that $\delta_{\overline{X}}(x, \omega_X/\mathfrak{m}_K\omega_X) \geq 1$.

The very definition of δ implies that $\delta_{\operatorname{Spec} \mathcal{O}_K}(\{\mathfrak{m}_K\}, \mathcal{O}_{\operatorname{Spec} \mathcal{O}_K}) \geq 1$. Therefore, equation (2.1) implies that $\delta_X(x, \omega_x) \geq 2$.

Now we note that X has a noetherian underlying topological space, X is clearly reduced, and $\omega_X^{\vee\vee}$ is a torsion-free $\mathcal{O}_X$-module. Therefore, [Sta19, Tag 0566] and Lemma 2.1.11 imply that $\delta_X(x, \omega_X^{\vee\vee}) \geq 1$.

Step 6*:* ω_X *is reflexive.* We need to show that the natural morphism

$$\alpha\colon \omega_X \to \omega_X^{\vee\vee}$$

is an isomorphism. It is clearly an isomorphism over the generic fiber as X_K is smooth, so ω_X is a line bundle, and both ω_X and $\omega_X^{\vee\vee}$ commute with open immersions. In particular, we see that α is injective as $\omega_X^{\vee\vee}$ is a torsion-free module.

It is also an isomorphism over the generic points in the special fiber. Indeed, there is an $\mathcal{O}_K$-smooth open subscheme $U \subset X$ containing all generic points in the special fiber of X. Then $\omega_X|_U \simeq \omega_U$ that is a line bundle by $\mathcal{O}_K$-smoothness of U. Therefore, α_U is an isomorphism as both ω_X and $\omega_X^{\vee\vee}$ commute with open immersions.

Finally, we show that α_x is an isomorphism for any non-generic point x in the special fiber. We have already shown that α is injective, so we have a short exact sequence

$$0 \to \omega_X \to \omega_X^{\vee\vee} \to \mathcal{Q} \to 0$$

for some coherent $\mathcal{O}_X$-module $\mathcal{Q}$. Then we know that $\delta_X(x, \omega_X) \geq 2$ and $\delta_X(x, \omega_X^{\vee\vee}) \geq 1$. So an easy cohomological consideration implies that $\delta_X(x, \mathcal{Q}) \geq 1$ for any non-generic point x in the special fiber. We also know that $\mathcal{Q}_y = 0$ for any generic point y in the special fiber and for any y in the generic fiber. This implies that $\delta_X(y, \mathcal{Q}) = \infty$ at these points. Therefore, $\delta_X(x, \mathcal{Q}) > 1$ for *any* $x \in X$. Therefore, $\mathcal{Q}$ is a coherent sheaf that has no weakly associated points. Thus, [Sta19, Tag 05AP] guarantees that $\mathcal{Q} \simeq 0$. □

Corollary 2.1.13. Let X be a separated, flat, finitely presented $\mathcal{O}_K$-scheme with smooth generic fiber X_K of pure dimension d, and geometrically reduced special fiber $\overline{X}$. Then the natural morphism $\omega_X \to j_*(\omega_{X^{\mathrm{sm}}})$ is an isomorphism, where $j\colon X^{\mathrm{sm}} \to X$ is the open immersion of the smooth locus of X into X.

Proof. This follows from Theorem 2.1.12 and Remark B.9. □

Theorem 2.1.14. Let $\mathfrak{X}$ is a separated admissible formal $\mathcal{O}_K$-scheme with smooth adic generic fiber $\mathfrak{X}_K$ of pure dimension d, and reduced special fiber $\overline{\mathfrak{X}}$. Then the dualizing sheaf $\omega_{\mathfrak{X}}$ is a reflexive $\mathcal{O}_{\mathfrak{X}}$-module, and the natural morphism $\omega_{\mathfrak{X}} \to j_{\mathfrak{U},*}(\omega_{\mathfrak{X}}|_{\mathfrak{U}}) \cap \omega_{\mathfrak{X}_K}$ is an isomorphism.

Proof. The question is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \operatorname{Spf} B$ is affine. Now we use [Tem17, Theorem 3.1.3] (it essentially boils down to [Elk73, Théoremèm 7 on page 582 and Remarque 2(c) on p.588] and [Tem08, Proposition 3.3.2]) that says that an affine rig-smooth formal scheme $\mathfrak{X} = \operatorname{Spf} B$ can be algebraized to an affine flat, finitely presented $\mathcal{O}_K$-scheme $X = \operatorname{Spec} A$ with the smooth generic fibre $\operatorname{Spec} A_K$. Now Lemma 2.1.8 implies that $\omega_{\mathfrak{X}} = c^*(\omega_X)$. Since the special fiber of X coincides with the special fiber of $\mathfrak{X}$, we conclude that it is reduced as well. Therefore, Theorem 2.1.12 implies that the $\mathcal{O}_X$-module ω_X is reflexive. The fact that its pullback $c^*(\omega_X)$ is reflexive boils down to the fact that $M \otimes_A \widehat{A}$ is reflexive over A, if M is a coherent and reflexive A-module. This is easily seen to hold true, as flatness of the morphism $A \to \widehat{A}$ implies that the natural morphism

$$\operatorname{Hom}_A(K, L) \otimes_A \widehat{A} \to \operatorname{Hom}_{\widehat{A}}(K \otimes_A \widehat{A}, L \otimes_A \widehat{A})$$

is an isomorphism for any finitely presented A-module K. The last assertion follows from Lemma B.8. □

2.2. **Trace Map for Proper Morphisms of Separated Admissible Formal Schemes.** In thise section, we discuss the definition of the trace map $\operatorname{Tr}_{\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\omega_{\mathfrak{X}} \to \omega_{\mathfrak{Y}}$ for a proper morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of separated, admissible formal $\mathcal{O}_K$-schemes with generic fibers of the same pure dimension d. Of course, it would be nice to develop good theory of $\mathfrak{f}^!$ functors for admissible formal $\mathcal{O}_K$-schemes such that $\operatorname{Tr}_{\mathfrak{f}}$ comes as $\mathcal{H}^{-d}$ of the well-defined trace morphism that comes from the adjunction $(\mathbf{R}\mathfrak{f}_*, \mathfrak{f}^!)$. But to the best of our knowledge such theory is not present in the literature and we do not develop this formalism here.

Before starting the construction, we recall that $\omega_{\mathfrak{Y}} := \lim_n \omega_{\mathfrak{Y}_n}$ comes with the canonical morphism $\omega_{\mathfrak{Y}} \to \omega_{\mathfrak{Y}_n}$ for each $n \geq 0$. This induces a morphism $\mathrm{BC}^n_{\omega_{\mathfrak{Y}}}\colon \omega_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n} \to \omega_{\mathfrak{Y}_n}$. Analogously, we have a canonical morphism $\mathrm{BC}^n_{\mathfrak{f}_*\omega_{\mathfrak{X}}}\colon \mathfrak{f}_*\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_n} \to \omega_{\mathfrak{X}_n}$.

Lemma 2.2.1. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of separated admissible formal $\mathcal{O}_K$-schemes with the adic generic fibers of the same pure dimension d. Then there is a unique trace map

$\mathrm{Tr}_{\mathfrak{f}} \colon \mathfrak{f}_* \omega_{\mathfrak{X}} \to \omega_{\mathfrak{Y}}$ such that, for any $n \geq 0$, the diagram

$$\begin{tikzcd} \mathfrak{f}_*(\omega_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n} \arrow[r, "\mathrm{BC}^n_{\mathfrak{f}_*\omega_{\mathfrak{X}}}"] \arrow[d, "\mathrm{Tr}_{\mathfrak{f}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n}"'] & \mathfrak{f}_{n,*}(\omega_{\mathfrak{X}_n}) \arrow[d, "\mathcal{H}^{-d}(\mathrm{Tr}_{\mathfrak{f}_n})"] \\ \omega_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n} \arrow[r, "\mathrm{BC}^n_{\omega_{\mathfrak{Y}}}"] & \omega_{\mathfrak{Y}_n}, \end{tikzcd} \tag{2.2}$$

where $\mathrm{Tr}_{\mathfrak{f}_n} \colon \mathbf{R}\mathfrak{f}_n \left(\omega^\bullet_{\mathfrak{X}_n}\right) \to \omega^\bullet_{\mathfrak{Y}_n}$ is the trace map in coherent duality, is commutative.

Proof. The uniqueness part is easy. Since $\omega_{\mathfrak{X}}$ and $\omega_{\mathfrak{Y}}$ are $\mathcal{O}_K$-flat by Lemma 2.1.9, we conclude that $\mathrm{BC}_{\mathfrak{f}_*\omega_{\mathfrak{X}}}$ and $\mathrm{BC}_{\omega_{\mathfrak{Y}}}$ are both injective. Then $\mathrm{Tr}_{\mathfrak{f}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n}$ is uniquely defined from the diagram (2.2). Then since both $\mathfrak{f}_*(\omega_{\mathfrak{X}})$ and $\omega_{\mathfrak{Y}}$ are coherent by [FK18, Theorem I.11.1.1], we see that $\mathrm{Tr}_{\mathfrak{f}} = \lim_n \mathrm{Tr}_{\mathfrak{f}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}_n}$ is uniquely defined.

We recall that $\mathfrak{f}_*$ commutes with all limits. In particular, there is a natural isomorphism $\mathfrak{f}_*(\omega_{\mathfrak{X}}) \to \lim_n \mathfrak{f}_*(\omega_{\mathfrak{X}_n})$. Thus, we can define $\mathrm{Tr}_{\mathfrak{f}} \colon \mathfrak{f}_*(\omega_{\mathfrak{X}}) \to \omega_{\mathfrak{Y}}$ as $\lim_n \mathcal{H}^{-d}(\mathrm{Tr}_{\mathfrak{f}_n})$. In order for this formula to make sense, we need to show that $\mathrm{Tr}_{\mathfrak{f}_n}$ are compatible for different n. This follows from [Sta19, Tag 0B6J] as $\mathfrak{Y}_{n-1}$ and $\mathfrak{X}_n$ are tor-independent over $\mathfrak{Y}_n$ by $\mathcal{O}_K$-flatness of both $\mathfrak{X}$ and $\mathfrak{Y}$. This construction defines a map $\mathrm{Tr}_{\mathfrak{f}} \colon \mathfrak{f}_*(\omega_{\mathfrak{X}}) \to \omega_{\mathfrak{Y}}$. We need to check that it commutes with base change. It suffices to show that the diagram

$$\begin{tikzcd} \mathfrak{f}_*(\omega_{\mathfrak{X}}) \arrow[r] \arrow[d, "\mathrm{Tr}_{\mathfrak{f}}"'] & \mathfrak{f}_{n,*}(\omega_{\mathfrak{X}_n}) \arrow[d, "\mathcal{H}^{-d}(\mathrm{Tr}_{\mathfrak{f}_n})"] \\ \omega_{\mathfrak{Y}} \arrow[r] & \omega_{\mathfrak{Y}_n} \end{tikzcd}$$

is commutative, but this follows from the construction. □

Remark 2.2.2. Suppose that $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ comes the ϖ-adic completion of a proper morphism $f \colon X \to Y$ of separated, flat, finitely presented $\mathcal{O}_K$-schemes. Then Lemma 2.1.8 gives identifications $\omega_{\mathfrak{X}} \simeq c_X^*(\omega_X)$ and $\omega_{\mathfrak{Y}} \simeq c_Y^*(\omega_Y)$, where $c_?$ are the completion morphisms. Moreover, [FK18, Theorem I.9.2.1] gives an identification $\mathfrak{f}_*(\omega_{\mathfrak{X}}) \simeq c_Y^*(f_*\omega_X)$. Then the trace map becomes the map

$$\mathrm{Tr}_{\mathfrak{f}} \colon c_Y^*(f_*\omega_X) \to c_Y^*(\omega_Y).$$

The construction of $\mathrm{Tr}_{\mathfrak{f}}$ in Lemma 2.2.1 implies that $\mathrm{Tr}_{\mathfrak{f}} = \lim \mathcal{H}^{-d}(\mathrm{Tr}_{\mathfrak{f}_n})$, so we conclude that $\mathrm{Tr}_{\mathfrak{f}} = c_Y^*(\mathrm{Tr}_f)$ under the identifications as above.

Lemma 2.2.3. Let $\mathfrak{f} \colon \mathfrak{X}'' \to \mathfrak{X}'$ and $\mathfrak{g} \colon \mathfrak{X}' \to \mathfrak{X}$ be two proper morphisms of separated admissible formal $\mathcal{O}_K$-schemes with the adic generic fibers of the same pure dimension d. Then the diagram

$$\begin{tikzcd} \mathfrak{g}_*\mathfrak{f}_*\omega_{\mathfrak{X}''} \arrow[r, "\mathfrak{g}_*(\mathrm{Tr}_{\mathfrak{f}})"] \arrow[d] & \mathfrak{g}_*\omega_{\mathfrak{X}'} \arrow[r, "\mathrm{Tr}_{\mathfrak{g}}"] & \omega_{\mathfrak{X}} \\ (\mathfrak{g} \circ \mathfrak{f})_*\omega_{\mathfrak{X}''} \arrow[urr, "\mathrm{Tr}_{\mathfrak{g}\circ\mathfrak{f}}"'] & & \end{tikzcd},$$

where the vertical arrow is the canonical identification of $\mathfrak{g}_* \circ \mathfrak{f}_*$ with $(\mathfrak{g} \circ \mathfrak{f})_*$, is commutative.

Proof. We firstly note the canonical identification $\mathbf{R}\mathfrak{g}_{n,*} \circ \mathbf{R}\mathfrak{f}_{n,*} \simeq \mathbf{R}(\mathfrak{g} \circ \mathfrak{f})_{n,*}$ implies via the usual adjunction properties that

$$\mathrm{Tr}_{\mathfrak{g}_n} \circ \mathbf{R}\mathfrak{g}_{n,*}\left(\mathrm{Tr}_{\mathfrak{f}_n}\right) \simeq \mathrm{Tr}_{(\mathfrak{g}\circ\mathfrak{f})_n}$$

Now we use the formal properties of derived limits to write:

$$\begin{aligned}
\mathfrak{g}_*\left(\mathrm{Tr}_{\mathfrak{f}}\right) \circ \mathrm{Tr}_{\mathfrak{g}} &\simeq \mathfrak{g}_*\left(\lim_n \mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{f}_n}\right)\right) \circ \lim_n \mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{g}_n}\right) \\
&\simeq \lim_n \left(\mathfrak{g}_{n,*}\left(\mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{f}_n}\right)\right)\right) \circ \lim_n \mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{g}_n}\right) \\
&\simeq \lim_n \left(\mathcal{H}^{-d}\left(\mathbf{R}\mathfrak{g}_{n,*}\left(\mathrm{Tr}_{\mathfrak{f}_n}\right)\right) \circ \mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{g}_n}\right)\right) \\
&\simeq \lim_n \left(\mathcal{H}^{-d}\left(\mathrm{Tr}_{(\mathfrak{g}\circ\mathfrak{f})_n}\right)\right) \\
&\simeq \mathrm{Tr}_{\mathfrak{g}\circ\mathfrak{f}} \ .
\end{aligned}$$

We emphasize that in the above equalities we crucially used that all complexes $\omega^\bullet_{\mathfrak{X}_n}, \omega^\bullet_{\mathfrak{X}'_n}$ and $\omega^\bullet_{\mathfrak{X}''_n}$ are concentrated in degrees $[-d, 0]$. ☐

2.3. **Explicit Trace Map.** This section is devoted to give a very explicit construction for the trace map in the case of a finite étale morphism between smooth, separated admissible formal $\mathcal{O}_K$-schemes.

Suppose that $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is a finite étale morphism of smooth, separated admissible formal $\mathcal{O}_K$-schemes of the same pure dimension d, then we can use the maps $r_{\mathfrak{Y}}$ and $r_{\mathfrak{X}}$ from Theorem 2.1.6 to identify $\omega^\bullet_{\mathfrak{X}}$, $\omega^\bullet_{\mathfrak{Y}}$ with $\widehat{\Omega}^d_{\mathfrak{X}}[d]$ and $\widehat{\Omega}^d_{\mathfrak{Y}}[d]$, respectively. The canonical morphism $\mathfrak{f}^*\widehat{\Omega}^d_{\mathfrak{Y}} \to \widehat{\Omega}^d_{\mathfrak{X}}$ is an isomorphism since $\mathfrak{f}$ is étale. So combining these identifications, we define the trace map $\mathrm{Tr}_{\mathfrak{f}}\colon \mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right) \to \widehat{\Omega}^d_{\mathfrak{Y}}$ by commutativity of the diagram

$$\begin{CD}
\mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right) @>{\mathfrak{f}_*(r_{\mathfrak{X}})}>> \mathfrak{f}_*\left(\omega_{\mathfrak{X}}\right) \\
@V{\mathrm{Tr}_{\mathfrak{f}}}VV @VV{\mathrm{Tr}_{\mathfrak{f}}}V \\
\widehat{\Omega}^d_{\mathfrak{Y}} @>{r_{\mathfrak{Y}}}>> \omega_{\mathfrak{Y}}
\end{CD}$$

We now want to give an explicit description for this trace map. We construct another morphism $\mathrm{Tr}^{\mathrm{expl}}_{\mathfrak{f}}\colon \mathfrak{f}_*\widehat{\Omega}^d_{\mathfrak{X}} \to \widehat{\Omega}^d_{\mathfrak{Y}}$ that we call *the explicit trace map* and show that it coincides with $\mathrm{Tr}_{\mathfrak{f}}$.

The trivial version of the projection formula for finite morphisms provides us with the canonical identification

$$\mathfrak{f}_*\left(\mathcal{O}_{\mathfrak{X}}\right) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}} \to \mathfrak{f}_*\left(\mathfrak{f}^*\widehat{\Omega}^d_{\mathfrak{Y}}\right) \ .$$

Combining it with the isomorphism $\mathfrak{f}^*\widehat{\Omega}^d_{\mathfrak{Y}} \simeq \widehat{\Omega}^d_{\mathfrak{X}}$, we get an canonical identification

$$\mathfrak{f}_*\left(\mathcal{O}_{\mathfrak{X}}\right) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}} \simeq \mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right).$$

Now we construct the map $\mathrm{Tr}^{\mathrm{expl}}_{\mathfrak{f}}\colon \mathfrak{f}_*\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{Y}}$. As f is finite, locally on $\mathfrak{Y}$, this is a morphism $\operatorname{Spf} B \to \operatorname{Spf} A$ for some flat, finite, finitely presented A-algebra B. Thus, B is a finite projective A-module, so we have the trace morphism $\mathrm{Tr}_{B/A}\colon B \to A$. Since Tr commutes with flat base change, this morphism glues to a well-defined morphism $\mathrm{Tr}'^{\mathrm{expl}}_{\mathfrak{f}}\colon \mathfrak{f}_*\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{Y}}$.

Finally, we use the same identifications as above to construct the explicit trace map $\mathrm{Tr}_{\mathfrak{f}}^{\mathrm{expl}}\colon \mathfrak{f}_*\widehat{\Omega}^d_{\mathfrak{X}} \to \widehat{\Omega}^d_{\mathfrak{Y}}$ as the morphism

$$\mathrm{Tr}_{\mathfrak{f}}^{\prime\mathrm{expl}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathrm{Id}\colon \mathfrak{f}_*\left(\mathcal{O}_{\mathfrak{X}}\right) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}} \simeq \widehat{\Omega}^d_{\mathfrak{Y}}.$$

Locally, in étale coordinates $U \xrightarrow{(z_1,\dots,z_d)} \widehat{\mathbf{A}}^d$, the maps looks like

$$f\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d \mapsto \mathrm{Tr}_{B/A}(f)\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d,$$

where $\mathrm{Tr}_{B/A}$ is the trace map for the flat, finite, finitely presented morphism $A \to B$ that comes from a morphism of affines $U = \mathrm{Spf}\, B \to \mathrm{Spf}\, A = \widehat{\mathbf{A}}^d$.

Lemma 2.3.1. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite étale morphism of smooth, separated admissible formal $\mathcal{O}_K$-schemes of pure dimension d. Then

$$\mathrm{Tr}_{\mathfrak{f}}, \mathrm{Tr}_{\mathfrak{f}}^{\mathrm{expl}}\colon \mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right) \to \widehat{\Omega}^d_{\mathfrak{Y}}$$

coincide.

Proof. We note that the morphism in the projection formula, $r_{\mathfrak{X}}$, $r_{\mathfrak{Y}}$, $\mathrm{Tr}_{\mathfrak{f}}$, and $\mathrm{Tr}_{\mathfrak{f}}^{\mathrm{expl}}$ commute with base change in $\mathfrak{Y}$. So it suffices to prove the analogous claim for each reduction $\mathfrak{f}_n$, where it is standard. We briefly mention how it is proven in the scheme case as it seems difficult to extract from the literature in this generality.

Firstly, it suffices to show that $\mathrm{Tr}_{\mathfrak{f}_n}^{\mathrm{expl}}$ and $\mathcal{H}^{-d}\left(\mathrm{Tr}_{\mathfrak{f}_n}\right)$ coincide after an étale base change. So one can assume that $\mathfrak{X}_n$ is a disjoint union of copies of union of connected components of $\mathfrak{Y}_n$. In this case the calculation is easy. □

2.4. **Dualizing Complexes on Smooth, Separated Rigid-Analytic Spaces.** We study properties of a dualizing module on a separated admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ with smooth adic generic fiber X of pure dimension d. In particular, we show that the generic fiber $(\omega_{\mathfrak{X}})_K$ can be canonically identified with the sheaf of top forms $\Omega^d_{\mathfrak{X}_K}$.

Lemma 2.4.1. Let $\mathfrak{X}$ is a separated admissible formal $\mathcal{O}_K$-scheme with smooth adic generic fiber $\mathfrak{X}_K$ of pure dimension d, and geometrically reduced special fiber $\overline{\mathfrak{X}}$. Then there is a canonical isomorphism $r_{\mathfrak{X}_K}\colon \Omega^d_{\mathfrak{X}_K} \to (\omega_{\mathfrak{X}})_K$ that coincides with the adic generic fiber of $r_{\mathfrak{X}^{\mathrm{sm}}}\colon \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}} \to \omega_{\mathfrak{X}^{\mathrm{sm}}}$ on $(\mathfrak{X}^{\mathrm{sm}})_K$.

Proof. Step 1: $(\omega_{\mathfrak{X}})_K$ is isomorphic to a line bundle. The claim is Zariski-local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\, B$ is affine. We use [Tem17, Theorem 3.1.3] to find an algebraization $X = \mathrm{Spec}\, A$ of $\mathfrak{X}$ with the smooth generic fiber and geometrically reduced special fiber. We can also assume that all connected components[10] of $\mathrm{Spec}\, A$ meet the special fiber as otherwise we can replace $\mathrm{Spec}\, A$ with the complement of this connected component without changing its completion. We note that this automatically implies that every irreducible $Z \subset \mathrm{Spec}\, A$ meets the special fiber. Suppose the contrary that Z does not meet the special fiber. Then Z is an irreducible component of smooth $\mathrm{Spec}\, A_K$. Therefore, it is open in $\mathrm{Spec}\, A_K$ and, thus, it is open in $\mathrm{Spec}\, A$. So Z is a connected component of $\mathrm{Spec}\, A$ that does not meet the special fiber, contradiction.

We note that the scheme X_K is of pure dimension d. Indeed, Lemma B.6 implies that X_k is of pure dimension d. Now pick any irreducible component $Z \subset X_K$ and a closed point $x \in Z$ that lies only on this irreducible component. Then we use that every irreducible component of X meets X_k

[10] Connected components of $\mathrm{Spec}\, A$ are clopen as the underlying topological space $|\mathrm{Spec}\, A|$ is noetherian.

to choose an open affine irreducible $U \subset X$ such that $x \in U$ and U_k is non-empty. Then [Gro66, Lemme 14.3.10] guarantees that $d = \dim U_k = \dim X_K$. And, therefore, $d = \dim_x X_K = \dim Z$.

Now we recall that there are 2 "analytifications" of X: we can either first take generic fiber and take its analytification, or we can consider the ϖ-adic completion of X and then take its adic generic fiber. We denote the former object by X_K^{an} and the latter one by $\mathfrak{X}_K = \widehat{X}_K$. Then we use [Con99, Theorem 5.3.1] to see that we have a commutative diagram

$$\begin{tikzcd} \mathfrak{X}_K \arrow[r, "\beta"] \arrow[d, "j"] & \mathfrak{X} \arrow[d, "c"] \\ X_K^{\mathrm{an}} \arrow[r, "\alpha"] & X \end{tikzcd}$$

and the morphism j is an open immersion. Now we use Lemma 2.1.8 and commutativity of this diagram to conclude that $(\omega_{\mathfrak{X}})_K \cong j^*\alpha^*\omega_X$. As j is an open immersion, it suffices to show that $\alpha^*\omega_X$ is a line bundle concentrated. We observe that α factor as the composition $X_K^{\mathrm{an}} \to X_K \to X$, so it is actually sufficient to show that the (usual) generic fiber of $(\omega_X)_K \cong \omega_{X_K}$ is a line bundle. Now we recall that the scheme X_K is smooth and of pure dimension d. Therefore, [Sta19, Tag 0BRT][11] implies that $\omega_{X_K} \cong \Omega^d_{X_K}$. In particular, it is isomorphic to a line bundle.

Step 2: $r_{\mathfrak{X}_K}$ is unique if it exists. First of all, we can assume that $\mathfrak{X}_K$ is connected by Lemma B.1. Now assume that we have two morphisms $f, g\colon \Omega^d_{\mathfrak{X}_K} \to (\omega_{\mathfrak{X}})_K$ that extend $(r_{\mathfrak{X}^{\mathrm{sm}}})_K$ from $(\mathfrak{X}^{\mathrm{sm}})_K$. Then $\varphi := f - g$ is a morphism of two line bundles such that it is zero on a non-empty, quasi-compact open $(\mathfrak{X}^{\mathrm{sm}})_K$. So the vanishing locus $\mathrm{V}(\varphi)$ is a Zariski-closed subset of $\mathfrak{X}$ that contains $(\mathfrak{X}^{\mathrm{sm}})_K$. Therefore, [Con99, Lemma 2.1.4] implies that $\mathrm{V}(\varphi) = \mathfrak{X}_K$ once we can show that $\mathfrak{X}^{\mathrm{sm}}$ is non-empty. Flatness of $\mathfrak{X}$ implies that $\overline{\mathfrak{X}^{\mathrm{sm}}} = \overline{\mathfrak{X}}^{\mathrm{sm}}$ and the smooth locus of $\overline{\mathfrak{X}}$ contains all generic points since it is geometrically reduced. So we conclude that $f = g$.

Step 3: Construction of $r_{\mathfrak{X}_K}$. We note that $\mathfrak{X}^{\mathrm{sm}}$ non-trivially intersects any non-empty open subset in $\mathfrak{X}$. Indeed, this can be checked on the special fiber, so it suffices to show that $\overline{\mathfrak{X}^{\mathrm{sm}}} = \overline{\mathfrak{X}}^{\mathrm{sm}}$ is dense in $\overline{\mathfrak{X}}$. This follows from the fact the smooth locus of any geometrically reduced finite type $k = \mathcal{O}_K/\mathfrak{m}_K$-scheme is dense. So by Step 2 it suffices to construct the map $r_{\mathfrak{X}_K}$ Zariski-locally on $\mathfrak{X}$. Thus we may and do assume that $\mathfrak{X} = \mathrm{Spf}\, A$ is affine. As above, we use the results of Temkin and Elkik to find an algebraization X of $\mathfrak{X}$ that has smooth generic fiber and geometrically reduced special fiber. Moreover, X is of pure relative dimension d as was shown in Step 1.

We denote the smooth locus of X by X^{sm}. Then $\mathcal{O}_K$-flatness of X implies that the p-adic completion of X^{sm} coincides with $\mathfrak{X}^{\mathrm{sm}}$ inside $\mathfrak{X}$. Moreover, we note that the morphism $X_K^{\mathrm{an}} \to X$ factors through X^{sm} as the generic fiber X_K is smooth. So we have commutative diagrams:

$$\begin{tikzcd} & \mathfrak{X}^{\mathrm{sm}}_K \arrow[r, "\beta'"] \arrow[d, "j'"] \arrow[dl, "j_{\mathfrak{X}^{\mathrm{sm}}_K \to \mathfrak{X}_K}"'] & \mathfrak{X}^{\mathrm{sm}} \arrow[d, "c'"] \\ \mathfrak{X}_K \arrow[r, "j"] & X_K^{\mathrm{an}} \arrow[r, "\alpha'"] & X^{\mathrm{sm}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{X}_K \arrow[rr, "\beta"] \arrow[d, "j"] & & \mathfrak{X} \arrow[d, "c"] \\ X_K^{\mathrm{an}} \arrow[r, "\alpha'"] \arrow[rr, bend right, "\alpha"'] & X^{\mathrm{sm}} \arrow[r, "j_{X^{\mathrm{sm}} \to X}"] & X \end{tikzcd}$$

with j and j' being open immersions. Now we use [Sta19, Tag 0BRT] again to get a canonical isomorphism $r_{X^{\mathrm{sm}}}\colon \Omega^d_{X^{\mathrm{sm}}} \to \omega_{X^{\mathrm{sm}}}$ that commutes with base change. This implies that the morphism

$$r_{X^{\mathrm{sm}}} \otimes_{\mathcal{O}_{X^{\mathrm{sm}}}} \mathcal{O}_{X_n^{\mathrm{sm}}} = r_{X^{\mathrm{sm}}} \otimes^{\mathbf{L}}_{\mathcal{O}_{X^{\mathrm{sm}}}} \mathcal{O}_{X_n^{\mathrm{sm}}}$$

[11] This proof does not really require properness of f.

is canonically identified with $r_{X_n^{\mathrm{sm}}}$. Therefore, Lemma 2.1.8 and the construction of $r_{\mathfrak{X}^{\mathrm{sm}}}$ in the proof of Theorem 2.1.6 imply that $c'^*(r_{X^{\mathrm{sm}}})$ is canonically identified with $r_{\mathfrak{X}^{\mathrm{sm}}}$.

Using Corollary 2.1.13, we see that the natural map $\omega_X \to j_{X^{\mathrm{sm}}\to X,*}(\omega_{X^{\mathrm{sm}}})$ is an isomorphism. So the $((-)^*, (-)_*)$ adjunction applied to

$$r_{X^{\mathrm{sm}}} : \Omega^d_{X^{\mathrm{sm}}} \to \omega_{X^{\mathrm{sm}}}$$

defines the map

$$r_X : \Omega^d_X \to j_{X^{\mathrm{sm}}\to X,*}(\omega_{X^{\mathrm{sm}}}) \simeq \omega_X$$

such that $j^*_{X^{\mathrm{sm}}\to X}(r_X) = r_{X^{\mathrm{sm}}}$.

Finally, we define the morphism $r_{\mathfrak{X}_K} : \widehat{\Omega}^d_{\mathfrak{X}_K} \to (\omega_{\mathfrak{X}})_K$ as the composition

$$\widehat{\Omega}^d_{\mathfrak{X}_K} \xrightarrow{j^*\alpha^*(r_X)} j^*\alpha^*\omega_X \xrightarrow{\sim} \beta^*c^*\omega_X \xrightarrow{\sim} (\omega_{\mathfrak{X}})_K$$

that, restricted on $\mathfrak{X}^{\mathrm{sm}}_K$, coincides[12] with the morphism

$$\begin{aligned} j^*_{\mathfrak{X}^{\mathrm{sm}}_K\to\mathfrak{X}_K} j^*\alpha^*(r_X) &= j^*_{\mathfrak{X}^{\mathrm{sm}}_K\to\mathfrak{X}_K}\beta^*c^*(r_X) \\ &= \beta'^*c'^*j^*_{X^{\mathrm{sm}}\to X}(r_X) \\ &= (r_{\mathfrak{X}^{\mathrm{sm}}})_K \end{aligned}$$

as noted above. This finishes the construction of $r_{\mathfrak{X}_K}$. $\square$

Remark 2.4.2. It is possible to argue along the same lines to show that $\Omega^d_X[d]$ is canonically isomorphic to $(\omega^\bullet_{\mathfrak{X}})_K$, but we do not discuss it here as we will never need this result.

The next goal is to construct the morphism $r_{\mathfrak{X}}$ for *any* admissible formal model of smooth X. In order to do this, we need to understand how a trace morphism constructed in Section 2.2 interacts with the operation of taking the adic generic fiber.

Definition 2.4.3. Let X be a smooth, separated rigid-analytic K-space of pure dimension d. And let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of two admissible formal models of X, i.e. the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathfrak{X}' \arrow[d, "\mathfrak{f}"] \\ & \mathfrak{X} \end{tikzcd}$$

is commutative. We define the *adic generic fiber of the Trace map*

$$(\mathrm{Tr}_{\mathfrak{f}})_K : (\omega_{\mathfrak{X}'})_K \to (\omega_{\mathfrak{X}})_K$$

as the composition

$$(\omega_{\mathfrak{X}'})_K \xrightarrow{\sim} (\mathfrak{f}_*\omega_{\mathfrak{X}'})_K \xrightarrow{(\mathrm{Tr}_{\mathfrak{f}})_K} (\omega_{\mathfrak{X}})_K$$

where the first map is the inverse of the natural base change map $(\mathfrak{f}_*\omega_{\mathfrak{X}'})_K \xrightarrow{\sim} (\omega_{\mathfrak{X}'})_K$. The base change is an isomorphism by Lemma C.9.

[12]Up to the canonical identifications.

Lemma 2.4.4. Let X be a smooth, separated rigid-analytic K-space of pure dimension d, and let $\mathfrak{f}\colon \mathfrak{X}'' \to \mathfrak{X}'$ and $\mathfrak{g}\colon \mathfrak{X}' \to \mathfrak{X}$ be two morphisms between admissible models of X, i.e. the diagram

$$\begin{tikzcd} & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ X \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{X}' \arrow[d, "\mathfrak{g}"] \\ & \mathfrak{X} \end{tikzcd}$$

is commutative. Then the diagram

$$\begin{tikzcd} (\omega_{\mathfrak{X}''})_K \arrow[r, "(\mathrm{Tr}_{\mathfrak{f}})_K"] \arrow[d, "\sim"] & (\omega_{\mathfrak{X}'})_K \arrow[r, "(\mathrm{Tr}_{\mathfrak{g}})_K"] & (\omega_{\mathfrak{X}})_K \\ (\omega_{\mathfrak{X}''})_K \arrow[urr, "(\mathrm{Tr}_{\mathfrak{g}\circ\mathfrak{f}})_K"'] & & \end{tikzcd},$$

is commutative.

Proof. This is an easy consequence of Lemma 2.2.3. We leave the details as an exercise to the reader. $\square$

Lemma 2.4.5. Let X be a smooth, separated rigid-analytic K-space of pure dimension d. And let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of two admissible formal models of X, i.e. the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathfrak{X}' \arrow[d, "\mathfrak{f}"] \\ & \mathfrak{X} \end{tikzcd}$$

is commutative. Then the adic generic fiber $(\mathrm{Tr}_{\mathfrak{f}})_K\colon (\omega_{\mathfrak{X}'})_K \to (\omega_{\mathfrak{X}})_K$ is an isomorphism.

Proof. First of all, we note that the formal schemes $\mathfrak{X}$ and $\mathfrak{X}'$ are separated by [BL93, Proposition 4.7], so $\omega^\bullet_{\mathfrak{X}}$ and $\omega^\bullet_{\mathfrak{X}'}$ make sense. We explain in Step 1 that $\mathfrak{f}$ is also automatically proper, so $\mathrm{Tr}_{\mathfrak{f}}$ does makes sense as well.

Step 1: Reduce to the case of an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$. We use [BL93, Claim (d) on page p.307] to find an admissible blow-up $\tau\colon \mathfrak{X}'' \to \mathfrak{X}'$ such that the composition $\mathfrak{X}'' \to \mathfrak{X}$ is also an admissible blow-up[13]. This firstly implies that $\mathfrak{f}$ is proper, and secondly Lemma 2.4.4 implies that it is sufficient to prove the claim for both $\mathfrak{X}'' \to \mathfrak{X}'$ and $\mathfrak{X}' \to \mathfrak{X}$. Thus, we can assume that $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blow-up.

Step 2: Proof in the case of an admissible blow-up. We can check that $(\mathrm{Tr}_{\mathfrak{f}})_K$ is an isomorphism Zariski-locally on $\mathfrak{X}$. So we may and do assume that $\mathfrak{X} = \mathrm{Spf}\, B$ is affine. Then [Tem17, Theorem 3.1.3] implies that $\mathfrak{X}$ can be algebraized to some flat, finitely presented affine scheme $\mathrm{Spec}\, A$ with smooth generic fiber. Let us say that $\mathfrak{X}' \to \mathfrak{X} = \mathrm{Spf}\, A$ is an admissible blow-up in an open finitely generated ideal $J \subset A$. As $B = \widehat{A}$ we see that $A/\varpi^n A = B/\varpi^n B$ for any $n \geq 1$. As J is open it contains $\varpi^n B$ for some n. This implies that we can find a finitely generated ideal $I \subset A$ such that $IB = J$. Then the morphism $\mathfrak{X}' \to \mathfrak{X}$ comes as the ϖ-adic completion of the admissible

[13]We separateness of $\mathfrak{X}$ here to conclude that the morphism ϕ constructed in [BL93] should be actually equal to $f \circ \tau$.

blow-up $g\colon \mathrm{Bl}_I(\mathrm{Spec}\, A) \to \mathrm{Spec}\, A$. Then we use Remark 2.2.2 to conclude that $c^*(\mathcal{H}^{-d}(\mathrm{Tr}_g))$, where $c\colon \mathrm{Spf}\, B \to \mathrm{Spec}\, A$ is the completion morphism, is a representative for $\mathrm{Tr}_{\mathfrak{f}}$. We recall that $\mathrm{Tr}_g\colon \mathbf{R}f_*\left(\omega^\bullet_{\mathrm{Bl}_I(\mathrm{Spec}\, A)}\right) \to \omega^\bullet_{\mathrm{Spec}\, A}$ is the trace maps from coherent duality. So it suffices to show that the schematic generic fiber of the morphism Tr_g is an isomorphism. But this is clear as the K-restriction of g is the identity morphism by its construction. □

Now we can define $r_{\mathfrak{X}_K}$ for any separated admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}$. We use [BLR95, Theorem 2.1] to find a finite admissible blow-up $\varpi\colon \mathfrak{X}' \to \mathfrak{X}$[14] that has reduced special fiber. Moreover, we note that [Lüt16, Proposition 3.4.1] guarantees that $\mathfrak{X}'$ is actually unique and coincides with "the normalization of $\mathfrak{X}$ in $\mathfrak{X}_K$". Then Lemma 2.4.5 implies that the Trace map $(\mathrm{Tr}_\pi)_K\colon (\omega_{\mathfrak{X}'})_K \to (\omega_{\mathfrak{X}})_K$ is an isomorphism.

Definition 2.4.6. We define the *comparison isomorphism* $r_{\mathfrak{X}_K}\colon \Omega^d_X \to (\omega_{\mathfrak{X}})_K$ as the composition

$$r_{\mathfrak{X}_K} := (\mathrm{Tr}_\pi)_K \circ r_{\mathfrak{X}'_K}.$$

Remark 2.4.7. One can show that $r_{\mathfrak{X}_K}$ provides an isomorphism $r_{\mathfrak{X}_K}\colon \Omega^d_X \to (\omega^\bullet_{\mathfrak{X}}[-d])_K$. We do not discuss it as we will never really need this in the paper.

Lemma 2.4.8. Let X be a smooth, separated, quasi-compact rigid-analytic K-space of pure dimension d. And let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of two admissible formal models of X, i.e. the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathfrak{X}' \arrow[d, "\mathfrak{f}"] \\ & \mathfrak{X} \end{tikzcd}$$

is commutative. Then the diagram

$$\begin{tikzcd} \Omega^d_X \arrow[r, "r_{\mathfrak{X}'_K}"] \arrow[dr, "r_{\mathfrak{X}_K}"] & (\omega_{\mathfrak{X}'})_K \arrow[d, "(\mathrm{Tr}_{\mathfrak{f}})_K"] \\ & (\omega_{\mathfrak{X}})_K \end{tikzcd}$$

is commutative.

Proof. Step 1. Reduce to the case when $\mathfrak{X}$ and $\mathfrak{X}'$ have reduced special fiber: We use [BLR95, Theorem 2.1] to find admissible blow-ups $\pi'\colon \widetilde{\mathfrak{X}}' \to \mathfrak{X}'$ and $\pi\colon \widetilde{\mathfrak{X}} \to \mathfrak{X}$ such that $\mathfrak{X}'$ and $\mathfrak{X}$ have reduced special fiber. We use the characterization of $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{X}}'$ from [Lüt16, Proposition 3.4.1] to conclude there is a natural morphism $\widetilde{\mathfrak{f}}\colon \widetilde{\mathfrak{X}}' \to \widetilde{\mathfrak{X}}$ making the diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[d, "\mathfrak{f}"] & \widetilde{\mathfrak{X}}' \arrow[l, "\pi'"'] \arrow[d, "\widetilde{\mathfrak{f}}"] \\ \mathfrak{X} & \widetilde{\mathfrak{X}} \arrow[l, "\pi"] \end{tikzcd}$$

[14]We recall that we asuume that K is algebraically closed, so $\mathfrak{X}'$ is already defined over $\mathcal{O}_K$.

commute. Then using the definition of $r_{\mathfrak{X}_K}$ and $r_{\mathfrak{X}'_K}$ we see that it suffices to show that the following diagram

$$\begin{tikzcd}
\Omega^d_X \arrow[r, "r_{\widetilde{\mathfrak{X}'_K}}"] \arrow[dr, "r_{\widetilde{\mathfrak{X}_K}}"'] & (\omega_{\widetilde{\mathfrak{X}}'})_K \arrow[r, "(\mathrm{Tr}_{\pi'})_K"] \arrow[d, "(\mathrm{Tr}_{\widetilde{\mathfrak{f}}})_K"] & (\omega_{\mathfrak{X}'})_K \arrow[d, "(\mathrm{Tr}_{\mathfrak{f}})_K"] \\
& (\omega_{\widetilde{\mathfrak{X}}})_K \arrow[r, "(\mathrm{Tr}_{\pi})_K"] & (\omega_{\mathfrak{X}})_K
\end{tikzcd}$$

is commutative. We now claim that the right commutative square is commutative. Indeed, it is generic fiber of the the diagram

$$\begin{tikzcd}
g_*\left(\omega_{\widetilde{\mathfrak{X}}'}\right) \arrow[r, "\widetilde{\mathfrak{f}}_*(\mathrm{Tr}_{\pi'})"] \arrow[d, "\pi'_*(\mathrm{Tr}_{\widetilde{\mathfrak{f}}})"] & \mathfrak{f}_*\left(\omega_{\mathfrak{X}'}\right) \arrow[d, "\mathrm{Tr}_{\mathfrak{f}}"] \\
\pi_*\left(\omega_{\widetilde{\mathfrak{X}}}\right) \arrow[r, "\mathrm{Tr}_{\pi}"] & \omega_{\mathfrak{X}}
\end{tikzcd},$$

where $\mathfrak{g} \coloneqq \mathfrak{f} \circ \pi' = \pi \circ \widetilde{\mathfrak{f}}$. Lemma 2.4.4 implies that is commutative, so the right square of the diagram 2.4 commutes. Thus it suffices to show that the left triangle commutes, so we may and do assume that $\mathfrak{X}$ and $\mathfrak{X}'$ have reduced special fibers.

