ALTERED LOCAL UNIFORMIZATION OF RIGID-ANALYTIC SPACES

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Abstract. We prove a version of Temkin's local altered uniformization theorem. We show that for any rig-smooth, quasi-compact and quasi-separated admissible formal $\mathcal{O}_K$-model $\mathfrak{X}$, there is a finite extension K'/K such that $\mathfrak{X}_{\mathcal{O}_{K'}}$ locally admits a rig-étale morphism $g\colon \mathfrak{X}' \to \mathfrak{X}_{\mathcal{O}_{K'}}$ and a rig-isomorphism $h\colon \mathfrak{X}'' \to \mathfrak{X}'$ with $\mathfrak{X}'$ being a successive semi-stable curve fibration over $\mathcal{O}_{K'}$ and $\mathfrak{X}''$ being a polystable formal $\mathcal{O}_{K'}$-scheme. Moreover, $\mathfrak{X}'$ admits an action of a finite group G such that $g\colon \mathfrak{X}' \to \mathfrak{X}_{\mathcal{O}_{K'}}$ is G-invariant, and the adic generic fiber $\mathfrak{X}'_{K'}$ becomes a G-torsor over its quasi-compact open image $U = g_{K'}(\mathfrak{X}'_{K'})$. Also, we study properties of the quotient map $\mathfrak{X}'/G \to \mathfrak{X}_{\mathcal{O}_{K'}}$ and show that it can be obtained as a composition of open immersions and rig-isomorphisms.

Contents

1. Introduction

1.1. **Historical Overview.** The stable modification theorem [DM69] of P. Deligne and D. Mumford says that for a discrete valuation ring R and a smooth proper curve X over the fraction field

K, there is a finite separable extension $K \subset K'$ such that the base change $X_{K'}$ extends to a semi-stable curve over the integral closure of R in K'. This theorem is crucial for many geometric and arithmetic applications since it allows one to reduce many questions about curves over the fraction field K of R to questions about *semi-stable curves* over a finite separable extension of K. For example, this theorem plays an important role in the proof of Mordell's Conjecture by G. Faltings (look at [CS86] for a discussion of this proof).

Over the years people tried to generalize this statement to a more general set-up. For example, a "Lemma of Gabber" [Del85] roughly says that the same result may be achieved over a quasi-compact and quasi-separated base[1] S after a base change along some proper and surjective morphism $S' \to S$. Another generalization was obtained by J. de Jong in [Jon97]: he shows that any projective relative curve $f: X \to S$ can be made semi-stable by means of a generically étale alteration $S' \to S$ and a further modification $X' \to X_{S'}$. This theorem has many important consequences in algebraic geometry (e.g. resolution of singularities up to an alteration), but it has a disadvantage that one can not control the étale locus of the morphism $S' \to S$ (and $X' \to X_{S'}$). For example, suppose that X is smooth (or just semi-stable) over an open $U \subset S$. Then de Jong's approach is not robust enough to allow one to choose an U-étale alteration $S' \to S$ such that $X_{S'}$ admits a semi-stable modification.

This difficulty was recently overcome by M. Temkin in [Tem11]. He shows there that given a relative curve (see Definition 2.1) $f: X \to S$ that is semi-stable over an (quasi-compact and schematically dense) open subset $U \subset S$, one can find an "U-étale covering" (see Definition 2.13) $S' \to S$ such that $X_{S'}$ admits a semi-stable modification. This proof uses completely new ideas (compared to all older proofs) related to the non-archimedean geometry. Roughly, Temkin uses the notion of a Riemann-Zariski space to reduce the case of an arbitrary quasi-compact and quasi-separated base S to the case of a spectrum of a complete rank-1 (possibly not discrete) valuation ring. He then treats this case using rigid-analytic techniques.

The case of higher dimensional families is much harder, and not much is known besides the characteristic 0 case. It is shown in [KKMS73] that given a characteristic 0 field k, a k-curve C, and a finite type morphism $X \to C$ that is smooth over the generic point of C, there is a finite morphism $C' \to C$ such that $X_{C'}$ admits a semi-stable modification. The main techniques in the proof are resolution of singularities and toroidal geometry. The case of higher dimensional base (and higher-dimensional fibers) was recently solved by K. Adiprasito, G. Liu and M. Temkin in [ALT19] using techniques based on the previous work of D. Abramovich and K. Karu [AK00].