Step 2. The proof under the assumption that $\mathfrak{X}$ and $\mathfrak{X}'$ have reduced special fibers: We use the same trick as in the proof of Lemma 2.4.1. We know that both $(\mathrm{Tr}_{\mathfrak{f}})_K \circ r_{\mathfrak{X}'_K}$ and $r_{\mathfrak{X}_K}$ are both morphisms of line bundles concentrated in degree $-d$. Thus we can use [Con99, Lemma 2.1.4] to say that it is sufficient to show that they are equal after passing to some dense open subset of $\mathfrak{X}$. Now we invoke Corollary B.4 to find some open $\mathfrak{U} \subset \mathfrak{X}^{\mathrm{sm}}$ such that $\mathfrak{f}$ is an isomorphism over $\mathfrak{U}$. Thus we have reduced to the case where $\mathfrak{f}$ is an isomorphism and $\mathfrak{X}$ is smooth. Then $r_{\mathfrak{X}_K}$ and $r_{\mathfrak{X}'_K}$ both come as generic fibers of the canonical isomorphisms $r_{\mathfrak{X}}\colon \widehat{\Omega}^d_{\mathfrak{X}} \to \omega_{\mathfrak{X}}$ and $r_{\mathfrak{X}'}\colon \widehat{\Omega}^d_{\mathfrak{X}'} \to \omega_{\mathfrak{X}'}$. Then canonicity of these maps implies that the diagram

$$\begin{tikzcd}
\Omega^d_X \arrow[r, "r_{\mathfrak{X}'_K}"] \arrow[dr, "r_{\mathfrak{X}_K}"'] & (\omega_{\mathfrak{X}'})_K \arrow[d, "(\mathrm{Tr}_{\mathfrak{f}})_K"] \\
& (\omega_{\mathfrak{X}})_K
\end{tikzcd}$$

is commutative because $\mathfrak{f}$ is an isomorphism. □

Now we define the trace map for a finite morphism of smooth, quasi-compact, separated rigid-analytic K-spaces of pure dimension d. We give a preliminary definition that depends on a choice of formal model and later we show that it is independent of any choices.

Definition 2.4.9. Let $\mathfrak{f}_K\colon X \to Y$ be a morphism as above, and let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be its formal model that is necessarily proper by [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]). Then we define the *trace morphism* $\mathrm{Tr}^{\mathfrak{f}}_{\mathfrak{f}_K}\colon \mathfrak{f}_{K,*}\left(\Omega^d_X\right) \to \Omega^d_Y$ as the unique morphism making the diagram

$$\begin{tikzcd}
\mathfrak{f}_{K,*}\left(\Omega^d_X\right) \arrow[r, "\mathfrak{f}_{K,*}(r_{\mathfrak{X}})"] \arrow[d, "\mathrm{Tr}^{\mathfrak{f}}_{\mathfrak{f}_K}"] & \mathfrak{f}_{K,*}\left(\omega_{\mathfrak{X}}\right)_K \arrow[d, "\left(\mathrm{Tr}_{\mathfrak{f}}\right)_K"] \\
\Omega^d_Y \arrow[r, "r_{\mathfrak{Y}}"] & \left(\omega_{\mathfrak{Y}}\right)_K
\end{tikzcd},$$

where the right Trace is defined using Lemma 2.2.1.

Lemma 2.4.10. Let $\mathfrak{f}_K\colon X \to Y$ be a finite morphism of smooth, quasi-compact, separated rigid-analytic K-spaces of pure dimension d. Then the morphism $\mathrm{Tr}^{\mathfrak{f}}_{\mathfrak{f}_K}\colon \mathfrak{f}_{K,*}(\Omega^d_X) \to \Omega^d_Y$ does not depend on a choice of a formal model $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$.

Proof. Suppose we have two formal models for $\mathfrak{f}_K\colon X \to Y$, we denote them by $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{f}'\colon \mathfrak{X}' \to \mathfrak{Y}'$. Then we use [BL93, Claims (c) and (d) on p. 307] to claim that there is model $\mathfrak{f}''\colon \mathfrak{X}'' \to \mathfrak{Y}''$ such that there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[d, "\mathfrak{f}"] & \mathfrak{X}'' \arrow[l] \arrow[r] \arrow[d, "\mathfrak{f}''"] & \mathfrak{X}' \arrow[d, "\mathfrak{f}'"] \\
\mathfrak{Y} & \mathfrak{Y}'' \arrow[l] \arrow[r] & \mathfrak{Y}'
\end{tikzcd},$$

where all horizontal maps are admissible blow-ups. Therefore, it suffices to prove the statement under the assumption that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\mathfrak{f}_K"] & \mathfrak{X} \arrow[l] \arrow[d, "\mathfrak{f}"] & \mathfrak{X}' \arrow[l, "\pi'"'] \arrow[d, "\mathfrak{f}'"] \\
Y & \mathfrak{Y} \arrow[l] & \mathfrak{Y}' \arrow[l, "\pi"']
\end{tikzcd},$$

where π and π' are some admissible blow-ups. Then we claim that there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{f}_{K,*}\left(\Omega^d_X\right) \arrow[r, "\mathfrak{f}_{K,*}\left(r_{\mathfrak{X}'_K}\right)"] \arrow[rr, bend left, "\mathfrak{f}_{K,*}\left(r_{\mathfrak{X}_K}\right)"] & \mathfrak{f}_{K,*}\left(\omega_{\mathfrak{X}'}\right)_K \arrow[r, "\mathfrak{f}_{K,*}\left(\left(\mathrm{Tr}_{\pi'}\right)_K\right)"] \arrow[d, "\left(\mathrm{Tr}_{\mathfrak{f}'}\right)_K"] & \mathfrak{f}_{K,*}\left(\omega_{\mathfrak{X}}\right)_K \arrow[d, "\left(\mathrm{Tr}_{\mathfrak{f}}\right)_K"] \\
\Omega^d_Y \arrow[r, "\left(r_{\mathfrak{Y}'}\right)_K"] \arrow[rr, bend right, "r_{\mathfrak{Y}_K}"'] & \left(\omega_{\mathfrak{Y}'}\right)_K \arrow[r, "\left(\mathrm{Tr}_{\pi}\right)_K"] & \left(\omega_{\mathfrak{Y}'}\right)_K
\end{tikzcd}$$

such that all horizontal traces are isomorphisms. Indeed, the top and bottom "triangles" commute by Lemma 2.4.8. And the square commutes by the argument similar to that of Lemma 2.4.4 that essentially boils down to Lemma 2.2.3. Now we use the very definition of $\mathrm{Tr}^{\mathfrak{f}}_{\mathfrak{f}_K}$ to see that it must coincide with $\mathrm{Tr}^{\mathfrak{f}'}_{\mathfrak{f}_K}$. ∎

Definition 2.4.11. Let $f\colon X \to Y$ be a finite morphism of smooth, separated, quasi-compact rigid-analytic K-spaces of pure dimension d. Then we define the *trace morphism*

$$\mathrm{Tr}_f \colon f_*\left(\Omega^d_X\right) \to \Omega^d_Y$$

as $\mathrm{Tr}^{\mathfrak{g}}_f$ for any choice of the formal model $\mathfrak{g}\colon \mathfrak{X} \to \mathfrak{Y}$ for the morphism f. Lemma 2.4.10 guarantees that this morphism is well-defined.

The last thing we want to discuss is the explicit model for the trace map in the case of a finite étale morphism of smooth rigid-spaces of the same dimension. Using the same procedure as in Section 2.3 to define *the explicit Trace map* $\mathrm{Tr}^{\mathrm{expl}}_f\colon f_*\left(\Omega^d_X\right) \to \Omega^d_Y$ for a finite étale map $f\colon X \to Y$ of smooth, separated, quasi-compact rigid-analytic K-spaces of pure dimension d.

Locally on Y, in étale coordinates $U \xrightarrow{z_1,\dots,z_d} \mathbf{D}^d$, the maps looks like

$$f\mathrm{d}z_1\wedge\dots\wedge\mathrm{d}z_d \mapsto \mathrm{Tr}_{B/A}(f)\mathrm{d}z_1\wedge\dots\wedge\mathrm{d}z_d,$$

where $\mathrm{Tr}_{B/A}$ is the trace map for the flat, finite, finitely presented morphism $A \to B$ that comes from a morphism of affinoids $\mathrm{Spa}(B,B^+) = f^{-1}(U) \to \mathrm{Spa}(A,A^+) = U$.

Lemma 2.4.12. Let $f\colon X \to Y$ be a finite étale morphism of smooth, separated, quasi-compact rigid-analytic K-spaces of pure dimension d. Then the morphisms

$$\mathrm{Tr}_f, \mathrm{Tr}^{\mathrm{expl}}_f \colon f_*\left(\Omega^d_X\right) \to \Omega^d_Y$$

coincide.

Proof. Step 1: Reduce to the case of f with a "nice" formal model. Firstly, we note that the claim is local on Y, so we can assume that Y (and, therefore, X) are affinoid. Say, $X = \mathrm{Spa}(B, B^\circ)$ and $Y = \mathrm{Spa}(A, A^\circ)$. Then [Lüt16, Theorem 3.4.2] guarantees that A° and B° are topologically finite type over $\mathcal{O}_K$[15] and special fibers are (geometrically) reduced. In particular,

$$\mathfrak{f}\colon \mathfrak{X} \coloneqq \mathrm{Spf}\, B^\circ \to \mathfrak{Y} \coloneqq \mathrm{Spf}\, A^\circ$$

is a formal model for $f\colon X \to Y$.

Since both A° and B° are topologically finite type and $\mathcal{O}_K$-flat, we conclude that they are topologically finitely presented over $\mathcal{O}_K$ by [Bos14, Corollary 7.3/5]. Therefore, the morphism $A^\circ \to B^\circ$ is topologically finitely presented. It is also integral by [BGR84, Theorem 6.4/1] because $A \to B$ is finite. Thus, we conclude that B° is finitely presented as an A°-module.

Now we note Tr_f and $\mathrm{Tr}^{\mathrm{expl}}_f$ are morphisms between vector bundles on Y, so [Con99, Lemma 2.1.4] guarantees that it suffices to check that these two maps coincide over some open subset $V \subset Y$ that meets each irreducible component of Y. Therefore, we are free to replace Y with adic generic fiber of any dense open subset in $\mathfrak{Y} = \mathrm{Spf}\, A^\circ$.

Now Lemma B.3 constructs an open dense affine subscheme $\mathfrak{U} \subset \mathfrak{Y}^{\mathrm{sm}}$ such that $\mathfrak{f}\colon \mathfrak{f}^{-1}(\mathfrak{U}) \to \mathfrak{U}$ is finite and flat. So we can replace Y with $\mathfrak{U}_K$ and $\mathfrak{Y}$ with $\mathfrak{U}$ to assume that $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is finite, flat and $\mathfrak{Y}$ is $\mathcal{O}_K$-smooth. Moreover, we can replace $\mathfrak{Y}$ with any dense open subset in the dense open subset $\mathfrak{Y} - \mathfrak{f}(\mathfrak{X} - \mathfrak{X}^{\mathrm{sm}})$ to also assume that $\mathfrak{X}$ is smooth.

Step 2. *Computation of* Tr_f *in the case of a "nice" formal model.* We firstly claim that

$$\mathfrak{f}_*\left(\widehat{\Omega}_{\mathfrak{X}}\right) \simeq \mathcal{H}om_{\mathfrak{Y}}(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{Y}}) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}}.$$

[15] Here we use again that K is algebraically closed by our assumption on K.

Lemma B.6 guarantees that $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ are of pure dimension d. So we use smoothness of $\mathfrak{Y}_n$ and the scheme case of Grothendieck Duality to conclude that

$$\begin{aligned}\mathbf{R}\mathfrak{f}_{n,*}\left(\omega^\bullet_{\mathfrak{X}_n}\right) &\simeq \mathbf{R}\mathfrak{f}_{n,*}\mathfrak{f}_n^!\left(\omega^\bullet_{\mathfrak{Y}_n}\right)\\ &\simeq \mathbf{R}\mathcal{H}om_{\mathfrak{Y}_n}\left(\mathfrak{f}_{n,*}\mathcal{O}_{\mathfrak{X}_n}, \omega^\bullet_{\mathfrak{Y}_n}\right)\\ &\simeq \mathbf{R}\mathcal{H}om_{\mathfrak{Y}_n}\left(\mathfrak{f}_{n,*}\mathcal{O}_{\mathfrak{X}_n}, \Omega^d_{\mathfrak{Y}_n}[d]\right)\\ &\simeq \left(\mathcal{H}om_{\mathfrak{Y}_n}\left(\mathfrak{f}_{n,*}\mathcal{O}_{\mathfrak{X}_n}, \Omega^d_{\mathfrak{Y}_n}\right)\right)[d].\end{aligned}$$

In particular, after taking $\mathcal{H}^{-d}(-)$, we use smoothness of $\mathfrak{X}_n$ to see that $\mathfrak{f}_{n,*}\left(\Omega^d_{\mathfrak{X}_n}\right) \simeq \mathcal{H}om_{\mathfrak{Y}_n}(\mathfrak{f}_{n,*}\mathcal{O}_{\mathfrak{X}_n}, \Omega^d_{\mathfrak{Y}_n})$ compatibly with base change. Therefore, we get that

$$\begin{aligned}\mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right) &= \lim_n \left(\mathcal{H}om_{\mathfrak{Y}_n}\left(\mathfrak{f}_{n,*}\mathcal{O}_{\mathfrak{Y}_n}, \Omega^d_{\mathfrak{Y}_n}\right)\right)\\ &\simeq \lim_n \left(\mathcal{H}om_{\mathfrak{Y}_n}\left(\mathfrak{f}_*\mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\mathcal{O}_{\mathfrak{Y}_n}, \widehat{\Omega}^d_{\mathfrak{Y}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\mathcal{O}_{\mathfrak{Y}_n}\right)\right)\\ &\simeq \lim_n \left(\mathcal{H}om_{\mathfrak{Y}}\left(\mathfrak{f}_*\mathcal{O}_{\mathfrak{Y}}, \widehat{\Omega}^d_{\mathfrak{Y}}\right)\otimes_{\mathcal{O}_{\mathfrak{Y}}}\mathcal{O}_{\mathfrak{Y}_n}\right)\\ &\simeq \mathcal{H}om_{\mathfrak{Y}}\left(\mathfrak{f}_*\mathcal{O}_{\mathfrak{Y}}, \widehat{\Omega}^d_{\mathfrak{Y}}\right)\\ &\simeq \mathcal{H}om_{\mathfrak{Y}}(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{Y}})\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^d_{\mathfrak{Y}}.\end{aligned}$$

The construction of the trace map as a limit implies that the trace map

$$\mathfrak{f}_*\left(\omega_{\mathfrak{X}}\right) \to \omega_{\mathfrak{Y}}$$

can be identified[16] with

$$\alpha\otimes_{\mathcal{O}_{\mathfrak{Y}}}\mathrm{Id}\colon \mathcal{H}om_{\mathfrak{Y}}(\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{Y}})\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^d_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{Y}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^d_{\mathfrak{Y}} \simeq \widehat{\Omega}^d_{\mathfrak{Y}},$$

where α is locally given by $\alpha(\phi) = \phi(1)$.

We claim that the following diagram

$$\begin{tikzcd}
\mathfrak{f}_*\mathfrak{f}^*\left(\widehat{\Omega}^d_{\mathfrak{Y}}\right) \arrow[r] \arrow[d] & \mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right) \arrow[r] \arrow[dr, "\mathrm{Tr}_{\mathfrak{f}}"] & \mathcal{H}om_{\mathfrak{Y}}\left(\mathfrak{f}_*\left(\mathcal{O}_{\mathfrak{X}}\right), \mathcal{O}_{\mathfrak{Y}}\right)\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^d_{\mathfrak{Y}} \arrow[d, "\alpha\otimes_{\mathcal{O}_{\mathfrak{Y}}}\mathrm{Id}"] \\
\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^d_{\mathfrak{Y}} \arrow[rr, "\mathrm{Tr}_{B^\circ/A^\circ}\otimes\mathrm{Id}"] & & \widehat{\Omega^d_{\mathfrak{Y}}}
\end{tikzcd}$$

commutes, where the left vertical arrow is an isomorphism from the projection formula, and the top left horizontal arrow is induced by the canonical map $\mathfrak{f}^*\left(\widehat{\Omega}^d_{\mathfrak{Y}}\right) \to \widehat{\Omega}^d_{\mathfrak{X}}$, and the top right map is the canonical isomorphism constructed above.

We do not give a full proof of the commutativity of this diagram, but only sketch the main ideas. The nature of all maps allows us easily to reduce the question to the case of smooth schemes over $\mathcal{O}_K/\varpi^n\mathcal{O}_K$ for all n. There, one uses [Sta19, Tag 0B6J] and standard approximation techniques to reduce to analogous statement over a noetherian base. In this situation, commutativity follows from [NS19, Theorem 9.2.14(ii)].

That being said, we finally see that the morphism

$$\mathrm{Tr}_f\colon f_*\left(\Omega^d_X\right) \to \Omega^d_Y$$

[16] since the same holds on the scheme level by [Sta19, Tag 0B6L]

comes as the generic fiber of the map

$$\mathfrak{f}_* (\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^d_{\mathfrak{Y}} \xrightarrow{\mathrm{Tr}_{B^\circ/A^\circ}} \widehat{\Omega^d_{\mathfrak{Y}}}$$

because rig-étaleness of $\mathfrak{f}$ implies that the map $\mathfrak{f}_*\mathfrak{f}^*\left(\widehat{\Omega}^d_{\mathfrak{Y}}\right) \to \mathfrak{f}_*\left(\widehat{\Omega}^d_{\mathfrak{X}}\right)$ is an isomorphism on adic generic fibers. Finally, we notice that $\mathrm{Tr}_{B^\circ/A^\circ}[1/\varpi] = \mathrm{Tr}_{B/A}$ to finish the proof. □

3. Construction of Faltings' Trace Map

3.1. Idea of the Construction. For the rest of the section, we fix a complete, algebraically closed, rank-1 valued field C of mixed characteristic $(0, p)$.

The main goal of this section is to the construction of Faltings' trace map

$$\mathrm{Tr}_{F,\mathfrak{X}} \colon \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \to \left(\omega^\bullet_{\mathfrak{X}_0}(-d)[-2d]\right)^a$$

for an admissible separated formal model $\mathfrak{X}$ with smooth generic fiber $X = \mathfrak{X}_C$ of pure dimension d.

Suppose that X is also proper, so [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]) implies that $\mathfrak{X}$ is proper as well. So we can combine Tr_F with the trace map in the almost Grothendieck Duality to get the trace map

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to (\mathcal{O}_C/p(-d)[-2d])^a\,.$$

This will define an almost pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \xrightarrow{(-)\cup(-)} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to (\mathcal{O}_C/p(-d)[-2d])^a$$

on any smooth and proper rigid C-space X. We will show that this pairing is always almost perfect in Section 5. This will be a crucial step in our proof of Poincaré Duality.

Now we say a few words about the construction of the Faltings' trace map. We explain the main ideas behind the construction:

(1) We use [Zav21a, Theorem 6.13.5] and Theorem 2.1.6 to note that $\mathbf{R}\nu_*\left(\mathcal{O}^+_X/p\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a$ and $\omega^\bullet_{\mathfrak{X}_0} \in \mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X}_0)$. So it actually suffices to define a map

$$\mathrm{Tr}^d_{F,\mathfrak{X}} \colon \mathrm{R}^d\nu_*\left(\mathcal{O}^+_X/p\right)^a \to \omega_{\mathfrak{X}_0}(-d)^a.$$

Moreover, since we also have $\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a$ and $\omega^\bullet_{\mathfrak{X}} \in \mathbf{D}^{[-d,0]}_{coh}(\mathfrak{X})$, it is actually sufficient to define a trace map

$$\mathrm{Tr}^{d,+}_{F,\mathfrak{X}} \colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a \to \omega_{\mathfrak{X}}(-d)^a.$$

since then $\mathrm{Tr}^d_{F,\mathfrak{X}}$ can be defined as the composition

$$\mathrm{R}^d\nu_*\left(\mathcal{O}^+_X/p\right)^a \xrightarrow{\sim} \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \xrightarrow{\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathrm{Id}} \omega_{\mathfrak{X}}(-d)^a \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \xrightarrow{\mathrm{BC}^a_{\omega_{\mathfrak{X}}(-d)}} \omega_{\mathfrak{X}_0}(-d).$$

(2) We use the Reduced Fiber Theorem [BLR95, Theorem 2.1] to find a finite rig-isomorphism $\mathfrak{f} \colon \mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ is an admissible formal $\mathcal{O}_C$-scheme with reduced special fiber. Then one uses the trace map $\mathrm{Tr}_{\mathfrak{f}} \colon \mathfrak{f}_*\left(\omega'_{\mathfrak{X}}\right) \to \omega_{\mathfrak{X}}$ to reduce to the case of $\mathfrak{X}'$. So we can replace $\mathfrak{X}$ with $\mathfrak{X}'$ to assume that the special fiber of $\mathfrak{X}$ is reduced.

(3) Under the extra assumption that $\mathfrak{X}$ has reduced special fiber, we define an honest map of $\mathcal{O}_{\mathfrak{X}}$-modules
$$\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right) \to \omega_{\mathfrak{X}}(-d).$$
The key idea is to use Theorem 2.1.14 to reduce the question of constructing $\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}$ to the smooth locus of $\mathfrak{X}$ and generic fiber $\mathfrak{X}_C$. On the smooth locus, we use a variation of the map from [BMS18] to construct $\mathrm{Tr}^{d,+}_{F,\mathfrak{X}^{\mathrm{sm}}}$. And we use the analogous map from [Sch13a] to construct $(\mathrm{Tr}^{d,+}_{F,\mathfrak{X}})_C$.

The rest of the section is devoted to fulfilling the plan above and consturting the desired trace map in full generality.

3.2. **The BMS Map for Non-Smooth Admissible Formal Models.** The main goal of this section is to recall the construction of the BMS map
$$\Phi^d_{\mathfrak{X}}\colon \widehat{\Omega}^d_{\mathfrak{X}}\{-d\} \to \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}$$
introduced in [BMS18]. The construction in [BMS18] is written under the assumption that the formal model $\mathfrak{X}$ is smooth, however it is not really needed in the construction. And it will be important for our purposes to define this map for non-smooth admissible formal $\mathcal{O}_C$-schemes.

We fix a smooth rigid-analytic C-space X of pure dimension d with an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$. Then functoriality of the cotangent complex provides us with the map
$$L_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p} \to \mathbf{R}\nu_*\left(L_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}\right)$$
that passing to the derived p-adic completions gives the map
$$\Phi''_{\mathfrak{X}}\colon \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p} \to \mathbf{R}\nu_*\left(\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}\right)\ .$$

The key idea of the BMS map is to realize $\mathcal{H}^0$ of this map in some explicit terms.

Lemma 3.2.1. Let X be a smooth rigid C-space with an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$. Then there is a natural isomorphism $\Omega^1_{\mathfrak{X}} \simeq \mathrm{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p})$.

Proof. We consider morphisms $\mathfrak{X} \to \mathrm{Spf}\,\mathcal{O}_C$ and $\mathrm{Spf}\,\mathcal{O}_C \to \mathrm{Spf}\,\mathbf{Z}_p$ that give us distinguished triangle
$$L_{\mathcal{O}_C/\mathbf{Z}_p} \otimes^{\mathbf{L}}_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X}} \to L_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p} \to L_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C}$$
that, after taking its derived p-adic completion, induces distinguished triangle
$$\widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p} \widehat{\otimes}^{\mathbf{L}}_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X}} \to \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p} \to \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C} \tag{3.1}$$

Now we analyze the terms of this distinguished triangle. We note that $\widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p} \simeq \mathcal{O}_C\{1\}[1]$ is a free $\mathcal{O}_C$-module concentrated in degree -1 by [SZ18, Theorem 3.1] and [GR03, Proposition 6.5.6] in case $\mathcal{O}_C = \widehat{\overline{Z}_p}$ and [Bha19, Remark 3.19] in general. Moreover, $\mathcal{O}_{\mathfrak{X}}$ is clearly a p-adically derived complete module on $\mathfrak{X}$, so $\widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p} \widehat{\otimes}^L_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_{\mathfrak{X}}\{1\}[1]$. Thus, the distinguished triangle (3.1) implies that $\mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p}) \xrightarrow{\sim} \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C})$. Finally, [GR03, Section 7.2.8] guarantees that the natural morphism $\widehat{\Omega}^1_{\mathfrak{X}} \to \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C})$ is an isomorphism. This gives a natural isomorphism
$$\Omega^1_{\mathfrak{X}} \xrightarrow{\sim} \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p}).$$
$\square$

Lemma 3.2.2. Let X be a smooth rigid C-space with an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$. Then there is a natural isomorphism

$$\mathcal{H}^0\left(\mathbf{R}\nu_*\left(\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}\right)\right) \simeq \mathrm{R}^1\nu_*\widehat{\mathcal{O}}^+_X\{1\}$$

Proof. We do a similar trick here: we consider the following morphisms of ringed spaces:

$$(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (\mathrm{Spa}(C, \mathcal{O}_C)_{\mathrm{an}}, \mathcal{O}_C) \to (\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)_{\mathrm{an}}, \mathbf{Z}_p).$$

This induce a distinguished triangle

$$L_{\mathcal{O}_C/\mathbf{Z}_p} \otimes^L_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X \to L_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p} \to L_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C}$$

that, after taking derived p-adic completion, induces a distinguished triangle

$$\widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p} \widehat{\otimes}^L_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X \to \widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p} \to \widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C}$$

The same argument using derived completeness of $\widehat{\mathcal{O}}^+_X$ (see [BMS18, Remark 5.5]) implies that $\widehat{L}_{\mathcal{O}_C/\mathbf{Z}_p} \widehat{\otimes}^{\mathbf{L}}_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_X \simeq \widehat{\mathcal{O}}^+_X\{1\}[1]$. Now [Bha19, Corollary 3.28] guarantees that $\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_C} \simeq 0$, so we have a natural isomorphism $\widehat{\mathcal{O}}^+_X\{1\}[1] \xrightarrow{\sim} \widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}$. Thus, we get an isomorphism

$$\mathcal{H}^0\left(\mathbf{R}\nu_*\left(\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}\right)\right) = \mathrm{R}^0\nu_*\left(\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p}\right) \xrightarrow{\sim} \mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\{1\}\right).$$

□

Lemmas 3.2.1 and 3.2.2 allows us to construct the morphism

$$\Phi''^1_{\mathfrak{X}} := \mathcal{H}^0(\Phi''_{\mathfrak{X}}) \colon \widehat{\Omega}^1_{\mathfrak{X}} \to \mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\{1\}\right)$$

that, after twisting by -1, gives us the map

$$\Phi'^1_{\mathfrak{X}} \colon \widehat{\Omega}^1_{\mathfrak{X}}\{-1\} \to \mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right).$$

We emphasize it again that this morphism is well-defined on any admissible formal model of a smooth space X, i.e. we do not need any smoothness assumptions on $\mathfrak{X}$.

Now we want to define Φ^n in "higher degrees". Basically, it is going to be the "composition" of $(\Phi'^1)^{\otimes n}$ with the n-th cup product map on $\mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_X)$ defined in [Sta19, Tag 0B68]. The problem is that the cup product is not always skew-symmetric on $\mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_X)$ if 2 is not invertible in $\mathbf{Z}_p$. The cup product is always anti-commutative by [Sta19, Tag 0FP5], i.e. $x \cup x = -x \cup x$ for any $x \in \mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_X)$. However, it *does not* imply that $x \cup x = 0$ unless 2 is invertible in $\mathbf{Z}_p$. So the issue occurs only in the case of residual characteristic 2. However, we want our discussion to be uniform in all characteristic, so we do the following trick.

We introduce the sheaf $\frac{\mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_X)}{\mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_X)[(\zeta_p-1)^\infty]}$ and denote the map induced by $\Phi'^1_{\mathfrak{X}}$ as

$$\Phi^1_{\mathfrak{X}} \colon \widehat{\Omega}^1_{\mathfrak{X}}\{-1\} \to \frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}.$$

The advantage of this gadget is that the natural morphism induced by the cup product

$$\left(\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)^{\otimes n} \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}$$

factors through

$$\overset{n}{\bigcup}\colon \bigwedge^{n}\left(\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)\to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\ .$$

Indeed, this is clear from [Sta19, Tag 0FP5] if the residual characteristic is not equal to 2. And if it does equal to 2, we see that $-2=\zeta_2-1$ so $\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}$ is 2-torsion free. Thus, anti-commutativity implies skew-symmetricity. This allows us to define

$$\Phi^n_{\mathfrak{X}}\colon \widehat{\Omega}^n_{\mathfrak{X}}\{-n\}\to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}$$

as the composition

$$\widehat{\Omega}^n_{\mathfrak{X}}\{-n\}\xrightarrow{\bigwedge^n(\Phi^1_{\mathfrak{X}})}\left(\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)\xrightarrow{\cup^n}\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}.$$

3.3. **Faltings' Trace Map on the Smooth Locus.** This section is devoted to the construction of the (integral) Faltings' map $\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)\to\omega_{\mathfrak{X}}(-d)$ on the smooth locus of $\mathfrak{X}$. We will denote this map by $\Psi_{\mathfrak{X}^{\mathrm{sm}}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)\to\omega_{\mathfrak{X}^{\mathrm{sm}}}(-d)$.

Unlike the Faltings' trace itself (that will be defined only in the almost category of sheaves), this map will be am honest map of $\mathcal{O}_{\mathfrak{X}}$-modules. Moreover, this map will be the crucial role in the construction of Faltings' trace on a general admissible formal $\mathcal{O}_C$-modle of a smooth rigid C-space.

The essential idea is to (slightly) modify the BMS map $\Phi^d_{\mathfrak{X}}$ on the smooth locus of $\mathfrak{X}$ to simultaneously reverse its direction and change Breuil-Kisin twists by Tate twists. In order to do this, we need to recall the main results from [BMS18] regarding the map $\Phi^n_{\mathfrak{X}^{\mathrm{sm}}}$.

For the rest of the section, we fix an admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ with smooth generic fiber $\mathfrak{X}_C$ of pure dimension d.

Lemma 3.3.1. The natural morphism

$$\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]}\to\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}\tag{3.2}$$

is an isomorphism for any $n\geq 1$. And the map $\Phi^n_{\mathfrak{X}^{\mathrm{sm}}}\colon \widehat{\Omega}^n_{\mathfrak{X}^{\mathrm{sm}}}\{-n\}\to\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ is an isomorphism onto $(\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ for $n\geq 1$.

Proof. Step 1. The map (3.2) is an isomorphism for any $n\geq 1$: We note that [BMS18, Theorem 8.3] guarantees that the complex $L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)\right)$ has finite, locally free cohomology sheaves.

On the other hand, we can use [BMS18, Lemma 6.4] to get an isomorphism

$$\mathcal{H}^n\left(L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)\right)\right) \xrightarrow{\sim} \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]}.$$

Combining these two results, we conclude that the sheaves $\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]}$ are already (ζ_p-1)-torsionfree. Thus, the map $\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]} \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ is indeed an isomorphism.

For future reference, we also note that this formally implies that

$$(\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]} = (\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$$

for $n \geq 1$.

Step 2. The map $\Phi^1_{\mathfrak{X}^{sm}}$ is an isomorphism onto $(\zeta_p-1)\left(\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}\right)$: The proofs of [BMS18, Theorem 8.3 and Theorem 8.7] show that the map $\Phi'^1_{\mathfrak{X}^{\mathrm{sm}}}\colon \widehat{\Omega}^1_{\mathfrak{X}^{\mathrm{sm}}}\{-1\} \to \mathrm{R}^1\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}$ uniquely factors through the map $\mathcal{H}^1\left(L\eta_{\zeta_p-1}(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \to \mathrm{R}^1\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}$ that comes from the natural map $\alpha\colon L\eta_{\zeta_p-1}(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}) \to \mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}$[17]. Moreover, it induces the isomorphism

$$\mathrm{BMS}^1_{\mathfrak{X}}\colon \widehat{\Omega}^1_{\mathfrak{X}^{\mathrm{sm}}}\{-1\} \xrightarrow{\sim} \mathcal{H}^1\left(L\eta_{\zeta_p-1}(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right).$$

As $\mathcal{H}^1(\alpha)$ induces an isomorphism of $\mathcal{H}^1(L\eta_{\zeta_p-1}(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}))$ onto

$$(\zeta_p-1)\mathrm{R}^1\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C} \simeq (\zeta_p-1)\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]} \simeq (\zeta_p-1)\frac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]},$$

where the last isomorphism comes from Step 1.

Step 3. The map $\Phi^n_{\mathfrak{X}^{sm}}$ is an isomorphism onto $(\zeta_p-1)^n\left(\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}\right)$ *for any* $n \geq 1$: We recall that [BMS18, Proposition 6.7] implies that there is the natural "cup-product"

$$\begin{tikzcd} L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\otimes^L_{\mathcal{O}_{\mathfrak{X}}} L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[r] \arrow[dr, "\cup"] & L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\otimes^L_{\mathcal{O}_{\mathfrak{X}}}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[d, "L\eta_{\zeta_p-1}(\cup)"] \\ & L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \end{tikzcd}.$$

[17]This map exists by [BMS18, Lemma 6.10] as $\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C} \in \mathbf{D}^{\geq 0}(\mathfrak{X})$ and $\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}$ is (ζ_p-1)-torsionfree.

It is clear from the constructions described in [BMS18, Proposition 6.7 and Lemma 6.10] that the diagram

$$\begin{tikzcd} L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\otimes^L_{\mathcal{O}_{\mathfrak{X}}} L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[r, "\cup"] \arrow[d, "\alpha\otimes^L_{\mathcal{O}_{\mathfrak{X}}}\alpha"] & L\eta_{\zeta_p-1}\left(\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[d, "\alpha"] \\ \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)\otimes^L_{\mathcal{O}_{\mathfrak{X}}}\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \arrow[r, "\cup"] & \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \end{tikzcd}$$

is commutative. This implies that the diagram

$$\begin{tikzcd} \left(\mathcal{H}^1\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\right)^{\otimes n} \arrow[r, "\cup^n"] \arrow[d, "\mathcal{H}^1(\alpha)^{\otimes n}"] & \mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[d, "\mathcal{H}^n(\alpha)"] \\ \left(\mathrm{R}^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)^{\otimes n} \arrow[r, "\cup^n"] \arrow[d] & \mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \arrow[d] \\ \left(\dfrac{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^1\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}\right)^{\otimes n} \arrow[r, "\cup^n"] & \dfrac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]} \end{tikzcd}$$

is commutative. Moreover, the proofs of [BMS18, Theorem 8.3 , Theorem 8.7, and Corollary 8.13(ii)] imply that the cup product on $\cup^n$ on $\left(\mathcal{H}^1\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\right)^{\otimes n}$ factors through the wedge power as the isomorphism

$$\cup^n\colon \Lambda^n\left(\mathcal{H}^1\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\right)\xrightarrow{\sim}\mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right).$$

Thus, the composition

$$\begin{tikzcd} \widehat{\Omega}^n_{\mathfrak{X}^{\mathrm{sm}}}\{-n\} \arrow[r, "\Lambda^n(\mathrm{BMS}^1)"] \arrow[rr, bend left, "\mathrm{BMS}^n"] & \Lambda^n\left(\mathcal{H}^1\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)\right) \arrow[r, "\cup^n"] & \mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \end{tikzcd} \tag{3.3}$$

is also an isomorphism. Commutativity of the diagram (3.3) implies that the diagram

$$\begin{tikzcd} \widehat{\Omega}^n_{\mathfrak{X}^{\mathrm{sm}}}\{-n\} \arrow[r, "\mathrm{BMS}^n"] \arrow[dr, "\Phi^n_{\mathfrak{X}^{\mathrm{sm}}}"] & \mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \arrow[d] \\ & \dfrac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]} \end{tikzcd}$$

is commutative.

It suffices to check that the vertical arrow $\mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ induces an isomorphism onto $(\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$. This morphism comes as the composition

$$\mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \xrightarrow{\mathcal{H}^n(\alpha)} \mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{(\zeta_p-1)^\infty-\text{torsion}}.$$

Now we note that the natural map $\mathcal{H}^n(\alpha)$ is an isomorphism onto $(\zeta_p-1)^n\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)$ as $\mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right)$ is already (ζ_p-1)-torsionfree. The quotient map

$$\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$$

induces the isomorphism $(\zeta_p-1)^n\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \xrightarrow{\sim} (\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ as $\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ is isomorphic to $\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)]}$ from Step 1. Combining these results we get that the map $\mathcal{H}^n\left(L\eta_{\zeta_p-1}\mathbf{R}\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C})\right) \to \frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ is an isomorphism onto $(\zeta_p-1)^n\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$. This finishes the proof. □

Now we almost ready to define $\Psi^d_{\mathfrak{X}^{\mathrm{sm}}}$. But before doing this, we remind the reader of the difference between Breuil-Kisin twists $\mathcal{O}_C\{n\}$ and Tate twists $\mathcal{O}_C(n)$.

There is always an injective homomorphism $\mathcal{O}_C(1) \to \mathcal{O}_C\{1\}$ with image equal to $(\zeta_p-1)\mathcal{O}_C\{1\}$. Then it is straighforward to see that this defines the short exact sequence

$$0 \to \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}\{-d\} \to \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d) \to \mathcal{Q} \to 0\ ,$$

with the image of $\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}\{-d\}$ in $\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d)$ equal to $(\zeta_p-1)^d\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d)$. In particular, $\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d)$ is a line bundle with a canonical isomorphism $\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}\{-d\} \simeq (\zeta_p-1)^d\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d)$.

Now we define an isomorphism

$$(\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}})^{-1}\colon \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d) \to \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}\ .$$

as the unique map that makes the diagram

$$\begin{tikzcd}
\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d) \arrow[r, dotted, "(\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}})^{-1}"] & \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]} \\
\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}\{-d\} \arrow[u] \arrow[r, "\Phi^d_{\mathfrak{X}^{\mathrm{sm}}}"] & (\zeta_p-1)^d\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}, \arrow[u]
\end{tikzcd}$$

commute. Firstly, we note the vertical maps are isomorphisms onto $(\zeta_p-1)^d\widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d)$ and $(\zeta_p-1)^d\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]}$ by the discussion above and the construction, respectively. And the horizontal map is an isomorphism by Lemma 3.3.1.

Thus, using that $\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{(\zeta_p-1)^\infty-\text{torsion}}$ is torsion-free, we see that we can uniquely "divide" the map $\Phi^d_{\mathfrak{X}^{\mathrm{sm}}}$ by $(\zeta_p-1)^d$ to define $(\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}})^{-1}$ that has to be an isomorphism. Namely, we define

$$(\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}})^{-1}(x) := \frac{\Phi^d_{\mathfrak{X}^{\mathrm{sm}}}((\zeta_p-1)^d x)}{(\zeta_p-1)^d}.$$

We denote the inverse of this map by

$$\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}}\colon \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right)[(\zeta_p-1)^\infty]} \to \widehat{\Omega}^d_{\mathfrak{X}^{\mathrm{sm}}}(-d).$$

Finally, we are ready to define the map $\Psi^d_{\mathfrak{X}^{\mathrm{sm}}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \omega_{\mathfrak{X}^{\mathrm{sm}}}(-d)$.

Definition 3.3.2. For an admissible separated formal $\mathcal{O}_C$-scheme $\mathfrak{X}$, we define the map

$$\Psi^d_{\mathfrak{X}^{\mathrm{sm}}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \omega_{\mathfrak{X}^{\mathrm{sm}}}(-d)$$

as the composition

$$\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]} \xrightarrow{\Psi'^d_{\mathfrak{X}^{\mathrm{sm}}}} \Omega^d_{\mathfrak{X}^{\mathrm{sm}}}(-d) \xrightarrow{r_{\mathfrak{X}^{\mathrm{sm}}}(-d)} \omega_{\mathfrak{X}^{\mathrm{sm}}}(-d) ,$$

where $r_{\mathfrak{X}^{\mathrm{sm}}}$ is the map from Theorem 2.1.6.

3.4. **Faltings' Trace Map on Generic Fiber.** In this section, we study generic fiber of the morphism

$$\Phi^d_{\mathfrak{X}}\colon \widehat{\Omega}^d_{\mathfrak{X}}\{-d\} \to \frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}.$$

Before we start proving the claim above, we recall some notation. For any rigid space X, we have a morphism of ringed sites $\lambda\colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (X_{\text{ét}}, \mathcal{O}^+_X)$, a morphism $p\colon (X_{\text{ét}}, \mathcal{O}^+_X) \to (X_{\text{an}}, \mathcal{O}^+_X)$. We denote the composition by $\mu\colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (X_{\text{an}}, \mathcal{O}^+_X)$.

We also recall that Scholze constructs an isomorphism

$$\mathrm{Sch}'^1_X\colon \Omega^1_{X_{\text{ét}}} \to \mathrm{R}^1\lambda_*\widehat{\mathcal{O}}_X(1)$$

in [Sch13a, Corollary 6.19, Remark 6.20] and [Sch13b, Lemma 3.24]. In particular, the sheaf $\mathrm{R}^1\lambda_*\widehat{\mathcal{O}}_X(1)$ is coherent, so vanishing of étale cohomology of coherent sheaves on affinoids implies that $\mathbf{R}\pi_*\mathrm{R}^1\lambda_*\widehat{\mathcal{O}}_X(1) = \pi_*\mathrm{R}^1\lambda_*\widehat{\mathcal{O}}_X(1)$. Thus, if we apply π_* to Sch'^1_X and twist it by -1, we get the map

$$\mathrm{Sch}^1_X\colon \Omega^1_X(-1) \to \mathrm{R}^1\mu_*\widehat{\mathcal{O}}_X.$$

Faithfully flat descent for coherent sheaves implies that $\pi^*(\mathrm{Sch}^1_X)(1) = \mathrm{Sch}'^1_X$. We define the map

$$\mathrm{Sch}^n_X\colon \Omega^n_X(-n) \to \mathrm{R}^n\mu_*\widehat{\mathcal{O}}_X$$

as the composition

$$\Omega^n_X(-n) \xrightarrow{\Lambda^n(\mathrm{Sch}^1)} \Lambda^n(\mathrm{R}^1\mu_*\widehat{\mathcal{O}}_X(1)) \xrightarrow{\cup^n} \mathrm{R}^n\mu_*\widehat{\mathcal{O}}_X\ .$$

We note that the cup product is alternating as 2 is invertible in C, so the cup product does factor through the wedge product. Now [Sch13b, Proposition 3.23] shows that the map

$$\bigwedge^n \left(\mathrm{R}^1\mu_*\widehat{\mathcal{O}}_X(1)\right) \xrightarrow{\cup^n} \mathrm{R}^n\mu_*\widehat{\mathcal{O}}_X$$

is an isomorphism, so the map Sch^n is an isomorphism for any n. In particular, all sheaves $\mathrm{R}^n\mu_*\widehat{\mathcal{O}}_X$ are coherent.

The main goal of this section is to show that $(\Phi^n_{\mathfrak{X}})_C$ can be canonically identified, up to a factor of $(\zeta_p-1)^n$, with the isomorphism $\mathrm{Sch}^n_X\colon \Omega^d_X(-n) \to \mathrm{R}^n\mu_*\widehat{\mathcal{O}}_X$. This will later allow us to glue Scholze's map on the generic fiber and Ψ^d on the smooth locus to define the (integral) Faltings' map $\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}\colon \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C}\right) \to \omega_{\mathfrak{X}}(-d)$ on an admissible formal $\mathcal{O}_C$-scheme with reduced special fiber and smooth generic fiber of pure dimension d.

We start by identifying the sides of both morphisms the source and target of the map:

$$(\Phi^n_{\mathfrak{X}})_C:\ \left(\widehat{\Omega}^n_{\mathfrak{X}}\{-n\}\right)_C \to \left(\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C.$$

We observe that $\left(\frac{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^n\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C$ is naturally isomorphic to $\mathrm{R}^n\mu_*\left(\widehat{\mathcal{O}}_X\right)$ by Lemma C.10. This identifies the target of the map $(\Phi^n_{\mathfrak{X}})_C$ with $\mathrm{R}^n\mu_*\left(\widehat{\mathcal{O}}_X\right)$.

Now we deal with the source of $(\Phi^n_{\mathfrak{X}})_C$. We consider the short exact sequence

$$0 \to \widehat{\Omega}^n_{\mathfrak{X}}\{-n\} \to \widehat{\Omega}^n_{\mathfrak{X}}(-n) \to \mathcal{Q} \to 0,$$

where the image of $\widehat{\Omega}^n_{\mathfrak{X}}\{-n\}$ is equal to $(\zeta_p-1)^d\widehat{\Omega}^n_{\mathfrak{X}}(-n)$. In partical, the cokernel $\mathcal{Q}$ is annihilated by $(\zeta_p-1)^d$, so we conclude that the morphism

$$\mathrm{Id}_{\widehat{\Omega}^n_{\mathfrak{X}}} \otimes_{\mathcal{O}_C} (\mathrm{dlog}^\vee_{\mathcal{O}_C})^{\otimes n}\colon \left(\widehat{\Omega}^n_{\mathfrak{X}}\{-n\}\right)_C \to \left(\widehat{\Omega}^n_{\mathfrak{X}}(-n)\right)_C$$

is an isomorphism onto $(\zeta_p - 1)^d \left(\widehat{\Omega}^n_{\mathfrak{X}}(-n)\right)_C$ by Lemma C.4. Finally, we identify $\left(\widehat{\Omega}^n_{\mathfrak{X}}(-n)\right)_C$ with $\Omega^n_X(-n)$ and similarly $\left(\widehat{\Omega}^n_{\mathfrak{X}}\{-n\}\right)_C$ with $\Omega^n_X\{-n\}$[18] to identify the source of $(\Phi^n_{\mathfrak{X}})_C$ with $\Omega^n_X\{-n\}$.

Theorem 3.4.1. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_C$-scheme with smooth adic generic fiber $X = \mathfrak{X}_C$. Then the diagram

$$\begin{tikzcd} \Omega^n_X(-n) \arrow[r, "\mathrm{Sch}^n_X"] & \mathrm{R}^n\mu_*\left(\widehat{\mathcal{O}}_X\right) \\ \Omega^n_X\{-n\} \arrow[u, "\mathrm{Id}_{\Omega^n_X}\otimes_{\mathcal{O}_C}(\mathrm{dlog}^\vee_{\mathcal{O}_C})^{\otimes n}"] \arrow[ur, "(\Phi^n_{\mathfrak{X}})_C"'] & \end{tikzcd} \tag{3.4}$$

is commutative for any $n \geq 1$. In particular, the map $(\Phi^n_{\mathfrak{X}})_C$ is an isomorphism for $n \geq 1$.

Before starting the proof of Theorem 3.4.1, we discuss a lemma that will be used in the proof.

Lemma 3.4.2. Let X be a smooth rigid C-space, and let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible formal $\mathcal{O}_C$-models of X. Then the morphisms

$$(\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p})_C \to (\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p})_C \text{ and } (\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C})_C \to (\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C})_C$$

are isomorphisms.

Proof. We recall that Lemma C.9 implies that the map $\mathcal{K}_C \to (\mathbf{R}\pi_*\mathcal{K})_C$ is an isomorphism for any $\mathcal{K} \in \mathbf{D}_{coh}(\mathfrak{X}')$. So it suffices to show that $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p}$ and $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C}$ have coherent cohomology sheaves.

The case of $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C}$ is done in [GR03, Proposition 7.2.10(i)], it shows that $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C} \in \mathbf{D}^-_{coh}(\mathfrak{X}')$. As for $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p}$, we recall that the proof of Lemma 3.2.1 showed that we have a distinguished triangle

$$\mathcal{O}_{\mathfrak{X}}\{1\}[1] \to \widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p} \to \widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C} .$$

So we conclude that $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p} \in \mathbf{D}^-_{coh}(\mathfrak{X}')$ as it holds for both $\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C}$ and $\mathcal{O}_{\mathfrak{X}}\{1\}[1]$. $\square$

The rest of the section is devoted to the proof of Theorem 3.4.1. We firstly show that this diagram commutes under the assumption that $\mathfrak{X}$ has the reduced special fiber, and then reduce to this case by means of the Reduced Fiber Theorem. For reader's convenience, we firstly discuss the proof under the extra assumption of reduced special fiber, and then deal with the general case.

Proof of Theorem 3.4.1 for a model $\mathfrak{X}$ with reduced special fiber: Step 1. Reduction to the case $n = 1$: We recall that both morphisms $\Phi^n_{\mathfrak{X}}$ and Sch^n are defined as the composition of the wedge power of $\Phi^1_{\mathfrak{X}}$ and Sch^1 with the cup product map. Since cup product commutes with derived base change, it is sufficient to show the claim for $n = 1$.

Step 2. Reduce to the case of a smooth model $\mathfrak{X}$ with "good coordinates": Firstly, we can assume that X is connected by replacing it with its connected component and corresponding connected component of $\mathfrak{X}$ (that is well-defined by Lemma B.1).

Now both $\Omega^1_X\{-1\}$ and $\mathrm{R}^1\mu_*\left(\widehat{\mathcal{O}}_X\right)$ are vector bundles, so the locus of fibers where the maps $\mathrm{Sch}^n_X \circ j$ and $(\Phi^n_{\mathfrak{X}})_C$ are equal on fibers is a Zariski-closed subset. SO it is enough to show that this Zariski-closed subset is equal to X because X is reduced.

[18] We note that the natural morphism $C\{-n\} \to C(-n)$ is an isomorphism, but it is not the identity morphism. We want to keep track of this difference to make arguments more canonical.

We use [Con99, Lemma 2.1.4] to see that it is sufficient to check the equality on some non-empty open subset $U \subset X$. In particular, it is sufficient to check equality on the open $\mathfrak{X}_C^{\mathrm{sm}}$ that is non-empty as $\mathfrak{X}$ has reduced special fiber. Therefore, we can assume that $\mathfrak{X}$ is smooth. Moreover, we can localize $\mathfrak{X}$ even further to assume that it is affine and has a finite étale morphism to $\widehat{\mathbf{G}}^d_{\mathcal{O}_C}$. The latter is achieved by [Bha18, Lemma 4.9].

Step 3. Proof in the case a smooth model $\mathfrak{X}$ with "good coordinates": So now we can assume that the formal $\mathcal{O}_C$-scheme $\mathfrak{X} = \operatorname{Spf} A$ is affine and comes with a finite étale morphism $\mathfrak{f}\colon \operatorname{Spf} A \to \widehat{\mathbf{G}}^d_{\mathcal{O}_C}$. Now we note that the map $\Phi^1_{\mathfrak{X}}$ is étale local on $\mathfrak{X}$ by [BMS18, Corollary 8.13] and the discussion after the proof of [BMS18, Lemma 8.16]. Therefore, the map $(\Phi^1_{\mathfrak{X}})_C$ is local on the étale topology of $\mathfrak{X}$[19]. The morphism Sch^1 is étale local on X by it construction as it comes from a morphism of coherent sheaves in the étale topology. Thus, we reduce the question to the case $\mathfrak{X} = \widehat{\mathbf{G}}^d_m$. In that case, we argue by an explicit computation after "trivializing" all sheaves involved in the diagram.

So, for $\mathfrak{X} = \widehat{\mathbf{G}}^d_m$, we choose coordinates $\mathfrak{X} = \operatorname{Spf} \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$ and denote

$$R \coloneqq \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle.$$

We consider the covering $\operatorname{Spf} R_\infty \to \operatorname{Spf} R$ with

$$R_\infty \coloneqq \mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle.$$

This map is a pro-étale covering on generic fibers and a $\Delta \coloneqq \mathbf{Z}_p(1)^{\oplus d}$-torsor.

The computation with the Koszul complexes in [Bha18, Lemma 4.6] shows that the natural map $\mathrm{H}^1_{\mathrm{cont}}(\Delta, C) \otimes_C R[1/p] \to \mathrm{H}^1_{\mathrm{cont}}(\Delta, R_\infty[1/p])$ is an isomorphism. The group $\mathrm{H}^1_{\mathrm{cont}}(\Delta, C) \cong C^d$, so $\mathrm{H}^1_{\mathrm{cont}}(\Delta, R_\infty[1/p]) \cong R[1/p]^d$. In particular, $\mathrm{H}^1_{\mathrm{cont}}(\Delta, R_\infty[1/p])$ is a finite $R[1/p]$-module.

Now the almost purity theorem implies[20] that the natural morphism

$$\mathrm{H}^1_{\mathrm{cont}}\widetilde{(\Delta, R_\infty}[1/p]) \to \mathrm{R}^1\mu_*\left(\widehat{\mathcal{O}}_X\right)$$

is an isomorphism. The left side makes sense as $\mathrm{H}^1_{\mathrm{cont}}(\Delta, R_\infty[1/p])$ was shown to be finite.

The diagram

$$\begin{tikzcd} \Omega^1_X(-1) \arrow[r, "\mathrm{Sch}^1_X"] & \mathrm{R}^1\mu_*\left(\widehat{\mathcal{O}}_X\right) \\ \Omega^1_X\{-1\} \arrow[u, "\mathrm{Id}\otimes \mathrm{dlog}^\vee_{\mathcal{O}_C}"] \arrow[ur, "(\Phi^1_{\mathfrak{X}})_C"'] & \end{tikzcd} \tag{3.5}$$

commutes if and only if the following diagram commutes:

$$\begin{tikzcd} \Omega^1_X \arrow[r, "(\mathrm{BMS}^1_{\mathfrak{X}})_C"] \arrow[dr, "\mathrm{Sch}'^1_X"'] & \mathrm{H}^1_{\mathrm{cont}}(\Delta, \mathcal{O}_C\{1\}) \otimes_{\mathcal{O}_C} R_\infty[1/p] \\ & \mathrm{H}^1_{\mathrm{cont}}(\Delta, \mathcal{O}_C(1)) \otimes_{\mathcal{O}_C} R_\infty[1/p] \arrow[u, "\mathrm{H}^1_{\mathrm{cont}}(\mathrm{dlog}_{\mathcal{O}_C})\otimes \mathrm{Id}"'] \end{tikzcd} \tag{3.6}$$

[19] We do not know at the moment if this map is étale local on $X = \mathfrak{X}_C$.

[20] Look at [Bha19, Theorem 3.29] for more details

All $R[1/p]$-modules involved in diagram (3.6) are isomorphic to $R[1/p]^d$ with bases on either module coming from the choice of coordinates (and the choice of roots of unity). It is enough to check that basis elements match, which by functoriality reduces to the case $d = 1$.

If $\mathfrak{X} = \widehat{\mathbf{G}}_m = \operatorname{Spf} \mathcal{O}_C\langle T^{\pm 1}\rangle = \operatorname{Spf} R$, we trivialize $\widehat{\Omega}^1_X \simeq R[1/p] \cdot \frac{\mathrm{d}T}{T}$. It suffices to show that

$$\mathrm{BMS}^1_{\mathfrak{X}}\left(\frac{\mathrm{d}T}{T}\right) = \left(\mathrm{H}^1_{\mathrm{cont}}(\mathrm{dlog}_{\mathcal{O}_C}) \otimes \mathrm{Id} \circ \mathrm{Sch}'^1_X\right)\left(\frac{\mathrm{d}T}{T}\right).$$

[BMS18, Proposition 8.15] shows that

$$\mathrm{BMS}^1_{\mathfrak{X}}\left(\frac{\mathrm{d}T}{T}\right) = \mathrm{dlog} \otimes 1 \in \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Z}_p(1), \mathcal{O}_C\{1\}) \otimes_{\mathcal{O}_C} R_\infty[1/p]$$

under identification $\mathrm{H}^1_{\mathrm{cont}}(\Delta, \mathcal{O}_C\{1\}) \simeq \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Z}_p(1), \mathcal{O}_C\{1\})$. Therefore, it suffices to show that

$$\mathrm{Sch}'^1_X\left(\frac{\mathrm{d}T}{T}\right) = \mathrm{Id} \otimes 1 \in \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Z}_p(1), \mathbf{Z}_p(1)) \subset \mathrm{H}^1_{\mathrm{cont}}(\Delta, \mathcal{O}_C(1)) \otimes_{\mathcal{O}_C} R_\infty[1/p].$$

Now we recall that [Sch13b, Lemma 3.24] guarantees that there is a commutative diagram

$$\begin{tikzcd}
(R[1/p])^\times \arrow[r, "\delta"] \arrow[d, "\mathrm{dlog}"] & \mathrm{H}^1_{\mathrm{cont}}(\Delta, \mathbf{Z}_p(1))[1/p] \arrow[d] \\
\Omega^1_{\operatorname{Spa}(R[1/p], R)} \arrow[r, "\mathrm{Sch}'^1_X"] & \mathrm{H}^1(\Delta, R_\infty[1/p]),
\end{tikzcd}$$

where the top horizontal map is the p-localization of the connecting map corresponding to the extension of sheaves

$$0 \to \widehat{\mathbf{Z}}_p(1) \to \lim_{\times p} \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

on $X_{\text{proét}}$. It now suffices to show that $\delta(T) = \mathrm{Id} \otimes 1 \in \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Z}_p(1), \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We recall that the connecting map can be computed via Čech complexes due to [Ros21, Proposition F.2.1]. Therefore, one can compute $\delta\left(\frac{\mathrm{d}T}{T}\right)$ using the covering of $X = \operatorname{Spa}(R[1/p], R)$ by an affinoid perfectoid $\widetilde{X} = \operatorname{Spa}(R_\infty[1/p], R_\infty)$. In this case, it is straighforward to see that

$$\delta(T) = \mathrm{Id} \otimes 1 \in \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Z}_p(1), \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

□

Proof of Theorem 3.4.1 for a general $\mathfrak{X}$. First of all, we can assume that $n = 1$ similarly to the proof in the case of a model with reduced special fiber.

Now we reduce the case of a general admissible formal model to the case of a model with the reduced fiber for $n = 1$. We recall that the Reduced Fiber Theorem [BLR95, Theorem 2.1] provides a finite rig-isomorphism $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has reduced special fiber. We know that diagram (3.4) commutes for $\mathfrak{X}'$, so the only thing we need to show is that $(\Phi^1_{\mathfrak{X}'})_C$ is equal to $(\Phi^1_{\mathfrak{X}})_C$. We recall the definition of $\Phi^1_{\mathfrak{X}}$. Namely, We considered the sequence of morphisms

$$\widehat{\Omega}^1_{\mathfrak{X}}[0] \xrightarrow{\alpha_{\mathfrak{X}}} \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C} \xleftarrow{\beta_{\mathfrak{X}}} \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p} \xrightarrow{\gamma_{\mathfrak{X}}} \mathbf{R}\nu_* \widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p} \simeq \mathbf{R}\nu_* \widehat{\mathcal{O}}^+_X\{1\}[1]$$

and showed that the first two arrows are isomorphisms on $\mathcal{H}^0$. So we defined $\Phi^1_{\mathfrak{X}}$ as the composition $\mathcal{H}^0(\gamma_{\mathfrak{X}}) \circ \mathcal{H}^0(\beta_{\mathfrak{X}})^{-1} \circ \mathcal{H}^0(\alpha_{\mathfrak{X}})$.

We denote by $\delta_{\mathfrak{X}} \colon (\mathbf{R}\nu_* \widehat{\mathcal{O}}^+_X\{1\}[1])_C \xrightarrow{\sim} \mathbf{R}\mu_* \widehat{\mathcal{O}}_X\{1\}[1]$ an isomorphism that comes from Lemma C.10. As taking the adic generic fiber of an $\mathcal{O}_{\mathfrak{X}}$-module is an exact functor we see that

$$(\Phi^1_{\mathfrak{X}})_C = \mathcal{H}^0(\delta_{\mathfrak{X}}) \circ \mathcal{H}^0(\gamma_{\mathfrak{X},C}) \circ \mathcal{H}^0(\beta_{\mathfrak{X},C})^{-1} \circ \mathcal{H}^0(\alpha_{\mathfrak{X},C})^{21}\ .$$

Now we consider the commutative diagram

$$\begin{tikzcd}
\Omega^1_{X/C} \arrow[r, "\mathcal{H}^0(\alpha_{\mathfrak{X}',C})"] \arrow[dr, "\mathcal{H}^0(\mathbf{R}\pi_*\alpha_{\mathfrak{X}',C})"'] & \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C})_C & \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p})_C \arrow[l, "\mathcal{H}^0(\beta_{\mathfrak{X}',C})"'] \arrow[r, "\mathcal{H}^0(\gamma_{\mathfrak{X}',C})"] & \mathcal{H}^0(\mathbf{R}\nu'_*\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p})_C \arrow[dr, "\mathcal{H}^0(\delta_{\mathfrak{X}'})"] & \\
& \mathcal{H}^0(\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C})_C \arrow[u] & \mathcal{H}^0(\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p})_C \arrow[l, "\mathcal{H}^0(\mathbf{R}\pi_*\beta_{\mathfrak{X}',C})"'] \arrow[u] \arrow[r, "\mathcal{H}^0(\mathbf{R}\pi_*\gamma_{\mathfrak{X}'})"] & \mathcal{H}^0(\mathbf{R}\nu_*\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p})_C \arrow[u] \arrow[r, "\mathcal{H}^0(\delta_{\mathfrak{X}})"] & \mathcal{H}^0(\mathbf{R}\mu_*\widehat{\mathcal{O}}_X\{1\}[1
\end{tikzcd} \tag{3.7}$$

where the vertical arrows are the natural morphisms induced by $\mathcal{K}_C \to (\mathbf{R}\pi_*\mathcal{K})_C$ defined for any $\mathcal{K} \in \mathbf{D}(\mathfrak{X}')$. The left triangle commutes by definition, the two middle squares commute by functoriality of the map $(\mathbf{R}\pi_*\mathcal{K})_C \to \mathcal{K}_C$, and the right triangle commutes by Lemma C.12. Lemma 3.4.2 implies that all vertical arrows are isomorphisms. Therefore, we see that $\mathcal{H}^0(\mathbf{R}\pi_*\beta_{\mathfrak{X}',C})$ is an isomorphism as the same holds for $\mathcal{H}^0(\beta_{\mathfrak{X}',C})$. We use this and commutativity of the diagram (3.7) to conclude that

$$(\Phi^1_{\mathfrak{X}'})_C = \mathcal{H}^0(\delta_{\mathfrak{X}}) \circ \mathcal{H}^0(\mathbf{R}\pi_*\gamma_{\mathfrak{X}',C}) \circ \mathcal{H}^0(\mathbf{R}\pi_*\beta_{\mathfrak{X}',C})^{-1} \circ \mathcal{H}^0(\mathbf{R}\pi_*\alpha_{\mathfrak{X}',C})$$

Now we consider another commutative square that relates $(\pi_*\Phi_{\mathfrak{X}'})_C$ to $(\Phi_{\mathfrak{X}})_C$:

$$\begin{tikzcd}
\Omega^1_{X} \arrow[r, "\mathcal{H}^0(\alpha_{\mathfrak{X},C})"] \arrow[dr, "\mathcal{H}^0(\mathbf{R}\pi_*\alpha_{\mathfrak{X}',C})"'] & \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C})_C \arrow[d] & \mathcal{H}^0(\widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathbf{Z}_p})_C \arrow[l, "\mathcal{H}^0(\beta_{\mathfrak{X},C})"'] \arrow[r, "\mathcal{H}^0(\gamma_{\mathfrak{X},C})"] \arrow[d] & \mathcal{H}^0(\mathbf{R}\nu_*\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p})_C \arrow[d, "\mathrm{Id}"] \arrow[dr, "\mathcal{H}^0(\delta_{\mathfrak{X}})"] & \\
& \mathcal{H}^0(\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C})_C & \mathcal{H}^0(\mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathbf{Z}_p})_C \arrow[l, "\mathcal{H}^0(\mathbf{R}\pi_*\beta_{\mathfrak{X}',C})"'] \arrow[r, "\mathcal{H}^0(\mathbf{R}\pi_*\gamma_{\mathfrak{X}',C})"] & \mathcal{H}^0(\mathbf{R}\nu_*\widehat{L}_{\widehat{\mathcal{O}}^+_X/\mathbf{Z}_p})_C \arrow[r, "\mathcal{H}^0(\delta_{\mathfrak{X}})"] & \mathcal{H}^0(\mathbf{R}\mu_*\widehat{\mathcal{O}}_X\{1\}[1]
\end{tikzcd}$$

where the left and the middle verticle arrows are defined by functoriality of the cotangent complex. We briefly explain why this diagram is commutative: the the middle squares commute by functoriality of the cotangent complex, and left triangle commutes because it comes as the generic fiber of the commutative diagram

$$\begin{tikzcd}
\widehat{\Omega}^1_{\mathfrak{X}}[0] \arrow[r] \arrow[d] & \widehat{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_C} \arrow[d] \\
\mathbf{R}\pi_*\widehat{\Omega}^1_{\mathfrak{X}'}[0] \arrow[r] & \mathbf{R}\pi_*\widehat{L}_{\mathcal{O}_{\mathfrak{X}'}/\mathcal{O}_C},
\end{tikzcd}$$

where the left vertical map is the identity morphism on generic fiber by Lemma C.9. So we see that

$$\begin{aligned}
(\Phi^1_{\mathfrak{X}'})_C &= \mathcal{H}^0(\delta_{\mathfrak{X}}) \circ \mathcal{H}^0(\mathbf{R}\pi_*\gamma_{\mathfrak{X}',C}) \circ \mathcal{H}^0(\mathbf{R}\pi_*\beta_{\mathfrak{X}',C})^{-1} \circ \mathcal{H}^0(\mathbf{R}\pi_*\alpha_{\mathfrak{X}',C}) \\
&= \mathcal{H}^0(\delta_{\mathfrak{X}}) \circ \mathcal{H}^0(\gamma_{\mathfrak{X},C}) \circ \mathcal{H}^0(\beta_{\mathfrak{X},C})^{-1} \circ \mathcal{H}^0(\alpha_{\mathfrak{X},C}) \\
&= (\Phi^1_{\mathfrak{X}})_C\ .
\end{aligned}$$

□

[21] For the ease of notation, we suppress the canonical identification of $(\mathbf{R}\nu_*\widehat{L}_{\widehat{\mathcal{O}}^+_X})_C$ with $\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\{1\}[1]$.

Now we are ready to define the (integral) Faltings' map on generic fiber that we will denote by $\Phi^d_X\colon \left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X\right)_C \to \omega_X(-d)$. Namely, Theorem 3.4.1 implies that the map

$$\left(\Phi^d_{\mathfrak{X}}\right)_C\colon \left(\widehat{\Omega}^d_{\mathfrak{X}}\{-d\}\right)_C \to \left(\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C$$

is an isomorphism of sheaves of finite, locally free $\mathcal{O}_X$-modules because the same holds for the map Sch^d_X. Thus, we can "divide" it by $(\zeta_p-1)^d$ (as ζ_p-1 is invertible in C) similarly to what is done just before Definition 3.3.2 to define

$$(\Psi'^d_X)^{-1}\colon \left(\widehat{\Omega}^d_{\mathfrak{X}}(-d)\right)_C \to \left(\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C .$$

This must be an isomorphism as well since $(\zeta_p-1)^d$ is invertible in C. So this defines the map

$$\Psi'^d_X\colon \left(\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C \to \left(\widehat{\Omega}^d_{\mathfrak{X}}(-d)\right)_C = \Omega^d_X(-d) .$$

Now we define the map Ψ^d_X as follows:

Definition 3.4.3. For an admissible separated formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ with smooth generic fiber $X=\mathfrak{X}_C$ of pure dimension d, we define the map $\Psi^d_X\colon \left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X\right)_C \to \omega_X(-d)$ as the composition

$$\left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X\right)_C \to \left(\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C \xrightarrow{\Psi'^d_X} \Omega^d_X(-d) \xrightarrow{r_{\mathfrak{X}_C}(-d)} (\omega_{\mathfrak{X}})_C(-d) ,$$

where $r_{\mathfrak{X}_C}$ is the map defined in Definition 2.4.6.

Remark 3.4.4. Under the canonical identifications $\delta_{\mathfrak{X}}\colon \left(\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)[(\zeta_p-1)^\infty]}\right)_C \xrightarrow{\sim} \mathrm{R}^d\mu_*\left(\widehat{\mathcal{O}}_X\right)$ and $(\omega_{\mathfrak{X}})_C \xrightarrow{r_{\mathfrak{X}_C}} \Omega^d_X$, the map Ψ^d_X becomes equal to Sch^d_X.

3.5. **Construction of the Faltings' Trace Map.** In this section, We define the Faltings' trace map on any admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fiber of pure dimension d. The main idea is to define it firstly on models with reduced special fiber using Theorem 2.1.14 and the constructions of $\Psi^d_{\mathfrak{X}^{\mathrm{sm}}}$ and Ψ^d_X that we constructed above. Then we consider the canonical finite rig-isomorphism $\mathfrak{X}'\to\mathfrak{X}$ such that the special fiber of $\mathfrak{X}'$ is reduced and roughly define the trace map on $\mathfrak{X}$ as the composition of the (derived) pushforward of the trace on $\mathfrak{X}'$ and the trace coming from Grothendieck duality.

Theorem 3.5.1. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme with smooth adic generic fiber $X=\mathfrak{X}_C$ and reduced special fiber $\overline{\mathfrak{X}}$. Suppose that X is of pure dimension d. Then there is a unique morphism

$$\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}\colon \mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X \to \omega_{\mathfrak{X}}(-d)$$

such that $\mathrm{Tr}^{d,+}_F|_{\mathfrak{X}^{\mathrm{sm}}}$ coincides with $\Psi^d_{\mathfrak{X}^{\mathrm{sm}}}$ and $(\mathrm{Tr}^{d,+}_F)_C$ coincides with Ψ^d_X.

Proof. We note that Theorem 2.1.14 guarantees that the natural map $\omega_{\mathfrak{X}} \to j_{\mathfrak{U},*}(\omega_{\mathfrak{X}}|_{\mathfrak{X}}^{\mathrm{sm}}) \cap \omega_{\mathfrak{X}_C}$ is an isomorphism. Therefore, it shows that $\mathrm{Tr}^{d,+}_{F,\mathfrak{X}}$ is uniquely defined by its restriction onto $\mathfrak{X}^{\mathrm{sm}}$ and its pullback on X.

Morever, Theorem 2.1.14 implies that it suffices to define separately two morphisms

$$(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X)_C \to (\omega_{\mathfrak{X}})_C(-d)$$

and

$$(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X)|_{\mathfrak{X}^{\mathrm{sm}}} = \mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{sm}}_C} \to \omega_{\mathfrak{X}^{\mathrm{sm}}}(-d)$$

that coincide on $(\mathfrak{X}^{\mathrm{sm}})_C$. in other words, we need to check that Ψ^d_X and $(\Psi^d_{\mathfrak{X}^{\mathrm{sm}}})_C$ are the same. This follows from their constructions and Theorem 3.4.1 (and Lemma 2.4.1 to guarantee that the identifications of differential forms and dualizing modules on $\mathfrak{X}^{\mathrm{sm}}$ and X agree on $(\mathfrak{X}^{\mathrm{sm}})_C$). □

Definition 3.5.2. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme such with smooth adic generic fiber of pure dimension d. We define the *integral Faltings' map* $\mathrm{Tr}^{+,d}_{F,\mathfrak{X}}\colon \mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X \to \omega_{\mathfrak{X}}(-d)$ as the composition

$$\mathrm{R}^d\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X \xrightarrow{e} \pi_*(\mathrm{R}^d\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X) \xrightarrow{\pi_*(\mathrm{Tr}^{+,d}_{F,\mathfrak{X}'})} \pi_*(\omega_{\mathfrak{X}'})(-d) \xrightarrow{\mathrm{Tr}_\pi(-d)} \omega_{\mathfrak{X}}(-d)$$

where e is the canonical edge map, $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ is a (unique) finite, rig-isomorphism with $\mathfrak{X}'$ having reduced special fiber coming from [BLR95, Theorem 2.1], $\mathrm{Tr}^{+,d}_{F,\mathfrak{X}'}$ is the Faltings' map constructed in Theorem 3.5.1, and Tr_π is the trace map constructed in Lemma 2.2.1.

Definition 3.5.3. Let $\mathfrak{X}$ be as in Definition 3.5.2. We define the *integral Faltings' trace*

$$\mathrm{Tr}^+_{F,\mathfrak{X}}\colon \left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)^a \to \omega^a_{\mathfrak{X}}(-d)[-d]$$

as the composition

$$\left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)^a \to \left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X\right)^a[-d] \xrightarrow{\mathrm{Tr}^{+,d}_F[-d]} \omega^a_{\mathfrak{X}}(-d)[-d],$$

where the first map is the projection of a complex on its top cohomology sheaf (we recall that $\left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})$ by [Zav21a, Theorem 6.13.6]).

Remark 3.5.4. Let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a rig-isomorphism of admissible separated formal $\mathcal{O}_C$-models with smooth adic generic fiber $X = \mathfrak{X}_C$ of pure dimension d. As π is finite, the functor π_* is almost exact on almost quasi-coherent modules, so the complex

$$\mathbf{R}\pi_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X\right)^a \simeq \pi_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a.$$

Therefore, the map $\mathrm{Tr}^+_{F,\mathfrak{X}'}\colon \left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X\right)^a \to \omega^a_{\mathfrak{X}'}(-d)[-d]$ induces the morphism

$$\pi_*(\mathrm{Tr}^+_{F,\mathfrak{X}'})\colon \pi_*(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X)^a \to \pi_*(\omega^a_{\mathfrak{X}'})(-d)[-d]$$

as $\pi_*(\omega_{\mathfrak{X}'})(-d)[-d]$ is concentrated in degree $-d$.

Lemma 3.5.5. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fiber of dimension d, and let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a (unique) finite, rig-isomorphism such that $\mathfrak{X}'$ has reduced special fiber. Then the map

$$\mathrm{Tr}^+_{F,\mathfrak{X}}\colon \left(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X\right) \to \omega^a_{\mathfrak{X}}(-d)[-d]$$

coincides with the following composition

$$\left(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X)\right)^a \xrightarrow{\sim} \pi_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X\right)^a \xrightarrow{\pi_*\left(\mathrm{Tr}^+_{F,\mathfrak{X}'}\right)} \pi_*(\omega_{\mathfrak{X}'})(-d)[-d] \xrightarrow{\mathrm{Tr}_\pi} \omega_{\mathfrak{X}}(-d)[-d].$$

Proof. The complex $\left(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X)\right)^a \in \mathbf{D}^{[0,d]_{acoh}}(\mathfrak{X})$ by [Zav21a, Theorem 6.13.6], so it suffices to show that $\mathcal{H}^d(-)$ of this composition coincides with $(\mathrm{Tr}^{+,d}_{F,\mathfrak{X}})^a$. This follows from the Definition 3.5.2 and Remark 3.5.4. ☐

Corollary 3.5.6. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme such that the adic generic fiber is smooth of pure dimension d. The the following diagram

$$\begin{tikzcd} (\mathbf{R}\nu_*\widehat{\mathcal{O}}^{+,a}_X)_C \arrow[r, "\delta_{\mathfrak{X}}"] \arrow[d, "(\mathrm{Tr}^+_{F,\mathfrak{X}})_C"] & \mathbf{R}\mu_*\widehat{\mathcal{O}}_X \arrow[r] & \mathrm{R}^d\mu_*\widehat{\mathcal{O}}_X[-d] \arrow[d, "(\mathrm{Sch}^d_X)^{-1}[-d]"] \\ (\omega^a_{\mathfrak{X}})_C(-d)[-d] \arrow[rr, "r_{\mathfrak{X}_C}(-d)[-d]"] & & \Omega^d_X(-d)[-d] \end{tikzcd}$$

is commutative.

Proof. If $\mathfrak{X}$ has the reduced special fiber this follows from the very definition of $\mathrm{Tr}^+_{F,\mathfrak{X}}$ Remark 3.4.4. If the special fiber of $\mathfrak{X}$ is not reduced, then one chooses a finite, rig-isomorphism $\mathfrak{X}' \to \mathfrak{X}$ and runs a standard argument to reduce the claim for $\mathfrak{X}$ to the claim for $\mathfrak{X}'$. It requires to show that certain diagrams commute, we leave details to the reader, and only mention that one needs to use Lemma 3.5.5, Lemma C.12, and Definition 2.4.6 to check the desired commutativities. ☐

Note that we could have defined the integral Faltings' trace as a map

$$\widetilde{\mathrm{Tr}}^+_F\colon \left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)^a \to \omega^{\bullet,a}_{\mathfrak{X}}(-d)[-2d]$$

by composing it with the natural morphism $\omega_{\mathfrak{X}}(-d)[-d] \to \omega^\bullet_{\mathfrak{X}}(-d)[-2d]$. However, as we do not have functorial enough construction of the dualizing complex, we prefer to stick to the version with dualizing modules. However, we do have good theory of Grothendieck duality mod-p, so we prefer to define the mod-p version of the Faltings' Trace as a map to the whole dualizing complex $\omega^\bullet_{\mathfrak{X}_0}(-d)[-2d]$.

Definition 3.5.7. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fiber X of pure dimension d. We define the *Faltings' trace*

$$\mathrm{Tr}_{F,\mathfrak{X}}\colon \mathbf{R}\left(\nu_*\mathcal{O}^+_X/p\right)^a \to \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$$

as the composition

$$\mathbf{R}\nu_*(\mathcal{O}^+_X/p)^a \xrightarrow{p^{-1}} \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \xrightarrow{\mathrm{Tr}_{F,\mathfrak{X}}\otimes^L_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}_0}} \omega^a_{\mathfrak{X}}(-d)[-d]\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}_0} \xrightarrow{\mathrm{BC}_{\omega_{\mathfrak{X}}}(-d)[-2d]} \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]\,,$$

where p^{-1} is the inverse of the isomorphism $p\colon \mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \xrightarrow{\sim} \mathbf{R}\nu_*(\mathcal{O}^+_X/p)$, and $\mathrm{BC}_{\omega_{\mathfrak{X}}}$[22] is the composition of the base change map defined before Lemma 2.2.1, and the natural "inclusion" $\omega_{\mathfrak{X}_0}(-d)[-d] \to \omega^\bullet_{\mathfrak{X}_0}(-d)[-2d]$.

[22]This slightly abuses the notations, but we hope that it does not cause any confusion in what follows.

Lemma 3.5.8. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fiber of pure dimension d, and let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a (unique) finite, rig-isomorphism such that $\mathfrak{X}'$ has reduced special fiber. Then the map

$$\mathrm{Tr}_{F,\mathfrak{X}}\colon \mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a \to \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$$

coincides with the following composition

$$\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a \xrightarrow{\sim} \pi_*\mathbf{R}\nu_{\mathfrak{X}',*}(\mathcal{O}_X^+/p)^a \xrightarrow{\pi_*(\mathrm{Tr}_{F,\mathfrak{X}'})} \pi_*(\omega^{\bullet,a}_{\mathfrak{X}'_0})(-d)[-2d] \xrightarrow{\mathrm{Tr}_{\pi_0}} \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d],$$

where the last map is the trace map in coherent duality[23].

Proof. Lemma essentially follows from Lemma 3.5.5, but we spell out some details here. We note the the following diagram

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a \arrow[r, "\sim"] & \pi_*\mathbf{R}\nu_{\mathfrak{X}',*}(\mathcal{O}_X^+/p)^a \arrow[r, "\pi_*(\mathrm{Tr}_{F,\mathfrak{X}'})"] & \pi_*(\omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d]) \arrow[r, "\mathrm{Tr}_\pi"] & \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d] \\
\mathbf{R}\nu_{\mathfrak{X},*}(\widehat{\mathcal{O}}_X^+)^a \arrow[r, "\sim"] \arrow[u] & \pi_*\mathbf{R}\nu_{\mathfrak{X}',*}(\widehat{\mathcal{O}}_X^+)^a \arrow[r, "\pi_*(\mathrm{Tr}^+_{F,\mathfrak{X}'})"] \arrow[u] & \pi_*(\omega^a_{\mathfrak{X}'}(-d)[-d]) \arrow[r, "\mathrm{Tr}_{\pi_0}"] \arrow[u, "\mathrm{BC}_{\pi_*(\omega_{\mathfrak{X}'})}(-d)[-2d]"'] & \omega^a_{\mathfrak{X}}(-d)[-d] \arrow[u, "\mathrm{BC}_{\omega_{\mathfrak{X}}}(-d)[-2d]"']
\end{tikzcd}$$

is commutative and the bottom composition $\mathbf{R}\nu_{\mathfrak{X},*}(\widehat{\mathcal{O}}_X^+)^a \to \omega^a_{\mathfrak{X}}(-d)[-d]$ is equal to $\mathrm{Tr}^+_{F,\mathfrak{X}}$. Indeed, the left square commutes for trivial reasons, the middle square commutes by the definition of $\mathrm{Tr}_{\mathcal{F},\mathfrak{X}'}$, and the right square commutes by Lemma 2.2.1. The bottom composition is equal to $\mathrm{Tr}^+_{F,\mathfrak{X}}$ by Lemma 3.5.5.

Now we tensor the bottom line agains $\mathcal{O}_{\mathfrak{X}_0}$ to get the commutative diagram:

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a \arrow[r, "\sim"] & \pi_*\mathbf{R}\nu_{\mathfrak{X}',*}(\mathcal{O}_X^+/p)^a \arrow[r, "\pi_*(\mathrm{Tr}_{F,\mathfrak{X}'})"] & \pi_*(\omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d]) \arrow[r, "\mathrm{Tr}_\pi"] & \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d] \\
\mathbf{R}\nu_{\mathfrak{X},*}(\widehat{\mathcal{O}}_X^+)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \arrow[u, "p"] \arrow[rrr, "\mathrm{Tr}^+_{F,\mathfrak{X}} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0}"] & & & \omega^a_{\mathfrak{X}}(-d)[-d] \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0}, \arrow[u, "\mathrm{BC}_{\omega_{\mathfrak{X}}}(-d)[-2d]"']
\end{tikzcd} \tag{3.8}$$

where we use that $\omega^a_{\mathfrak{X}}(-d)[-d] \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0} \simeq \omega^a_{\mathfrak{X}}(-d)[-d] \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0}$ by $\mathcal{O}_C$-flatness of $\omega_{\mathfrak{X}}$ that comes from Lemma 2.1.9.

Since the composition of the top row in diagram (3.8) is equal to $\mathrm{Tr}_{F,\mathfrak{X}}$, we conclude that

$$\mathrm{Tr}_{F,\mathfrak{X}} \coloneqq \mathrm{BC}_{\omega_{\mathfrak{X}}}(-d)[-2d] \circ (\mathrm{Tr}^+_{F,\mathfrak{X}} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_0}) \circ p^{-1}$$

finishing the proof. $\square$

3.6. **Functoriality of the Faltings' Trace.** The main goal of this section is to show that, for a rig-finite, rig-étale morphism $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ of admissible formal $\mathcal{O}_C$-schemes with smooth generic fibers, Faltings' trace map for $\mathfrak{X}'$ and $\mathfrak{X}$ are related by means of the pro-étale and coherent trace maps for $\mathfrak{f}$.

Recall that for any $\mathfrak{f}$ as above there is the trace map the pro-étale trace map

$$\mathrm{Tr}^+_{\mathrm{Zar},\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'}\right)^a \to \left(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X\right)^a.$$

[23]We note that $\mathbf{R}\pi_*(\omega^\bullet_{\mathfrak{X}'_0}) \simeq \pi_*(\omega^\bullet_{\mathfrak{X}'_0})$ as π is finite and $\omega^\bullet_{\mathfrak{X}'_0} \in \mathbf{D}^+_{coh}(\mathfrak{X}'_0)$.

constructed in Appendix D.

Lemma 3.6.1. Let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible separated formal $\mathcal{O}_C$-schemes with smooth generic fibers of pure dimension d. Suppose that the adic generic fiber $f\colon X' \to X$ is finite étale. Then the following diagram

$$\begin{tikzcd}
\mathbf{R}\mathfrak{f}_* \left(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'}\right)^a \arrow[r, "\mathbf{R}\mathfrak{f}_*(\mathrm{Tr}^+_{F,\mathfrak{X}'})"] \arrow[d, "\mathrm{Tr}^+_{\mathrm{Zar},\mathfrak{f}}"] & \mathfrak{f}_*\omega^a_{\mathfrak{X}'}(-d)[-d] \arrow[d, "\mathrm{Tr}_{\mathfrak{f}}"] \\
\left(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_{X'}\right)^a \arrow[r, "\mathrm{Tr}^+_{F,\mathfrak{X}}"] & \omega^a_{\mathfrak{X}}(-d)[-d]
\end{tikzcd}$$

commutes.

Proof. Lemma 2.1.9 implies that the module $\omega_{\mathfrak{X}}(-d)$ is $\mathcal{O}_C$-flat, it suffices to check that the diagram commutes after taking the generic fiber. Then Lemma D.11 and Corollary 3.5.6 guarantee that it is enough to check that the diagram the diagram

$$\begin{tikzcd}
f_{\mathrm{an},*} \circ \mathrm{R}^d\mu_{X',*}\widehat{\mathcal{O}}_X \arrow[r, "f_{\mathrm{an},*}\left(\mathrm{Sch}^d_{X'}\right)^{-1}"] \arrow[d, "\mathcal{H}^d(\mathrm{Tr}_{\mathrm{an},f})"] & f_{\mathrm{an},*}(\Omega^d_{X'})(-d) \arrow[d, "\mathrm{Tr}_f(-d)"] \\
\mathrm{R}^d\mu_{X,*}\widehat{\mathcal{O}}_X \arrow[r, "(\mathrm{Sch}^d_X)^{-1}"] & \Omega^d_X(-d)
\end{tikzcd} \tag{3.9}$$

commutes.

This claim can be checked locally on X, so we use [Sch13a, Lemma 5.2] (or [Hub96, Corollary 1.6.10], [Zav21a, Corollary D.5], and the standard embedding $\mathbf{D}^n \subset \mathbf{T}^n$ as a rational subset $|X_i - 1| \leq |p|$) to find a covering of X by affinoids U_i such that each U_i admits a map $g_i\colon U_i \to \mathbf{T}^d$ that is a composition of finite étale maps and rational embeddings. So we may and do assume that $X = \mathrm{Spa}(A, A^+)$ is affinoid with a morphism $g\colon X \to \mathbf{T}^d$ as above. We note that X' is automatically affinoid because it is finite over X, we say $X' = \mathrm{Spa}(B, B^+)$.

Now we choose a standard affinoid perfectoid covering $\mathbf{T}^d_\infty \to \mathbf{T}^d$[24] that is a $\Delta \coloneqq \mathbf{Z}_p(1)^d$-torsor. Then the pullback $X_\infty \coloneqq X \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$ defines an affinoid perfectoid covering of X that is a Δ-torsor. Similarly, we define an affinoid perfectoid $X'_\infty \coloneqq X' \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$. Suppose that the associated adic spaces are given by $\widehat{X}_\infty = \mathrm{Spa}(A_\infty, A^+_\infty)$ and $\widehat{X'}_\infty = \mathrm{Spa}(B_\infty, B^+_\infty)$. Then the discussion just before Theorem D.14 implies that

$$f_{\mathrm{an},*} \circ \mathrm{R}^d\mu_{X',*}\widehat{\mathcal{O}}_{X'} \cong \widetilde{\mathrm{H}^d_{cont}(\Delta, B_\infty)} = \widetilde{\mathrm{H}^d_{cont}(\Delta, B)} = \widetilde{\mathrm{H}^d_{cont}(\Delta, C)} \otimes_C B$$

and

$$\mathrm{R}^d\mu_{X,*}\widehat{\mathcal{O}}_X \cong \widetilde{\mathrm{H}^d_{cont}(\Delta, A_\infty)} = \widetilde{\mathrm{H}^d_{cont}(\Delta, A)} = \widetilde{\mathrm{H}^d_{cont}(\Delta, C)} \otimes_C A.$$

Theorem D.14 implies that the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{H}^d_{\mathrm{cont}}(\Delta, C)} \otimes_C B \arrow[r] \arrow[d, "\mathrm{Id}\otimes\widetilde{\mathrm{Tr}}_{B/A}"] & f_{\mathrm{an},*}\mathrm{R}^d\mu_{X',*}\widehat{\mathcal{O}}_{X'} \arrow[d, "\mathcal{H}^d(\mathrm{Tr}_{an,f})"] \\
\widetilde{\mathrm{H}^d_{\mathrm{cont}}(\Delta, C)} \otimes_C A \arrow[r] & \mathrm{R}^d\mu_{X,*}\widehat{\mathcal{O}}_X
\end{tikzcd}$$

[24]Look at the discussion before Theorem D.14 for the definition of $\mathbf{T}^d_\infty$

commutes. Using the identification $\mathrm{H}^d_{cont}(\Delta, C) \simeq C$ coming from the Koszul complex, we conclude that $\mathcal{H}^d(\mathrm{Tr}_{\mathrm{an},f})$ can be identified with the trace map $\widetilde{\mathrm{Tr}_{B/A}}\colon \widetilde{B} \to \widetilde{A}$.