To the best of our knowledge, not much is known in the case of finite or mixed characteristic schemes. However, there are some positive results in this direction. M. Temkin proved in [Tem17] that for any valuation ring R of rank-1[2] and a finite type flat R-scheme X with a smooth fiber over the fraction field K, there is a finite extension of valued fields $K \subset K'$ with ring of integers R' such that the base change $X_{R'}$ admits a K'-étale covering (see Definition 2.13 and Remark 2.15) $X' \to X_{R'}$ with a strictly semi-stable R'-scheme X'. The proof is based on his previous work on the Stable Modification Theorem and Relative Riemann-Zariski spaces in [Tem11]. We note that this is pretty far from the case of the actual Semi-Stable Reduction Theorem since the argument is local on X, so it can not control properness of $X' \to X_{R'}$. Also, usually the algorithm produces X' that is not birational to $X_{R'}$.

[1]The result is stated in a weaker form, but using the "spreading out" techniques developed in [Gro66] and [TT90] the case of a qcqs base can be reduced to [Del85, Lemma 1.6].

[2]The case of a discrete valuation ring R was obtained before in the work of U. Hartl [Har03].

It is important to note that Temkin's result actually shows more. Namely, his theorem applies not only to usual R-schemes, but also to formal R-schemes (in the case of a complete R). In particular, it is shown in [Tem17, Theorem 3.3.1] that given a complete valuation ring R of rank-1 and a rig-smooth admissible formal R-scheme $\mathfrak{X}$, there is a finite valued field extension $K \subset K'$ with the valuation ring R' such that $\mathfrak{X}_{R'}$ admits a rig-étale covering (see Definition B.8) $\mathfrak{X}' \to \mathfrak{X}_{R'}$ with a strictly semi-stable admissible formal R'-scheme $\mathfrak{X}'$.

1.2. **Our results.** In this paper, we prove a version of the local altered uniformization theorem for smooth rigid-analytic varieties following the ideas from [Tem17]. We start with proving a slightly more refined version of Temkin's Stable Modification Theorem [Tem11, Theorem 2.3.3] that will be very useful for our later purposes:

Theorem 1.1. Let $U \subset S$ be a schematically dense quasi-compact open subset of a quasi-compact and quasi-separated scheme S, and let $f\colon X \to S$ be an S-curve that is semi-stable over U (see Definition 2.4). Then there exist

- A projective U-modification (see Definition 2.8) $h\colon S' \to S$ with a finite open Zariski covering $\cup_{i=1}^n V'_i = S'$ by quasi-compact opens $V'_i \subset S'$.
- A finite group G_i and a finite, finitely presented, faithfully flat, and U-étale G_i-invariant morphism $t_i\colon W'_i \to V'_i$ for each $i \leq n$. In particular, the morphism $t\colon W' = \sqcup_{i=1}^n W'_i \to S$ is a U-étale covering (see Definition 2.13).

satisfying the following properties:

(1) The induced morphisms $t_{i,U}\colon W'_{i,U} \to V'_{i,U}$ are G_i-torsors.

(2) Each $X_{W'_i}$ admits a W'_i-stable U-modification (see Definition 2.10) $g_i\colon X'_i \to X_{W'_i}$.

Remark 1.2. We actually prove a slightly more general result. Namely, Theorem 1.1 holds under fewer assumptions on U (see Theorem 5.6 and Setup 5.1). We only mention here that, in particular, Theorem 1.1 holds if U is the generic point of a quasi-compact, quasi-separated integral scheme S.

The main difference between our theorem and [Tem11, Theorem 2.3.3] is that we gain a better control over the U-étale morphism (see Definition 2.12) $W' \to S$[3] such that $X_{W'}$ admits a stable U-modification. Namely, our $W' \to S$ has the form

$$W' := \bigsqcup_{i=1}^n W'_i \to \bigsqcup_{i=1}^n V'_i \to S$$

that it is not merely an abstract U-étale covering but is a disjoint union of G_i-invariant finite, finitely presented and faithfully flat morphisms over the Zariski open covering of a projective U-modification $\bigcup_{i=1}^n V'_i = S' \to S$. Moreover, the U-restriction of each $W'_i \to S$ becomes a G_i-torsor over its open image. We later use these further properties in a crucial way.

Theorem 1.1 allows us to prove a version of Temkin's local altered uniformization Theorem [Tem17, Theorem 3.3.1]. We now recall its statement. For any smooth, quasi-compact and quasi-separated rigid-analytic space X over a complete non-archimedean field K and an admissible formal model $\mathfrak{X}$ over $\operatorname{Spf} \mathcal{O}_K$, there is a finite separable field extension $K \subset K'$ and a rig-étale map $g\colon \mathfrak{X}' \to \mathfrak{X}_{\mathcal{O}_{K'}}$ such that $\mathfrak{X}'$ is strictly semi-stable over $\operatorname{Spf} \mathcal{O}_{K'}$. In this article, we weaken the condition on $\mathfrak{X}'$, but we get some control over the structure of the morphism $g\colon \mathfrak{X}' \to \mathfrak{X}_{\mathcal{O}_{K'}}$ instead. Namely, in our version of Temkin's local altered uniformization result, $\mathfrak{X}'$ will not be (strictly)

[3] W' is denoted as S' in [Tem11, Theorem 2.3.3].

semi-stable over $\operatorname{Spf} \mathcal{O}_{K'}$, but it rather will be only polystable over $\operatorname{Spf} \mathcal{O}_{K'}$. In exchange, we will get a better control over the generic fiber of $\mathfrak{X}'$ (in terms of an action of a finite group).

Theorem 1.3. Let X be a quasi-compact and quasi-separated smooth rigid-analytic space over $\operatorname{Spa}(K, \mathcal{O}_K)$ with a given admissible quasi-compact formal model $\mathfrak{X}$. Then there is a finite Galois extension $K \subset K'$, a finite extension $K' \subset K''$, a finite number of morphisms of admissible formal $\mathcal{O}_{K'}$-schemes (resp. $\mathcal{O}_{K''}$-schemes) $g_i \colon \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ and $h_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_{i,\mathcal{O}_{K''}}$, such that

- Each $\mathfrak{X}'_i$ admits an action of a finite group G_i such that $g_i \colon \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ is G_i-invariant for each i.
- The morphism $g \colon \mathfrak{X}' := \sqcup_i \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ is a rig-étale covering (see Definition B.5).
- On the generic fiber, each $\mathfrak{X}'_i$ becomes a G_i-torsor over its (quasi-compact) open image in the adic generic fiber $X_{K'} = \mathfrak{X}_{K'}$.
- Each $\mathfrak{X}'_i$ is formally quasi-projective over $\mathcal{O}_{K'}$ (see Definition B.14) and has a structure of a successive formal semi-stable rig-smooth curve fibration (see Definition B.9).
- Each $h_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_{i,\mathcal{O}_{K''}}$ is a rig-isomorphism, and $\mathfrak{X}''_i$ is rig-smooth, polystable formal $\mathcal{O}_{K''}$-scheme (see Definition B.12).
- If $\mathcal{O}_K$ is discretely valued, one can choose $\mathfrak{X}''_i$ to be strictly semi-stable.

Let us now explain the main advantage of our version of the local uniformization theorem. One of the disadvantages of [Tem17, Theorem 3.3.1] is that there the admissible formal $\mathcal{O}_{K'}$-scheme $\mathfrak{X}'$ does not give a model of the rigid-analytic space $X_{K'}$. Instead, what one gets is only that the morphism $\mathfrak{X}'_{K'} \to X_{K'}$ is an étale covering. While it is good enough to study questions that are local on X, it is usually not strong enough for question that are only Zariski local on $\mathfrak{X}$. Theorem 1.3 is better suited for these type of questions. In order to see this, we need to exploit the group action achieved in Theorem 1.3. Namely, it allows us to define quotients $\mathfrak{X}_i := \mathfrak{X}'_i/G_i$ that come with natural maps $\varphi_i \colon \mathfrak{X}_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$. Then the assumptions on $\mathfrak{X}_i$ in Theorem 1.3 imply that the $\mathfrak{X}_i$ are admissible formal $\mathcal{O}_{K'}$-models of an open covering of $X_{K'}$. It turns out that the maps φ_i can be controlled in a reasonable way *integrally*, this plays an important role in our paper [Zav21a] where we give a proof of Poincaré Duality for étale $\mathbf{F}_p$ cohomology groups on a smooth and proper p-adic rigid space.