Now, all objects in the diagram (3.9) are coherent sheaves, so we can check commutativity on the global sections. We see that the map

$$g\colon X \to \mathbf{T}^d = \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$$

defines "coordinates" $z_i \coloneqq g^\sharp(T_i)$ on X, and

$$\mathrm{H}^0(X, \Omega^d_X) = A\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}\right)$$

and, similarly,

$$\mathrm{H}^0(X', \Omega^d_{X'}) = B\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}\right).$$

So we finally trivialize the sheaf $\mathcal{O}_X(-d)$ by choosing compatible system of p-power roots of unity ζ_{p^n}, and use Theorem 2.4.12 to argue that the question boils down to showing commutativity of the diagram:

$$\begin{tikzcd}
B \arrow[r, "(\mathrm{Sch}^d_{X'})^{-1}"] \arrow[d, "\mathrm{Tr}_{B/A}"] & B\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}\right) \arrow[d, "\mathrm{Tr}_{B/A}\cdot\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}\right)"] \\
A \arrow[r, "(\mathrm{Sch}^d_{X'})^{-1}"] & A\left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}\right).
\end{tikzcd}$$

It suffices to show that both $(\mathrm{Sch}^d_{X'})^{-1}$ and $(\mathrm{Sch}^d_X)^{-1}$ send 1 to $\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \dots \wedge \frac{\mathrm{d}z_d}{z_d}$. This computation is analogous to the one done at the end of the proof of Theorem 3.4.1 for $\mathfrak{X}'$ with reduced special fiber. ☐

Corollary 3.6.2. Let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible separated formal $\mathcal{O}_C$-schemes with smooth generic fibers of pure dimension d. Suppose that the generic fiber $f\colon X' \to X$ is finite étale. Then the diagram

$$\begin{tikzcd}
\mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\mathcal{O}^+_{X'}/p\right)^a \otimes^L \mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\mathcal{O}^+_{X'}/p\right)^a \arrow[r, "\cup"] \arrow[d, shift left=8em, "\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}"] & \mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\mathcal{O}^+_{X'}/p\right)^a \arrow[r, "\mathbf{R}\mathfrak{f}_*(\mathrm{Tr}_{F,\mathfrak{X}})"] \arrow[d, "\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}"] & \mathbf{R}\mathfrak{f}_*\left(\omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d]\right) \arrow[d, "\mathrm{Tr}_{\mathfrak{f}}"] \\
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \otimes^L \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \arrow[r, "\cup"] \arrow[u, shift left=8em, "\mathrm{Res}_{\mathfrak{f}}"] & \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \arrow[r, "\mathrm{Tr}_{F,\mathfrak{X}}"] & \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]
\end{tikzcd}$$

is commutative in $\mathbf{D}(\mathfrak{X})^a$ (see Section 1.4 for the precise meaning of this commutativity).

Proof. Commutativity of the right square follows directly from Lemma 3.6.1. So the only question is to show commutativity of the following square:

$$\begin{tikzcd}
\mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\mathcal{O}^+_{X'}/p\right) \otimes^L \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right) \arrow[r, "\cup"] \arrow[d, "\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}"] & \mathbf{R}\mathfrak{f}_*\left(\mathbf{R}\nu_{\mathfrak{X}',*}\mathcal{O}^+_{X'}/p\right) \arrow[d, "\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}"] \\
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right) \otimes^L \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right) \arrow[r, "\cup"] \arrow[u, "\mathrm{Res}_{\mathfrak{f}}"] & \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right).
\end{tikzcd}$$

This square, in turn, can be identified with the square

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \otimes^L \mathbf{R}\nu_{\mathfrak{X},*}f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \arrow[r, "\mathbf{R}\nu_{\mathfrak{X},*}(-\cup-)"] & \mathbf{R}\nu_{\mathfrak{X},*}f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \arrow[d, "\mathbf{R}\nu_{\mathfrak{X},*}(\mathrm{Tr}_{\text{proét},f})"] \\
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_{X}^+/p\right) \otimes^L \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_{X}^+/p\right) \arrow[u, "\mathbf{R}\nu_{\mathfrak{X},*}(\mathrm{Res}_f)\ \uparrow\ \ \mathbf{R}\nu_{\mathfrak{X},*}(\mathrm{Tr}_{\text{proét},f})\downarrow"] \arrow[r, "\cup"] & \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_{X}^+/p\right).
\end{tikzcd}$$

Thus, it is sufficient to show that the diagram

$$\begin{tikzcd}
f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \otimes^L f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \arrow[r, "\cup"] & f_{\text{proét},*}\left(\mathcal{O}_{X'}^+/p\right) \arrow[d, "\mathrm{Tr}_{\text{proét},f}"] \\
\mathcal{O}_X^+/p \otimes^L \mathcal{O}_X^+/p \arrow[u, "\mathrm{Res}_f\ \uparrow\ \ \mathrm{Tr}_{\text{proét},f}\downarrow"] \arrow[r, "\cup"] & \mathcal{O}_X^+/p.
\end{tikzcd}$$

This equality can now be checked pro-étale locally, so one can assume that f is a split étale cover. In this case, the claim is obvious. ☐

We also check that the Faltings' trace map is étale local on $\mathfrak{X}$. More precisely, we have the following result:

Lemma 3.6.3. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an étale map of admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fibers of pure dimension d. Denote the generic fiber of $\mathfrak{f}$ by $f\colon X \to Y$. Then the diagram

$$\begin{tikzcd}
\mathbf{L}\mathfrak{f}^*(\mathbf{R}\nu_{Y,*}\widehat{\mathcal{O}}_Y^+)^a \arrow[r, "\mathbf{L}\mathfrak{f}^*\mathrm{Tr}_{F,\mathfrak{Y}}"] \arrow[d] & \mathbf{L}\mathfrak{f}^*\omega_{\mathfrak{Y}}^a(-d)[-d] \arrow[d] \\
\left(\mathbf{R}\nu_{X,*}\widehat{\mathcal{O}}_X^+\right)^a \arrow[r, "\mathrm{Tr}_{F,\mathfrak{X}}"] & \omega_{\mathfrak{X}}^a(-d)[-d]\ ,
\end{tikzcd} \tag{3.10}$$

where the left vertical arrow is the base change map and the right vertical arrow is the map from Lemma 2.1.7, is commutative in $\mathbf{D}(\mathfrak{X})^a$ with vertical arrows being almost isomorphisms.

Proof. We note that $\mathbf{L}\mathfrak{f}^*\omega_{\mathfrak{Y}}(-d)[-d] \simeq \mathfrak{f}^*\omega_{\mathfrak{Y}}(-d)[-d]$ as $\mathfrak{f}^*$ is flat. So Lemma 2.1.7 guarantees that the right vertical map is an isomorphism. Moreover, [Zav21a, Theorem 6.13.5] guarantees that the left vertical map is an almost isomorphism. So both vertical maps are isomorphisms in $\mathbf{D}(\mathfrak{X})^a$. It suffices to show that the diagram commutes.

Now we use that $(\mathbf{R}\nu_{Y,*}\widehat{\mathcal{O}}_Y^+)^a$ is concentrated in degrees $[0,d]$ by [Zav21a, Theorem 6.13.6]. Therefore, flatness of $\mathfrak{f}$ implies that the same holds for $\mathbf{L}\mathfrak{f}^*(\mathbf{R}\nu_{Y,*}\widehat{\mathcal{O}}_Y^+)^a$. So it suffices to show that both morphisms

$$\mathfrak{f}^*(\mathrm{R}^d\nu_{Y,*}\widehat{\mathcal{O}}_Y^+)^a \to \omega_{\mathfrak{X}}^a(-d)$$

coincide.

Now we recall that $\omega_{\mathfrak{X}}^a(-d)$ is $\mathcal{O}_C$-flat by Lemma 2.1.9. Therefore, equality of these maps can be checked on the generic fiber. However, on the generic fiber the diagram (3.10) can be identified with

$$\begin{tikzcd}
f^*(\mathbf{R}\mu_*\widehat{\mathcal{O}}_Y) \arrow[r, "f^*(\mathrm{Sch}_Y^d)^{-1}"] \arrow[d] & f^*\Omega_Y^d(-d)[-d] \arrow[d] \\
\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \arrow[r, "(\mathrm{Sch}_X^d)^{-1}"] & \Omega_X^d(-d)[-d]
\end{tikzcd}$$

So the question reduces to show that $(\mathrm{Sch}^d_X)^{-1}$ is étale local on X. However, this follows from its definition as this map comes as the restriction of the inverse of the map

$$\Omega^d_{X_{\text{ét}}}(-d) \to \mathbf{R}\lambda_* \widehat{\mathcal{O}}^+_X$$

that is étale local by the construction as it is defined on the étale site of X. $\square$

4. Local Duality

4.1. **Overview.** For the rest of the section, we fix a complete, algebraically closed, rank-1 valued field C of mixed characteristic $(0, p)$.

The main goal of Section 4 is to show that the Faltings' trace $\mathrm{Tr}_{F,\mathfrak{X}} \colon \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \to \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$ induced an *almost perfect pairing*

$$\mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \xrightarrow{\cup} \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \xrightarrow{\mathrm{Tr}_{F,\mathfrak{X}}} \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d], \tag{4.1}$$

i.e. the *duality morphism*

$$D_{\mathfrak{X}} \colon \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}_0}\left(\mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]\right)$$

is an isomorphism in $\mathbf{D}(\mathfrak{X}_0)^a$.

Definition 4.1.1. Let $\mathfrak{X}$ be a separated admissible formal $\mathcal{O}_C$-scheme with smooth generic fiber $X = \mathfrak{X}_C$ of pure dimension d. Then the paring (4.1) is called the *Faltings pairing* with the corresponding duality morphism

$$D_{\mathfrak{X}} \colon \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}_0}\left(\mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]\right)$$

Theorem 4.1.2. (Theorem 4.5.8) Let $\mathfrak{X}$ be as in Definition 4.1.1. Then Faltings' pairing is almost perfect.

The essential idea of the proof is to use [Zav21b, Theorem 1.4] to reduce the case of a general $\mathfrak{X}$ to the case of a polystable model $\mathfrak{X}$. In this situation, one can relate $\mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a$ to the continuous group cohomology of $\mathbf{Z}_p(1)^d$ that can be explicitly understood by means of the Koszul complex.

We now explain the main steps involved in the the proof in more detail:

(1) We firstly show the claim for a polystable formal $\mathcal{O}_C$-schemes (see Definition 4.2.1) with smooth generic fiber. The proof in this case is very similar to the calculations in [ČK19, §3]. Namely, we choose an explicit affinoid perfectoid cover of the model polystable formal scheme. This covering turns out to be a $\Delta \simeq \mathbf{Z}_p(1)^d$-torsor that allows us to reduce the claim to the duality statement for the continuous cohomology groups of Δ. This, in turn, can be dealt with explicitly using Koszul complexes.

(2) Then we use [Zav21b, Theorem 1.4] to relate any admissible formal $\mathcal{O}_C$-scheme with smooth generic fiber of dimension d to polystable formal $\mathcal{O}_C$-schemes. Roughly, [Zav21b, Theorem 1.4] says that, locally on $\mathfrak{X}'$, any such $\mathfrak{X}'$ is isomorphic to a polystable formal scheme up to rig-isomorphisms and quotients by a finite group with free action on the generic fiber. So, we only need to show that the property that $D_{\mathfrak{X}}$ is an isomorphism descends through rig-isomorphisms and "good quotients" by finite groups.

(3) We show that, if $\mathfrak{X}' \to \mathfrak{X}$ is a rig-isomorphism of admissible formal $\mathcal{O}_C$-schemes with smooth generic fiber of pure dimension d and $D_{\mathfrak{X}'}$ is an isomorphism. Then $D_{\mathfrak{X}}$ is also an isomorphism. The key input here is the almost version of the Grothendieck Duality.

(4) Finally, we show that, if $\mathfrak{X}' \to \mathfrak{X}$ is a "good" quotient by a finite group G of admissible formal $\mathcal{O}_C$-schemes with smooth generic fiber of pure dimension d and $D_{\mathfrak{X}'}$ is an isomorphism. Then $D_{\mathfrak{X}'}$ is also an isomorphism. The key input here is the almost version of the Grothendieck Duality and duality betweenn homotopy invariants and coinvariants for an action of a finite group G.

4.2. **Preliminaries on Polystable Models.** We recall the definition of polystable models from [Zav21b]. There are slightly non-equivalent definitions in the literature. But we will always stick to this definition in this paper.

Definition 4.2.1. An admissible rig-smooth formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ is called *polystable* if étale locally it admits an étale morphism

$$\mathfrak{U} \to \operatorname{Spf} \frac{\mathcal{O}_C\langle t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}\rangle}{t_{1,0}\cdots t_{1,n_1} - \varpi_1, \dots, t_{l,0}\cdots t_{l,n_l} - \varpi_l)}$$

for some $\varpi_i \in \mathcal{O}_C \setminus \{0\}$.

Definition 4.2.2. We say that the formal $\mathcal{O}_C$-scheme

$$\operatorname{Spf} \frac{\mathcal{O}_C\langle t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}\rangle}{(t_{1,0}\cdots t_{1,n_1} - \varpi_1, \dots, t_{l,0}\cdots t_{l,n_l} - \varpi_l)}$$

is a *model polystable formal $\mathcal{O}_C$-scheme.*

Before that the pairing (4.1) is almost perfect for polystable $\mathfrak{X}$, we need to understand better the complexes $\omega^\bullet_{\mathfrak{X}_0}$ and $\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a$. The two main goals in to reformulate the condition of pairing (4.1) being almost perfect fully in terms of $\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a$ (i.e. without any reference to the Faltings' trace and $\omega^\bullet_{\mathfrak{X}_0}$), and then reformulate it in terms of the continuous group cohomology for a profinite group $\Delta = \mathbf{Z}_p(1)^d$.

We start by showing that the dualizing complex $\omega^\bullet_{\mathfrak{X}_0}$ on a poly-stable model of dimension d is a line bundle concentrated in degree $-d$. Then we show that the same result holds for $\omega^\bullet_{\mathfrak{X}}$. This will eventually allow us to show that the Faltings' trace induces an isomorhism

$$\frac{\mathrm{R}^d\nu_*\left(\mathcal{O}_X^+/p\right)^a}{\mathrm{R}^d\nu_*\left(\mathcal{O}_X^+/p\right)[1-\zeta_p]^a} \to \omega^a_{\mathfrak{X}_0}(-d).$$

This will, in turn, imply that pairing (4.1) is almost perfect on a polystable $\mathfrak{X}$ if and only if the pairing

$$\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a \xrightarrow{\cup} \mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a \to \frac{\mathrm{R}^d\nu_*\left(\mathcal{O}_X^+/p\right)^a}{\mathrm{R}^d\nu_*\left(\mathcal{O}_X^+/p\right)[1-\zeta_p]^a} \tag{4.2}$$

is almost perfect. And now almost perfectness of the pairing (4.2) is purely a question about $\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a$ that has no reference to the Faltings' trace or the dualizing module $\omega^\bullet_{\mathfrak{X}_0}$.

Then we consruct certain explicit pro-étale covers of the model polystable formal schemes and use them to relate $\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a$ to group continuous group cohomology.

4.2.1. *Dualizing Complex on Polystable Models.*

Lemma 4.2.3. Let $\mathfrak{X}$ be a separated polystable formal $\mathcal{O}_C$-scheme. Then $\omega^\bullet_{\mathfrak{X}_n}$ is isomorphic to a line bundle in degree $-\dim \mathfrak{X}_n = -\dim \overline{X}$.

Proof. We prove the claim for $n = 0$, the proof is absolutely the same for $n \geq 1$. Recall that the relative dualizing complex commutes with étale base change by [Sta19, Tag 0FWI]. So it suffices to treat the model example

$$\mathfrak{X}_0 = \operatorname{Spf} \frac{\mathcal{O}_C/p[t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots t_{l,n_l}]}{(t_{1,0} \dots t_{1,n_1} - \varpi_1, \dots, t_{l,0} \dots t_{l,n_l} - \varpi_l)} \ .$$

This can be realizing as a closed subscheme of

$$i \colon \mathfrak{X}_0 \to Y_0 \coloneqq \mathbf{A}^{\dim \mathfrak{X}_0 + l}_{\mathcal{O}_C/p}$$

defined by the ideal $I \coloneqq (t_{1,0} \dots t_{1,n_1} - \varpi_1, \dots, t_{l,0} \dots t_{l,n_l} - \varpi_l)$. It can be easily seen that the sequence $\{t_{1,0} \dots t_{1,n_1} - \varpi_1, \dots, t_{l,0} \dots t_{l,n_l} - \varpi_l\}$ is regular, thus the ideal I is Koszul-regular by [Sta19, Tag 063E].

Now we note that $\omega^\bullet_{Y_0} \cong \Omega^{\dim Y_0}_{Y_0}[\dim Y_0]$ since Y is smooth. Moreover, we have a (non-canonical) isomorphism $\Omega^{\dim Y_0}_{Y_0}[\dim Y_0] \cong \mathcal{O}_{Y_0}[\dim Y_0]$ since Y_0 is an affine space. Now we recall that $\omega_{\mathfrak{X}_0} \cong i^!(\omega^\bullet_{Y_0})$, so [Sta19, Tag 0BR0] provides an identification

$$\omega^\bullet_{\mathfrak{X}_0} \cong \left(\bigwedge^l \left(\mathcal{I}/\mathcal{I}^2\right)^\vee \right) [\dim \mathfrak{X}_0 + l - l] = \left(\Lambda^l \left(\mathcal{I}/\mathcal{I}^2\right)^\vee\right) [\dim \mathfrak{X}_0] \ .$$

The last observation is that the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank l by the proof of [Sta19, Tag 063H]. $\square$

Lemma 4.2.4. Let $\mathfrak{X}$ be a separated polystable formal $\mathcal{O}_C$-scheme. Then $\omega^\bullet_{\mathfrak{X}}$ is isomorphic to a line bundle in degree $-\dim \mathfrak{X}_0$.

Proof. We recall that $\omega^\bullet_{\mathfrak{X}} \cong \mathbf{R}\lim \omega^\bullet_{\mathfrak{X}_n}$ and $\omega^\bullet_{\mathfrak{X}_n} = \omega_{\mathfrak{X}_n}[\dim \mathfrak{X}_n]$ by Lemma 4.2.3. Now the isomorphism $\omega^\bullet_{\mathfrak{X}_{n+1}} \otimes^L_{\mathcal{O}_{\mathfrak{X}_{n+1}}} \mathcal{O}_{\mathfrak{X}_n} \cong \omega^\bullet_{\mathfrak{X}_n}$ reduces to the isomorphism

$$\omega_{\mathfrak{X}_{n+1}} \otimes_{\mathcal{O}_{\mathfrak{X}_{n+1}}} \mathcal{O}_{\mathfrak{X}_n} \cong \omega_{\mathfrak{X}_n} \ .$$

In particular, the natural map of coherent sheaves $\omega_{\mathfrak{X}_{n+1}} \to \omega_{\mathfrak{X}_n}$ is surjective. Thus, it is surjective on global sections over affines. Therefore, $\mathbf{R}\lim_n \omega_{\mathfrak{X}_n} \cong \lim_n \omega_{\mathfrak{X}_n}$ by [Sta19, Tag 0A09], vanishing of higher cohomology groups on affines, and the surjectivity established above.

Overall, we see that

$$\omega^\bullet_{\mathfrak{X}} \cong \mathbf{R}\lim \omega^\bullet_{\mathfrak{X}_n} \cong (\lim_n \omega_{\mathfrak{X}_n})[\dim \mathfrak{X}_0].$$

The only thing we are left to show is that $\omega_{\mathfrak{X}} = \lim_n \omega_{\mathfrak{X}_n}$ is a line bundle. The question is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Now [FK18, Proposition I.3.4.1 and Proposition I.4.8.1] ensure that $\lim_n \omega_{\mathfrak{X}_n}$ is adically quasi-coherent with finite flat global sections $M = \omega_{\mathfrak{X}}(\mathfrak{X})$. Then [Bos14, Theorem 7.3/4] implies that M is finitely presented over A, so M is a projective A-module. Thus $\omega_{\mathfrak{X}} = M^\Delta$ is a vector bundle. Then it is clear that it must be of rank-1 as $\omega_{\mathfrak{X}}/\varpi\omega_{\mathfrak{X}} \cong \omega_{\mathfrak{X}_0}$ a line bundle on $\mathfrak{X}_0$. $\square$

4.2.2. *Pro-étale Cover of Model Polystable Formal Scheme.* In this subsection we construct an explicit "cover" of a the model polystable formal scheme such that on the adic generic fiber it become a pro-finite étale cover by an affinoid perfectoid. The case $\mathfrak{X} = \mathbf{G}^n_m$ was done in [BMS18] and the case of a semi-stable model[25] was done in [ČK19, §3]. The main difference between our approach and the approach in [ČK19, §3] is that we write some formulas in a more canonical way. It will both later simply the proofs and the notation.

[25]With some extra mild assumptions.

For the rest of the subsection, we fix a ring

$$R^+ \coloneqq \frac{\mathcal{O}_C \langle t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l} \rangle}{(t_{1,0} \dots t_{1,n_1} - \varpi_1, \dots, t_{l,0} \dots t_{l,n_l} - \varpi_l)}$$

and define rings

$$R_m^+ \coloneqq \frac{\mathcal{O}_C \left\langle t_{1,0}^{1/p^m}, \dots, t_{1,n_1}^{1/p^m}, \dots, t_{l,0}^{1/p^m}, \dots, t_{l,n_l}^{1/p^m} \right\rangle}{\left(t_{1,0}^{1/p^m} \dots t_{1,n_1}^{1/p^m} - \varpi_1^{1/p^m}, \dots, t_{l,0}^{1/p^m} \dots t_{l,n_l}^{1/p^m} - \varpi_l^{1/p^m}\right)},$$

where we implicitly choose a compatible sequence of roots $\varpi_i^{1/p^m} \in \mathcal{O}_C$ for all $i = 1, \dots, l$.

We clearly have maps $R^+ \to R_m^+$ that just send $t_{i,j} \in R^+$ to $t_{i,j}$ considered as elements of R_m^+. It is easy to see that the associated map $\operatorname{Spf} R_m^+ \to \operatorname{Spf} R^+$ is finite and rig-étale, i.e. its adic generic fiber is étale as a map of rigid spaces. We define the rational version of those rings as $R_m \coloneqq R_m^+[1/p]$. As the notation suggests, R_m^+ is integrally closed in R_m and is actually equal to R_m°, the set of power-bounded elements.

Lemma 4.2.5. The ring R_m^+ is integrally closed in R_m for any $m \geq 0$. Moreover, we have an equality $R_m^+ = R_m^\circ$, where $(-)^\circ$ stands for the set of power-bounded elements.

Proof. We note that the adic generic fiber $\operatorname{Spa}(R_m, R_m^\circ)$ of $\operatorname{Spf} R_m^+$ is smooth by construction. In particular, it is (geometrically) reduced. Thus, [Lüt16, Proposition 3.4.1] implies that $R_m^+ = R_m^\circ$. Now we just note that the set of powerbounded elements R_m° is always integrally closed in R_m. This finishes the proof. ☐

Note that a group $\Delta_m \coloneqq \prod_{i=1}^l \mu_{p^m}^{n_i}(C)$ admits a continuous R^+-linear action on R_m^+. Namely, we can realize Δ_m as a subgroup

$$\Delta_m = \left\{ (\varepsilon_{1,i_1})_{i_1=1}^{n_1} \times \dots \times (\varepsilon_{l,i_l})_{il=1}^{n_l} \in \times \left(\prod_{i=1}^{l} \mu_{p^m}^{n_i}(C) \right) \mid \prod_{j=0}^{n_i} \varepsilon_{i,j} = 1 \ \forall i = 1, \dots l \right\} \subset \prod_{i=1}^{l} \mu_{p^m}^{n_i+1}(C)$$

In this presentation an element $\varepsilon \in \Delta_m$ acts on R_m^+ as the multiplication of each coordinate $t_{i,j}^{1/p^m}$ by the corresponding root of unity $\varepsilon_{i,j}$. This action is clearly seen to be continuous and R^+-linear. Moreover, one can see that the adic generic fiber

$$\operatorname{Spa}(R_m, R_m^+) \to \operatorname{Spa}(R, R^+)$$

is finite étale Δ_m-torsor.

We now define the ring $R_\infty^+ \coloneqq (\operatorname{colim}_m R_m^+)^\wedge$ where the completion is p-adic, and its rational version $R_\infty \coloneqq R_\infty^+[1/p]$. Our first goal is to see that (R_∞, R_∞^+) is a perfectoid pair.

We note that abstractly as an $\mathcal{O}_C$-module we have have the decomposition

$$R_\infty^+ \cong \widehat{\bigoplus}_{(d_{i,j})_{i=1,j=1}^{l,n_i} \in \mathbf{Z}[\frac{1}{p}]_{\geq 0}, \forall i=1,\dots,l \exists j \in [0,n_i] \text{ such that } d_{i,j}=0} \mathcal{O}_C t_{1,0}^{d_{1,0}} \dots t_{l,n_l}^{d_{l,n_l}} \tag{4.3}$$

This formula might look quite unpleasant, but later we will see how to write this formula in a more conceptual way.

Lemma 4.2.6. The ring R_∞^+ is a p-torsionfree integral perfectoid, and the pair (R_∞, R_∞^+) is a perfectoid pair.

Proof. What we really need to show here is that R^+_∞ is p-torsionfree, integrally closed in R_∞ and the map $\varphi\colon R^+_\infty/p^{1/p} \to R^+_\infty/p$ induced by $x \mapsto x^p$ is an isomorphism. Indeed, this would imply that (R_∞, R^+_∞) is a Huber pair and R^+_∞ is an integral perfectoid. Thus, [BMS18, Lemma 3.20] would imply that R_∞ is perfectoid, so (R_∞, R^+_∞) would be a perfectoid pair.

The fact R^+_∞ is p-torsionfree can be either seen directly from decomposition (4.3), or seen from a more general fact that p-adic completion of a p-torsionfree algebra is p-torsionfree. Lemma 4.2.5 implies that $\operatorname{colim} R^+_m$ is integrally closed in $\operatorname{colim} R_m$. Therefore, [Bha, Lemma 5.1.2] guarantees that R^+_∞ is integrally closed in R_∞. Finally, we can see that the map φ is an isomorphism directly from the decomposition 4.3. $\square$

We define the profinite group Δ as the inverse limit

$$\Delta := \lim_m \Delta_m \subset \lim_m \prod_{i=1}^{l} \mu_{p^m}^{n_i+1}(C) = \mathbf{Z}_p(1)^{(\sum_{i=1}^l n_i)+l} = \mathbf{Z}_p(1)^{\dim X + l}\ . \tag{4.4}$$

We note the group Δ is isomorphic to $\mathbf{Z}_p(1)^{\dim X}$ as can be seen from its construction. Moreover, after choosing a compatible sequence of p-power roots of unity $(\zeta_p, \zeta_{p^2}, \dots)$ it becomes isomorphic to $\mathbf{Z}_p^{\dim X}$. By passing to the limit, we see that Δ admits a continuous R^+-linear action on R^+_∞ that makes R into Galois Δ-torsor. So we see that

$$\text{“}\lim_n \operatorname{Spa}(R_m, R^+_m)\text{”} \to \operatorname{Spa}(R, R^+)$$

is a Galos Δ-torsor (in particular, it is a pro-finite étale cover) of $\operatorname{Spa}(R, R^+)$ by an affinoid perfectoid with associated adic space $\operatorname{Spa}(R_\infty, R^+_\infty)$. So we summarize the discussion in the following lemma:

Lemma 4.2.7. In the notation as above, the map “$\lim_n \operatorname{Spa}(R_m, R^+_m)$” $\to \operatorname{Spa}(R, R^+)$ is a pro-étale Δ-torsor by an affinoid perfectoid with the associated adic space $\operatorname{Spa}(R_\infty, R^+_\infty)$.

Now we discuss how to rewrite decomposition (4.3) in a more conceptual way. We start the discussion by defining the split $\mathcal{O}_C$-torus T by its functor of points

$$T(S) = \left\{ (x_{1,0}, \dots, x_{1,n_1}, \dots, x_{l,0}, \dots, x_{l,n_l}) \in \mathbf{G}_m^{\dim X + l}(S) \mid \prod_{j=0}^{n_i} x_{i,j} = 1\ \forall i = 1, \dots l \right\} \subset \mathbf{G}_m^{\dim X + l}$$

One easily sees that T is abstractly isomorphic to $\mathbf{G}_m^{\dim X}$, but there is not preferred isomorphism. The group scheme T admits an $\mathcal{O}_C$-action on $\operatorname{Spf} R^+$ defined on regular functions as

$$(x).t_{i,j} = x_{i,j}t_{i,j} \text{ for } x \in T.$$

Denote by $X(T) := \operatorname{Hom}_{\mathcal{O}_C-\mathrm{gp}}(T, \mathbf{G}_m)$ the character group of T. Then we have a canonical decomposition of R^+ by weights:

$$R^+ \simeq \widehat{\bigoplus}_{\chi \in X(T)} V_\chi \tag{4.5}$$

with V_χ being a one-dimensional free $\mathcal{O}_C$-module that corresponds to the character χ. And the multiplication in R^+ is induced by canonical isomorphisms

$$V_\chi \otimes V_{\chi'} \xrightarrow{\sim} V_{\chi+\chi'}\text{[26]}\ .$$

More explicitly, we recall that the inclusion $T \subset \mathbf{G}_m^{\dim X + l}$ induces the surjection

$$\mathbf{Z}^{\dim X + l} \simeq X(\mathbf{G}_m^{\dim X + l}) \to X(T).$$

[26] We use additive notation for elements of $X(T)$.

So a direct summand V_χ explicitly corresponds to rank-1 free $\mathcal{O}_C$-module

$$\mathcal{O}_C \prod_{i=1}^{n} t_{i,0}^{a_{i,0}-a_{i,j_i}} \dots t_{i,n_i}^{a_{i,n_i}-a_{i,j_i}} \subset R^+,$$

where $(a_{i,j}) \in \mathbf{Z}^{\dim X + l}$ is any lift of χ and, for each i, j_i is the integer such that a_{i,j_i} is the smallest element among all $\{a_{i,j}\}_{j=0}^{n_i}$.

Lemma 4.2.8. In the notation as above, there is a canonical decomposition

$$R^+ \simeq \widehat{\bigoplus}_{\chi \in X(T)} V_\chi,$$

where each V_χ is a free $\mathcal{O}_C$-module of rank-1. And the multiplication map on R^+ induce isomorphisms

$$V_\chi \otimes V_{\chi'} \xrightarrow{\sim} V_{\chi+\chi'}$$

for each $\chi, \chi' \in X(T)$.

We now rewrite decomposition (4.3) in terms of characters of T, similar to decomposition (4.5). It turns out that basically the same decomposition holds with the difference is that the sum is taken over $X(T)[1/p]$ instead of $X(T)$.

Namely, we claim that there is a decomposition

$$R^+_\infty = \widehat{\bigoplus}_{\chi \in X(T)[1/p]} V_\chi$$

but we now need to explain what we mean by V_χ as T does not any longer acts on R^+_∞. One way to do this is to define V_χ explicitly. We already have a definition for $\chi \in X(T)$, we extend it for $\chi \in X(T)[1/p]$ as follows. Suppose that $p^m\chi \in X(T)$ and

$$V_{p^m\chi} = \mathcal{O}_C t_{1,0}^{a_{1,0}} \dots t_{l,n_l}^{a_{l,n_l}} \text{ with the condition that at least one } a_{i,j} = 0 \ \forall i$$

then we define

$$V_\chi \coloneqq \mathcal{O}_C t_{1,0}^{a_{1,0}/p^m} \dots t_{l,n_l}^{a_{l,n_l}/p^m}$$

where each V_χ being a one-dimensional free $\mathcal{O}_C$-module.

This definition clearly does not depend on a choice of m. Moreover, one sees directly that this decomposition recovers decomposition (4.3) and the multiplication on R^+_∞ is induced by the canonical isomorphisms

$$V_\chi \otimes V_{\chi'} \xrightarrow{\sim} V_{\chi+\chi'} \ .$$

The next thing we note is that, for each class $\chi \in X(T)[1/p]$, there is a canonical R^+-module structure on $\widehat{\bigoplus}_{\chi' \in \chi X(T)} V_\chi$. This can be seen from decomposition (4.5) and the multiplication structure on R^+_∞ in terms of $V_{\chi'}$ established above. Moreover, the induced map

$$V_\chi \otimes_{\mathcal{O}_C} R^+ \to \widehat{\bigoplus}_{\chi' \in \chi X(T)} V_\chi$$

is an isomorphism for all $\chi \in X(T)[1/p]$. This gives us the R^+-*module* decomposition

$$R^+_\infty = \widehat{\bigoplus}_{\overline{\chi} \in X(T)[1/p]/X(T)} (V_{\overline{\chi}})_{R^+} \tag{4.6}$$

where $(V_{\overline{\chi}})_{R^+} \coloneqq V_\chi \otimes_{\mathcal{O}_C} R^+$ that does not depend on a choice of $\chi \in X(T)$ by the discussion above.

Lemma 4.2.9. In the notation as above, there are canonical decompositions

$$R^+_\infty \simeq \widehat{\bigoplus}_{\chi \in X(T)[1/p]} V_\chi, \ R^+_\infty = \widehat{\bigoplus}_{\overline{\chi} \in X(T)[1/p]/X(T)} (V_{\overline{\chi}})_{R^+}$$

where each V_χ (resp. $(V_{\overline{\chi}})_{R^+}$) is a free $\mathcal{O}_C$-module (resp. R^+-module) of rank-1. And the multiplication map on R^+_∞ induce isomorphisms

$$V_\chi \otimes_{\mathcal{O}_C} V_{\chi'} \xrightarrow{\sim} V_{\chi+\chi'}, \ (V_{\overline{\chi}})_{R^+} \otimes_{R^+} (V_{\overline{\chi'}})_{R^+} \xrightarrow{\sim} (V_{\overline{\chi+\chi'}})_{R^+}$$

for each $\chi, \chi' \in X(T)[1/p]$.

The last thing we want to understand is the action of Δ on R^+_∞ in terms of decomposition (4.6). In order to do this, we also need to redefine Δ in terms of T as well. So, from now on, we define

$$\Delta \coloneqq \mathrm{T}_p(T) \simeq \mathrm{Hom}\left(X\left(T\right)\left[1/p\right]/X\left(T\right), \mu_{p^\infty}\left(C\right)\right) \ .$$

We use the standard inclusion $T \subset \mathbf{G}_m^{\dim X + l}$ to get an injection

$$\Delta \subset \mathrm{T}_p(\mathbf{G}_m^{\dim X+l}) \simeq \mathbf{Z}_p(1)^{\dim X + l}$$

that recovers the definition of Δ from the formula (4.4). So, in particular, it acts by R^+-linear continuous automorphisms on R^+_∞. It preserves the decomposition 4.6 as it acts trivially on R^+. And, moreover, we see that the action of Δ on $(V_{\overline{\chi}})_{R^+}$ is given by

$$\gamma.x = \gamma(\overline{\chi})x \text{ for any } \gamma \in \Delta = \mathrm{Hom}\left(X\left(T\right)\left[1/p\right]/X\left(T\right), \mu_{p^\infty}\left(C\right)\right), \ x \in (V_{\overline{\chi}})_{R^+}.$$

Lemma 4.2.10. In the notation as above, the action of $\gamma \in \Delta = \mathrm{Hom}\left(X\left(T\right)\left[1/p\right]/X\left(T\right), \mu_{p^\infty}\left(C\right)\right)$ on $(V_\chi)_{R^+}$ is given by

$$\gamma.x = \gamma(\overline{\chi})x.$$

4.2.3. *Almost Computation of $\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X$ for a Framed Polystable Formal Scheme.* The ultimate goal of this section is to almost (in the technical sense) compute $\mathcal{O}_{\mathfrak{X}}$-modules $\mathrm{R}^i\nu_*\widehat{\mathcal{O}}^+_X$ in terms of the continuous cohomology of the profinite group Δ defined in the previous section. We will proceed in this section under the assumption that $\mathfrak{X}$ is "framed", i.e. $\mathfrak{X} = \mathrm{Spf}\, S^+$ is affine and admits an étale map $\mathfrak{X} \to \mathrm{Spf}\, R^+$ to a model polystable formal $\mathcal{O}_C$-scheme $\mathrm{Spf}\, R^+$.

The main ingredients of the computation are the almost purity theorem and the pro-étale cover "$\lim_m \mathrm{Spa}(R_m, R_m^+)$" $\to \mathrm{Spa}(R, R^+)$ constructed in the previous section. This cover defines a pro-étale cover of $X = \mathfrak{X}_C = \mathrm{Spa}(S, S^\circ)$ given by $X_\infty \coloneqq$ "$\lim_m \mathrm{Spa}(S_m, S_m^\circ)$" $\to \mathrm{Spa}(S, S^+)$ with

$$S_m^+ \coloneqq R_m^+ \widehat{\otimes}_{R^+} S^+ \simeq R_m^+ \otimes_{R^+} S^+, \ S_m \coloneqq S_m^+[1/p].$$

Remark 4.2.11. We note that $S_m^+ \simeq R_m^+ \widehat{\otimes}_{R^+} S^+$ because R_m^+ is finite, projective over R^+, and so $R_m^+ \otimes_{R^+} S^+$ is already p-adically complete.

We define

$$S^+_\infty \coloneqq (\mathrm{colim}_m\, S_m^+)^\wedge \cong R^+_\infty \widehat{\otimes}_{R^+} S^+, \ S_\infty \coloneqq S^+_\infty[1/p],$$

where $(-)^\wedge$ stands for p-adic completion. We claim that $X_\infty \to X$ is a pro-étale cover by an affinoid perfectoid with the associated adic space $\mathrm{Spa}(S_\infty, S_\infty)$.

Lemma 4.2.12. Under the notation as above, there is an equality $S_m^+ = S_m^\circ$ for any m. The pair (S_∞, S^+_∞) is a perfectoid pair with an integral perfectoid S^+_∞.

Proof. We recall that the map $\mathrm{Spf}\, S_m^+ \to \mathrm{Spf}\, R_m^+$ is étale for any m. Thus, the special fiber of $\mathrm{Spf}\, S_m^+$ is (geometrically) reduced for all m, and the adic generic fiber is smooth. So [Lüt16, Proposition 3.4.1] implies that $S_m^+ = S_m^\circ$. The second claim follows from [Zav21a, Corollary 6.4.5]. □

We note that the pro-étale cover

$$X_\infty := \text{“}\lim_m \mathrm{Spa}(S_m, S_m^+)\text{”} \to \mathrm{Spa}(S, S^+) = X$$

is a Δ-torsor over $\mathrm{Spa}(S, S^+)$ as a base change of the Δ-torsor “$\lim_m \mathrm{Spa}(R_m, R_m^+) \to \mathrm{Spa}(R, R^+)$. So the proof of [Sch13a, Lemma 5.6] identifies the Čech complex associated with the sheaf $\widehat{\mathcal{O}}_X^+$ and the cover $X_\infty \to X$ with $\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+)$. This induces a morphism

$$\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+) \to \mathbf{R}\Gamma(X, \widehat{\mathcal{O}}_X^+)$$

that, in turn, induces a morphism

$$\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+)^\Delta \to \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}_X^+\right).$$

by [Zav21a, Theorem 6.13.6].

Lemma 4.2.13. The map $\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+)^\Delta \to \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}_X^+\right)$ is an almost isomorphism. Similarly, the natural map $\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+/p) \to \mathbf{R}\nu_*(\mathcal{O}_X^+/p)$ is an almost isomorphism.

Proof. We note that [Zav21a, Theorem 6.13.6] ensures that the complex $\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}_X^+\right)$ is in $\mathbf{D}^b_{acoh}(\mathfrak{X})$ and the natural morphism

$$\mathbf{R}\Gamma(X, \widehat{\mathcal{O}}_X^+)^\Delta \to \mathbf{R}\nu_*\widehat{\mathcal{O}}_X^+$$

Thus, it is sufficient to show that the map

$$\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+) \to \mathbf{R}\Gamma(X, \widehat{\mathcal{O}}_X^+)$$

is almost isomorphism. The standard argument with the Čech-to-derived spectral sequence reduces the claim to show that $\mathrm{H}^q(X_\infty^{j/X}, \widehat{\mathcal{O}}_X^+) = 0$ for any $q > 0, j \geq 1$. Each of $X_\infty^{j/X}$ is an affinoid perfectoid, so the claim follows from [Sch13a, Lemma 4.10]. The proof for $\mathcal{O}_X^+/p$ is the same. ☐

Remark 4.2.14. The almost coherence of the complex $\mathbf{R}\nu_*\widehat{\mathcal{O}}_X^+$ proven in [Zav21a, Theorem 6.13.6] implies that $\mathbf{R}\Gamma_{cont}(\Delta, S_\infty^+) \in \mathbf{D}^b_{acoh}(S^+)$. This can be also seen directly using decomposition (4.3).

4.2.4. *Computation of $L\eta_{1-\zeta_p}\left(\mathbf{R}\nu_*\widehat{\mathcal{O}}_X^+\right)$ for a Framed Polystable Formal Scheme.* The main goal of this section is to compute $L\eta_{1-\zeta_p}\mathbf{R}\nu_*\left(\widehat{\mathcal{O}}_X^+\right)$ without the prefix “up to almost zero”. This computation will imply that $\frac{\mathrm{R}^d\nu_*(\widehat{\mathcal{O}}_X^+)}{\mathrm{R}^d\nu_*(\widehat{\mathcal{O}}_X^+)[1-\zeta_p]}$ is a line bundle on $\mathfrak{X}$. This will be one of the crucial inputs to show that the Faltings' map induces an isomorphism

$$\frac{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}_X^+\right)}{\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}_X^+\right)[1-\zeta_p]} \to \omega_{\mathfrak{X}}(-d)$$

on any polystable formal $\mathcal{O}_C$-scheme.

We fix the notation of the previous subsection, $\mathfrak{X} = \mathrm{Spf}\, S$ is a framed polystable formal $\mathcal{O}_C$-scheme with an étale morphism $\mathfrak{X} \to \mathrm{Spf}\, R$, and constructed at the beginning of Section 4.2.3 explicit pro-étale cover $X_\infty = \text{“}\lim_m \mathrm{Spa}(S_m, S_m^+)\text{”} \to X$ that is a Δ-torsor.

We start by a more refined study of the cohomology groups $\mathrm{H}^i_{\mathrm{cont}}(\Delta, S_\infty^+)$.

Lemma 4.2.15. The natural map $\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty) \otimes^L_{R^+} S^+ \to \mathbf{R}\Gamma_{cont}(\Delta, S^+_\infty)$ is an isomorphism. In particular, the natural maps

$$\mathrm{H}^i_{cont}(\Delta, R^+_\infty) \otimes_{R^+} S^+ \to \mathrm{H}^i_{cont}(\Delta, S^+_\infty)$$

are isomorphisms for $i \geq 0$.

Proof. It follows from Lemma E.3 and Remark 4.2.14 that guarantees that all $\mathrm{H}^i_{cont}(\Delta, R^+_\infty)$ are almost finitely presented over R^+. $\square$

The following lemma will be one of the key calculations to get rid of almost mathematics in the almost isomorphism $\mathbf{R}\Gamma_{cont}(\Delta, S^+)^\Delta \simeq \mathbf{R}\nu_*\left(\widehat{\mathcal{O}}^+_X\right)$ after applying $L\eta_{1-\zeta_p}$.

Lemma 4.2.16. For $i \geq 0$, the cohomology groups $\mathrm{H}^i_{cont}(\Delta, S^+_\infty)$ have no $\mathfrak{m}$-torsion, and there is a decomposition

$$\mathrm{H}^i_{cont}(\Delta, S^+_\infty) = \mathrm{H}^i_{cont}(\Delta, S^+) \bigoplus N_i$$

where N_i is an S^+-module is annihilated by $1-\zeta_p$ with $\mathrm{H}^i_{cont}(\Delta, S^+) \simeq \bigwedge^i_{S^+}(S^+)^d$.

Proof. First of all, we use Lemma 4.2.15 to get isomorphisms

$$\mathrm{H}^i_{cont}(\Delta, S^+_\infty) \cong \mathrm{H}^i_{cont}(\Delta, R^+_\infty) \otimes_{R^+} S^+$$

for all $i \geq 0$. So it suffices to prove the claim for $S^+ = R^+$.

Now we use decomposition (4.6) to get the Δ-stable decomposition

$$R^+_\infty = \bigoplus_{\overline{\chi} \in X(T)[1/p]/X(T)} (V_{\overline{\chi}})_{R^+}$$

We rewrite this decomposition as $R^+_\infty = R^+ \oplus M_\infty$ where $M_\infty = \bigoplus_{\overline{\chi}\neq 0 \in X(T)[1/p]/X(T)} (V_{\overline{\chi}})_{R^+}$. Therefore, we get

$$\mathrm{H}^i_{cont}(\Delta, R^+_\infty) = \mathrm{H}^i_{cont}(\Delta, R^+) \bigoplus \mathrm{H}^i_{cont}(\Delta, M_\infty)\ .$$

Using Corollary E.5, it is sufficient to show

- $\mathrm{H}^i_{cont}(\Delta, R^+) = \bigwedge^n_{R^+}(R^+)^d$, and
- $\mathrm{H}^i_{cont}(\Delta, (V_{\overline{\chi}})_{R^+})$ is killed by $1-\zeta_p$ and has no $\mathfrak{m}$-torsion for $\overline{\chi} \neq 0$.

So we study cohomology of each $(V_{\overline{\chi}})_{R^+}$ separately. We recall that $V_{\overline{\chi}} \simeq V_\chi \otimes_{\mathcal{O}_C} R^+$ where V_χ is a one-dimensional free $\mathcal{O}_C$-module, and Δ acts as

$$\gamma.v = \gamma(\overline{\chi})v \text{ for each } \gamma \in \Delta = \mathrm{Hom}(X(T)[1/p]/X(T), \mu_{p^\infty}(C)) \text{ and } v \in V_{\overline{\chi}}.$$

So by applying Lemma E.3 twice, we see that it is sufficient to show that

- $\mathrm{H}^i_{cont}(\Delta, \mathcal{O}_C) = \bigwedge^n_{\mathcal{O}_C}(\mathcal{O}_C)^d$, and
- $\mathrm{H}^i_{cont}(\Delta, V_\chi)$ is a finitely presented $\mathcal{O}_C$-module that is killed by $1-\zeta_p$ and has no $\mathfrak{m}$-torsion for $\chi \neq 0$.

We choose a trivialization $T \simeq \mathbf{G}^d_m$ and a choice of compatible system of p-power roots of unity $(1, \zeta_p, \zeta_{p^2}, \dots)$ to trivialize $\Delta \cong \mathbf{Z}^d_p$ with basis elements $\gamma_1, \dots, \gamma_d$. Then any character χ has a decomposition $\chi = (\chi_1, \dots, \chi_n)$ where $\chi_i \in X(\mathbf{G}_{m,i})$, so $\chi_i(\gamma_j) = 1$ for $i \neq j$. Moreover, we now use [BMS18, Lemma 7.3] to recall that we have an isomorphism

$$K^\bullet \coloneqq K\left(\mathcal{O}_C; \gamma_1\left(\chi_1\right) - 1, \dots, \gamma_d\left(\chi_d\right) - 1\right) \cong \mathbf{R}\Gamma_{cont}\left(\Delta, V_\chi\right)$$

where $K(\mathcal{O}_C, \gamma_1(\chi_1)-1, \ldots, \gamma_d(\chi_d)-1)$ is the Koszul complex $\otimes^{\bullet}_{i=1,\ldots,d}(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C)$. Now we deal with the two cases separately.

Case 1: Character χ is the trivial character. In this case, all the differentials in the complex $K^\bullet$ are zero. Thus we easily see that

$$\mathrm{H}^i\left(K^\bullet\right) \simeq \bigwedge^i(\mathcal{O}_C^d)\ .$$

Case 2: Character χ is a non-trivial character. Then we choose i such that $\gamma_i(\chi_i)-1$ has minimal p-adic valuation. Then by the choice of i, multiplication by $\gamma_j(\chi_j)-1$ is homotopic to zero on $K(\mathcal{O}_C; \gamma_i(\chi_i)-1)$ for any j. So [Sta19, Tag 0663] and [Sta19, Tag 0628] ensure that $K^\bullet$ is a finite direct sum of shifts of $(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C)$. So, it is sufficient to show that cohomology of the complex $(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C)$ are finitely presented, (ζ_p-1)-torsion, and have no $\mathfrak{m}$-torsion.

The first two claims follow from the computation $\mathrm{H}^0(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C) = 0$, $\mathrm{H}^1(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C) = \mathcal{O}_C/(\gamma_i(\chi_i)-1)\mathcal{O}_C$, and the fact that $\gamma_i(\chi_i)$ is a p-power root of unity. Now the last claim follows from the fact the the complex $(\mathcal{O}_C \xrightarrow{\gamma_i(\chi_i)-1} \mathcal{O}_C)$ is defined over the ring of integers of $\mathbf{Q}_p\left(\gamma_i(\chi_i)-1\right)$, so its cohomology groups cannot have any $\mathfrak{m}$-torsion as they are defined over some discretely valued field (see [ČK19, Lemma 3.5]). □

Lemma 4.2.17. Let $\mathfrak{X} = \operatorname{Spf} S^+$ be a framed poly-stable $\mathcal{O}_C$-model. Then the natural map

$$L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right)\right) \to L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma(X, \widehat{\mathcal{O}}^+_X)\right)$$

is an (honest) isomorphism.

Proof. We know that the map $\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right) \to \mathbf{R}\Gamma(X, \widehat{\mathcal{O}}^+_X)$ is an almost isomorphism by Lemma 4.2.13. Now [BMS18, Lemma 8.11(ii)] says that in order to show that the map

$$L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right)\right) \to L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma(X, \widehat{\mathcal{O}}^+_X)\right)$$

is an isomorphism, it is sufficient to show $\mathrm{H}^i_{cont}(\Delta, S^+_\infty)$ and $\frac{\mathrm{H}^i_{cont}(\Delta,S^+_\infty)}{\mathrm{H}^i_{cont}(\Delta,S^+_\infty)[1-\zeta_p]}$ have no non-zero $\mathfrak{m}$-torsion for all n. Lemma 4.2.16 shows that $\mathrm{H}^i_{cont}(\Delta, S^+_\infty)$ has no non-zero $\mathfrak{m}$-torsion, and also that $\frac{\mathrm{H}^i_{cont}(\Delta,S^+_\infty)}{\mathrm{H}^i_{cont}(\Delta,S^+_\infty)[1-\zeta_p]}$ is finite free. In particular, it has no $\mathfrak{m}$-torsion. This finishes the proof. □

Theorem 4.2.18. Let $\mathfrak{X} = \operatorname{Spf} S^+$ be a framed poly-stable $\mathcal{O}_C$-model. Then the natural map

$$L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right)^\Delta\right) \to L\eta_{1-\zeta_p}\left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)$$

is an (honest) isomorphism.

Proof. In order to get the claim, we need to show that, for any non-zero $f \in S^+$, the natural map

$$L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right)\widehat{\otimes}_{S^+}S^+_{\{f\}}\right) \to L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma\left(X\left(\frac{f}{1}\right), \widehat{\mathcal{O}}^+_{X_{\left(\frac{f}{1}\right)}}\right)\right)$$

is an isomorphism. Lemma 4.2.15 guarantees that

$$\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_\infty\right)\widehat{\otimes}_{S^+}S^+_{\{f\}} \simeq \mathbf{R}\Gamma_{cont}\left(\Delta, S^+_{\{f\},\infty}\right)\ .$$

So what we really need to show is that the natural map

$$L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, S^+_{\{f\},\infty}\right)\right) \to L\eta_{1-\zeta_p}\left(\mathbf{R}\Gamma\left(X\left(\frac{f}{1}\right), \widehat{\mathcal{O}}^+_{X_{\left(\frac{f}{1}\right)}}\right)\right)$$

is an isomorphism. This follows from Lemma 4.2.17. ☐

Corollary 4.2.19. Under the assumption as above, $\mathrm{R}^i\nu_*\widehat{\mathcal{O}}^+_X[1-\zeta_p] = \mathrm{R}^i\nu_*\widehat{\mathcal{O}}^+_X[(1-\zeta_p)^\infty]$ and $\frac{\mathrm{R}^i\nu_*\widehat{\mathcal{O}}^+_X}{\mathrm{R}^i\nu_*\widehat{\mathcal{O}}^+_X[1-\zeta_p]}$ is locally free of rank $\binom{d}{i}$.

Proof. This follows from Theorem 4.2.18 and Lemma 4.2.16. ☐

Theorem 4.2.20. Let $\mathfrak{X} = \operatorname{Spf} S^+$ be a framed poly-stable $\mathcal{O}_C$-model. Then the Faltings' map

$$\mathrm{Tr}^{+,d}_{F,\mathfrak{X}} \colon \mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X \to \omega_{\mathfrak{X}}(-d)$$

induces the (honest) isomorphism

$$t^+_{\mathfrak{X}} \colon \frac{\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X}{(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X)[1-\zeta_p]} \xrightarrow{\sim} \omega_{\mathfrak{X}}(-d) \simeq \omega^\bullet_{\mathfrak{X}}(-d)[d]$$

Proof. First of all, Lemma 4.2.4 guarantees that $\omega_{\mathfrak{X}}(-d) \simeq \omega^\bullet_{\mathfrak{X}}(-d)[-d]$ is a line bundle. Moreover, Corollary 4.2.19 ensures that $\frac{\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X}{(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X)[1-\zeta_p]}$ is a line bundle. In particular, both sheaves are reflexive. Now note that $\mathfrak{X}$ has (geometrically) reduced special fiber and smooth generic fiber. Thus, Corollary B.4 says that it is sufficient to check that the map $t^+_{\mathfrak{X}}$ is an isomorphism on the smooth locus $\mathfrak{X}^{\mathrm{sm}}$ and the generic fiber $\mathfrak{X}_C$.

After unravelling the definition, the former case follows [BMS18, Theorem 8.3] and the latter case follows from [Sch13b, Proposition 3.23]. ☐

Corollary 4.2.21. Let $\mathfrak{X} = \operatorname{Spf} S^+$ be a framed poly-stable $\mathcal{O}_C$-model. Then the Faltings' trace map

$$\mathcal{H}^d(\mathrm{Tr}_{F,\mathfrak{X}}) \colon \mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a \to \omega^a_{\mathfrak{X}_0}(-d)$$

induces the isomorphism

$$t_{\mathfrak{X}} \colon \frac{\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a}{\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a[1-\zeta_p]} \xrightarrow{\sim} \omega^a_{\mathfrak{X}_0}(-d) \simeq \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[d]$$

of $\mathcal{O}^a_{\mathfrak{X}_0}$-modules.

Proof. We recall that $\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X$ is almost concentrated in degrees $[0,d]$ by [Zav21a, Theorem 6.13.6]. Thus $\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a \simeq \mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right)^a/p\left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X\right)^a$. Moreover, Theorem 4.2.20 and Lemma 4.2.4 imply that, locally on $\mathfrak{X}$,

$$\mathrm{R}^d\nu_*\left(\widehat{\mathcal{O}}^+_X\right) \cong \mathcal{L} \oplus \mathcal{F},$$

where $\mathcal{L}$ is a line a bundle and $\mathcal{F}$ is $(1-\zeta_p)$-torsion sheaf. This implies that

$$\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a \simeq \mathcal{L}^a/p\mathcal{L}^a \oplus \mathcal{F}^a/p\mathcal{F}^a \text{ and } \left(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X/p\right)^a[1-\zeta_p] \simeq \mathcal{F}^a/p\mathcal{F}^a.$$

We now recall that $\mathcal{H}^d(\mathrm{Tr}_{F,\mathfrak{X}}) \simeq^a \mathrm{Tr}^+_{F,\mathfrak{X}}/p$ and that $\omega_{\mathfrak{X}_0}(-d) \simeq \omega_{\mathfrak{X}}(-d)/p\omega_{\mathfrak{X}}(-d)$ by Theorem 2.1.6.

Finally, Theorem 4.2.20 implies that

$$t_{\mathfrak{X}} \colon \mathcal{L} \simeq \frac{\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X}{(\mathrm{R}^d\nu_*\widehat{\mathcal{O}}^+_X)[1-\zeta_p]} \to \omega_{\mathfrak{X}}(-d)$$

is an isomorphism. Therefore,

$$t_{\mathfrak{X}_0} \colon \mathcal{L}^a/p\mathcal{L}^a \simeq \frac{\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a}{\mathrm{R}^d\nu_*(\mathcal{O}^+_X/p)^a[1-\zeta_p]} \to \omega^a_{\mathfrak{X}_0}(-d)$$

is an (almost) isomorphism as well. $\square$

4.3. **Almost Duality in Group Cohomology.** We show that, for any model polystable $\mathcal{O}_C$-model $\mathfrak{X} = \operatorname{Spf} R^+$ of dimension d, the cohomology groups $\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p)$ is an almost self dual complex via the explicit trace map

$$t^+_{R^+}\colon \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty) \to \mathrm{H}^d_{cont}(\Delta, R^+)[-d] \simeq R^+[-d]$$

that comes from the map $\mathrm{H}^d_{cont}(\Delta, R^+_\infty) \to \mathrm{H}^d_{cont}(\Delta, R^+)$ induced by the projection $R^+_\infty \to R^+$ coming from decomposition (4.6). There is also an evident mod-p version of the trace map

$$t_{R^+}\colon \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \to \mathrm{H}^d_{cont}(\Delta, R^+/p)[-d] \simeq \left(R^+/p\right)[-d]$$

The main goal of this section is to show that the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \otimes^L_{R^+/p} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \xrightarrow{\cup} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \xrightarrow{t} \left(R^+/p\right)[-d]$$

is an almost perfect, i.e. the natural map

$$D_{R^+}\colon \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \to \mathbf{R}\mathrm{Hom}_{R^+/p}\left(\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p), \left(R^+/p\right)[-d]\right)$$

is an almost isomorphism.

Remark 4.3.1. We will see in the proof of Theorem 4.3.4 that D is not an (honest) isomorphism, and is merely an almost isomorphism. This will be enough for all our purposes, but we want to emphasize the important of using almost mathematics at this place.

Lemma 4.3.2. Let $\chi \in X(T)[1/p]$ be any "rational" character of T. Then the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}) \otimes^L_{R^+} \mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+}) \to \mathbf{R}\Gamma_{cont}(\Delta, R^+) \to R^+[-d]$$

is a perfect pairing (see the discussion before Lemma 4.2.9 for the defitinion of $(V_{\overline{\chi}})_{R^+}$).

Proof. As the statement is symmetric in χ and $-\chi$, it is sufficient to show that the map

$$\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}) \to \mathbf{R}\mathrm{Hom}_{R^+}(\mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+}), R^+[-d])$$

is an isomorphism.

We recall that $V_{\overline{\chi}}$ comes as the base change $V_{\overline{\chi}} \simeq V_\chi \otimes_{\mathcal{O}_C} R^+$. So we have a commutative diagram

$$\begin{tikzcd}
\left(\mathbf{R}\Gamma_{cont}(\Delta, V_\chi) \otimes^L_{\mathcal{O}_C} R^+\right) \otimes^L_{R^+} \left(\mathbf{R}\Gamma_{cont}(\Delta, V_{-\chi}) \otimes^L_{\mathcal{O}_C} R^+\right) \arrow[r, "\cup \otimes_{\mathcal{O}_C} R^+"] \arrow[d] & \mathbf{R}\Gamma_{cont}(\Delta, \mathcal{O}_C) \otimes^L_{\mathcal{O}_C} R^+ \arrow[r, "t^+_{\mathcal{O}_C} \otimes_{\mathcal{O}_C} R^+"] \arrow[d] & R^+[-d] \arrow[d] \\
(\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}) \otimes^L_{R^+} (\mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+})) \arrow[r, "\cup"] & \mathbf{R}\Gamma_{cont}(\Delta, R^+) \arrow[r, "t^+_{R^+}"] & R^+[-d]
\end{tikzcd} \tag{4.7}$$

with $t^+_{\mathcal{O}_C}\colon \mathbf{R}\Gamma_{cint}(\Delta, \mathcal{O}_C) \to \mathrm{H}^d_{cont}(\Delta, \mathcal{O}_C)[-d] \simeq \mathcal{O}_C[-d]$ being the natural projection map. All vertical maps in diagram (4.7) are isomorphisms by Lemma E.3[27]. Thus, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma_{cont}(\Delta, V_\chi) \otimes^L_{\mathcal{O}_C} R^+ \arrow[r] \arrow[dr] & \mathbf{R}\mathrm{Hom}_{R^+}(\mathbf{R}\Gamma_{cont}(\Delta, V_{-\chi}) \otimes_{\mathcal{O}_C} R^+, R^+[-d]) \\
& \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}(\mathbf{R}\Gamma_{cont}(\Delta, V_{-\chi}), \mathcal{O}_C[-d]) \otimes^L_{\mathcal{O}_C} R^+ \arrow[u]
\end{tikzcd}$$

[27]And the calculation from the proof of Lemma 4.2.16 that shows that cohomology groups of $\mathbf{R}\Gamma_{cont}(\Delta, V_\chi)$ are almost finitely presented over $\mathcal{O}_C$.

The vertical map is an isomorphism by [Sta19, Tag 0ATK]. Thus, it is sufficient to show that the diagonal map is an isomorphism, i.e. the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, V_\chi) \otimes^L_{R^+} \mathbf{R}\Gamma_{cont}(\Delta, V_{-\chi}) \xrightarrow{\cup} \mathbf{R}\Gamma_{cont}(\Delta, \mathcal{O}_C) \xrightarrow{t^+_{\mathcal{O}_C}} \mathcal{O}_C[-d]$$

is perfect.

We know choose a trivialization of the torus $T \cong \mathbf{G}_m^d$ and a compatible trivialization of $\Delta \cong \prod_{i=1}^d \Delta_i \cong \mathbf{Z}_p^d$ using some choice of p-power roots of unity $(1, \zeta_p, \zeta_{p^2}, \dots)$. Then any rational character $\chi \in X(T)[1/p]$ can be written as

$$\chi = (\chi_1, \dots, \chi_d),$$

where $\chi_i \in X(\mathbf{G}_{m,i})[1/p]$ is a rational character of $\mathbf{G}_{m,i}$. Then using the argument with Koszul complexes from [BMS18, Lemma 7.3], we easily see that the natural map $\otimes^L_{\mathcal{O}_C} \mathbf{R}\Gamma_{cont}(\Delta_i, V_{\chi_i}) \to \mathbf{R}\Gamma_{cont}(\Delta, V_\chi)$ is an isomorphism, and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma_{cont}\left(\Delta, V_\chi\right) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, V_{-\chi}\right), \mathcal{O}_C[-d]\right) \\
\otimes^L_{\mathcal{O}_C} \mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{\chi_i}\right) \arrow[u] \arrow[r] \arrow[dr] & \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\otimes^L_{\mathcal{O}_C} \mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{-\chi_i}\right), \mathcal{O}_C[-d]\right) \arrow[u] \\
& \otimes^L_{\mathcal{O}_C} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{-\chi_i}\right), \mathcal{O}_C[-1]\right) \arrow[u]
\end{tikzcd}$$

with vertical maps being isomorphisms. Therefore, it suffices to show that the the map

$$\otimes^L_{\mathcal{O}_C} \mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{\chi_i}\right) \to \otimes^L_{\mathcal{O}_C} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{-\chi_i}\right), \mathcal{O}_C[-1]\right)$$

is an isomorphism. Then it is sufficiesnt to show that the map

$$\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{\chi_i}\right) \to \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{-\chi_i}\right), \mathcal{O}_C[-1]\right)$$

is an isomorphism for each $i = 1, \dots, d$.

Now we choose the standard identification of $\mathbf{R}\Gamma_{cont}(\Delta_i, V_{\pm\chi_i})$ with the Koszul complex

$$L_i^\bullet \coloneqq (\mathcal{O}_C \xrightarrow{\gamma_i(\pm\overline{\chi_i})-1} \mathcal{O}_C).$$

Then the map $\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{\chi_i}\right) \to \mathbf{R}\mathrm{Hom}_{\mathcal{O}_C}\left(\mathbf{R}\Gamma_{cont}\left(\Delta_i, V_{-\chi_i}\right), \mathcal{O}_C[-1]\right)$ becomes the map

$$\begin{tikzcd}
\mathcal{O}_C \arrow[r, "\gamma(\overline{\chi})-1"] \arrow[d, "\mathrm{Id}"] & \mathcal{O}_C \arrow[d, "\gamma(\overline{\chi})^{-1}"] \\
\mathcal{O}_C \arrow[r, "1-\gamma(\overline{\chi})^{-1}"] & \mathcal{O}_C
\end{tikzcd}$$

that is easily seen to be a quasi-isomorphism as $\gamma(\overline{\chi})$ is invertible in $\mathcal{O}_C$. $\square$

Corollary 4.3.3. Let $\chi \in X(T)[1/p]$ be any rational character of T. Then the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p) \otimes^L_{R^+} \mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+}/p) \to \mathbf{R}\Gamma_{cont}(\Delta, R^+/p) \to R^+/p[-d]$$

is a perfect pairing.

Proof. This follows from Lemma 4.3.2 by applying the derived tensor product $-\otimes^L_R R^+/p$ to the perfect pairing in Lemma 4.3.2. One uses [Sta19, Tag 0E1W] and [Sta19, Tag 0ATK] to make sure that

$$\mathbf{R}\mathrm{Hom}_{R^+}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, (V_{-\overline{\chi}})_{R^+}\right), R^+[-d]\right) \simeq \mathbf{R}\mathrm{Hom}_{R^+/p}\left(\mathbf{R}\Gamma_{cont}\left(\Delta, (V_{-\overline{\chi}})_{R^+}/p\right), R^+/p[-d]\right).$$

□

Theorem 4.3.4. Let $\mathfrak{X} = \mathrm{Spf}\, R^+$ be a model polystable formal $\mathcal{O}_C$-scheme. Then the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \otimes^L_{R^+/p} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \xrightarrow{\cup} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \xrightarrow{t} R^+/p[-d]$$

is almost perfect.

Proof. We note that decomposition (4.6) induces the decomposition

$$R^+_\infty/p = \bigoplus_{\overline{\chi}\in X(T)[1/p]/X(T)} (V_{\overline{\chi}})_{R^+}/p\ .$$

As the continuous cohomology of a profinite group with discrete coefficients commute with infinite direct sums, we get the decomposition

$$\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \simeq \bigoplus_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p)\ .$$

This also induces the decomposition

$$\mathbf{R}\mathrm{Hom}_{R^+/p}(\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p), R^+/p[-d]) \simeq \prod_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\mathrm{Hom}_{R^+/p}(\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p), R^+/p[-d])\ .$$

Now, we note that the trace map is induced by the projection projection $R^+_\infty/p \to V_{\overline{0}}/p$, so we see that the induced pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, V_{\overline{\chi}}/p) \otimes^L_{R^+/p} \mathbf{R}\Gamma_{cont}(\Delta, V_{\overline{\chi'}}/p) \to R^+/p[-d]$$

is zero if $\overline{\chi} \neq -\overline{\chi'}$, and the pairing from Corollary 4.3.3 if $\overline{\chi} = -\overline{\chi'}$. In other words, the duality map

$$D_{R^+}\colon \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \to \mathbf{R}\mathrm{Hom}_{R^+/p}\left(\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p), \left(R^+/p\right)[-d]\right)$$

can be identified with the map

$$\bigoplus_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p) \to \prod_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\mathrm{Hom}_{R^+/p}(\mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+}/p), R^+/p[-d]) \tag{4.8}$$

that component-wise is the duality map coming from Corollary 4.3.3. In particular, Corollary 4.3.3 implies that the map (4.8) is an isomorphism component-wise on each component. However, this map is clearly not an isomorphism as we have an infinite direct sum on the left side and infinite direct product on the right side. We claim that this map is an almost isomorphism.

That being said, it is sufficient to show that the natural "inclusion"

$$\bigoplus_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p) \to \prod_{\overline{\chi}\in X(T)[1/p]/X(T)} \mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p) \tag{4.9}$$

is an almost isomorphism.

As always, we now fix a trivialization $T \cong \mathbf{G}^d_m$ and a compatible trivialization

$$\Delta \simeq \mathrm{Hom}\left(X(T)[1/p]/X(T), \mu_{p^\infty}(C)\right) \cong \mathbf{Z}_p^d$$

with generators $\gamma_1, \dots, \gamma_d$. If $\overline{\chi} = (\overline{\chi_1}, \dots, \overline{\chi_d}) \in X(T)[1/p]/X(T)$ with $\gamma_i(\chi_i) - 1$ having the smallest p-adic valuation, then cohomology groups of $\mathbf{R}\Gamma_{cont}(\Delta, (V_{\overline{\chi}})_{R^+}/p)$ are annihilated by $\gamma_i(\overline{\chi}_i) - 1 = \zeta_{p^v}^a - 1$ where v is the order of $\gamma_i(\overline{\chi}_i) \in \mu_{p^\infty}(C)$ and $(a,p) = 1$. There are only finitely many $\overline{\chi}_i \in X(\mathbf{G}_{m,i})[1/p]/X(\mathbf{G}_{m,i})$ such that the order of $\gamma_i(\overline{\chi_i}) \le v$ for any fixed v.

Now we recall that $v_p(\zeta_{p^v}^a - 1) = \frac{1}{p^v - p^{v-1}}$ is approaching 0 as v grows. Thus, for each $\epsilon > 0$, cohomology groups of $\mathbf{R}\Gamma_{cont}(\Delta, (V_{-\overline{\chi}})_{R^+}/p)$ are annihilated by p^ϵ for all but finitely many rational characters $\overline{\chi}$. Thus, the map (4.9) is indeed an almost isomorphism as this map is an isomorphism term-wise, and, for each $\epsilon > 0$, all but finitely many terms are p^ϵ-torsion. □

4.4. **Local Duality for $\mathbf{R}\nu_* \left(\mathcal{O}_X^+/p\right)^a$ on Polystable Models.** We are now finally ready to show that Faltings' trace is almost perfect on any (rig-smooth) polystable formal $\mathcal{O}_C$-scheme $\mathfrak{X}$. Given the work already done in previous subsections, the proof is quite easy. We firstly use Lemma 3.6.3 to reduce to the case of a model polystable formal $\mathcal{O}_C$-scheme. Then we use Corollary 4.2.21 and Lemma 4.2.13 to reduce the original question to the almost perfectness of the certain pairing on continuous cohomology groups of the profinite group Δ. And this problem was already solved in Theorem 4.3.4.

Theorem 4.4.1. Let $\mathfrak{X}$ be a rig-smooth separated polystable formal $\mathcal{O}_C$-scheme with the adic generic fiber $X = \mathfrak{X}_C$ of pure dimension d. Then the pairing

$$\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \xrightarrow{\mathrm{Tr}_{F,\mathfrak{X}}} \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$$

is almost perfect.

Proof. Step 1: We reduce to a model polystable formal $\mathcal{O}_C$-scheme $\mathfrak{X} = \operatorname{Spf} R^+$. Since the question is symmetric in both variables, it is sufficient to show that the induced map

$$\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}_0}(\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])$$

is an almost isomorphism. The question is local on $\mathfrak{X}$, so we can assume that there is a tower of étale maps

$$\begin{tikzcd} & \mathfrak{U} \arrow[dl, "\mathfrak{g}"'] \arrow[dr, "\mathfrak{f}"] & \\ \mathfrak{X} & & \operatorname{Spf} R^+ \end{tikzcd}$$

with $\operatorname{Spf} R^+$ a model polystable formal $\mathcal{O}_C$-scheme. Now note that since f_0 is étale (in particular, it is flat) and $\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \in \mathbf{D}^b_{acoh}(X)$ by [Zav21a, Theorem 6.13.5]. Therefore, [Zav21a, Theorem 6.13.5, Lemma 4.4.10, and Lemma 2.9.12] and flatness of the morphism $R^+ \to \mathcal{O}_{\mathfrak{U}}(\mathfrak{U})$ imply that the natural morphisms

$$\mathbf{L}\mathfrak{f}_0^*\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a \to \mathbf{R}\nu_{\mathfrak{U},*}(\mathcal{O}_U^+/p)^a$$

$$\mathbf{L}\mathfrak{f}_0^*\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}_0}(\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{U}_0}(\mathbf{L}\mathfrak{f}_0^*\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{O}_X^+/p)^a, \mathbf{L}\mathfrak{f}_0^*\omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])$$

are isomorphisms. Moreover, Lemma 3.6.3[28] implies that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{L}\mathfrak{f}_0^*\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \arrow[r] \arrow[dd] & \mathbf{L}\mathfrak{f}_0^*\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}_0}(\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \arrow[d] \\
 & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{U}_0}(\mathbf{L}\mathfrak{f}_0^*\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a, \mathbf{L}\mathfrak{f}_0^*\omega^{\bullet,a}_{\mathfrak{X}_0}(-d)) \arrow[d] \\
\mathbf{R}\nu_{\mathfrak{U},*}(\mathcal{O}_U^+/p)^a \arrow[r] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{U}_0}(\mathbf{R}\nu_{\mathfrak{U},*}(\mathcal{O}_U^+/p)^a, \omega^{\bullet,a}_{\mathfrak{U}_0}(-d)[-2d])
\end{tikzcd}$$

is commutative with vertical maps being isomorphisms. Thus, the (almost) faithfully flat base change implies that it is sufficient to prove the pairing is perfect for $\mathfrak{X} = \mathfrak{U}$. Then the same argument reduces the situation to the case $\mathfrak{X} = \operatorname{Spf} R^+$.

Step 2: We reduce to the almost duality in group cohomology. Now we note that Corollary 4.2.21 implies that the trace map $\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$ can be identified with the map

$$\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a[-d] \to \frac{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a}{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a[1-\zeta_p]}[-d] \simeq \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d] \ .$$

Therefore, we can reformulate the almost perfectness of the pairing induces by the Faltings' trace intrinsically in terms of $\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a$. More precisely, it is sufficient to show that the map

$$\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \xrightarrow{\cup} \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \frac{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a}{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a[1-\zeta_p]}$$

is an almost perfect pairing. Now Lemma 4.2.13 and Corollary 4.2.21 ensure that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma_{cont}\widetilde{(\Delta, R^+_\infty/p)}^a \otimes^L_{\mathcal{O}^a_{\mathfrak{X}_0}} \mathbf{R}\Gamma_{cont}\widetilde{(\Delta, R^+_\infty/p)}^a \arrow[r, "\cup"] \arrow[d] & \mathbf{R}\Gamma_{cont}\widetilde{(\Delta, R^+_\infty/p)}^a \arrow[r] \arrow[d] & \mathrm{H}^d_{cont}\widetilde{(\Delta, R^+/p)}[-d]^a \arrow[d] \\
\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}^a_{\mathfrak{X}_0}} \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \arrow[r] & \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \arrow[r] & \frac{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a}{\mathrm{R}^d\nu_*(\mathcal{O}_X^+/p)^a[1-\zeta_p]}
\end{tikzcd}$$

with vertical maps being isomorphisms. Now we use [Zav21a, Lemma 4.4.10 and Theorem 6.13.5] to reduce the question to the almost perfectness of the pairing

$$\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \otimes^L_{R^+/p} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \xrightarrow{\cup} \mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p) \to \mathrm{H}^d_{cont}(\Delta, R^+/p)[-d] \ .$$

Step 3: We show that $\mathbf{R}\Gamma_{cont}(\Delta, R^+_\infty/p)$ is almost self-dual. This part was already done in Theorem 4.3.4. □

4.5. **Local Duality on a General Rig-Smooth Admissible Formal $\mathcal{O}_C$-scheme.** The main goal of this section is to generalize Theorem 4.4.1 to any admissible separated formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ with smooth generic fiber $X = \mathfrak{X}_C$ of pure dimension d.

The main idea of our proof is to reduce to the polystable case using the version of the local uniformization result from [Zav21b]. Let us recall the main result of *loc. cit.* in the form we need it:

[28] Its mod-p version that is a formal consequence of Lemma 3.6.3 itself.

Theorem 4.5.1. ([Zav21b, Theorem 1.3]) Let $\mathfrak{X}$ be an admissible[29] formal $\mathcal{O}_C$-scheme with smooth generic fiber $\mathfrak{X}_C$. Then there is a finite set $(\mathfrak{X}_i, \mathfrak{f}_i)_{i\in I}$ of admissible formal $\mathcal{O}_C$-schemes with morphisms $\mathfrak{f}_i\colon \mathfrak{X}_i \to \mathfrak{X}$ such that

- The set $(\mathfrak{X}_i, \mathfrak{f}_i)$ can be obtained from $\mathfrak{X}$ as a composition of open Zariski coverings and rig-isomorphisms.
- Each $\mathfrak{X}_i$ is a geometric quotient of an admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}'_i$ by an action of a finite group G_i such that the quotient map $\mathfrak{g}_{i,C}\colon \mathfrak{X}'_{i,C} \to \mathfrak{X}_{i,C}$ is a G_i-torsor.
- Each $\mathfrak{X}'_i$ admits a rig-isomorphism $\pi_i\colon \mathfrak{X}''_i \to \mathfrak{X}'_i$ with a rig-smooth, polystable formal $\mathcal{O}_C$-scheme $\mathfrak{X}''_i$.

So, in order to show that the Faltings' pairing

$$\mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \to \mathbf{R}\nu_*(\mathcal{O}_X^+/p)^a \xrightarrow{\mathrm{Tr}_{F,\mathfrak{X}}} \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]$$

is almost perfect, it suffices to show that almost perfectness of Faltings' pairing descends through rig-isomorphisms and "good quotients" (as in Theorem 4.5.1) because we have alrady established its almost perfectness on polystable models in Theorem 4.4.1.

Lemma 4.5.2. Let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a rig-isomorphism of admissible separated formal $\mathcal{O}_C$-schemes with smooth generic fibers of pure dimension d. Suppose that Faltings' pairing is almost perfect on $\mathfrak{X}'$. Then the same holds on $\mathfrak{X}$.

Proof. We note that Corollary 3.6.2 implies that the diagram

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_X^+/p\right)^a \arrow[r, "D_{\mathfrak{X}}"] \arrow[dd, "\sim"] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_X^+/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \arrow[d, "\sim"] \\
 & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\pi_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}_{X'}^+/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \\
\mathbf{R}\pi_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}_{X'}^+/p\right)^a \arrow[r, "\mathbf{R}\pi_*(D_{\mathfrak{X}'})"] & \mathbf{R}\pi_*\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}'}(\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}_{X'}^+/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d]), \arrow[u]
\end{tikzcd}$$

where the right bottom vertical map is the isomorphism from [Zav21a, Lemma 5.5.6]. Other horizontal maps are also clearly isomorphisms, and the bottom horizontal morphism is an isomorphism by assumption. Therefore, the top horizontal morphism is an isomorphism as well. ☐

Now we deal with the case of quotients. We need to recall the definition of homotopy invariants and co-invariants on a space with a trivial action of a finite group G.

Definition 4.5.3. Let $(X, \mathcal{O}_X)$ be a ringed space with a trivial action of a finite group G, and let $\mathcal{F} \in \mathbf{D}(\mathcal{O}_X[G])$. Then the complex of *homotopy invariants* (resp. *homotopy co-invariants*) is $\mathcal{F}^{hG} := \mathcal{F} \otimes^L_{\mathcal{O}_X[G]} \mathcal{O}_X$ (resp. $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X[G]}(\mathcal{O}_X, \mathcal{F})$) where $\mathcal{O}_X$ is given the structure of an $\mathcal{O}_X[G]$-module via the surjection $\mathcal{O}_X[G] \to \mathcal{O}_X$.

Remark 4.5.4. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a G-invariant morphism for some finite group action G on X. Then the complex $\mathbf{R}f_*(\mathcal{O}_X)$ is naturally an object of $\mathbf{D}(\mathcal{O}_Y[G])$.

[29] In this paper, we assume that any admissible formal scheme is quasi-compact and quasi-separated by definition

Lemma 4.5.5. Let X be a rigid space over C with an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$, and $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ a geometric quotient map for an $\mathcal{O}_C$-action of a finite group $G \curvearrowright \mathfrak{X}'$ that induces a free action on the generic fiber $\mathfrak{X}'_C = X'$. Then the pro-étale trace map

$$\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)$$

induces an isomorphism

$$\left(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)\right)_{hG} \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right).$$

Similarly, the restriction map $\mathrm{Res}_{\mathfrak{f}}\colon \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right) \to \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)$ induces an isomorphism

$$\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right) \to \left(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)\right)^{hG}.$$

Proof. We note that

$$\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right) \simeq \mathbf{R}\nu_{\mathfrak{X},*}\mathbf{R}f_*\left(\mathcal{O}^+_{X'}/p\right)$$

where $f\colon X' \to X$ is the generic fiber of $\mathfrak{f}$. Now we note that

$$\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}} \simeq \mathbf{R}t_{\mathfrak{X},*}(\mathrm{Tr}_{\text{ét},f}),$$
$$\mathrm{Res}_{\mathfrak{f}} \simeq \mathbf{R}t_{\mathfrak{X},*}(\mathrm{Res}_f),$$

where $t\colon (X_{\text{ét}}, \mathcal{O}^+_X/p) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}_0})$ is the natural morphism of ringed sites. Since both homotopy invariants and homotopy co-invariants commute with $\mathbf{R}t_*$, it suffices to show that the maps

$$\left(\mathbf{R}f_{\text{ét},*}\mathcal{O}^+_{X'}/p\right)_{hG} \to \mathcal{O}^+_X/p,$$
$$\mathcal{O}^+_X/p \to \left(\mathbf{R}f_{\text{ét},*}\mathcal{O}^+_{X'}/p\right)^{hG}.$$

are isomorphisms.

The claim is étale local on X, so we can assume that f is a split finite étale morphism. Both claims are trivial in this case. □

Lemma 4.5.6. Let $(X, \mathcal{O}_X)$ be a ringed space, and G a finite group. Let $\mathcal{F} \in \mathbf{D}(\mathcal{O}_X[G])$ and $\mathcal{G} \in \mathbf{D}(\mathcal{O}_X)$. Then the natural map $\mathbf{R}\underline{\mathcal{H}om}_X(\mathcal{F}, \mathcal{G})^{hG} \to \mathbf{R}\underline{\mathcal{H}om}_X(\mathcal{F}_{hG}, \mathcal{G})$ is an isomorphism.

Proof. Unravelling the definitions, we see that one has to show that the natural map

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X[G]}(\mathcal{O}_X, \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F} \otimes^L_{\mathcal{O}_X[G]} \mathcal{O}_X, \mathcal{G})$$

is an isomorphism. This is just the standard (derived) tensor-hom adjunction. □

Lemma 4.5.7. Let $\mathfrak{X}$ be an admissible separated formal $\mathcal{O}_C$-scheme with smooth generic fiber of pure dimension d, and $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ a G-invariant $\mathcal{O}_C$-morphism for an $\mathcal{O}_C$-linear action of a finite group G on $\mathfrak{X}'$. Suppose that $\mathfrak{f}$ on generic fibers $f\colon X' \to X$ is a G-torsor and that Faltings' pairing is almost perfect on $\mathfrak{X}'$. Then the same holds on $\mathfrak{X}$.

Proof. We note that Corollary 3.6.2 implies that the diagram

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \arrow[r, "D_{\mathfrak{X}}"] \arrow[dd, "\mathrm{Res}_{\mathfrak{f}}"] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \arrow[d] \\
& \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \\
\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a \arrow[r, "\mathbf{R}\mathfrak{f}_*(D_{\mathfrak{X}'})"] & \mathbf{R}\mathfrak{f}_*\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}'}(\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d]), \arrow[u]
\end{tikzcd}$$

where the right bottom vertical map is the isomorphism from [Zav21a, Lemma 5.5.6], and the right top vertical map is induced by

$$\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right).$$

Now we observe that the bottom horizontal arrow is an isomorphism by the assumption, and the right bottom arrow is an isomorphism as notes above. Therefore, we can take its inverse to get the commutative diagram

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \arrow[r, "D_{\mathfrak{X}}"] \arrow[d, "\mathrm{Res}_{\mathfrak{f}}"] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \arrow[d] \\
\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a \arrow[r, "\mathbf{R}\mathfrak{f}_*(D_{\mathfrak{X}'})"] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]),
\end{tikzcd} \tag{4.10}$$

with the bottom arrow an almost isomorphism. Now note that both

$$\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a \text{ and } \mathbf{R}\mathfrak{f}_*\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}'}(\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}'_0}(-d)[-2d])$$

are naturally elements of $\mathbf{D}(\mathcal{O}_X[G])^a$ by Remark 4.5.4.

Moreover, we note that morphisms in the diagram (4.10) are all G-equivariant if $\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a$ and $\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])$ are endowed with the trivial G-action. This implies that the diagram (4.10) induces the commutative diagram:

$$\begin{tikzcd}
\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a \arrow[r, "D_{\mathfrak{X}}"] \arrow[d] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \arrow[d] \\
\left(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a\right)^{hG} \arrow[r, "(\mathbf{R}\mathfrak{f}_*(D_{\mathfrak{X}'}))^{hG}"] \arrow[d] & \left(\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])\right)^{hG} \arrow[d] \\
\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a \arrow[r, "\mathbf{R}\mathfrak{f}_*(D_{\mathfrak{X}'})"] & \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]),
\end{tikzcd}$$

Now we note that $(\mathbf{R}\mathfrak{f}_*(D_{\mathfrak{X}'}))^{hG}$ is an isomorphism since $D'_{\mathfrak{X}'}$ is. Moreover, the top left vertical arrow is an isomorphism by Lemma 4.5.5. Thus, it suffices to show that the top right vertical arrow

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) \to \left(\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])\right)^{hG}$$

is an isomorphism. Lemma 4.5.6 ensures that it suffices to show that the composition

$$\begin{aligned}\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d]) &\to \left(\mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])\right)^{hG} \\ &\to \mathbf{R}\underline{al\mathcal{H}om}_{\mathfrak{X}}(\left(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)^a\right)_{hG}, \omega^{\bullet,a}_{\mathfrak{X}_0}(-d)[-2d])\end{aligned}$$

is anisomorphism. This, in turn, follows from Lemma 4.5.5 as already the morphism

$$\left(\mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right)\right)_{hG} \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)$$

is an isomorphism. □

Theorem 4.5.8. Let $\mathfrak{X}$ be a separated admissible formal $\mathcal{O}_C$-scheme with smooth generic fiber $X = \mathfrak{X}_C$ of pure dimension d. Then the Faltings' pairing is almost perfect on $\mathfrak{X}$.

Proof. We use Theorem 4.5.1 to find admissible formal $\mathcal{O}_C$-schemes $\mathfrak{X}_i$, $\mathfrak{X}'_i$, $\mathfrak{X}''_i$ and morphisms $\mathfrak{f}_i\colon \mathfrak{X}_i \to \mathfrak{X}$, $\mathfrak{g}_i\colon \mathfrak{X}'_i \to \mathfrak{X}_i$, and $\pi_i\colon \mathfrak{X}''_i \to \mathfrak{X}'_i$ such that

(1) The set $(\mathfrak{X}_i, \mathfrak{f}_i)$ can be obtained from $\mathfrak{X}$ as a composition of open Zariski coverings and rig-isomorphisms.

(2) Each $\mathfrak{g}_i \colon \mathfrak{X}'_i \to \mathfrak{X}_i$ is a geometric quotient map for an action of a finite G_i, and the map is a G_i-torsor over the generic fiber.

(3) $\pi_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_i$ is a rig-isomorphism, and $\mathfrak{X}''_i$ is a rig-smooth, polystable formal $\mathcal{O}_C$-scheme.

Step 1. Prove the claim for each $\mathfrak{X}''_i$: This was already done in Theorem 4.4.1 as $\mathfrak{X}''_i$ is assumed to be rig-smooth and polystable. We only note that $X''_i = \mathfrak{X}''_{i,C}$ is indeed pure of dimension d as it is étale over X.