Theorem 1.4. Let $\mathfrak{X}$ be an admissible, quasi-compact and quasi-separated formal $\mathcal{O}_K$-scheme with the smooth generic fiber $\mathfrak{X}_K$. Then there is a finite Galois extension $K \subset K'$, and a finite extension $K' \subset K''$, a finite set $(\mathfrak{X}_i, \varphi_i)_{i \in I}$ of quasi-compact, quasi-separeted admissible formal $\mathcal{O}_{K'}$-schemes with morphisms $\varphi_i \colon \mathfrak{X}_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ such that

- The set $(\mathfrak{X}_i, \varphi_i)$ can be obtained from $\mathfrak{X}_{\mathcal{O}_{K'}}$ as a "composition of open Zariski coverings and rig-isomorphisms" (see Definition 8.2).
- Each $\mathfrak{X}_i$ is a geometric quotient (see Remark 1.5) of an admissible formal $\mathcal{O}_{K'}$-scheme $\mathfrak{X}'_i$ by an action of a finite group G_i such that the map $p_{i,K'} \colon \mathfrak{X}'_{i,K'} \to \mathfrak{X}_{i,K'}$ is a G_i-torsor.
- Each $\mathfrak{X}'_i$ has a structure of a formal semi-stable, rig-smooth successive curve fibration over $\operatorname{Spf} \mathcal{O}_{K'}$.
- Each $\mathfrak{X}'_{i,\mathcal{O}_{K''}}$ admits a rig-isomorphism $h_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_{i,\mathcal{O}_{K''}}$ with a rig-smooth, polystable formal $\mathcal{O}_{K''}$-scheme $\mathfrak{X}''_i$.
- If $\mathcal{O}_K$ is discretely valued, one can choose $\mathfrak{X}''_i$ to be strictly semi-stable.

Remark 1.5. The formalism of (geometric) quotients in the set-up of admissible formal schemes and (strongly noetherian) adic spaces is developed from scratch in [Zav21b]. We briefly give the definition of geometric quotient in Section 8 (see Definition 8.1) and refer to [Zav21b] for a detailed discussion of the subject.

Let us say a few words how one can use Theorem 1.4 in particular, in particular clarify its role in [Zav21a]. Suppose we want to show some statement on a(ny) admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ of a smooth, quasi-compact, quasi-separated rigid space X over an algebraically closed non-archimedean field C. Then Theorem 1.4 implies that it suffices to show that this statement is Zariski-local on $\mathfrak{X}$, descends along rig-isomorphisms and geometric quotients by an action of a finite group G acting freely on generic fiber, and prove the claim for *polystable* $\mathfrak{X}$. This strategy is used in [Zav21a] to show that particularly defined Faltings' trace map defines an almost (in the technical) perfect pairing on any admissible formal $\mathcal{O}_C$-model of a smooth X by reducing to the case of a polystable model where one can verify it by hands. This, together with the construction of the Faltings' trace map, are the two key parts in the proof of Poincaré Duality.

One can think of Theorem 1.4 as a useful local substitute for a much stronger polystable reduction conjecture:

Conjecture 1.6. [ALPT19, Rigid Version of Conjecture 1.2.6] Let $\mathcal{O}$ be a complete rank-1 valuation ring with the algebraically closed fraction field K, and let $\mathfrak{X}$ be a flat, topologically finitely presented formal $\mathcal{O}$-scheme whose adic generic fiber $\mathfrak{X}_K$ is smooth. Then there is a rig-isomorphism $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ is polystable.

Remark 1.7. The conjecture is formulated in the most optimistic way. For instance, it does not even assume that X is separated. This generality is probably motivated by Temkin's Stable Modification Theorem [Tem11, Theorem 2.3.3], where also no separatedness assumptions are required.