Step 2. Prove the claim for each $\mathfrak{X}'_i$: We recall that the morphism $\pi_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_i$ is a rig-isomorphism between rig-smooth formal $\mathcal{O}_C$-models with generic fibers of pure dimension d. Then Lemma 4.5.2 ensures that the Faltings trace is almost perfect on $\mathfrak{X}'_i$ if it is almost perfect on $\mathfrak{X}''_i$.

Step 3. Prove the claim for each $\mathfrak{X}_i$: The proof is similar to Step 2. The only difference is that we use Lemma 4.5.7 instead of Lemma 4.5.2.

Step 4. Prove the claim for $\mathfrak{X}$: We note that the set $(\mathfrak{X}_i, \mathfrak{f}_i)$ can be obtained from $\mathfrak{X}$ as the composition of rig-isomorphisms and open Zariski coverings. Thus, in order to show that the Faltings' pairing is almost perfect on $\mathfrak{X}$, it suffice to show that this property descends through Zariski open coverings and rig-isomorphisms. The first claim is trivial as the question is Zariski local on $\mathfrak{X}$ by design. The fact that almost perfectness of the Faltings pairing descends through rig-isomorphisms was proven in Lemma 4.5.2. □

5. Global Duality

5.1. **Overview.** For the rest of the section, we fix a rank-1 valued field K of mixed characteristic $(0, p)$. We denote its completed algebraic closure by $C \coloneqq \widehat{\overline{K}}$.

Section 5 has two main goals. The first one is roughly to construct a trace map

$$\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \to \mathcal{O}_C^a/p(-d)[-2d]$$

on a smooth proper rigid C-space of pure dimension d, and show that this makes the complex $\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a$ almost self-dual. The second goal is to deduce Poincaré duality for the constant étale sheaf $\underline{\mathbf{F}}_p$ on X. Now we discuss both of our goals in more detail.

We firstly construct a global trace map

$$\mathrm{Tr}_X \colon \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \to \mathcal{O}_C^a/p(-d)[-2d].$$

The construction is quite formal. We choose an admissible formal model $\mathfrak{X}$ that is automatically proper by [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]). Then Tr_X comes by applying the almost version of Grothendieck Duality [Zav21a, §5.5] to the Faltings' trace map

$$\mathrm{Tr}_{F,\mathfrak{X}} \colon \mathbf{R}\nu_* \left(\mathcal{O}_X^+/p\right)^a \to \omega^\bullet_{\mathfrak{X}_0}(-d)[-2d].$$

This pairing a priori depends on a choice of $\mathfrak{X}$, but it will later be shown to be independent of this choice. Almost perfectness of the pairing induced by Tr_X will then formally follow from Theorem 4.5.8 and the almost version of Grothendieck Duality.

Theorem 5.1.1. Let X be a smooth proper rigid C-space of pure dimension d. Then the pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \xrightarrow{-\cup-} \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \xrightarrow{\mathrm{Tr}_X} \mathcal{O}_C^a/p(-d)[-2d]$$

is almost perfect.

Then we discuss the trace map map in étale cohomology. Now suppose that X is a smooth proper rigid K-space of pure dimension d. Then we will use the trace map

$$t_X\colon \mathbf{R}\Gamma(X_C, \mathbf{F}_p(d)) \to \mathbf{F}_p[-2d]$$

constructed by Berkovich[30] in [Ber93] to define the pairing

$$\mathbf{R}\Gamma(X_C, \mathbf{F}_p) \otimes^L_{\mathbf{F}_p} \mathbf{R}\Gamma(X_C, \mathbf{F}_p(d)) \xrightarrow{-\cup-} \mathbf{R}\Gamma(X_C, \mathbf{F}_p(d)) \xrightarrow{t^B_X} \mathbf{F}_p[-2d].$$

We will show that this pairing is perfect. The essential idea of this proof is to use the primitive comparison theorem

$$\mathrm{H}^i(X_C, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}^a_C/p \simeq \mathrm{H}^i(X_C, \mathcal{O}^+_{X_C}/p)^a$$

to reduce this claim to Theorem 5.1.1. This will recover Poincaré Duality on X.

Theorem 5.1.2. Let X be a smooth proper rigid K-space of pure dimension d. Then the pairing

$$\mathrm{H}^i(X_C, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}(X_C, \mathbf{F}_p(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \xrightarrow{\mathrm{H}^{2d}(t_X(d))} \mathbf{F}_p$$

is perfect for $i \geq 0$.

5.2. **Global Almost Duality.** In this section, we define the almost trace map

$$\mathrm{Tr}_X\colon \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to \mathcal{O}^a_C/p(-d)[-2d]$$

for any smooth and proper rigid C-space X of pure dimension d. Then we show that the associated pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \to \mathcal{O}^a_C/p(-d)[-2d]$$

is almost perfect.

We start with the construction of the trace map. We pick an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ that is automatically proper by [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]), and consider the associated morphism of ringed topoi

$$(X_{\text{proét}}, \mathcal{O}^+_X/p) \xrightarrow{\nu} (\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}).$$

Now we consider the Faltings' trace map constructed in Definition 3.5.7:

$$\mathrm{Tr}_{F,\mathfrak{X}}\colon \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a \to \omega^{\bullet;a}_{\mathfrak{X}_0}(-d)[-2d] \simeq f^!_0\left(\mathcal{O}^a_{\operatorname{Spec}\mathcal{O}_C/p}\right)(-d)[-2d]$$

where $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf}\mathcal{O}_C$ is the structure morphism. The $(\mathbf{R}f_*, f^!)$-adjunction in the almost world [Zav21a, Theorem 5.5.5] gives the morphism

$$\mathrm{Tr}^{\mathfrak{X}}_X\colon \mathbf{R}\Gamma(X, \mathcal{O}^+_X/p)^a \simeq \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_* \left(\mathcal{O}^+_X/p\right)^a) \to \mathcal{O}^a_C/p(-d)[-2d].$$

This construction, a priori, depends on the choice of an admissible model $\mathfrak{X}$. We show that actually this construction is canonically independent of this choice.

Lemma 5.2.1. Let X be as above. Then the trace maps $\mathrm{Tr}^{\mathfrak{X}}_X$ and $\mathrm{Tr}^{\mathfrak{X}'}_X$ are canonically identified for any two choices of admissible formal $\mathcal{O}_C$-models $\mathfrak{X}$ and $\mathfrak{X}'$.

[30]Berkovich defines the trace map in terms of Berkovich spaces. We will translate his trace map into the language of adic spaces.

Proof. Suppose we have two admissible formal models $\mathfrak{X}$ and $\mathfrak{X}'$ of the rigid space X. We choose another formal $\mathcal{O}_C$-model $\mathfrak{X}''$ of X that dominates both $\mathfrak{X}$ and $\mathfrak{X}'$. Therefore, for the purpose of proving that $\mathrm{Tr}_X^{\mathfrak{X}}$ can be identified with $\mathrm{Tr}_X^{\mathfrak{X}'}$, it is enough to assume that there is a rig-isomorphism $\pi\colon \mathfrak{X}' \to \mathfrak{X}$.

Now Lemma 3.6.1 implies that the following diagram commutes

$$\begin{tikzcd}
\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \arrow[r, "\sim"] & \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}_X^+/p\right)^a\right) \arrow[r, "\sim"] \arrow[d, "\mathbf{R}\Gamma(\mathfrak{X}_0, \mathrm{Tr}_{F,\mathfrak{X}})"] & \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\pi_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}_X^+/p\right)^a\right) \arrow[d, "\mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\pi_*(\mathrm{Tr}_{F,\mathfrak{X}'}))"] \\
\mathcal{O}_C^a/p(-d)[-2d] & \mathbf{R}\Gamma\left(\mathfrak{X}_0, \omega_{\mathfrak{X}_0}^{\bullet,a}(-d)[-2d]\right) \arrow[l, "\mathrm{Tr}_{\mathfrak{f}_0}"'] & \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\pi_*\omega_{\mathfrak{X}'_0}^{\bullet,a}(-d)[-2d]\right). \arrow[l, "\mathbf{R}\Gamma(\mathfrak{X}_0, \mathrm{Tr}_{\pi_0})"']
\end{tikzcd}$$

The inner square defines the morphism $\mathrm{Tr}_X^{\mathfrak{X}}\colon \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \to \mathcal{O}_C^a/p(-d)[-2d]$, and the outer square defines the morphism $\mathrm{Tr}_X^{\mathfrak{X}'}$ because $\mathrm{Tr}_{\mathfrak{f}_0} \circ \mathbf{R}\Gamma(\mathfrak{X}_0, \mathrm{Tr}_{\pi_0}) = \mathrm{Tr}_{\mathfrak{f}'_0}$. Therefore, $\mathrm{Tr}_X^{\mathfrak{X}} \simeq \mathrm{Tr}_X^{\mathfrak{X}'}$.

Now we need to show that this identification does not depend on a choice of a model $\mathfrak{X}''$ that dominates both $\mathfrak{X}$ and $\mathfrak{X}''$. This is done by a standard argument by choosing a model $\mathfrak{X}'''$ that dominates two possible choices $\mathfrak{X}''$ and $\mathfrak{X}''_2$. Details are left to the reader. □

Definition 5.2.2. We define the *trace map*

$$\mathrm{Tr}_X\colon \mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a \to \mathcal{O}_C^a/p(-d)[-2d]$$

as in Lemma 5.2.1. In particular, it does not depend on a choice of the model $\mathfrak{X}$.

The trace map defines the pairing

$$\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \xrightarrow{-\cup-} \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \xrightarrow{\mathrm{Tr}_X} \mathcal{O}_C^a/p(-d)[-2d]. \tag{5.1}$$

We show that this pairing is almost perfect, i.e. the *duality map*

$$\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \to \mathbf{R}\mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a, \omega_{\mathfrak{X}_0}^{\bullet,a}(-d)[-2d]\right)$$

is an almost isomorphism[31].

Theorem 5.2.3. Let X be a smooth proper rigid C-space of pure dimensiond d. Then the pairing (5.1) is almost perfect.

Proof. We need to show that the duality morphism

$$D_X\colon \mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a \to \mathbf{R}\mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a, \mathcal{O}_C^{/}p(-d)[-2d]\right)$$

is an almost isomorphism. We note that [Sta19, Tag 0FP6] gives that the diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p) \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p) \arrow[r, "\sim"] \arrow[d, "\cup"] & \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_*\mathcal{O}_X^+/p) \otimes^L_{\mathcal{O}_C/p} \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_*\mathcal{O}_X^+/p) \arrow[d, "\cup"] \\
\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p) & \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_*\mathcal{O}_X^+/p \otimes^L_{\mathcal{O}_{\mathfrak{X}_0}} \mathbf{R}\nu_*\mathcal{O}_X^+/p) \arrow[l, "\mathbf{R}\Gamma(\mathfrak{X}_0, -\cup-)"']
\end{tikzcd}$$

[31] This pairing is symmetric, so this definition coincides with the definition from Section 1.4

is commutative. As the trace map in Grothendieck duality is defined via cup products, this formally implies that the diagram

$$\begin{tikzcd} \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a\right) \arrow[r, "\mathbf{R}\Gamma(\mathfrak{X}_0, D_{\mathfrak{X}})"] \arrow[d, "\sim"] & \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}_0}}\left(\mathbf{R}\nu_*\left(\mathcal{O}_X^+/p\right)^a, \omega^{\bullet, a}_{\mathfrak{X}_0}(-d)[-2d]\right)\right) \arrow[d] \\ \mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)^a \arrow[r, "D_X"] & \mathbf{R}\mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a, \mathcal{O}_C^a/p(-d)[2d]\right) \end{tikzcd}$$

is commutative, where the right vertical arrow is the almost isomorphism coming from the $(\mathbf{R}f_*, f^!)$-adjunction in the almost world [Zav21a, Theorem 5.5.5]. Now we note that the top horizontal arrow is an almost isomorphism by Theorem 4.5.8, and the left horizontal map is an almost isomorphism for tautological reasons. This implies that D_X is almost isormorphism. □

Now we want to show a non-derived analogue of Theorem 5.2.3. We recall that $\mathbf{R}\Gamma(X, \mathcal{O}_X^+/p)$ is almost concetrated in degrees $[0, 2d]$ by [Zav21a, Corollary 6.13.5]. Therefore, Tr_X induces a morphism

$$\mathrm{Tr}_X^d \colon \mathrm{H}^{2d}(X, \mathcal{O}_X^+/p) \to \mathcal{O}_C/p(-d)$$

This, in turn, induces the pairing

$$\mathrm{H}^i(X, \mathcal{O}_X^+/p) \otimes_{\mathcal{O}_C/p} \mathrm{H}^{2d-i}(X, \mathcal{O}_X^+/p) \xrightarrow{-\cup-} \mathrm{H}^{2d}(X, \mathcal{O}_X^+/p) \xrightarrow{\mathrm{Tr}_X^d} \mathcal{O}_C/p(-d). \tag{5.2}$$

Theorem 5.2.4. Let X be a smooth proper rigid C-space of pure dimension d. Then the pairing (5.2) is almost perfect for every $i \geq 0$.

Proof. Theorem 5.2.3 says that the morphism

$$D_X \colon \mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a \to \mathbf{R}\mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a, \mathcal{O}_C^a/p(-d)[-2d]\right)$$

is an almost isomorphism. Now we recall the primitive comparison theorem [Sch13a, Theorem 5.1] that says that the natural map

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \to \mathrm{H}^i(X, \mathcal{O}_X^+/p)$$

is an almost isomorphism for any $i \geq 0$. This implies that $\mathrm{H}^i(X, \mathcal{O}_X^+/p)^a$ are almost projective over $\mathcal{O}_C/p$ by [Zav21a, Lemma 2.2.6]. Therefore, we get isomorphisms

$$\mathrm{H}^i\left(\mathbf{R}\mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathbf{R}\Gamma\left(X, \mathcal{O}_X^+/p\right)^a, \mathcal{O}_C^a/p(-d)[-2d]\right)\right) \simeq \mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathrm{H}^{2d-i}\left(X, \mathcal{O}_X^+/p\right)^a, \mathcal{O}_C^a/p(-d)\right).$$

Thus, Theorem 5.2.3 implies that the natural map

$$\mathrm{H}^i(X, \mathcal{O}_X^+/p)^a \to \mathrm{alHom}_{\mathcal{O}_C/p}(\mathrm{H}^{2d-i}(X, \mathcal{O}_X/p)^a, \mathcal{O}_C^a/p(-d))$$

is an almost isomorphism, i.e. the pairing (5.2) is almost perfect. □

5.3. **Berkovich Trace.** We recall the trace map in étale cohomology constructed in [Ber93]. Berkovich defined the trace map for a smooth morphism of Berkovich spaces, we transfer his construction to the language of adic spaces.

For the rest of the section, we fix a ring $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for some $n > 0$.

Definition 5.3.1. We say that a morphism of rigid K-spaces $f \colon X \to Y$ is pure of dimension d if, for every maximal point $y \in Y$, the fiber X_y is empty or a rigid $K(Y)$-variety of dimension d.

Lemma 5.3.2. Let $f \colon X \to Y$ be a partially proper morphism of rigid K-spaces of pure dimension d. Then $\mathrm{R}^i f_!(\mathcal{F}) = 0$ for any $\mathcal{F} \in \mathrm{Shv}(X_{\mathrm{\acute{e}t}}, \Lambda)$ and $i > 2d$.

Proof. [Hub96, Proposition 5.3.11] implies that $\mathrm{R}^i f_! (\mathcal{F}) = 0$ for any $i > \dim.\mathrm{tr}(f) = \sup_{y\in Y}(\dim.\mathrm{tr} f^{-1}(y))$. Now the result follows from [Hub96, Lemma 1.8.5 and Lemma 1.8.6] that ensure that

$$\dim.\mathrm{tr}(f) = \sup_{y\in Y_{\max}} (\dim.\mathrm{tr} f^{-1}(y)) = \sup_{y\in Y_{\max}} \dim X_y = d.$$

□

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth partially proper morphisms of taut rigid K-spaces of pure dimension d and e, respectively. Suppose we are given three homomorphisms $\alpha\colon \mathrm{R}^{2d} f_!\Lambda_X(d) \to \Lambda_Y$, $\beta : R^{2e} g_!\Lambda_Y(e) \to \Lambda_Z$, and $\gamma\colon \mathrm{R}^{2(d+e)}(g\circ f)_!\Lambda_X(d+e) \to \Lambda_Z$. Using the Leray spectral sequence and Lemma 5.3.2, we get an isomorphism

$$\mathrm{R}^{2(d+e)}(g \circ f)_!\Lambda_X(d + e) \simeq \mathrm{R}^{2e} g_!(\mathrm{R}^{2d} f_!\Lambda_X(d))(e)$$

This defines a morphism

$$\mathrm{R}^{2(d+e)}(g \circ f)_!\Lambda_X(d + e) \simeq \mathrm{R}^{2e} g_!(\mathrm{R}^{2d} f_!\Lambda_X(d)(e)) \xrightarrow{\mathrm{R}^{2d} g_!(\alpha)(e)} \mathrm{R}^{2e} g_!\Lambda_Y(e) \xrightarrow{\beta} \Lambda_Z$$

that is denoted by $\beta\Box\alpha$. We say that α, β, and γ are *compatible with the composition* if $\gamma = \beta\Box\alpha$.

Theorem 5.3.3. Let $f\colon X \to Y$ be a partially proper smooth morphism of pure dimension d. Then one can define the trace map $t_f\colon \mathrm{R}^{2d} f_!\Lambda_X(d) \to \Lambda_Y$ satisfying the following properties:

(1) t_f is compatible with taking geometric fibers over maximal points,
(2) t_f is compatible with composition, i.e. if $g\colon Y \to Z$ is another smooth, partially proper morphism of pure dimension e, then $t_g\Box t_f = t_{g\circ f}$,
(3) if $d = 0$, then $t_f\colon f_!\Lambda_X \to \Lambda_Y$ is the map coming from the adjunction $(f_!, f^*)$.
(4) t_f is surjective if all fibers of f are non-empty.

Proof. We note that $\mathrm{R}^i f_!\Lambda_X(d)$ is an overconvergent sheaf on Y by [Hub96, Corollary 8.2.4]. Therefore, (1) ensures that it suffices to construct trace locally on Y because a map of overconvergent sheaves is uniquely defined by a map on stalks over maximal points. So we may and do assume that Y is affinoid. In particular, Remark A.3 gives that Y is taut and X is also taut as it is partially proper over Y.

We consider the associated morphism of Berkovich spaces $u(f)\colon u(X) \to u(Y)$. Lemma A.7 implies that it is a smooth morphism of pure dimension d. Therefore, [Ber93, Theorem 7.2.1] constructs the morphism

$$t_f^B\colon \mathrm{R}^{2d} u(f)_!\Lambda_{u(X)} \to \Lambda_{u(Y)}$$

with all desired properties. Now we use Theorem A.11 to define

$$t_f\colon \mathrm{R}^{2d} f_!\Lambda_X \to \Lambda_Y$$

as $\theta_Y^*(t_f^B)$.

It is easy to see that t_f^B satisfies (1), (2), and (3) as t_f^B does[32]. Finally, we know that t_f^B is surjective if all fibers of $u(f)$ are non-empty. Therefore, in order to show (4), it suffices to show that $u(f)$ is surjective when f is. But this is clear as topologically $u(f)$ can be identified with the morphism $f_{\max}\colon X_{\max} \to Y_{\max}$. □

[32] Use Lemma A.13 to show (3).

5.4. **Poincaré Duality for $\mathbf{F}_p$-coefficients.** The main goal of this section is to show Poincaré Duality for $\mathbf{F}_p$-coefficients on smooth proper rigid K-space X.

We start by discussing the essential idea of the proof. We firstly show that the Berkovich trace map $t_X\colon \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \to \mathbf{F}_p$ is an isomorphism for connected X. Then we use the primitive comparison theorem to reduce duality for $\mathbf{F}_p$-coefficients to almost duality for $\mathcal{O}^+_{X_C}/p$-coefficient with *any* trace map $\mathrm{H}^{2d}(X_C, \mathcal{O}^+_{X_C}/p)^a \to \mathcal{O}^a_C/p$ that is an almost isomorphism. This allows us to use the trace constructed in Section 5.2 for the $\mathcal{O}^+_{X_C}/p$ duality. So everything will essentially boil down to Theorem 5.2.3.

Theorem 5.4.1. Let X be a smooth proper geometrically connected rigid K-space of pure dimension d. Then the trace map $t_X\colon \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \to \mathbf{F}_p$ is an isomorphism.

Proof. Step 0. Reduce to the case $K = C$ is algebraically closed: It suffices to show that t_X is an isomorphism after a base change to $C = \widehat{\overline{K}}$ that is algebraically closed. Theorem 5.3.3(1) ensures that t_X commutes with that base change. Therefore, we may and do assume that $K = C$ is algebraically closed.

Step 1. Proof for $K = C$: Theorem 5.3.3 already shows that t_X is surjective. Therefore, it suffices to show that $\mathrm{H}^{2d}(X, \mathbf{F}_p(d))$ is a 1-dimensional vector space over $\mathbf{F}_p$. Using the primitive comparison theorem, we conclude that

$$\mathrm{H}^i(X, \mathbf{F}_p(d)) \otimes_{\mathbf{F}_p} \mathcal{O}^a_C/p \simeq \mathrm{H}^i(X, (\mathcal{O}^+_X/p)(d))^a$$

is an almost isomorphism for all i.

Now we note that the classification of finitely presented torsion $\mathcal{O}_C$-modules [Sch13a, Proposition 2.10] guarantees that it suffices to show that $\mathrm{H}^{2d}(X, (\mathcal{O}^+_X/p)(d))^a \simeq \mathcal{O}^a_C/p$.

We know from $\mathrm{H}^{2d}(X, (\mathcal{O}^+_X/p)(d))$ is almost dual to $\mathrm{H}^0(X, \mathcal{O}^+_X/p)$ by Theorem 5.2.3. Now we use again the primitive comparison theorem again to conclude that

$$\mathrm{H}^0(X, \mathcal{O}^+_X/p)^a \simeq \mathrm{H}^0(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}^a_C/p \simeq \mathcal{O}^a_C/p$$

because X is (geometrically) connected. Therefore,

$$\mathrm{H}^{2d}(X, (\mathcal{O}^+_X/p)(d))^a \simeq \mathrm{alHom}_{\mathcal{O}_C/p}(\mathrm{H}^0(X, \mathcal{O}^+_X/p)^a, \mathcal{O}^a_C/p) \simeq^a \mathrm{alHom}_{\mathcal{O}_C/p}(\mathcal{O}^a_C/p, \mathcal{O}^a_C/p) \simeq \mathcal{O}^a_C/p.$$

□

Theorem 5.4.2. Let X be a smooth and proper rigid K-variety of pure dimension d. Then the pairing

$$\mathrm{H}^i(X_C, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}(X_C, \mathbf{F}_p(d)) \to \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p$$

is a perfect Galois-equivariant pairing.

Proof. Similar to Theorem 5.4.1, we can assume that $K = C$ is algebraically closed. Then X is a disjoint union of its (geometrically) connected components $X = \sqcup_{i=1}^n X_i$. Theorem 5.3.3(3) ensures that t_X is equal to

$$\sum_{i=1}^n t_{X_i}\colon \bigoplus_{i=1}^n \mathrm{H}^{2d}(X_i, \mathbf{F}_p(d)) = \mathrm{H}^{2d}(X, \mathbf{F}_p(d)) \to \mathbf{F}_p.$$

Therefore, we may and do assume that X is connected and $K = C$ is algebraically closed.

Then Theorem 5.4.1 gives that t_X is an isomorphism. Thus, it suffices to show that the natural morphism

$$\mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_p}(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p))$$

induced by the cup produce is an isomorphism. The main advantage of this reformulation is that it is independent of the construction of the trace map.

Step 1. We show that $h^i := \dim_{\mathbf{F}_p} \mathrm{H}^i(X, \mathbf{F}_p)$ *and* $h^{2d-i} := \dim_{\mathbf{F}_p} \mathrm{H}^{2d-i}(X, \mathbf{F}_p)$ *are the same*: The primitive comparison theorem ensures that

$$\mathrm{H}^i(X, \mathcal{O}_X^+/p)^a \simeq \mathrm{H}^i(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C^a/p \simeq (\mathcal{O}_C^a/p)^{h^i}$$

And Theorem 5.2.3 implies that

$$\mathrm{H}^i(X, \mathcal{O}_X^+/p) \simeq^a \mathrm{alHom}_{\mathcal{O}_C/p}\left(\mathrm{H}^{2d-i}(X, (\mathcal{O}_X^+/p)(d))^a, \mathcal{O}_C^a/p\right) \simeq^a \mathcal{O}_C^a/p^{h_{2d-i}}.$$

Thus, [Sch13a, Proposition 2.10] guarantees that $h^i = h^{2d-i}$.

Step 2. We show that the map $\mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_p}\left(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p)\right)$ *is injective*: We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(X, \mathbf{F}_p) \arrow[r, "d_1"] \arrow[d] & \mathrm{Hom}_{\mathbf{F}_p}\left(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p)\right) \arrow[d] \\
\mathrm{H}^i(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \arrow[r, "d_2"] \arrow[dd, "\gamma"] & \mathrm{Hom}_{\mathbf{F}_p}\left(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p)\right) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \arrow[d, "\alpha"] \\
 & \mathrm{Hom}_{\mathcal{O}_C/p}\left(\mathrm{H}^{2d-i}(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p, \mathrm{H}^{2d}(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p\right) \arrow[d, "\beta"] \\
\mathrm{H}^i(X, \mathcal{O}_X^+/p) \arrow[r, "d_3"] & \mathrm{Hom}_{\mathcal{O}_C/p}\left(\mathrm{H}^{2d-i}(X, \mathcal{O}_X^+/p), \mathrm{H}^{2d}(X, \mathcal{O}_X^+/p)\right)
\end{tikzcd}$$

We note that α is easily seen to be an isomorphism as $\mathrm{H}^{2d-i}(X, \mathbf{F}_p)$ is of finite dimension, β and γ are almost isomorphisms by the primitive comparison theorem. The map d_3 is almost isomorphism by Theorem 5.2.3 and [Zav21a, Proposition 2.2.1]. This implies that d_2 is an almost isomorphism. This, in turn, ensures that

$$d_1 \colon \mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_p}(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p))$$

is injective as otherwise the kernel of d_2 is not almost zero.

Step 3. We show that the map $\mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_p}\left(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p)\right)$ *is an isomorphism*: Steps 1 and 2 imply that the morphism

$$\mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{Hom}_{\mathbf{F}_p}(\mathrm{H}^{2d-i}(X, \mathbf{F}_p), \mathrm{H}^{2d}(X, \mathbf{F}_p))$$

is an injective $\mathbf{F}_p$-linear morphism of finite dimensional $\mathbf{F}_p$-vectors spaces of the same dimension. Therefore, it must be an isomorphism. □

We also show a version of Poincaré Duality for $\mathbf{F}_p$-local systems. For this, we recall that there is a natural "evaluation" morphism

$$\mathbf{L} \otimes \mathbf{L}^\vee \to \mathbf{F}_p$$

for any étale $\mathbf{F}_p$-local system $\mathbf{L}$ on X. Then we can combine it with the cup-product to get a functorial in $\mathbf{L}$ morphism

$$\mathrm{H}^i(X_C, \mathbf{L}) \otimes \mathrm{H}^{2d-i}(X_C, \mathbf{L}^\vee) \to \mathrm{H}^{2d}(X_C, \mathbf{L} \otimes \mathbf{L}^\vee(d)) \to \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d))$$

Theorem 5.4.3. Let X be a smooth and proper rigid K-variety of pure dimension d, and $\mathbf{L}$ is an étale $\mathbf{F}_p$-local system on X. Then the pairing

$$\mathrm{H}^i(X_C, \mathbf{L}) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}(X_C, \mathbf{L}^\vee(d)) \to \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p$$

is a perfect Galois-equivariant pairing.

Proof. If $\mathbf{L} = f_*\mathbf{L}'$ for some finite étale morphism $f\colon X' \to X$, then Poincaré Duality for the local system $\mathbf{L}$ is equivalent to Poincaré Duality for the local system $\mathbf{L}'$. Now using method de la trace (argue as in [Sta19, Tag 03SH]), we can find a finite étale morphism $f\colon X' \to X$ of degree r prime to p such that

$$\mathbf{L}|_{X'}$$

is a successive extension of constant $\mathbf{F}_p$-local systems on X'. Now using the trace morphism for finite étale morphisms, we see that the composition

$$\mathbf{L} \to f_*\left(\mathbf{L}|_{X'}\right) \to \mathbf{L}$$

is equal to the multiplication by r map, in particular it is invertible. Thus $\mathbf{L}$ is a direct summand of $f_*\left(\mathbf{L}|_{X'}\right)$, and so it suffices to show the claim for $f_*\left(\mathbf{L}|_{X'}\right)$. An argument above implies that, furthermore, it suffices to show the claim for $\mathbf{L}|_{X'}$ and X'. So we can assume that $\mathbf{L}$ is a successive extension of constant local systems. In this case, the claim follows from the 2-out-3 property and Theorem 5.4.2. □

5.5. **Poincaré Duality for p-adic Coefficients.** The main goal of this section is to generalize Theorem 5.4.2 to $\mathbf{Z}/p^n\mathbf{Z}$, $\mathbf{Z}_p$, and $\mathbf{Q}_p$-coefficients.

Lemma 5.5.1. Let X be a proper rigid C-space of pure dimension d. Then $\mathbf{R}\Gamma\left(X, \mathbf{Z}/p^n\mathbf{Z}\right) \in \mathbf{D}^{[0,2d]}_{perf}\left(\mathbf{Z}/p^n\mathbf{Z}\right)$ and $\mathbf{R}\Gamma(X, \mathbf{Z}_p) \in \mathbf{D}^{[0,2d]}_{perf}\left(\mathbf{Z}_p\right)$. Furthermore, a natural morphism

$$\mathrm{H}^i(X, \mathbf{Z}_p) \to \lim_n \mathrm{H}^i(X, \mathbf{Z}/p^n\mathbf{Z})$$

is an isomorphism.

Proof. We note that

$$\mathbf{R}\Gamma\left(X, \mathbf{Z}/p^n\mathbf{Z}\right) \otimes^L_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{F}_p \simeq \mathbf{R}\Gamma\left(X, \mathbf{F}_p\right) \in \mathbf{D}^{[0,2d]}_{coh}\left(\mathbf{F}_p\right) = \mathbf{D}^{[0,2d]}_{perf}\left(\mathbf{F}_p\right).$$

by [Sch13b, Theorem 3.17] (or [Zav21a, Theorem 1.1.2]). Therefore, [Sta19, Tag 07LU] ensures that $\mathbf{R}\Gamma(X, \mathbf{Z}/p^n\mathbf{Z}) \in \mathbf{D}_{perf}(\mathbf{Z}/p^n\mathbf{Z})$. An easy argument with Nakayama's lemma ensures that

$$\mathbf{R}\Gamma(X, \mathbf{Z}/p^n\mathbf{Z}) \in \mathbf{D}^{[0,2d]}_{perf}(\mathbf{Z}/p^n\mathbf{Z}).$$

The same argument shows that $\mathbf{R}\Gamma(X, \mathbf{Z}_p) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{Z}_p)$. The final claim follows from a Milnor exact sequence and a standard Mittag-Leffler argument. □

Lemma 5.5.2. Let X be a proper rigid C-space of pure dimension d. Then $\mathbf{R}\Gamma(X, \mathbf{Q}_p) \simeq \mathbf{R}\Gamma(X, \mathbf{Z}_p)[1/p]$.

Proof. We recall that, by definition,

$$\mathbf{R}\Gamma(X, \mathbf{Z}_p) \simeq \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathbf{Z}}_p)$$

for the pro-étale sheaf $\widehat{\mathbf{Z}}_p = \lim_n \underline{\mathbf{Z}/p^n\mathbf{Z}}$. Similarly,

$$\mathbf{R}\Gamma(X, \mathbf{Q}_p) \simeq \mathbf{R}\Gamma(X_{\text{proét}}, \widehat{\mathbf{Q}}_p)$$

for the pro-étale sheaf $\widehat{\mathbf{Q}}_p = \widehat{\mathbf{Z}}_p[1/p]$. Therefore, it suffices to show that the cohomology groups of $X_{\text{proét}}$ commute with filtered colimits. This, in turn follows, follows from [Sch13a, Proposition 3.12], [Sta19, Tag 0739], and the fact that X is quasi-compact and quasi-separated. □

Theorem 5.5.3. Let X be a smooth proper rigid K-variety of pure dimension d. Then the pairing

$$\mathbf{R}\Gamma(X_C, \mathbf{Z}/p^n\mathbf{Z}) \otimes^L_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{R}\Gamma(X_C, \mathbf{Z}/p^n\mathbf{Z}(d)) \xrightarrow{-\cup-} \mathbf{R}\Gamma(X_C, \mathbf{Z}/p^n\mathbf{Z}(d)) \xrightarrow{t_{X,\mathbf{Z}/p^n\mathbf{Z}}} \mathbf{Z}/p^n\mathbf{Z}$$

is a perfect Galois-equivariant pairing.

Proof. Similarly to the proof of Theorem 5.4.2, we can assume that $K = C$ is algebraically closed.

We consider the duality morphism

$$D_{X,\mathbf{Z}/p^n\mathbf{Z}} \colon \mathbf{R}\Gamma\left(X_C, \mathbf{Z}/p^n\mathbf{Z}\right) \to \mathbf{R}\mathrm{Hom}_{\mathbf{Z}/p^n\mathbf{Z}}\left(\mathbf{R}\Gamma\left(X_C, \mathbf{Z}/p^n\mathbf{Z}(d)\right), \mathbf{Z}/p^n\mathbf{Z}\right).$$

We need to show that this morphism is an isomorphism. Lemma 5.5.1 implies that

$$D_{X,\mathbf{Z}/p^n\mathbf{Z}} \colon \mathbf{R}\Gamma(X, \mathbf{Z}/p^n\mathbf{Z}) \to \mathbf{R}\mathrm{Hom}_{\mathbf{Z}/p^n\mathbf{Z}}\left(\mathbf{R}\Gamma(X, \mathbf{Z}/p^n\mathbf{Z}(d)), \mathbf{Z}/p^n\mathbf{Z}\right)$$

is map of perfect $\mathbf{Z}/p^n\mathbf{Z}$ complexes. Therefore, it suffices to show it is an isomorphism after tensoring with $\mathbf{F}_p$.

Now we note that that the square

$$\begin{tikzcd} \mathrm{H}^{2d}(X, \mathbf{Z}/p^n\mathbf{Z}(d)) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{F}_p \arrow[r, "t_{X,\mathbf{Z}/p^n\mathbf{Z}\otimes_{\mathbf{Z}/p^n\mathbf{Z}}\mathbf{F}_p}"] \arrow[d, "\sim"] & \mathbf{F}_p \\ \mathrm{H}^{2d}(X, \mathbf{F}_p(d)) \arrow[ur, "t_{X,\mathbf{F}_p}"'] & \end{tikzcd}$$

is an isomorphism by the construction of the trace map. Therefore, the square

$$\begin{tikzcd} \mathbf{R}\Gamma\left(X, \mathbf{Z}/p^n\mathbf{Z}\right) \otimes^L_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{F}_p \arrow[r, "D_{X,\mathbf{Z}/p^n\mathbf{Z}}\otimes^L_{\mathbf{Z}/p^n\mathbf{Z}}\mathbf{F}_p"] \arrow[dd, "\gamma"] & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}/p^n\mathbf{Z}}\left(\mathbf{R}\Gamma\left(X, \mathbf{Z}/p^n\mathbf{Z}(d)\right), \mathbf{Z}/p^n\mathbf{Z}\right) \otimes^L_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{F}_p \arrow[d, "\alpha"] \\ & \mathbf{R}\mathrm{Hom}_{\mathbf{F}_p}\Big(\mathbf{R}\Gamma\left(X, \mathbf{Z}/p^n\mathbf{Z}(d)\right) \otimes^L_{\mathbf{Z}/p^n\mathbf{Z}} \mathbf{F}_p, \mathbf{F}_p\Big) \arrow[d, "\beta"] \\ \mathbf{R}\Gamma\left(X, \mathbf{F}_p\right) \arrow[r, "D_{X,\mathbf{F}_p}"] & \mathbf{R}\mathrm{Hom}_{\mathbf{F}_p}\left(\mathbf{R}\Gamma\left(X, \mathbf{F}_p(d)\right), \mathbf{F}_p\right) \end{tikzcd}$$

is commutative. Now we note that α is an isomorphism by [Sta19, Tag 0A6A] as $\mathbf{R}\Gamma(X, \mathbf{Z}/p^n\mathbf{Z}(d))$ is perfect by Lemma 5.5.1. Clearly, β, γ are isomorphisms, and $D_{X,\mathbf{F}_p}$ is an isomorphism by Theorem 5.4.2. This shows that $D_{X,\mathbf{Z}/p^n\mathbf{Z}}\otimes^L_{\mathbf{Z}/p^n\mathbf{Z}}\mathbf{F}_p$ is an isomorphism, and therefore so is $D_{X,\mathbf{Z}/p^n\mathbf{Z}}$. □

Definition 5.5.4. The *trace map*

$$t_{X,\mathbf{Z}_p} \colon \mathrm{H}^{2d}(X_C, \mathbf{Z}_p(d)) \to \mathbf{Z}_p$$

for a smooth proper rigid K-space X of pure dimension d, is defined as following:

$$t_{X,\mathbf{Z}_p} \coloneqq \lim_n t_{X,\mathbf{Z}/p^n\mathbf{Z}} \colon \mathrm{H}^{2d}(X_C, \mathbf{Z}_p(d)) = \lim_n \mathrm{H}^{2d}(X_C, \mathbf{Z}/p^n\mathbf{Z}(d)) \to \mathbf{Z}_p.$$

Theorem 5.5.5. Let X be a smooth proper rigid K-variety of pure dimension d. Then the pairing

$$\mathbf{R}\Gamma(X_C, \mathbf{Z}_p) \otimes^L_{\mathbf{Z}_p} \mathbf{R}\Gamma(X_C, \mathbf{Z}_p(d)) \xrightarrow{-\cup-} \mathbf{R}\Gamma(X_C, \mathbf{Z}_p(d)) \xrightarrow{t_{X,\mathbf{Z}_p}} \mathbf{Z}_p$$

is a perfect Galois-equivariant pairing.

Proof. The proof of Theorem 5.5.3 works almost verbatim. □

Definition 5.5.6. The *trace map* $t_{X,\mathbf{Q}_p}\colon \mathrm{H}^{2d}(X_C, \mathbf{Q}_p(d)) \to \mathbf{Q}_p$ for a smooth proper rigid K-space X of pure dimension d, is defined as following:

$$t_{X,\mathbf{Q}_p} \coloneqq t_{X,\mathbf{Z}_p}[1/p]\colon \mathrm{H}^{2d}(X_C, \mathbf{Q}_p(d)) = \mathrm{H}^{2d}(X_C, \mathbf{Z}_p(d))[1/p] \to \mathbf{Q}_p.$$

Theorem 5.5.7. Let X be a smooth proper rigid K-variety of pure dimension d. Then the pairing

$$\mathbf{R}\Gamma(X_C, \mathbf{Q}_p) \otimes^L_{\mathbf{Q}_p} \mathbf{R}\Gamma(X_C, \mathbf{Q}_p(d)) \xrightarrow{-\cup-} \mathbf{R}\Gamma(X_C, \mathbf{Q}_p(d)) \xrightarrow{t_{X,\mathbf{Q}_p}} \mathbf{Q}_p$$

is a perfect Galois-equivariant pairing.

Proof. This follows easily from Theorem 5.5.5 and Lemma 5.5.2. □

Corollary 5.5.8. Let X be a smooth proper rigid K-variety of pure dimension d. Then the pairing

$$\mathrm{H}^i(X_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathrm{H}^{2d-i}(X_C, \mathbf{Q}_p(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}(X_C, \mathbf{Q}_p(d)) \xrightarrow{t_{X,\mathbf{Q}_p}} \mathbf{Q}_p$$

is a perfect Galois-equivariant pairing.

Proof. This follows directly from Theorem 5.5.7 and the fact that $\mathbf{Q}_p$ is a field. □

Appendix A. Adic Spaces and Berkovich Spaces

We fix a complete, rank-1 valuation field K with a pseudo-uniformizer ϖ.

Definition A.1. A morphism of adic space $f\colon X \to Y$ is called *partially proper* if f is locally of finite type, and satisfies the valuative criterion: for every complete microbial valuation ring k^+ and a commutative diagram

$$\begin{tikzcd} \mathrm{Spa}(k, k^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spa}(k, k^+) \arrow[r] & Y \end{tikzcd}$$

there is a unique lifting $\mathrm{Spa}(k, k^+) \to X$ making the diagram commute.

Definition A.2. An adic space X is called *taut* if X is quasi-separated and, for every quasi-compact subset U of X, the closure $\overline{U}$ of U in X is quasi-compact.

A morphism $f\colon X \to Y$ of adic spaces is called *taut* if it is quasi-compact and, for every taut open subspace U of Y, the inverse image $f^{-1}(U)$ is taut.

Remark A.3. [Hub96, Lemma 5.1.3, 5.1.4] Any quasi-compact and quasi-separated space is taut. In particular, any affinoid space is taut.

A map between taut adic spaces is taut, and any partially proper morphism is taut.

Definition A.4. The category of *adic spaces locally of finite type over* $\mathrm{Spa}(K, \mathcal{O}_K)$ is denoted by (A).

The category of *taut adic spaces locally of finite type over* $\mathrm{Spa}(K, \mathcal{O}_K)$ is denoted by $(A)'$.

We recall that any point $x \in X$ on an analytic adic space X has a unique rank-1 generalization x_{gen}. The topological space $X_{\max}$ is defined to be the set of rank-1 points of X with the strongest topology such that the map

$$\omega_X\colon X \to X_{\max}$$
$$x \mapsto x_{\mathrm{gen}}$$

is continuous. In other words, we endow $X_{\max}$ with the quotient topology from X.

Definition A.5. The category of *hausdorff strictly K-analytic Berkovich spaces*[33] is denoted by (An).

We recall that [Hub96, Proposition 8.3.1 and 5.6.2] guarantees that the functor

$$s'\colon (An) \to (A)'.$$

constructed in [Hub96, §8.3] is an equivalence of categories. The quasi-inverse of this functor is denoted by

$$u\colon (A)' \to (An)^{34}.$$

We summarize the main properties of this functor below:

Facts A.6.
1. u sends an open immersion $U = \mathrm{Spa}(A, A^\circ) \subset X$ to an affinoid domain $\mathcal{M}(A) \subset u(X)$.
2. u sends a morphism $f\colon \mathrm{Spa}(B, B^\circ) \to \mathrm{Spa}(A, A^\circ)$ to the corresponding $u(f)\colon \mathcal{M}(B) \to \mathcal{M}(A)$

[33] Look at [Ber93, §1.2.] for the precise definition.

[34] Look at [Hen16] for a more direct construction of u.

(3) the underlying topological space of $u(X)$ is functorially identified with $X_{\max}$.

Lemma A.7. Let $f\colon X \to Y$ be a morphism in (A'). Suppose that f is partially proper (resp. proper, resp. étale, resp. smooth). Then $u(f)$ is closed[35] (resp. proper, resp. quasi-étale, resp. quasi-smooth[36]).

Proof. Suppose f is partially proper. Then it suffices to show that the morphism of germs[37]

$$(u(X), x) \to (u(Y), u(f)(x))$$

is closed[38] for any $x \in u(X) = X_{\max}$. Now [Tem00, Theorem 4.1.] gives that this map is closed if and only if the morphism of reductions

$$(\widetilde{u(X), x}) \to (\widetilde{u(Y), f}(x))$$

is proper, i.e. it induces the bijection[39]

$$(\widetilde{u(X), x}) \to \mathbf{P}_{\widetilde{\mathcal{H}(f(x))}/\widetilde{K}} \times_{\mathbf{P}_{\widetilde{\mathcal{H}(x)}/\widetilde{K}}} (\widetilde{u(Y), f}(x))^{40}.$$

Now we use [Tem00, Remark 2.6] to see that the underlying topological space of $(\widetilde{u(X), x})$ coincides with $\omega_X^{-1}(x) \subset X$ and similarly for $(\widetilde{u(Y), f}(x))$. So bijectivity of the above map follows from [Hub96, Corollary 1.3.9].

Suppose f is proper. For the purpose of showing that $u(f)$ is proper, it suffices to treat the case of affinoid $Y = \mathrm{Spa}(A, A^\circ)$. Then X is quasi-compact quasi-separated rigid K-space, so f has a formal model $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$. Now [Hub96, Remark 1.3.18(ii)] guarantees that f is proper if and only if so is $\mathfrak{f}$. Then [Tem00, Corollary 4.4] guarantees that $u(f)$ is proper.

Suppose f is étale. Then [Duc18, p. 5.2.10] and property (1) imply that it suffices to assume that X and Y are affinoids. Therefore, [Hub96, Lemma 2.2.8] show that f can be written as $f = g \circ j$, where $j\colon X \to \mathrm{Spa}(B, B^\circ)$ is an open immersion and $g : \mathrm{Spa}(B, B^\circ) \to \mathrm{Spa}(A, A^\circ)$ is a finite étale morphism. It is clear from property (1) that $u(j)$ is an analytic domain, in particular, it is quasi-étale. And $u(g)\colon \mathcal{M}(B) \to \mathcal{M}(A)$ is clearly finite étale as so is $A \to B$.

Suppose f is smooth. We use [Duc18, p. 5.2.10], property (1), and [Hub96, Corollary 1.6.10] to reduce to the case $f\colon X \to Y$ is a map of affinoids that factors as

$$X \xrightarrow{g} \mathbf{D}^N_Y \to Y,$$

where g is étale. Then we see that $u(f)$ can be written as the composition

$$u(X) \xrightarrow{u(g)} \mathbf{D}^N_{u(Y)} \to u(Y)$$

with $u(g)$ being quasi-étale. Therefore, $u(f)$ is quasi-smooth as the n-dimensional disc $\mathbf{D}^N_{u(Y)}$ is quasi-smooth over $u(Y)$. $\square$

[35] It is called boundaryless in [Duc18]

[36] Look at [Duc18, Definition 5.2.4 and Definition 5.2.6] for the definitions of quasi-étale and quasi-smooth morphisms of Berkovich spaces.

[37] Look at [Ber93, §3.4] for the definition of germs of Berkovich spaces.

[38] By definition, this means that the map is induced by a map $X' \to u(Y)$, where X' is an open neighborhood of x in $u(X)$.

[39] Look at [Tem00, p.1] for the definition of a proper morphism of reductions.

[40] $\mathbf{P}_{\widetilde{\mathcal{H}(f(x))}/\widetilde{K}}$ is the set of all valuations of the residue field of $\mathcal{H}(f(x))$ that is trivial on the residue field of K. In [Tem00], this is denoted just by $\mathbf{P}_{\widetilde{\mathcal{H}(f(x))}}$.

Corollary A.8. Let $f\colon X \to Y$ be a morphism in $(A)'$. Suppose that f is partially proper and étale (resp. partially proper and smooth). Then $u(f)$ is étale (resp. smooth).

Proof. We recall that [Duc18, Corollary 5.4.8] (and [Ber99, Remark 9.7] to make it work in a non-necessary good strictly K-analytic case) implies that a morphism is strictly K-analytic spaces is smooth (resp. étale) if and only if it is quasi-smooth (resp. quasi-étale) and closed. Therefore, the claim follows from Lemma A.7. □

Definition A.9. The *strict étale site* $X_{\text{ét}.s}$ of a strictly K-analytic space X is a site such that

(1) the underlying category $(\text{Ét}/X)_s$ of $X_{\text{ét}.s}$ is the full subcategory of (An) consisting of all X-objects Y such that the structure morphism $f\colon Y \to X$ is an étale morphism of K-analytic spaces,

(2) A family $(Y_i \xrightarrow{f_i} Y)$ is a covering of Y when $Y = \cup_{i\in I} f_i(Y_i)$.

Remark A.10. We note that Berkovich considers a different étale site $X_{\text{ét}}$ in [Ber93]. He allows all étale X-spaces Y that are not necessary K-strict. However, one can easily show that the natural morphism of sites

$$X_{\text{ét}} \to X_{\text{ét}.s}$$

induces an isomorphism of corresponding topoi. Therefore, all results of [Ber93] still hold true if one uses $X_{\text{ét}.s}$ instead of $X_{\text{ét}}$[41].

If X is a taut rigid space over K, there is a natural morphism of sites

$$\theta_X\colon X_{\text{ét}} \to u(X)_{\text{ét}.s}$$

induced by the functor

$$(\text{Ét}/u(X))_s \to \text{Ét}/X$$
$$(Y \to u(X)) \mapsto \left(s'(Y) \to s'(u(X)) = X\right).$$

Now suppose $f\colon X \to Y$ be a morphism in $(A)'$. Then there is a commutative diagram

$$\begin{tikzcd} X_{\text{ét}} \arrow[r, "\theta_X"] \arrow[d, "f"] & u(X)_{\text{ét}.s} \arrow[d, "u(f)"] \\ Y_{\text{ét}} \arrow[r, "\theta_Y"] & u(Y)_{\text{ét}.s}. \end{tikzcd} \tag{A.1}$$

We want to relate $\theta_Y^*\mathbf{R}u(f)_!K$ to $\mathbf{R}f_!\theta_X^*K$ for a partially proper f.

Theorem A.11. Let $f\colon X \to Y$ be a partially proper morphism in $(A)'$. Then there is a natural isomorphism

$$\alpha_f(K)\colon \theta_Y^*\mathbf{R}u(f)_!K \to \mathbf{R}f_!\theta_X^*K$$

for any $K \in \mathbf{D}^+(u(X)_{\text{ét}.s}, \mathbf{Z})$.

Proof. We note that $u(f)$ is closed since f is partially proper by Lemma A.7, so [Hub96, Proposition 8.3.6][42] proves the claim. □

[41] One uses [Tem04, Corollary 4.10] to ensure that the category of strictly K-analytic Berkovich spaces is a full faithful subcategory of the category of all K-analytic Berkovich spaces.

[42] Proposition is written in terms of Tate rigid spaces, but the actual proof shows the statement on the level of corresponding adic spaces

Now suppose that $f\colon X\to Y$ be a partially proper étale morphism in $(A)'$. The functor $f_!$ is a left adjoint to f^* by [Hub96, Lemma 2.7.6]. In particular, there is a trace map

$$t_f\colon f_!f^*\mathcal{F}\to\mathcal{F}$$

for any $\mathcal{F}\in\mathcal{A}b(Y_{\text{ét}})$.

Similarly, Corollary A.8 guarantees that $u(f)\colon u(X)\to u(Y)$ is an étale morphism. Therefore, the functor $u(f)_!$ is a left adjoint to $u(f)^*$ by [Ber93, Remark 5.4.2(ii)]. So, for any $\mathcal{G}\in\mathcal{A}b(u(Y)_{\text{ét}.s})$, there is a trace map

$$t^B_{u(f)}\colon u(f)_!u(f)^*\mathcal{G}\to\mathcal{G}.$$

We want to compare these trace maps using Lemma A.12.

Lemma A.12. [Hub96, Theorem 8.3.5] Let X be a taut rigid K-space. Then the functor

$$\theta_X^*\colon\mathcal{A}b(u(X)_{\text{ét}.s})\to\mathcal{A}b(X_{\text{ét}})$$

induces an equivalence of categories

$$\theta_X^*\colon\mathcal{A}b(u(X)_{\text{ét}.s})\to\mathcal{A}b_{overconv}(X_{\text{ét}})^{43}$$

Diagram (A.1) implies that there is a natural isomorphism of functors $f^*\circ\theta_Y^*\simeq\theta_X^*\circ u(f)^*$.

Lemma A.13. Let $f\colon X\to Y$ be a partially proper étale morphism of taut rigid K-spaces. Then for any $\mathcal{G}\in\mathcal{A}b(u(Y)_{\text{ét}})$ the following diagram is commutative

$$\begin{tikzcd}
\theta_Y^*u(f)_!u(f)^*\mathcal{G} \arrow[r, "\theta_Y^*t^B_{u(f)}"] \arrow[d, "\alpha_f(u(f)^*\mathcal{G})"] & \theta_Y^*\mathcal{G} \\
f_!\theta_X^*u(f)^*\mathcal{G} \arrow[r, "\sim"] & f_!f^*\theta_Y^*\mathcal{G}. \arrow[u, "t_f"']
\end{tikzcd}$$

Proof. The trace map $t^B_{u(f)}$ comes as the co-unit of the adjunction

$$\mathcal{A}b(u(Y)_{\text{ét}.s}) \underset{f^*}{\overset{f_!}{\leftrightarrows}} \mathcal{A}b(u(X)_{\text{ét}.s}).$$

Similarly, the trace map t_f evaluated on any overconvergent sheaf comes as the counit of the adjunction

$$\mathcal{A}b(Y_{\text{ét}}) \underset{f^*}{\overset{f_!}{\leftrightarrows}} \mathcal{A}b(X_{\text{ét}})^{44}.$$

Now Lemma A.13 and Theorem A.11 formally imply the claim. ∎

Appendix B. Some Facts from Rigid Geometry

For the rest of the appendix, we fix a complete rank-1 valuation ring $\mathcal{O}_K$ with a pseudo-uniformizer ϖ, maximal ideal $\mathfrak{m}$, and residue field k.

Lemma B.1. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with generic fiber $\mathfrak{X}_K=X$ and reduced special fiber $\overline{\mathfrak{X}}$. Suppose that $T\subset X$ a connected component of X. Then there is a connected component $\mathfrak{T}\subset\mathfrak{X}$ such that $\mathfrak{T}_K=T$.

[43] $\mathcal{A}b_{overconv}(X_{\text{ét}})$ is the category of overconvergent étale sheaves on X, i.e. sheaves of abelian groups $\mathcal{F}$ such that, for every specialization $u\colon\eta_1\to\eta_2$ of geometric points of X, the map $u^*(\mathcal{F})\colon\mathcal{F}_{\eta_2}\to\mathcal{F}_{\eta_1}$ is an isomorphism.

Proof. It suffices to show every idempotent of $\Gamma(X, \mathcal{O}_X)$ lifts to an idempotent of $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Since $\mathcal{O}_X$ is a sheaf on X and $\mathfrak{X}$ is $\mathcal{O}_K$-flat, it is easy to see that it suffices to treat the case of an affine admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X} = \mathrm{Spf}\, A$. In other words, we need to show that any idempotent of $A[1/\varpi]$ sits in A.

Pick any idempotent $e \in A[1/\varpi]$. Since $e^2 = e$, it is clearly power-bounded, i.e. $e \in A[1/\varpi]^\circ$. Then [Lüt16, Proposition 3.4.1][45] ensures that $A[1/\varpi]^\circ = A$, so $e \in A$. □

Remark B.2. Lemma B.1 is false if special fiber of A is not necessarily reduced. Consider $A = \mathbf{Z}_p[X]/(X^2 - pX)$ and $\mathfrak{X} = \mathrm{Spf}\, A$. Then

$$\overline{X} = \mathrm{Spec}\, \mathbf{F}_p[X]/(X^2)$$

is a one (non-reduced) point. But

$$\mathfrak{X}_{\mathbf{Q}_p} = \mathrm{Spa}\left(\mathbf{Q}_p[X]/(X^2 - pX), \mathbf{Q}_p[X]/(X^2 - pX)^\circ\right)$$

is two (reduced) points since $\mathbf{Q}_p[X]/(X^2 - pX) \simeq \mathbf{Q}_p[X]/(X) \oplus \mathbf{Q}_p[X]/(X - p)$.

Lemma B.3. Let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of admissible formal $\mathcal{O}_K$-schemes. Suppose that the adic generic fiber $\mathfrak{f}_K \colon \mathfrak{X}_K \to \mathfrak{Y}_K$ is a finite morphism of rigid space and that the special fiber of $\mathfrak{Y}$ is geometrically reduced. Then there is an open dense formal subscheme $\mathfrak{U} \subset \mathfrak{Y}^{\mathrm{sm}}$ in the smooth locus of $\mathfrak{Y}$ such that the restriction $\mathfrak{f}|_{\mathfrak{X}_{\mathfrak{U}}} \colon \mathfrak{X}_{\mathfrak{U}} \to \mathfrak{U}$ is finite and flat.

Proof. As $\mathfrak{Y}$ is $\mathcal{O}_K$-flat, we conclude that the smooth locus $\mathfrak{Y}^{\mathrm{sm}} \subset \mathfrak{Y}$ coincides with the smooth locus in the special fiber. Then we use that the special fiber of $\mathfrak{Y}$ is geometrically reduced to conclude that the locus $\overline{\mathfrak{Y}}^{\mathrm{sm}}$ is schematically dense in $\overline{\mathfrak{Y}}$, so we can replace $\mathfrak{Y}$ with $\mathfrak{Y}^{\mathrm{sm}}$ to assume that $\mathfrak{Y}$ is smooth.

Now [Gro66, Theoreme 14.44] implies that the locus $\{x \in \overline{\mathfrak{X}} \mid \bar{\mathfrak{f}} \text{ is flat at } x\}$ is open. Reducedness of $\overline{\mathfrak{Y}}$ guarantees that this locus contains all point over the generic points of $\overline{\mathfrak{Y}}$. Then a standard argument using that properness of $\bar{\mathfrak{f}}$ guarantees that there is a dense $U \subset \overline{Y}$ such that $\bar{\mathfrak{f}}|_{\bar{\mathfrak{f}}^{-1}(U)} \colon \bar{\mathfrak{f}}^{-1}(U) \to U$ is flat. The open subscheme $U \subset \overline{\mathfrak{Y}}$ defines an open dense subscheme $\mathfrak{U} \subset \mathfrak{Y}$. The fiber-by-fiber flatness criterion implies that the morphism $\mathfrak{f}_0 \colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ is flat over $\mathfrak{U}_0$, and then [Bos14, Lemma 8.2/1] guarantees that $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ is flat over a dense open $\mathfrak{U} \subset \mathfrak{Y}$. So we can replace $\mathfrak{Y}$ with $\mathfrak{U}$ to assume that $\mathfrak{f}$ is a flat morphism and $\mathfrak{Y}$ is smooth.

We note that the special fibers $\overline{\mathfrak{Y}}$ and $\overline{\mathfrak{X}}$ are both of pure dimension d by [FK18, Corollary II.10.1.11]. Thus, every irreducible component of both finite type k-schemes $\overline{\mathfrak{Y}}$ and $\overline{\mathfrak{X}}$ has dimension d. This implies that $\bar{\mathfrak{f}}^{-1}(\eta)$ is discrete for any generic point $\eta \in \overline{\mathfrak{Y}}$, so it is quasi-finite over generic points of $\overline{\mathfrak{Y}}$. Now $\bar{\mathfrak{f}}$ is proper by [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]), so we can use [Gro66, Corollaire 13.1.5] to conclude that there is an open, schematically dense subset $\overline{\mathfrak{U}} \subset \overline{Y}$ such that $\mathfrak{f}$ is quasi-finite over. As $\bar{\mathfrak{f}}$ is proper, it is automatically finite over $\overline{\mathfrak{U}}$ by [Gro66, Théorème 8.11.1]. Now we use [FK18, Proposition 4.2.3] to conclude that $\mathfrak{f}$ is finite over that $\mathfrak{U}$. □

Corollary B.4. Let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of admissible formal $\mathcal{O}_K$-schemes with geometrically reduced special fibers. Suppose that the adic generic fiber $\mathfrak{f}_K \colon \mathfrak{X}_K \to \mathfrak{Y}_K$ is an isomorphism. Then there is an open dense formal subscheme $\mathfrak{U} \subset \mathfrak{Y}^{\mathrm{sm}}$ in the smooth locus of $\mathfrak{Y}$ such that the restriction $\mathfrak{f}|_{\mathfrak{X}_{\mathfrak{U}}} \colon \mathfrak{X}_{\mathfrak{U}} \to \mathfrak{U}$ is an isomorphism.

[45]We note that the proof there uses that A_K is reduced only to ensure that any $g \in A_K$ with $|g|_{sup} < 1$ is topologically nilpotent. But this is always true by [Bos14, Corollary 3.1/18].

Proof. We use lemma B.3 to find open dense $\mathfrak{U} \subset \mathfrak{Y}^{\mathrm{sm}}$ such that $\mathfrak{f}|_{\mathfrak{X}_{\mathfrak{U}}} \colon \mathfrak{X}_{\mathfrak{U}} \to \mathfrak{U}$ is finite. We claim that it is an isomorphism, it is sufficient to check locally. So we can assume that $\mathfrak{Y} = \mathrm{Spf}\, A$ is affine and the morphism is given by a finite morphism $\mathrm{Spf}\, B \to \mathrm{Spf}\, A$. As f_K is an isomorphism, we conclude that the map $A[1/\varpi] \to B[1/\varpi]$ is an isomorphism. Then the map $A \to B$ must be an isomorphism by [Lüt16, Proposition 3.4.1]. □

Lemma B.5. Let $X = \mathrm{Spa}(A, A^\circ)$ be an affinoid rigid K-space of pure dimension d. Then $\dim A = d$.

Proof. This follows from [FK18, Proposition II.10.1.9 and Corollary II.10.1.10]. Namely, *loc. cit.* guarantees that

$$\dim A = \dim X = \sup_{x \in X^{\mathrm{cl}}} (\dim \mathcal{O}_{X,x}) = d.$$

□

Lemma B.6. Let X be a rigid K-space of pure dimension d, and $\mathfrak{X}$ its formal $\mathcal{O}_K$-model. Then $\overline{\mathfrak{X}}$ is a k-scheme of pure dimension d.

Proof. We note that $\overline{\mathfrak{X}}$ is of pure dimension d if and only if any affine open $U \subset \overline{\mathfrak{X}}$ is of dimension d. We note that U defines an affine open $\mathfrak{U} = \mathrm{Spf}\, B \subset \mathfrak{X}$ with $U = \mathrm{Spec}\, B/\mathfrak{m}B$. Then [FK18, Corollary II.10.1.11] and Lemma B.5 imply that

$$\dim U = \dim B/\mathfrak{m}B = \dim B_K = \dim \mathfrak{U}_K = d.$$

Therefore, $\overline{\mathfrak{X}}$ is of pure dimension d. □

Lemma B.7. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with reduced special fiber, and $\mathcal{E}$ a vector bundle on $\mathfrak{X}$. Then, for any dense open $\mathfrak{U} \subset \mathfrak{X}$, the natural morphism $\mathcal{E} \to j_{\mathfrak{U},*}(\mathcal{F}|_{\mathfrak{U}})$ is injective.

Proof. The claim can be checked on global sections over all open affine formal subschemes of $\mathfrak{X}$. Thus, it suffices to show that $\mathcal{E}(\mathfrak{X}) \to \mathcal{E}(\mathfrak{U})$ is injective for an affine $\mathfrak{X} = \mathrm{Spf}\, A$ and a dense open $\mathfrak{U} \subset \mathfrak{X}$.

We can check this claim after shrinking $\mathfrak{U}$, so the Prime Avoidance Lemma [Sta19, Tag 00DS], applied to the special fiber, allows us to find a dense open $\mathrm{Spf}\, A_{\{f\}} \subset \mathfrak{U}$ for some $f \in A$. Therefore, we can assume that dense open $\mathfrak{U} = \mathrm{Spf}\, A_{\{f\}}$ is a princial open. Suppose that $\mathcal{E} \simeq P^{\Delta}$ for some finite, projective A-module P. Then what we need to check is that the natural map $P \to P_{\{f\}}$ is injective. We note that both sides are ϖ-adically complete and the limit functor is left-exact, so it suffcies to show that the morphism

$$P/\varpi^n P \to P_f/\varpi^n P_f$$

is injective for any $n \geq 1$. Now we use [Sta19, Tag 05CB] to see that the morphism

$$P/\varpi^n P \to \prod_{\mathfrak{p} \in \mathrm{WeakAss}(P/\varpi^n P)} (P/\varpi^n P)_{\mathfrak{p}}$$

is injective. So it is enough to show that this map factors through $P/\varpi^n P \to P_f/\varpi^n P_f$. Equivalently, we need to show that $\overline{f} \notin \mathfrak{p}$ for any weakly associated prime $\mathfrak{p} \subset P/\varpi^n P$ and any $n \geq 1$. This is equivalent to show that $\mathrm{WeakAss}(P/\varpi^n P) \subset \mathrm{D}(\overline{f})$ for any $n \geq 1$. Now we use [RG71, Proposition 3.4.3], applied to the morphism $\mathrm{Spec}\, A/\varpi^n A \to \mathrm{Spec}\, \mathcal{O}_K/\varpi^n \mathcal{O}_K$, $\mathcal{M} = \widetilde{(P/\varpi^n P)}$, and $\mathcal{N} = \mathcal{O}_{\mathrm{Spec}\, \mathcal{O}_K/\varpi^n \mathcal{O}_K}$, to see that $\mathrm{WeakAss}(P/\varpi^n P) = \mathrm{WeakAss}(P/\mathfrak{m}P)$ as subsets of $|\mathrm{Spec}\, A/\varpi^n A| = |\mathrm{Spec}\, A/\mathfrak{m}|$. In particular, we conclude that we can check that the inclusion $\mathrm{WeakAss}(P/\varpi^n P) \subset \mathrm{D}(\overline{f})$ on the special fiber! Now we use [Sta19, Tag 0EMA] and the assumption that $\mathrm{Spec}\, A/\mathfrak{m}$

is reduced to conclude that $\mathrm{WeakAss}(A/\mathfrak{m}A) = \{\text{Generic point of } \operatorname{Spec} A/\mathfrak{m}\}$ and $\mathrm{D}(\overline{f})$ contains every generic point of $\operatorname{Spec} A/\mathfrak{m}$ as $\operatorname{Spf} A_{\{f\}}$ is dense. So, we see that $\mathrm{WeakAss}(P/\varpi^n P) \subset \mathrm{D}(\overline{f})$ for any $n \geq 1$. This finishes the proof. $\square$

Lemma B.8. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme, and let $\mathcal{F} \in \mathbf{Coh}(\mathfrak{X})$ be a reflexive module. Suppose that adic generic fiber $X = \mathfrak{X}_K$ is smooth, special fiber $\overline{\mathfrak{X}}$ is reduced, and $\mathfrak{U} \subset \mathfrak{X}$ is an open containing all generic points of the special fiber. Then the natural morphism $\mathcal{F} \to j_{\mathfrak{U},*}\mathcal{F}_{\mathfrak{U}} \cap \mathcal{F}_K$ is an isomorphism.

Proof. The question is local, so we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine with the unique generic point in the special fiber η. In this case, we see that the dual module $\mathcal{F}^\vee \coloneqq \mathcal{H}om_{\mathfrak{X}}(\mathcal{F}, \mathcal{O}_{\mathfrak{X}})$ is coherent, so we find some presentation

$$\mathcal{O}_{\mathfrak{X}}^n \to \mathcal{O}_{\mathfrak{X}}^m \to \mathcal{F}^\vee \to 0 \ .$$

Thus we can use reflexivity of $\mathcal{F}$ to get a "co-presentation"

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathfrak{X}}^m \to \mathcal{O}_{\mathfrak{X}}^n \ .$$

Now we that the functor $\mathcal{G} \mapsto j_{\mathfrak{U},*}\mathcal{G}_{\mathfrak{U}} \cap \mathcal{G}_K$ is left exact, so we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{X}}^m \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{X}}^n \arrow[d] \\
0 \arrow[r] & j_{\mathfrak{U},*}\mathcal{F}_{\mathfrak{U}} \cap \mathcal{F}_K \arrow[r] & j_{\mathfrak{U},*}(\mathcal{O}_{\mathfrak{X}}^m)_{\mathfrak{U}} \cap (\mathcal{O}_{\mathfrak{X}}^m)_K \arrow[r] & j_{\mathfrak{U},*}(\mathcal{O}_{\mathfrak{X}})_{\mathfrak{U}}^n \cap (\mathcal{O}_{\mathfrak{X}})_K^n
\end{tikzcd}$$

that shows that it is sufficient to show the result for $\mathcal{O}_{\mathfrak{X}}^m$ for any m. As the functor $\mathcal{G} \mapsto j_{\mathfrak{U},*}\mathcal{G}_{\mathfrak{U}} \cap \mathcal{G}_K$ clearly commutes with finite direct sums, it is enough to show the claim for $\mathcal{O}_{\mathfrak{X}}$.

Injectivity of the map $\mathcal{O}_{\mathfrak{X}} \mapsto j_{\mathfrak{U},*}\mathcal{O}_{\mathfrak{U}}$: This follows from Lemma B.7 applied to $\mathcal{E} = \mathcal{O}_{\mathfrak{X}}$.

The map $\mathcal{O}_{\mathfrak{X}} \mapsto j_{\mathfrak{U},*}\mathcal{O}_{\mathfrak{U}} \cap (\mathcal{O}_{\mathfrak{X}})_K$ *is an isomorphism*: The previous step clearly implies that the map is injective, so we only need to check surjectivity. As open affines form a base of topology on $\mathfrak{X}$, it suffices to show that the morphism

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \cap \mathcal{O}_{\mathfrak{X}_K}(\mathfrak{X}_K)$$

is surjective. More concretely, we need to show that the morphism

$$A \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \cap A_K$$

is surjective. We pick any element $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \cap A_K$ and want to show that f lives in A[46]. We note that there is some $n \geq 0$ such that $\varpi^n f \in A$. Then we consider the map $A/\varpi^n \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})/\varpi^n$, the proof of Lemma B.7 implies that this map is injective. However, $\varpi^n f$ lies in its kernel as $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$. This implies that $\varpi^n f = \varpi^n g$ for some $g \in A$. This, in turn, implies that $f = g$ as A is $\mathcal{O}_K$-flat. This shows that $f \in A$. $\square$

Remark B.9. A similar argument shows that Lemma B.8 holds for schemes over $\mathcal{O}_K$. Namely, suppose that X a flat, finitely presented $\mathcal{O}_K$-scheme with smooth generic fiber, reduced special fiber, and $\mathcal{F}$ a coherent, reflexive $\mathcal{O}_X$-module. Then the natural morphism $j_*\left(\mathcal{F}|_{X^{\mathrm{sm}}}\right) \to \mathcal{F}$ is an isomorphism, where $j\colon X^{\mathrm{sm}} \to X$ is the inclusion of the smooth locus of X into X.

[46]As we already know injectivity, it does make sense to ask if $f \in A$.

Appendix C. Generic Fibers of $\mathcal{O}_{\mathfrak{X}}$-modules

We collect some results about generic fibers of certain $\mathcal{O}_{\mathfrak{X}}$-modules on an admissible, formal model $\mathfrak{X}$. Even though all the results seem to be very standard it is hard to find a precise reference.

For the rest of the section, we fix a complete rank-1 valuation field K with the valuation ring $\mathcal{O}_K$ and a pseudo-uniformizer ϖ. We also fix $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathfrak{X}_K$. Then we have a morphism of ringed sites

$$\mathrm{sp}_{\mathfrak{X}} \colon (X_{\mathrm{an}}, \mathcal{O}_X) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}}) \ .$$

that is flat by [FK18, Corollary II.5.1.4].

Definition C.1. The *generic fiber* of $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$ is the complex $\mathcal{F}_C \coloneqq \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}(\mathcal{F})$.

Remark C.2. Flatness of $\mathrm{sp}_{\mathfrak{X}}$ guarantees that $\mathcal{F}_C \simeq \mathrm{sp}^*_{\mathfrak{X}}\mathcal{F}$ for $\mathcal{F} \in \mathbf{Mod}_{\mathfrak{X}}$.

Definition C.3. An $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is ϖ^∞-torsion if the natural morphism

$$\mathrm{colim}_n\, \mathcal{F}[p^n] \to \mathcal{F}$$

is an isomorphism.

Lemma C.4. Let $\mathcal{F}$ be a ϖ^∞-torsion $\mathcal{O}_{\mathfrak{X}}$-module. Then $\mathcal{F}_K$ is the zero sheaf. More generally, if $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$ with ϖ^∞-torsion cohomology sheaves, then $\mathcal{F}_K$ is the zero complex.

Proof. It suffices to show that $\mathcal{F}_K \simeq 0$ for a ϖ^∞-torsion $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. Since $\mathrm{sp}^*_{\mathfrak{X}}$ commutes with colimits, we see that the natural morphism

$$\mathrm{colim}_n (\mathcal{F}[p^n])_K \to \mathcal{F}_K$$

is an isomorphism. Now, clearly $\mathcal{F}[p^n]_K \simeq 0$. And so $\mathcal{F}_K \simeq 0$. ☐

Corollary C.5. The functor $(-)_K \colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(X)$ canonically descends to the functor

$$(-)_K \colon \mathbf{D}(\mathfrak{X})^a \to \mathbf{D}(X).$$

Proof. [Zav21a, Definition 3.4.2 and Theorem 3.4.9] guarantee that $\mathbf{D}(\mathfrak{X})^a$ is the Verdier quotient of $\mathbf{D}(\mathfrak{X})$ by the subcategory of $\mathbf{D}_{\Sigma_{\mathfrak{X}}}(\mathfrak{X})$ of complexes with almost zero cohomology sheaves. Therefore, the results follows from the universal properties of Verdier quotients, Lemma C.4, and the observation that any almost zero $\mathcal{O}_{\mathfrak{X}}$-module is ϖ^∞-torsion. ☐

Lemma C.6. The functors $(-)_K \colon \mathbf{D}(\mathfrak{X})^a \to \mathbf{D}(X)$ induces a functor $(-)_K \colon \mathbf{D}_{acoh}(\mathfrak{X})^a \to \mathbf{D}_{coh}(X)$. More precisely, if $\mathfrak{X} = \mathrm{Spf}\, A$ is affine and $\mathcal{F} \simeq M^\Delta$ for some $M \in \mathbf{Mod}^{acoh}_A$. Then the natural morphism $\widetilde{M[1/\varpi]} \to \mathcal{F}_K$ is an isomorphism.

Proof. The claim is local on $\mathfrak{X}$, so we may assume that $\mathfrak{X} = \mathrm{Spf}\, A$ is affine. Then using flatness of $\mathrm{sp}_{\mathfrak{X}}$ and Lemma C.4, we easily reduce to the case an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. Now [Zav21a, Lemma 4.6.1] guarantees that $\mathcal{F} \simeq M^\Delta$ for some almost coherent A-module M. Clearly, $M[1/\varpi]$ is a finite $A[1/\varpi]$-module. So it suffices to show that $\mathcal{F}_K \simeq \widetilde{M[1/\varpi]}$.

Choose a finitely presented A-module N and map $N \to M$ with kernel and cokernel annihilated by ϖ. Then [Zav21a, Lemma 4.5.14] guarantees that $N^\Delta \to M^\Delta$ has kernel and cokernel is annihilated by ϖ. Therefore, Lemma C.4 guarantees that

$$(N^\Delta)_K \simeq (M^\Delta)_K \simeq \mathcal{F}_K.$$

Now [FK18, Theorem II.5.3.1] implies that

$$\mathcal{F}_K \simeq (N^\Delta)_K$$

is coherent. Therefore, the natural morphism

$$\Gamma(\widetilde{X, \mathcal{F}_K}) \to \mathcal{F}_K$$

is an isomorphism. And now

$$\Gamma(\widetilde{X, \mathcal{F}_K}) \simeq \Gamma(\widetilde{X, (N^\Delta)_K}) \simeq N[1/\varpi] \simeq M[1/\varpi].$$

□

Lemma C.7. Let $\mathcal{F} \in \mathbf{D}^+_{coh}(X)$, then the natural morphism $\mathcal{F} \to (\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{F})_K$ is an isomorphism.

Proof. First of all, we use flatness of $\mathrm{sp}_{\mathfrak{X}}$ and the convergent spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p\,\mathrm{sp}_{\mathfrak{X},*}(\mathcal{H}^q(\mathcal{F})) \Rightarrow \mathrm{R}^{p+q}\,\mathrm{sp}_{\mathfrak{X},*}(\mathcal{F})$$

to reduce the question to the case of a coherent sheaf $\mathcal{F}$ concentrated in degree 0. Then we use that vanishing of higher cohomology groups of coherent sheaves on affinoid spaces [FK18, Proposition 6.5.1] to conclude that $\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{F} = \mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{F}$. Thus, we only need to show that the natural map

$$\mathcal{F} \to (\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{F})_K$$

is an isomorphism.

The claim is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\,A$ is affine. Then $\mathcal{F}$ is a coherent sheaf on an affinoid space, so it admits a presentation

$$\mathcal{O}_X^n \to \mathcal{O}_X^m \to \mathcal{F} \to 0$$

Note that the kernels of the maps $\mathcal{O}_X^m \to \mathcal{F}$ and $\mathcal{O}_X^n \to \mathcal{O}_X^m$ are coherent, so the higher derived functors $\mathrm{R}^n\,\mathrm{sp}_{\mathfrak{X},*}$ vanish on these sheaves. Therefore, we conclude that the sequence

$$\mathrm{sp}_{\mathfrak{X},*}\,(\mathcal{O}_X^n) \to \mathrm{sp}_{\mathfrak{X},*}\,(\mathcal{O}_X^m) \to \mathrm{sp}_{\mathfrak{X},*}\,(\mathcal{F}) \to 0$$

is exact. Now we use exactness of $\mathrm{sp}_{\mathfrak{X}}^*$ to show that we have an exact sequence

$$\left(\mathrm{sp}_{\mathfrak{X},*}(\mathcal{O}_X^n)\right)_K \to \left(\mathrm{sp}_{\mathfrak{X},*}(\mathcal{O}_X^m)\right)_K \to \left(\mathrm{sp}_{\mathfrak{X},*}(\mathcal{F})\right)_K \to 0\ .$$

Using the both $\mathrm{sp}_{\mathfrak{X},*}$ and $\mathrm{sp}_{\mathfrak{X}}^*$ commute with finite direct sums, we conclude that it is sufficient to check the claim for $\mathcal{F} = \mathcal{O}_X$.

Now we see that the very definition of $(-)_K$ implies that the natural morphism

$$\mathcal{O}_X \to (\mathcal{O}_{\mathfrak{X}})_K$$

is an isomorphism. Therefore, it suffices to show that the natural map $(\mathcal{O}_{\mathfrak{X}})_K \to (\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{O}_X)_K$ is an isomorphism. However, we notice that the map $\mathcal{O}_{\mathfrak{X}} \to \mathrm{sp}_{\mathfrak{X},*}\,\mathcal{O}_X$ is injective with the cokernel $\mathcal{Q}$ of ϖ^∞-torsion. Thus Lemma C.4 implies that $\mathcal{Q}_K = 0$, so the map $(\mathcal{O}_{\mathfrak{X}})_K \to (\mathrm{sp}_{\mathfrak{X},*}\,\mathcal{O}_X)_K$ is an isomorphism as $(-)_K$ is exact. □

Lemma C.8. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of topologically finitely presented formal $\mathcal{O}_K$-schemes. Then a natural map $\mathcal{F} \to \mathbf{R}\lim_n \tau^{\geq -n}\mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$. Moreover, there is an integer N such that $\mathcal{H}^j(\mathbf{R}\mathfrak{f}_*\mathcal{F}) \to \mathcal{H}^j(\mathbf{R}\mathfrak{f}_*\tau^{\geq -n}\mathcal{F})$ is an isomorphism for any $j \geq N - n$ and any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$.

Proof. We note that [Zav21a, Lemma 5.1.7] implies that the underlying topological space of $\mathfrak{X}$ is of finite Krull dimension as $\dim \operatorname{Spf} \mathcal{O}_K = 0$. Moreover, $\dim \mathfrak{U} \leq \dim \mathfrak{X}$ for any open $\mathfrak{U} \subset \mathfrak{X}$. Thus [Sta19, Tag 0A3G] implies that the assumptions of [Sta19, Tag 0D6S] for $\mathcal{A} = \mathbf{Mod}_{\mathfrak{X}}$ are satisfied, so we can conclude that the map $\mathcal{F} \to \mathbf{R}\lim_n \tau^{\geq -n}\mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$. The second part of the lemma follows similarly from [Zav21a, Lemma 5.1.7] and [Sta19, Tag 0D6U]. □

Now we consider a morphism $\pi \colon \mathfrak{X}' \to \mathfrak{X}$ of admissible formal $\mathcal{O}_K$-schemes with generic fiber $\pi_K \colon X' \to X$. Then we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{sp}_{\mathfrak{X}'}"] \arrow[d, "\pi_K"] & \mathfrak{X}' \arrow[d, "\pi"] \\ X \arrow[r, "\mathrm{sp}_{\mathfrak{X}}"] & \mathfrak{X}. \end{tikzcd}$$

This defines a natural base change map

$$\mathbf{L}\,\mathrm{sp}_{\mathfrak{X}}^*\,\mathbf{R}\pi_*\mathcal{F} \to \mathbf{R}\pi_{K,*}\mathbf{L}\,\mathrm{sp}_{\mathfrak{X}'}^*\,\mathcal{F}$$

for any $\mathcal{F} \in \mathbf{D}(\mathfrak{X}')$. This map can be rewritten in our notation as the morphism

$$\left(\mathbf{R}\pi_*\mathcal{F}\right)_K \to \mathbf{R}\pi_{K,*}\left(\mathcal{F}_K\right).$$

Similarly, the same map $\left(\mathbf{R}\pi_*\mathcal{F}\right)_K \to \mathbf{R}\pi_{K,*}\left(\mathcal{F}_K\right)$ can be defined for any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})^a$.

Lemma C.9. Suppose that $\pi \colon \mathfrak{X}' \to \mathfrak{X}$ is a proper map. Then the base change morphism

$$\left(\mathbf{R}\pi_*\mathcal{F}\right)_K \to \mathbf{R}\pi_{K,*}\left(\mathcal{F}_K\right)$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}_{acoh}(\mathfrak{X}')^a$ or $\mathcal{F} \in \mathbf{D}_{acoh}(\mathfrak{X}')$.

Proof. The claim is local on $\mathfrak{X}$, so we may and do assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Now we use the functor $(-)_! \colon \mathbf{D}_{acoh}(\mathfrak{X}')^a \to \mathbf{D}_{qc,acoh}(\mathfrak{X}')$ and Lemma C.4 to reduce to the case $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X}')$. Then we use exactness of the functor $(-)_K$ and Lemma C.8 to reduce the question to the case $\mathcal{F} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}')$. Finally, we use the spectral sequences

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p\pi_*(\mathcal{H}^q(\mathcal{F})) \Rightarrow \mathrm{R}^{p+q}\pi_*(\mathcal{F}),$$
$$\mathrm{E}_2'^{p,q} = \mathrm{R}^p\pi_{K,*}(\mathcal{H}^q(\mathcal{F}_K)) \Rightarrow \mathrm{R}^{p+q}\pi_{K,*}(\mathcal{F}_K)$$

and exactness of $(-)_K$ to reduce to the case of an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}'}$-module $\mathcal{F}$.

We reduce the claim to showing that the natural morphisms

$$(\mathrm{R}^i\pi_*\mathcal{F})_K \to \mathrm{R}^i\pi_{K,*}(\mathcal{F}_K)$$

are isomorphisms for any adically quasi-coherent, almost coherent $\mathcal{F}$ and $i \geq 0$. We note that [Lüt90, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]) implies that π_K is automatically proper. Therefore, [Zav21a, Theorem 5.1.7] implies that the $\mathcal{O}_{\mathfrak{X}}$-modules $\mathbf{R}^i\pi_*\mathcal{F}$ are adically quasi-coherent, almost coherent and the natural morphisms

$$\mathrm{H}^i(\mathfrak{X}', \mathcal{F})^{\Delta} \to \mathrm{R}^i\pi_*\mathcal{F}$$

are isomorphisms. Therefore, Lemma C.6 implies that the sheaves $(\mathrm{R}^i\pi_*\mathcal{F})_K$ are coherent and canonically isomorphic to $\mathrm{H}^i\widetilde{(\mathfrak{X}', \mathcal{F})}[1/\varpi]$. Likewise, [FK18, Theorem 7.5.19] (and its proof) guarantees that $\mathrm{R}^i\pi_{K,*}\mathcal{F}_K$ are coherent and the natural maps

$$\mathrm{H}^i\widetilde{(X', \mathcal{F}_K)} \to \mathrm{R}^i\pi_{K,*}\mathcal{F}_K$$

are isomorphisms for all $i \geq 0$. Therefore, the question is reduced to showing that the natural map

$$\mathrm{H}^i(\mathfrak{X}', \mathcal{F})[1/\varpi] \to \mathrm{H}^i(X', \mathcal{F}_K)$$

is an isomorphism for any adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ and any $i \geq 0$.

We choose a covering $\mathfrak{X}' = \cup_{j=1}^N \mathfrak{U}_j$ of $\mathfrak{X}'$ by open affines $\mathfrak{U}_j$. [FK18, Theorem I.7.1.1] implies that cohomology groups of $\mathcal{F}$ can be computed by means of the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ associated to $\mathcal{F}$.

The complex $C^\bullet(\mathfrak{U}_K, \mathcal{F}_K)$ is equal to $C^\bullet(\mathfrak{U}, \mathcal{F})[1/\varpi]$ by Lemma C.6. Since coherent sheaves do not have higher cohomology groups on affinoid spaces, we conclude that

$$\mathrm{H}^i(\mathfrak{X}', \mathcal{F})[1/\varpi] = \mathrm{H}^i(C^\bullet(\mathfrak{U}, \mathcal{F})[1/\varpi]) = \mathrm{H}^i(C^\bullet(\mathfrak{U}_K, \mathcal{F}_K)) = \mathrm{H}^i(\mathfrak{X}'_K, \mathcal{F}_K)$$

for $i \geq 0$. This finishes the argument. $\square$

Now we assume that $K = C$ is a complete, algebraically closed rank-1 valued field of mixed characteristic $(0, p)$, and $\mathfrak{X}$ an admissible formal $\mathcal{O}_C$-model with smooth generic fiber $X = \mathfrak{X}_C$. We consider morphisms of ringed topoi

$$\begin{aligned} \mu \colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) &\to (X_{\text{an}}, \mathcal{O}^+_X), \\ \nu \colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) &\to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), \\ \mathrm{sp}_{\mathfrak{X}} \colon (X_{\text{an}}, \mathcal{O}_X) &\to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), \end{aligned}$$

and the integral version of the specialization map

$$\mathrm{sp}^+_{\mathfrak{X}} \colon (X_{\text{an}}, \mathcal{O}^+_X) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Lemma C.10. Let X be a smooth rigid C-space with an admissible formal model $\mathfrak{X}$. Then the natural maps

$$\left(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X\right)_C \to \left(\mathbf{R}\nu_*\widehat{\mathcal{O}}_X\right)_C \text{ and } \left(\mathbf{R}\nu_*\widehat{\mathcal{O}}_X\right)_C \to \mathbf{R}\mu_*\widehat{\mathcal{O}}_X$$

are isomorphisms.

Proof. We recall that $\widehat{\mathcal{O}}_X \simeq \widehat{\mathcal{O}}^+_X[1/p]$, so the cokernel $\widehat{\mathcal{O}}^+_X \to \widehat{\mathcal{O}}_X$ is p^∞-torsion. Thus, the cone of the map $\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X \to \mathbf{R}\nu_*\widehat{\mathcal{O}}_X$ has p^∞-torsion cohomology sheaves. Therefore, Lemma C.4 implies that $(\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X)_C \to (\mathbf{R}\nu_*\widehat{\mathcal{O}}_X)_C$ is an isomorphism.

The map $(\mathbf{R}\nu_*\widehat{\mathcal{O}}_X)_C \to \mathbf{R}\mu_*\widehat{\mathcal{O}}_X$ comes as the adjunction

$$\eta \colon \left(\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}(\mathbf{R}\mu_*\widehat{\mathcal{O}}_X)\right)_C = \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}(\mathbf{R}\mu_*\widehat{\mathcal{O}}_X) \to \mathbf{R}\mu_*\widehat{\mathcal{O}}_X$$

We note that the complex $\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \in \mathbf{D}^b_{coh}(X)$ by [Sch13b, Proposition 3.23], so Lemma C.7 implies that η is an isomorphism. $\square$

Definition C.11. We denote by $\delta_{\mathfrak{X}} \colon (\mathbf{R}\nu_*\widehat{\mathcal{O}}^+_X)_C \to \mathbf{R}\mu_*\widehat{\mathcal{O}}_X$ the isomorphism that is the composition of isomorphisms from Lemma C.10.