1.3. **Ideas of the main proofs.** We now explain the plan of our proof that we follow in the paper. Basically, we follow the ideas of [Tem10], [Tem11], and [Tem17], but unfortunately, we do not know how to use the results of those papers directly to achieve our form of the local uniformization result. Instead we slightly modify all the mains proofs in [Tem10], [Tem11] and [Tem17]. At the end we use the main result of [ALPT19] as an important input. Namely, we use the following strategy:

- We prove a (slightly) refined version of a stable modification theorem for relative curves over a finite rank valuation rings. The main input here is [Tem10, Proposition 4.5.1].
- We generalize this refined version of a stable modification theorem for relative curves over semi-valuation rings of finite rank. Here the main input is [Tem11, Section 2] and the previous bullet point.
- We finally prove Theorem 1.1. We follow the ideas of [Tem11, Theorem 2.3.3, Step 2] but we use the result from the previous bullet point as our input over a semi-valuation base.
- We prove a schematic version of our local altered uniformization result. This is the most difficult step, we use the ideas from the proof of [Tem17, Corollary 2.4.2]. Namely, we present our S-scheme X (after some U-admissible blow up) as a successive curve fibration over a base. Then we apply our version of the stable modification theorem on each level of this successive curve fibration, and finally we use [ALPT19, Theorem 5.2.16] to find a polystable model. In the case of a discretely valued $\mathcal{O}_K$, we use results from [KKMS73] instead.

The key points here are flattening techniques from [RG71], our formulation of relative stable modification theorem for curves (which we establish in the previous bullet point), and [ALPT19, Theorem 5.2.16] (resp. [KKMS73] in discretely valued case).

- We reduce Theorem 1.3 to the schematic local altered uniformization theorem by means of [Elk73] and [Tem08, Proposition 3.3.2].
- To prove Theorem 1.4, we study the construction of the local altered uniformization in more detail. The main input here is the observation that all the constructions above are quite explicit, so we can understand the quotients $\mathfrak{X}'_i/G_i$ "by hands". We also use the results on geometric quotients from [Zav21b] to ensure certain properties of these quotients.

The last thing that we want to emphasize is that the main reason behind the fact that we can rather 'easily' implement group actions in Temkin's local uniformization result is the uniqueness of stable modification over normal bases [Tem10, Theorem 1.2]. This makes certain constructions automatically G-equivariant and simplifies the exposition significantly.

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2. Terminology

We will freely use a lot of definitions related to the notion of a "(relative) curve". It seems that different people denote slightly different things by this name. For example, there is no consensus whether a (relative) curve should have geometrically connected fibers. Since it is actually very important for our purposes that we allow curve to be disconnected (and also be non-proper) we decide to explain here some relevant notations.

Another thing that we recall here is a number of definitions from [Tem17] that also seem to vary from author to author. So, in order to get rid of any ambiguities in the definitions we decided to spell them out in this section.

Definition 2.1. We say that a morphism of scheme $f\colon X \to S$ is a *relative curve* (or *curve fibration*), if it is finitely presented[4] flat morphism and all non-empty fibers X_s are of pure relative dimension 1.

Remark 2.2. As mentioned above, we do not impose any properness assumption on f. Moreover, we do not require fibers of f to be (geometrically) connected. This is an important technical point that will help us a lot later. Also, we do not even require f to be separated, but this generality will be of no real interest to us in this paper.

Definition 2.3. We say that a morphism $f\colon X \to \operatorname{Spec} k$ is a *semi-stable curve*, if it is a curve over $\operatorname{Spec} k$ (in the sense of Definition 2.1) and for each closed point $x \in X_{\bar{k}}$ the completed local ring $\widehat{\mathcal{O}_{X_{\bar{k}},x}}$ is isomorphic to $\bar{k}[\![T]\!]$ or $\bar{k}[\![U,V]\!]/(UV)$.

Definition 2.4. We say that a morphism $f\colon X \to S$ is a *semi-stable relative curve* (or *semi-stable curve fibration*), if f is a relative curve (in the sense of Definition 2.1) and for all $s \in S$ the fiber $f_s\colon X_s \to \operatorname{Spec} k(s)$ is a semi-stable curve (in the sense of Definition 2.3).

[4] We recall that a finitely presented morphism is required to be quasi-separated.

Lemma 2.5. Let $f\colon X \to S$ be a semi-stable relative curve, then for any point $x \in X$ either f is smooth at x, or there is an open affine neighborhood $\operatorname{Spec} B \subset X$ containing x and an open affine neighborhood $\operatorname{Spec} A \subset S$ containing $y = f(x)$, such that there exists a diagram of pointed schemes

$$\begin{array}{ccc} & (\operatorname{Spec} C, s) & \\ {}^{g}\swarrow & & \searrow^{h} \\ (\operatorname{Spec} B, x) & & \left(\operatorname{Spec} \frac{A[U,V]}{(UV-a)}, \{y, 0, 0\}\right), \end{array}$$

where g and h are étale and a lies in the ideal of y.

Proof. The proof is standard, we only give a reference to [FK88, Ch. III, Proposition 2.7]. □

For the rest of the definitions in this section, we fix a quasi-compact quasi-separated scheme S and a subset $|U| \subset |S|$ that schematically dense, quasi-compact, closed under generalization, and *admits a schematic structure*. What we mean by this is that there is a quasi-compact scheme U and a schematically dominant monomorphism $i\colon U \to S$ that can be topologically identified with the (topological) embedding $|U| \subset |S|$.

Two main types of examples of such U are quasi-compact schematically dense open subschemes of S and the generic points of an integral scheme S.

Definition 2.6. We say that an S-scheme X is *U-admissible* if a scheme X_U is schematically dense in X.

Remark 2.7. We note that that any finitely presented, flat S-scheme is U-admissible.

Definition 2.8. We say that a map of two S-schemes $f\colon X' \to X$ is a *U-modification*, if X' is U-admissible and f is proper morphism such that the base change $f_U\colon X'_U \to X_U$ is an isomorphism.

Remark 2.9. For technical reasons, it will be more convenient not to assume that X or X' are finitely presented in Definition 2.8 and Definition 2.6. However, the main examples of interest will be finitely presented.

Definition 2.10. Let $h\colon S' \to S$ be a map of quasi-compact, quasi-separated schemes. We say that a morphism $f\colon X' \to X$ of relative S'-curves is an *S'-stable U-modification* if it is a $h^{-1}(U)$-modification, X' is a semi-stable relative curve over S', and for all geometric points $\overline{s}' \in S'$ all contracted irreducible components $Z \subset X'_{\overline{s}'}$ that are isomorphic to $\mathbf{P}^1_{k(\overline{s}')}$ intersect the singular locus $X'_{\overline{s}'}$ in at least 3 points.

Remark 2.11. If $S = S'$, we will call S-stable U-modifications simply as stable U-modifications.

Definition 2.12. We say that a map of two S-schemes $f\colon X' \to X$ is a *U-étale morphism* (resp. *U-smooth morphism*), if it is a separated, finitely presented morphism such that the base change $f_U\colon X'_U \to X_U$ is étale (resp. smooth) and X' is U-admissible.

Definition 2.13. We say that a map of two S-schemes $f\colon X' \to X$ is a *U-étale covering*, if it is U-étale and for any valuation ring R any morphism $\phi\colon \operatorname{Spec} R \to X$ taking the generic point to X_U lifts to a morphism $\phi'\colon \operatorname{Spec} R' \to X'$ where R' is some valuation ring such that $\operatorname{Frac}(R')/\operatorname{Frac}(R)$ is finite.

The two main examples of U-étale coverings are U-modifications and faithfully flat U-étale morphisms.

Lemma 2.14. Let S, U be as above, and let $h\colon S' \to S$ be either a U-modification or a faithfully flat U-étale morphism. Then h is a U-étale covering (in the sense of Definition 2.13).

Proof. Let us firstly consider the case of a U-modification morphism $h\colon S' \to S$. Then it is clearly U-étale and S' is an U-admissible scheme by the very definition of an U-modification. The condition about lifting maps from valuation rings is trivially satisfied by the valuative criterion of properness.