Lemma C.12. Let X be a smooth rigid-analytic C-space, and let $\pi \colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of two admissible formal $\mathcal{O}_C$-models of X, i.e. the diagram

$$\begin{tikzcd} & (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'}) \arrow[d, "\pi"] \\ (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \arrow[ur, "\nu'"] \arrow[r, "\nu"] & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{tikzcd}$$

is commutative. Then so is the diagram

$$\begin{tikzcd}
(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X)_C \arrow[rr, "\delta_{\mathfrak{X}'}"] & & \mathbf{R}\mu_*\widehat{\mathcal{O}}_X \\
(\mathbf{R}\pi_*\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X)_C \arrow[u] \arrow[r, "\sim"] & (\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X)_C, \arrow[ur, "\delta_{\mathfrak{X}}"] &
\end{tikzcd} \tag{C.1}$$

where the left vertical maps is the base change isomorphism[47] from Lemma C.9.

Proof. We recall the definition of the map $\delta_{\mathfrak{X}}$. Firstly, we identify the functors $\mathbf{R}\nu_*$ with $\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}\circ\mathbf{R}\mu_*$ where $\nu\colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (X_{\mathrm{an}}, \mathcal{O}^+_X)$. And then $\delta_{\mathfrak{X}}$ comes as the composition

$$\mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}^+_{\mathfrak{X},*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}^+_X \xrightarrow{\alpha_{\mathfrak{X}}} \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \xrightarrow{\eta_{\mathfrak{X}}} \mathbf{R}\mu_*\widehat{\mathcal{O}}_X\ ,$$

where the map $\alpha_{\mathfrak{X}}$ is just the natural map induces by the localization map $\widehat{\mathcal{O}}^+_X \to \widehat{\mathcal{O}}_X$ and the second map comes from the adjunction map $\mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*} \to \mathrm{Id}$. Thus, in order to see that the diagram (C.1) commutes, it is sufficient to see that the diagram

$$\begin{tikzcd}
\mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}'}\,\mathbf{R}\,\mathrm{sp}^+_{\mathfrak{X}',*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}^+_X \arrow[r, "\alpha_{\mathfrak{X}'}"] & \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}'}\,\mathbf{R}\,\mathrm{sp}_{\mathfrak{X}',*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \arrow[dr, "\eta_{\mathfrak{X}'}"] & \\
\mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\pi_*\mathbf{R}\,\mathrm{sp}^+_{\mathfrak{X}',*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}^+_X \arrow[u] \arrow[r] \arrow[d, "\sim"] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(1)"] & \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\pi_*\mathbf{R}\,\mathrm{sp}_{\mathfrak{X}',*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \arrow[u] \arrow[d, "\sim"] \arrow[r, phantom, "(3)"] & \mathbf{R}\mu_*\widehat{\mathcal{O}}_X \\
\mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}^+_{\mathfrak{X},*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}^+_X \arrow[r, "\alpha_{\mathfrak{X}}"] & \mathbf{L}\,\mathrm{sp}^*_{\mathfrak{X}}\,\mathbf{R}\,\mathrm{sp}_{\mathfrak{X},*}\,\mathbf{R}\mu_*\widehat{\mathcal{O}}_X \arrow[ur, "\eta_{\mathfrak{X}}"] &
\end{tikzcd}$$

commutative. We now briefly explain why this diagram is commutative. Square (1) commutes by tautological reasons, diagram (2) commutes by functoriality of the base change morphisms. The tricky part part is to check commutativity of the square (3), however this is a standard exercise on adjoint functors, so we leave this to the reader. □

Appendix D. Pro-Étale Trace Maps

For the rest of the section, we fix a complete, algebraically closed rank-1 valued field C of mixed characteristic $(0, p)$. We also fix a finite étale morphism $f\colon X' \to X$ of rigid C-spaces.

The main goal for this section is to construct trace maps associated with such f.

We note that such f is proper, so [Hub96, Definition 5.2.1 and Propositions 5.2.4] imply that $f_* = f_!\colon \mathcal{A}b(X'_{\text{ét}}) \to \mathcal{A}b(X_{\text{ét}})$. Therefore, there is an adjunction (f_*, f^*) for any finite étale f.

Definition D.1. We define the *étale trace map*

$$\mathrm{Tr}_{\text{ét},f,\mathcal{F}}\colon f_*f^{-1}\mathcal{F} \to \mathcal{F}$$

for any $\mathcal{F} \in \mathcal{A}b(X_{\text{ét}})$ to be the counit of the adjunction (f_*, f^*).

[47] $\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_X \in \mathbf{D}_{acoh}(\mathfrak{X})$ by [Zav21a, Theorem 6.13.6]

We will be particularly interested in the étale traces

$$\mathrm{Tr}_{\text{ét},f,\mathcal{O}_X^+}\colon f_*\left(\mathcal{O}_{X'}^+\right) \to \mathcal{O}_X^+, \text{ and}$$

$$\mathrm{Tr}_{\text{ét},f,\mathcal{O}_X^+/p^n}\colon f_*\left(\mathcal{O}_{X'}^+/p\right) \to \mathcal{O}_X^+/p.$$

Remark D.2. If there is no ambiguity, we will often denote any of these maps just by $\mathrm{Tr}_{\text{ét},f}$.

Now we want to generalize the construction of the trace map to the pro-étale sheaves $\widehat{\mathcal{O}}^+$ and $\widehat{\mathcal{O}}$.

We recall that we have a commutative diagram:

$$\begin{tikzcd}
(X'_{\text{proét}}, \widehat{\mathcal{O}}^+_{X'}) \arrow[r, "\lambda_{X'}"] \arrow[d, "f_{\text{proét}}"] \arrow[rr, bend left, "\mu_{X'}"] & (X'_{\text{ét}}, \mathcal{O}^+_{X'}) \arrow[r] \arrow[d, "f_{\text{ét}}"] & (X'_{\text{an}}, \mathcal{O}^+_{X'}) \arrow[d, "f_{\text{an}}"] \\
(X_{\text{proét}}, \widehat{\mathcal{O}}^+_{X}) \arrow[r, "\lambda_{X}"] \arrow[rr, bend right, "\mu_{X}"'] & (X_{\text{ét}}, \mathcal{O}^+_{X}) \arrow[r] & (X_{\text{an}}, \mathcal{O}^+_{X})
\end{tikzcd}$$

Now [Sch13a, Corollary 3.17(ii)] guarantees that $f_{\text{proét},*}\left(\mathcal{O}^+_{X'}/p^n\right) \simeq \lambda_X^{-1}(f_{\text{ét},*}\mathcal{O}^+_{X'}/p^n)$.

Definition D.3. We define the *(mod-p^n) pro-étale trace map*

$$\mathrm{Tr}_{\text{proét},f,\mathcal{O}_X^+/p^n}\colon f_{\text{proét},*}\left(\mathcal{O}^+_{X'}/p^n\right) \to \mathcal{O}^+_X/p^n$$

as $\mathrm{Tr}_{\text{proét},f} := \lambda_X^{-1}\left(\mathrm{Tr}_{\text{proét},f,\mathcal{O}_X^+/p^n}\right)$

Remark D.4. If there is no ambiguity, we will often denote the trace map from Definition D.3 just by $\mathrm{Tr}_{\text{proét},f}$.

Lemma D.5. Let $f\colon X' \to X$ be a finite étale morphism of rigid-analytic spaces. Then the functor $f_{\text{proét},*}(-)$ is exact, i.e. $\mathbf{R}f_{\text{proét},*}\mathcal{F}$ is concentrated in degree 0 for any $\mathcal{F} \in \mathcal{A}(X'_{\text{proét}})$.

Proof. The claim is (pro-)étale local on X, so we can assume that $X' \to X$ is a split finite étale morphism. In this case, the claim is trivial. □

We recall that the sheaf $\widehat{\mathcal{O}}^+_X$ is defined as $\lim_n \mathcal{O}^+_X/p^n$ on $X_{\text{proét}}$, and $\widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}^+_X[1/p]$. Therefore, Lemma D.5 implies that

$$\mathbf{R}f_{\text{proét},*}(\widehat{\mathcal{O}}^+_{X'}) \simeq f_{\text{proét},*}(\widehat{\mathcal{O}}^+_{X'}) \simeq \lim_n f_{\text{proét},*}(\mathcal{O}^+_{X'}/p^n), \text{ and}$$

$$\mathbf{R}f_{\text{proét},*}(\widehat{\mathcal{O}}_{X'}) \simeq f_{\text{proét},*}(\widehat{\mathcal{O}}_{X'}) \simeq f_{\text{proét},*}(\widehat{\mathcal{O}}^+_{X'})[1/p]^{48}.$$

Definition D.6. We define the *(integral) pro-étale trace*

$$\mathrm{Tr}^+_{\text{proét},f}\colon f_{\text{proét},*}(\widehat{\mathcal{O}}^+_{X'}) \to \widehat{\mathcal{O}}^+_X$$

as $\mathrm{Tr}^+_{\text{proét},f} := \lim_n \mathrm{Tr}_{\text{proét},f,\mathcal{O}_X^+/p^n}$.

[48]Here we are using that the pro-étale site of an affinoid is coherent by [Sch13a, Proposition 3.12], so cohomology commute with filtered colimits.

Definition D.7. We define the *(rational) pro-étale trace*

$$\mathrm{Tr}_{\mathrm{pro\acute{e}t},f}\colon f_{\mathrm{pro\acute{e}t},*}(\widehat{\mathcal{O}}_{X'}) \to \widehat{\mathcal{O}}_X$$

as $\mathrm{Tr}_{\mathrm{pro\acute{e}t},f} := \mathrm{Tr}^+_{\mathrm{pro\acute{e}t},f}[1/p]$.

Definition D.8. We also define *pro-étale traces*

$$\mathrm{Tr}_{\mathrm{an},f}\colon \mathbf{R}f_{\mathrm{an},*}\mathbf{R}\mu_{X',*}\left(\widehat{\mathcal{O}}_{X'}\right) \to \mathbf{R}\mu_{X,*}\left(\widehat{\mathcal{O}}_X\right)$$

as

$$\mathrm{Tr}_{\mathrm{an},f} := \mathbf{R}\mu_{X,*}\left(\mathrm{Tr}_{\mathrm{pro\acute{e}t},f}\right) : \mathbf{R}\mu_{X,*}\left(f_{\mathrm{pro\acute{e}t},*}\widehat{\mathcal{O}}_{X'}\right) \to \mathbf{R}\mu_{X,*}\left(\widehat{\mathcal{O}}_X\right).$$

Now suppose that $f\colon X' \to X$ comes as generic fiber of a morphism $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ between admissible formal $\mathcal{O}_C$-models. Then we have a commutative diagram

$$\begin{tikzcd}
(X'_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_{X'}) \arrow[r, "\nu_{\mathfrak{X}',*}"] \arrow[d, "f_{\mathrm{pro\acute{e}t}}"] & (\mathfrak{X}'_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}'}) \arrow[d, "\mathfrak{f}"] \\
(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X) \arrow[r, "\nu_{\mathfrak{X},*}"] & (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}}).
\end{tikzcd}$$

Definition D.9. We define the *pro-étale trace*

$$\mathrm{Tr}^+_{\mathrm{Zar},\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\widehat{\mathcal{O}}^+_{X'}\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\widehat{\mathcal{O}}^+_X\right)$$

as

$$\mathrm{Tr}^+_{\mathrm{Zar},\mathfrak{f}} := \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathrm{Tr}_{\mathrm{pro\acute{e}t},f}\right) : \mathbf{R}\nu_{\mathfrak{X},*}\left(f_{\mathrm{pro\acute{e}t},*}\widehat{\mathcal{O}}^+_{X'}\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\widehat{\mathcal{O}}^+_X\right).$$

Definition D.10. Similarly, we define the *pro-étale trace*

$$\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}}\colon \mathbf{R}\mathfrak{f}_*\mathbf{R}\nu_{\mathfrak{X}',*}\left(\mathcal{O}^+_{X'}/p\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right)$$

as

$$\mathrm{Tr}_{\mathrm{Zar},\mathfrak{f}} := \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathrm{Tr}_{\mathrm{pro\acute{e}t},f,\mathcal{O}^+_X/p}\right) : \mathbf{R}\nu_{\mathfrak{X},*}\left(f_{\mathrm{pro\acute{e}t},*}\mathcal{O}^+_{X'}/p\right) \to \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{O}^+_X/p\right).$$

Lemma D.11. Let $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible formal $\mathcal{O}_C$-schemes such that its generic fiber $f\colon X' \to X$ is finite and étale. Then the following diagram

$$\begin{tikzcd}
(\mathbf{R}\mathfrak{f}_* \circ \mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'})_C \arrow[r] \arrow[d, "(\mathrm{Tr}^+_{\mathrm{Zar},\mathfrak{f}})_C"] & \mathbf{R}f_{\mathrm{an},*}(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'})_C \arrow[r, "\mathbf{R}f_{\mathrm{an},*}(\delta_{\mathfrak{X}'})"] & \mathbf{R}f_{\mathrm{an},*} \circ \mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_X \arrow[d, "\mathrm{Tr}_{\mathrm{an},f}"] \\
(\mathbf{R}\nu_{\mathfrak{X},*}\widehat{\mathcal{O}}^+_X)_C \arrow[rr, "\delta_{\mathfrak{X}}"] & & \mathbf{R}\mu_{X,*}\widehat{\mathcal{O}}_X
\end{tikzcd}$$

is commutative, and the base change map $(\mathbf{R}\mathfrak{f}_* \circ \mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'})_C \to \mathbf{R}f_{\mathrm{an},*}(\mathbf{R}\nu_{\mathfrak{X}',*}\widehat{\mathcal{O}}^+_{X'})$ is an isomorphism. Moreover, the top left arrow is an isomorphism.

Proof. Commutativity is easy but tedious and left to the reader. The base change result follows from Lemma C.9 and almost coherence of $\mathbf{R}\nu_{\mathfrak{X}',*}\left(\widehat{\mathcal{O}}^+_{X'}\right)$ provided by [Zav21a, Theorem 6.13.6]. □

Now our goal is to explicitly understand the pro-étale trace

$$\mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_X \xrightarrow{\mathrm{Tr}_{\mathrm{an},f}} \mathbf{R}\mu_{X,*}\widehat{\mathcal{O}}_X.$$

We relate it to the "usual" trace map in the affinoid case.

Lemma D.12. Let $f\colon X' = \mathrm{Spa}(B, B^+) \to X = \mathrm{Spa}(A, A^+)$ be a finite étale morphism of affinoid rigid C-spaces. Then the trace maps

$$\mathrm{Tr}_{\text{ét}, f}(X)\colon B = (f_* \mathcal{O}_{X'})(X) \to A = \mathcal{O}_X(X)$$

coincides with the explicit trace map

$$\mathrm{Tr}_{B/A}\colon B \to A.$$

Proof. Without loss of generality we can assume that X is connected. In particular, f is surjective in this case[49]. Then if f is split, the claim is trivial. So it suffices to reduce to the split case.

We can find a surjective finite étale morphism $Y \to X$ such that $Y' := Y \times_X X'$ splits over Y. The rigid spaces Y and Y' are affinoid, and

$$\mathcal{O}(Y') \simeq \mathcal{O}_Y(Y) \widehat{\otimes}_A B \simeq \mathcal{O}_Y(Y) \otimes_A B$$

as A and $\mathcal{O}(Y)$ are strongly noetherian and B is finite over A. Moreover, the map $A \to \mathcal{O}(Y)$ is faithfully flat as $A \to \mathcal{O}(Y)$ is finite étale and the map $Y \to X$ is surjective. As the formation of the explicit trace map and the étale trace map commute with étale base change, it is sufficient to check the equality on Y that has been already done above. $\square$

Lemma D.13. Let $f\colon X' = \mathrm{Spa}(B, B^+) \to X = \mathrm{Spa}(A, A^+)$ be a finite étale morphism of affinoid rigid C-spaces, $U \to X$ a pro-étale morphism with $\widehat{U} = \mathrm{Spa}(R, R^+)$ being an affinoid perfectoid, and $U' = U \times_X X'$. Then U' is an affinoid perfectod with $\widehat{U'} \simeq \mathrm{Spa}(S, S^+)$ with $S = R \otimes_A B$, and the pro-étale trace map

$$\mathrm{Tr}_{\text{proét}, f}(U)\colon S = \left(f_{\text{proét},*} \widehat{\mathcal{O}}_{X'}\right)(U) \to R = \widehat{\mathcal{O}}_X(U)$$

coincides with the explicit trace map

$$\mathrm{Tr}_{S/R}\colon S \to R.$$

Proof. The proof is essentially the same as that of Lemma D.12, but there are certain complications due to the fact that perfectoid spaces are not strongly noetherian.

First of all, we note that [Sch13a, Lemma 4.5] implies that U' is affinoid perfectoid, and that $\widehat{U'} = \widehat{U} \times_X X' = \mathrm{Spa}(S, S^+)$ for $S \cong R \widehat{\otimes}_A B$, so we need to show that $R \otimes_A B$ is already complete.

We use the notion of natural topology to prove completeness of $R \otimes_A B$. We refer to [Zav21c, Appendix B.3] for a self-contained discussion of this notion. We note that [Zav21c, Lemma B.3.4] implies that the topology on B coincides with the natural A-module topology. Then [Zav21c, Lemma B.3.5] implies that the topology on the topologized tensor product $R \otimes_A B$ coincides with the natural R-module topology. Now we note that B is finite étale over A, so it is direct summand of a finite free A-module A^N. Therefore, $R \otimes_A B$ is a direct summand of a finite free module R^N. Clearly, R^N is complete in its natural topology. Thefore, so is $R \otimes_A B$ as its direct summand.

As for the claim on traces, we note that we can assume that X is connected. In particular that f is surjective[50]. If f splits over X, the claim is trivial. So we reduce to this case. The reduction is identical to that in Lemma D.12 given that U' is affinoid perfectoid with $\widehat{U}' \simeq \mathrm{Spa}(S, S^+)$ for $S \simeq R \otimes_A B$. $\square$

[49]Or X' is empty, but this case is trivial.

[50]Or X' is empty, but this case is trivial

The last goal of this Appendix is to explicitly understand the pro-étale trace morphism

$$\mathrm{Tr}_{\mathrm{an},f}\colon \mathbf{R}f_{\mathrm{an},*}\mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_{X'} \to \mathbf{R}\mu_{X,*}\widehat{\mathcal{O}}_X$$

for a smooth rigid C-space X.

We note that $\mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_{X'} \in \mathbf{D}^b_{coh}(X')$ and $\mathbf{R}\mu_{X,*}\widehat{\mathcal{O}}_X \in \mathbf{D}^b_{coh}(X)$ by [Sch13b, Proposition 3.23]. Therefore, $\mathbf{R}f_{\mathrm{an},*} \circ \mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_{X'} \in \mathbf{D}^b_{coh}(X)$ because $f\colon X' \to X$ is finite. So the trace map induces mo

$$\mathcal{H}^i(\mathrm{Tr}_{\mathrm{an},f})\colon \mathcal{H}^i(\mathbf{R}f_{\mathrm{an},*} \circ \mathbf{R}\mu_{X',*}\widehat{\mathcal{O}}_{X'}) \cong f_{\mathrm{an},*}(\mathrm{R}^i\mu_{X',*}\widehat{\mathcal{O}}_{X'}) \to \mathrm{R}^i\mu_{X,*}\widehat{\mathcal{O}}_X$$

are morphisms of coherent sheaves. Therefore, if X is affinoid we use canonical isomorphisms $\mathrm{H}^i(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X) \to \mathrm{R}^i\nu_{X,*}\widehat{\mathcal{O}}_X$ and $\mathrm{H}^i(X'_{\mathrm{proét}}, \widehat{\mathcal{O}}_{X'}) \to f_{\mathrm{an},*}\mathrm{R}^i\nu_{X',*}\widehat{\mathcal{O}}_{X'}$ to see that $\mathcal{H}^i(\mathrm{Tr}_{\mathrm{an},f})$ is uniquely determined by

$$\mathcal{H}^i(\mathrm{Tr}_{\mathrm{an},f})(X)\colon \mathrm{H}^i(X'_{\mathrm{proét}}, \widehat{\mathcal{O}}_{X'}) \to \mathrm{H}^i(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X).$$

Our goal is to give an explicit description for this map under the assumption that X admits "good coordinates", i.e. a map to the d-dimensional torus $\mathbf{T}^d_C$ that is a composition of rational embeddings and finite étale morphisms.

Before we give this description, we need to recall the standard pro-étale covering of the torus

$$\mathbf{T}^d = \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$$

by an affinoid perfectoid. We define

$$\mathbf{T}^d_\infty = \text{“}\lim_m \mathrm{Spa}(C\langle T_1^{\pm 1/p^m}, \dots, T_d^{\pm 1/p^m}\rangle, \mathcal{O}_C\langle T_1^{\pm 1/p^m}, \dots, T_d^{\pm 1/p^m}\rangle)\text{”}.$$

The natural map $\mathbf{T}^d_\infty \to \mathbf{T}^d$ is a pro-étale covering that is a $\Delta \coloneqq \mathbf{Z}_p(1)^d$-torsor. Moreover, $\mathbf{T}^d_\infty$ is an affinoid perfectoid with

$$\widehat{\mathbf{T}}^d_\infty = \mathrm{Spa}(C\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle, \mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle).$$

Now [Sch13a, Lemma 4.5] guarantees that $X_\infty \coloneqq X \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$ and $X'_\infty \coloneqq X' \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$ are affinoid perfectoid spaces that are pro-étale Δ-torsors over X and X', respectively. Using the the vanishing of the higher cohomology groups

$$\mathrm{H}^i(X_\infty, \widehat{\mathcal{O}}_X) = 0 \text{ for } i \geq 1$$

$$\mathrm{H}^i(X'_\infty, \widehat{\mathcal{O}}_{X'}) = 0 \text{ for } i \geq 1$$

that follows from [Sch13a, Lemma 4.10(v)], we conclude that one can compute $\mathrm{H}^i(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X)$ and $\mathrm{H}^i(X'_{\mathrm{proét}}, \widehat{\mathcal{O}}_{X'})$ using the Čech complexes for the pro-étale covers $X_\infty \to X$ and $X'_\infty \to X'$, respectively. Now [Sch13a, Lemma 3.16] implies that we can identify the corresponding Čech comlexes with $\mathbf{R}\Gamma_{cont}(\Delta, A_\infty)$ and $\mathbf{R}\Gamma_{cont}(\Delta, B_\infty)$, where $\widehat{X}_\infty = \mathrm{Spa}(A_\infty, A_\infty^+)$ and $\widehat{X'}_\infty = \mathrm{Spa}(B_\infty, B_\infty^+)$. So there are natural isomorphisms

$$\mathrm{H}^i_{cont}(\Delta, A_\infty) \to \mathrm{H}^i(X, \widehat{\mathcal{O}}_X)$$

and

$$\mathrm{H}^i_{cont}(\Delta, B_\infty) \to \mathrm{H}^i(X', \widehat{\mathcal{O}}_{X'})$$

with $B_\infty \simeq A_\infty \otimes_A B$ by Lemma D.13. In particular, B_∞ is finite étale over A_∞.

Theorem D.14. Let $f\colon X' = \mathrm{Spa}(B, B^+) \to X = \mathrm{Spa}(A, A^+)$ be a finite étale morphism of smooth affinoid rigid C-spaces. Suppose that $X \to \mathbf{T}^d$ is morphism that is a composition of rational embeddings and finite étale morphisms. Then the following diagram

$$\begin{tikzcd}
\mathrm{H}^i_{cont}\widetilde{(\Delta, B)} \arrow[r, "\mathrm{H}^i_{cont}\widetilde{(\Delta, \mathrm{Tr}}_{B/A})"] \arrow[d] & \mathrm{H}^i_{cont}\widetilde{(\Delta, A)} \arrow[d] \\
\mathrm{H}^i_{cont}\widetilde{(\Delta, B_\infty)} \arrow[r, "\mathrm{H}^i_{cont}\widetilde{(\Delta, \mathrm{Tr}}_{B_\infty/A_\infty})"] \arrow[d] & \mathrm{H}^i_{cont}\widetilde{(\Delta, A_\infty)} \arrow[d] \\
\mathrm{H}^i(\widetilde{X'_{\text{proét}}}, \widehat{\mathcal{O}}_{X'}) = \mathrm{H}^i(X_{\text{proét}}, \widetilde{f_{\text{proét},*}}\widehat{\mathcal{O}}_{X'}) \arrow[r, "\mathrm{H}^i(X_{\text{proét}}\widetilde{, \mathrm{Tr}}_{\text{proét}, f})"] \arrow[d] & \mathrm{H}^i(\widetilde{X_{\text{proét}}}, \widehat{\mathcal{O}}_X) \arrow[d] \\
f_{\mathrm{an},*}\mathrm{R}^i\nu_{X',*}\widehat{\mathcal{O}}_{X'} \arrow[r, "\mathcal{H}^i(\mathrm{Tr}_{\mathrm{an},f})"] & \mathrm{R}^i\nu_{X,*}\widehat{\mathcal{O}}_X
\end{tikzcd}$$

is commutative with vertical arrow being isomorphisms.

Proof. We note that we have already explained above that all vertical maps are isomorphisms except for the two top vertical ones. However, [Sch13a, Lemma 5.5] guarantees that the maps $\mathrm{H}^i_{cont}(\Delta, A) \to \mathrm{H}^i_{cont}(\Delta, A_\infty)$ and $\mathrm{H}^i_{cont}(\Delta, B) \to \mathrm{H}^i_{cont}(\Delta, B_\infty)$ are isomorphisms for $i \geq 0$. So the top vertical maps are isomorphisms.

Commutativity of the top and bottom squares is obvious. So it suffices to show that the square

$$\begin{tikzcd}
\mathrm{H}^i_{cont}\widetilde{(\Delta, B_\infty)} \arrow[r, "\mathrm{H}^i_{cont}\widetilde{(\Delta, \mathrm{Tr}}_{B_\infty/A_\infty})"] \arrow[d] & \mathrm{H}^i_{cont}\widetilde{(\Delta, A_\infty)} \arrow[d] \\
\mathrm{H}^i(\widetilde{X'_{\text{proét}}}, \widehat{\mathcal{O}}_{X'}) = \mathrm{H}^i(X_{\text{proét}}, \widetilde{f_{\text{proét},*}}\widehat{\mathcal{O}}_{X'}) \arrow[r, "\mathrm{H}^i(X_{\text{proét}}\widetilde{, \mathrm{Tr}}_{\text{proét}, f})"] & \mathrm{H}^i(\widetilde{X_{\text{proét}}}, \widehat{\mathcal{O}}_X)
\end{tikzcd}$$

is commutative. As this is a map of coherent sheaves, it is sufficient to show that the square

$$\begin{tikzcd}
\mathrm{H}^i_{cont}(\Delta, B_\infty) \arrow[r, "\mathrm{H}^i_{cont}(\Delta, \mathrm{Tr}_{B_\infty/A_\infty})"] \arrow[d] & \mathrm{H}^i_{cont}(\Delta, A_\infty) \arrow[d] \\
\mathrm{H}^i(X'_{\text{proét}}, \widehat{\mathcal{O}}_{X'}) = \mathrm{H}^i(X_{\text{proét}}, \widetilde{f_{\text{proét},*}}\widehat{\mathcal{O}}_{X'}) \arrow[r, "\mathrm{H}^i(X_{\text{proét}}, \mathrm{Tr}_{\text{proét}, f})"] & \mathrm{H}^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X)
\end{tikzcd}$$

is commutative. This fits into the diagram

$$\begin{tikzcd}
& \mathrm{H}^i_{cont}(\Delta, B_\infty) \arrow[r, "\mathrm{H}^i_{cont}(\Delta, \mathrm{Tr}_{B_\infty/A_\infty})"] \arrow[d] & \mathrm{H}^i_{cont}(\Delta, A_\infty) \arrow[d] \\
\check{\mathrm{H}}^i(X'_\infty/X', \widehat{\mathcal{O}}_{X'}) = & \check{\mathrm{H}}^i(X_\infty/X, f_{\text{proét},*}\widehat{\mathcal{O}}_{X'}) \arrow[r, "\check{\mathrm{H}}^i(X_\infty/X, \mathrm{Tr}_{\text{proét},f})"] \arrow[d] & \check{\mathrm{H}}^i(X_\infty/X, \widehat{\mathcal{O}}_X) \arrow[d] \\
\mathrm{H}^i(X'_{\text{proét}}, \widehat{\mathcal{O}}_{X'}) = & \mathrm{H}^i(X_{\text{proét}}, f_{\text{proét},*}\widehat{\mathcal{O}}_{X'}) \arrow[r, "\mathrm{H}^i(X_{\text{proét}}, \mathrm{Tr}_{\text{proét},f})"] & \mathrm{H}^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X)\ ,
\end{tikzcd}$$

where the maps $\check{\mathrm{H}}^i(X'_\infty/X', \widehat{\mathcal{O}}_{X'}) \to \mathrm{H}^i(X'_{\text{proét}}, \widehat{\mathcal{O}}_{X'})$ and $\check{\mathrm{H}}^i(X_\infty/X, \widehat{\mathcal{O}}_X) \to \mathrm{H}^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$ are the functorial edge morphisms from the Čech cohomology groups to the derived cohomology groups. Now we note that the lower square commutes by functoriality of the edge maps as both maps are induced by the pro-étale trace map $\mathrm{Tr}_{\text{proét},f}\colon f_{\text{proét},*}\widehat{\mathcal{O}}_{X'} \to \widehat{\mathcal{O}}_X$. Commutativity of the upper square follows from Lemma D.13. ◻

Appendix E. Continuous Group Cohomology of Profinite Groups

We recall some general lemmas about continuous cohomology of profinite groups that seem to be difficult to extract from the literature. We find the easiest way to think about $\mathbf{R}\Gamma_{cont}(G, -)$ as the cohomology on the site $BG_{\text{proét}}$ introduced in [BS15, Section 4.3]. We recall the main facts about this construction.

Let G be a profinite group. The objects of $BG_{\text{proét}}$ are profinite continuous G-sets[51]. A family of maps $\{Y_i \to Y\}$ in $BG_{\text{proét}}$ is a covering family if any quasi-compact open in Y is mapped onto by a quasi-compact open in $\bigsqcup_i Y$. Each continuous G-module M defines a sheaf of abelian groups $\mathcal{F}_M$ on $BG_{\text{proét}}$ by the rule

$$\mathcal{F}_M(U) := \mathrm{Hom}_{cont,G}(U, M) \text{ for } U \in BG_{\text{proét}}.$$

Moreover, [BS15, Lemma 4.3.8] provides the functorial morphism

$$\mathbf{R}\Gamma_{cont}(G, M) \to \mathbf{R}\Gamma(BG_{\text{proét}}, \mathcal{F}_M)$$

that is not always an isomorphism. But it is an isomorphism for a sufficiently large class of continuous G-modules M according to [BS15, Lemma 4.3.9]. In particular, it is an isomorphism for any discrete and p-adically complete module M.

Lemma E.1. Let G be a profinite group, then the topos $BG_{\text{proét}}$ is replete and coherent. In particular, cohomology groups $\mathrm{H}^i(BG_{\text{proét}}, -)$ commute with infinite sums in $\mathcal{A}b(BG_{\text{proét}})$.

Proof. In order to show that that $BG_{\text{proét}}$ is replete, if suffices to show that it is locally weakly contractible in the sense of [BS15, Definition 3.2.1]. This reduction is explained in [BS15, Proposition 3.2.3].

Now we show that $BG_{\text{proét}}$ is locally weakly contractible. It suffices to show that any G-profinite set S admits a cover $S' \to S$ such that any cover of S' is split. We define $S' := G \times \beta(S)$ where $p\colon \beta(S) \to S$ is the Stone-Cech compactification of S. And the map $\pi\colon S' \to S$ is defined as $\pi(g, s') = g.p(s')$, the map is clearly surjective and G-equivariant. Moreover, $\beta(S)$ is extremally disconnected space by [BS15, Example 2.4.6]. In particular, it is profinite so $S' \in BG_{\text{proét}}$.

[51] With some set-theoretic restrictions as in [BS15, Remark 4.1.2]

Now we want to show that any G-equivariant continuous surjection $f\colon S'' \to S'$ admits a G-equivariant continuous section. We consider $\beta(S) = \{e\} \times \beta(S)$ as a closed subset of S', then the extremally disconnected condition guarantees that $f^{-1}(\beta(S)) \to \beta(S)$ admits a continuous section s'. We define the map $s\colon G \times \beta(S) \to S''$ as $s(g,x) = g.s'(x)$. This is easily seen to be a continuous G-equivariant section.

It is easy to see that any representable object h_S is quasi-compact and quasi-separated in $\mathrm{Shv}(BG_{\text{proét}})$. So the topos is coherent. The fact that $\mathrm{H}^i(BG_{\text{proét}}, -)$ commutes with infinite sums follows from [Gro72, Exp.VI, §5, Corollarie 5.2] and the fact that H^i commutes with finite sums (=finite products). ☐

Lemma E.2. Let G be a profinite group, and let $A \to B$ be a continuous flat morphism of p-adically complete, p-torsionfree modules. Suppose that M be any p-adically complete, p-torsion free A-module with a continuous A-linear action of G. Then the complexes $\mathbf{R}\Gamma_{cont}(G, M)$ and $\mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B)$ are derived p-adically complete, and the natural morphism

$$\mathbf{R}\Gamma_{cont}(G, M)\widehat{\otimes}^L_A B \to \mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B)$$

is an isomorphism where the completion on the left hand side should be understood in the derived sense.

Proof. The p-adic completeness assumption on M implies that the natural map $\mathbf{R}\Gamma_{cont}(G, M) \to \mathbf{R}\Gamma(BG_{\text{proét}}, \mathcal{F}_M)$ is an isomorphism and the same holds for $M\widehat{\otimes}_A B$. Also one can easily see that the sequence

$$0 \to \mathcal{F}_M \xrightarrow{p^n} \mathcal{F}_M \to \mathcal{F}_{M/p^nM} \to 0 \tag{E.1}$$

is exact in $BG_{\text{proét}}$ for any $n > 0$. We also claim that the map

$$\mathcal{F}_M \to \mathbf{R}\lim_n \mathcal{F}_{M/p^nM} \tag{E.2}$$

is an isomorphism. The functor $\mathcal{F}_{(-)}$ is limit preserving by [BS15, Lemma 4.3.2], so $\mathcal{F}_M \simeq \lim_n \mathcal{F}_{M/p^nM}$. Finally, [BS15, Proposition 3.1.10] guarantees that $\lim_n \mathcal{F}_{M/p^nM} \cong \mathbf{R}\lim_n \mathcal{F}_{M/p^nM}$ as $BG_{\text{proét}}$ is replete by Lemma E.1.

Now (E.1) and (E.2) formally imply that

$$[\mathbf{R}\Gamma(BG_{\text{proét}}, \mathcal{F}_M)/p^n] \cong \mathbf{R}\Gamma(BG_{\text{proét}}, \mathcal{F}_{M/p^n}),$$

$$\mathbf{R}\Gamma(BG_{\text{proét}}, \mathcal{F}_M) \cong \mathbf{R}\lim_n [\Gamma(BG_{\text{proét}}, \mathcal{F}_M)/p^n],$$

where $[X/p^n]$ stands for the cone of $X \xrightarrow{p^n} X$. This, in turn, shows that we have isomorphisms

$$[\mathbf{R}\Gamma_{cont}(G, M)/p^n] \simeq \mathbf{R}\Gamma_{cont}(G, M/p^n), \tag{E.3}$$

$$\mathbf{R}\Gamma_{cont}(G, M) \simeq \mathbf{R}\lim_n [\mathbf{R}\Gamma_{cont}(G, M)/p^n]. \tag{E.4}$$

The same holds for $M\widehat{\otimes}_A B$ as it is p-torsionfree by flatness of the map $A \to B$. In particular, Equation (E.4) shows that both $\mathbf{R}\Gamma_{cont}(G, M)$ and $\mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B)$ are indeed derived p-adically complete. So the natural map $\mathbf{R}\Gamma_{cont}(G, M) \otimes^L_A B \to \mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B)$ induces the map

$$\mathbf{R}\Gamma_{cont}(G, M)\widehat{\otimes}^L_A B \to \mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B)\ .$$

We now show that it is an isomorphism. Firstly, note that $[A/p^n] \simeq A/p^n$, $[B/p^n] \simeq B/p^n$ as these rings are p-torsionfree, and the maps $A/p^n \to B/p^n$ are flat for any $n \geq 1$. The claim now follows from the sequence of isomorphisms

$$\begin{aligned}
\mathbf{R}\Gamma_{cont}\left(G, M\right)\widehat{\otimes}^L_A B &\simeq \mathbf{R}\lim_n \left(\left[\mathbf{R}\Gamma_{cont}\left(G, M\right)/p^n\right] \otimes_{A/p^n} B/p^n\right) && \text{[Sta19, Tag 0920]}\\
&\simeq \mathbf{R}\lim_n \left(\mathbf{R}\Gamma_{cont}\left(G, M/p^n\right) \otimes_{A/p^n} B/p^n\right) && \text{Equation (E.3)}\\
&\simeq \mathbf{R}\lim_n \left(\mathbf{R}\Gamma_{cont}\left(G, M/p^n \otimes_{A/p^n} B/p^n\right)\right) && \text{[Zav21a, Lemma 6.6.8]}\\
&\simeq \mathbf{R}\lim_n \mathbf{R}\Gamma_{cont}\left(G, \left(M\widehat{\otimes}_A B\right)/p^n\right) && \text{[Sta19, Tag 05GG]}\\
&\simeq \mathbf{R}\lim_n [\mathbf{R}\Gamma_{cont}\left(G, M\widehat{\otimes}_A B\right)/p^n] && \text{Equation (E.4)}\\
&\simeq \mathbf{R}\Gamma_{cont}\left(G, M\widehat{\otimes}_A B\right) && \text{[Sta19, Tag 0920]}
\end{aligned}$$

□

Lemma E.3. Let G be a profinite group, $A \to B$ a flat morphism of topologically finitely presented $\mathcal{O}_C$-algebras, and M a p-adically complete, p-torsionfree A-module with continuous A-linear G-action. Suppose that $\mathrm{H}^i_{cont}(G, M)$ are almost finitely generated over A for each $i \geq 0$. Then the natural morphism

$$\mathrm{H}^i_{cont}(G, M) \otimes_A B \to \mathrm{H}^i_{cont}(G, M\widehat{\otimes}_A B)$$

is an isomorphism for each $i \geq 0$.

Proof. We can apply Lemma E.2 to see that

$$\mathbf{R}\Gamma_{cont}(G, M)\widehat{\otimes}^L_A B \simeq \mathbf{R}\Gamma_{cont}(G, M\widehat{\otimes}_A B).$$

Therefore, it suffices to show that the complex $\mathbf{R}\Gamma_{cont}(G, M) \otimes^L_A B$ is already derived p-adically complete.

We use [Sta19, Tag 091P] and [Sta19, Tag 091T] to ensure that it suffices to show that all cohomology modules of

$$\mathbf{R}\Gamma_{cont}(G, M) \otimes^L_A B$$

are p-adically complete. Using that the map $A \to B$ is flat, it is enough to show that all $\mathrm{H}^i_{cont}(G, M) \otimes_A B$ are p-adically complete.

Since $\mathrm{H}^i_{cont}(G, M)$ is almost finitely presented over A for any $i \geq 0$, we conclude that

$$\mathrm{H}^i_{cont}(G, M) \otimes_A B$$

is almost finitely generated over B. So we apply [Zav21a, Lemma 2.12.7] to guarantee that

$$\mathrm{H}^i_{cont}(G, M) \otimes_A B$$

are p-adically complete for all $i \geq 0$. □

Lemma E.4. Let G be a profinite group, and let $\{M_i\}_{i\in I}$ be p-adically complete, p-torsion free continuous G-modules. Then the natural map

$$\widehat{\bigoplus_I}^L \mathbf{R}\Gamma_{cont}(G, M_i) \to \mathbf{R}\Gamma_{cont}(G, \widehat{\bigoplus_I} M_i)$$

is an isomorphism, where the completion on the left hand side should be understood in the derived sense.

Proof. The proof is similar to Lemma E.2. We use the argument with derived completions to reduce to the case of p^n-torsion G-modules M_i. In this case all derived completions disappear, so we can use Lemma E.1 to finish the proof. □

Corollary E.5. [ČK19, Lemma 3.6] Let G be a profinite group, let $\{M_I\}_{i\in I}$ be p-adically complete, p-torsion free, continuous G-modules. Suppose that $\oplus_I \mathrm{H}^n_{cont}(G, M_i)$ has bounded p^∞-torsion for a fixed integer $n \geq 0$. Then the natural map $\oplus_I \mathrm{H}^n_{cont}(G, M_i) \to \mathrm{H}^n_{cont}(G, \widehat{\oplus}_I M_i)$ induces an isomorphism

$$\widehat{\bigoplus}_I \mathrm{H}^n_{cont}(G, M_i) \to \mathrm{H}^n_{cont}(G, \widehat{\bigoplus}_I M_i) .$$

In particular, $\mathrm{H}^n_{cont}(G, \widehat{\oplus}_I M_i) \to \prod_I \mathrm{H}^n_{cont}(G, M_i)$ is injective.

Proof. We use Lemma E.4 and [Sta19, Tag 0BKE] to get the spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i\left(\widehat{\oplus}^L_I \mathrm{H}^j_{cont}(G, M_i)\right) \Rightarrow \mathrm{H}^{i+j}_{cont}(G, \widehat{\oplus}_I M_i)$$

The derived completion has cohomological dimesion 1, so $\mathrm{E}_2^{i,j} = 0$ for $i \notin \{-1, 0, 1\}$. Thus, it suffices to show that $E^{0,n} = \widehat{\oplus}_I \mathrm{H}^n_{cont}(G, M_i)$ and $\mathrm{E}^{-1,n+1} = E^{1,n-1} = 0$.

We note that $\oplus_I \mathrm{H}^n_{cont}(G, M_i)$ has bounded p^∞-torsion by the assumption. Thus its derived completion coincides with the usual completion, this proves that $E^{0,n} = \widehat{\oplus}_I \mathrm{H}^n_{cont}(G, M_i)$.

Now we deal with $E^{1,n-1}$. It is equal to $\lim^1_n \left(\oplus_I \mathrm{H}^{n-1}_{cont}(G, M_i)/p^n\right)$. The transition maps are surjective in this system, thus the $\lim^1$ term vanishes by the Mittag-Leffler criterion.

Finally, we show that $E^{-1,n+1} = 0$. By definition, it is equal to $\mathrm{H}^{-1}\left(\widehat{\oplus}^L_I \mathrm{H}^{n+1}_{cont}(G, M_i)\right)$. We note that $\oplus_I \mathrm{H}^{n+1}_{cont}(G, M_i)$ is a subgroup of the group $\prod_I \mathrm{H}^{n+1}_{cont}(G, M_i)$. So since derived completion has cohomological dimension 1, we see that

$$\mathrm{H}^{-1}\left(\widehat{\oplus}^L_I \mathrm{H}^{n+1}_{cont}(G, M_i)\right) \subset \mathrm{H}^{-1}\left(\widehat{\prod}^L_I \mathrm{H}^{n+1}_{cont}(G, M_i)\right)$$

so it suffices to show vanishing of the right side. The key is that the group $\prod_I \mathrm{H}^{n+1}_{cont}(G, M_i)$ is already derived p-adically complete as the (derived) product[52] of derived abelian groups. In particular, the former group is concentrated in degree 0, thus $\widehat{\prod}^L_I \mathrm{H}^{n+1}_{cont}(G, M_i) = \prod_I \mathrm{H}^{n+1}_{cont}(G, M_i)$, so $\mathrm{H}^{-1}\left(\widehat{\prod}^L_I \mathrm{H}^{n+1}_{cont}(G, M_i)\right) = 0$. □

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