Now let us deal with the case of faithfully flat U-étale morphisms. Again, the only condition we need to check is the condition about lifting maps from valuation rings. Suppose that we are given a morphism $\phi\colon \operatorname{Spec} R \to S$ such that R is a valuation ring and ϕ sends the generic point of $\operatorname{Spec} R$ into U. We want to lift it to a morphism $\phi'\colon \operatorname{Spec} R' \to S'$ in a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \xrightarrow{\phi'} & S' \\ \downarrow & & \downarrow{\scriptstyle h} \\ \operatorname{Spec} R & \xrightarrow{\phi} & S. \end{array}$$

such that R' is also a valuation ring and the extension of fraction fields $\operatorname{Frac}(R')/\operatorname{Frac}(R)$ is finite. Since ϕ sends the generic point of $\operatorname{Spec} R$ into U, we conclude that $U \times_S \operatorname{Spec} R$ is schematically dense in $\operatorname{Spec} R$ as R is reduced. Note that surjectivity and flatness are preserved by arbitrary base change. Therefore, it is sufficient to assume that $S = \operatorname{Spec} R$ for the purposes of the proof.

Consider the closed point $s \in \operatorname{Spec} R$ and choose an arbitrary point $s' \in S'$ in a fiber over s. We find find some open affine subscheme $V' \subset S'$ that contains the point s'. The Going-Down Theorem ([Mat86, Theorem 9.5]) shows that there is a point $\eta' \in V'$ that maps to the generic point $\eta \in \operatorname{Spec} R = S$. Moreover, this point can be chosen to be a generic point of V'.

We claim that $\mathcal{O}_{S',\eta'}$ is a field. The only thing we need to check is that S' is reduced at η', but this is automatic because $\operatorname{Spec} R$ is reduced and $h\colon S' \to \operatorname{Spec} R$ is étale over η. Let us denote a prime ideal corresponding to s' by $\mathfrak{p}$. Then U-admissibility of S' implies that $\mathcal{O}_{S',s'}$ is a subring of a field $\mathcal{O}_{S',\eta'}$, and [Mat86, Theorem 10.2] guarantees that there is a valuation ring $R' \subset \mathcal{O}_{S',\eta'}$ that dominates $\mathcal{O}_{S',s'}$. It defines a morphism $\phi'\colon \operatorname{Spec} R' \to S'$ as a composition

$$\operatorname{Spec} R' \to \operatorname{Spec} \mathcal{O}_{S',s'} \to S'.$$

Then it is clear that the diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \xrightarrow{\phi'} & S' \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \xrightarrow{\phi} & S \end{array}$$

is commutative. The finiteness $\operatorname{Frac}(R')/\operatorname{Frac}(R) = \mathcal{O}_{S',\eta'}/\operatorname{Frac}(R)$ follows from quasi-finiteness (over U) of the morphism $f\colon S' \to \operatorname{Spec} R$. □

Remark 2.15. In the case $S = \operatorname{Spec} \mathcal{O}_K$ for some valuation ring K and $U = \operatorname{Spec} K$, we will usually say K-admissible (resp. K-modification, resp. K-étale, resp. K-smooth) instead of U-admissible (resp. U-modification, resp. U-étale, resp. U-smooth).

We recall another two technical definitions that will play an important role in this paper:

Definition 2.16. Let $\mathcal{O}_K$ be a discrete valuation ring. A finitely presented $\mathcal{O}_K$-scheme X is called *strictly semi-stable* if Zariski-locally it admits an étale morphism

$$U \to \operatorname{Spec} \frac{\mathcal{O}_K[t_0, \dots, t_l]}{(t_0 \cdots t_m - \pi)}$$

for some integers $m \leq l$, and a uniformizer $\pi \in \mathfrak{m}_K \setminus \mathfrak{m}_K^2$.

Definition 2.17. [ALPT19] Let $\mathcal{O}_K$ be a rank-1 valuation ring. A finitely presented $\mathcal{O}_K$-scheme X is called *ALPT-polystable* if étale locally it admits an étale morphism

$$U \to \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}, \dots, t_{0,n_0}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

for some $\pi_i \in \mathfrak{m}_K$. The $\mathcal{O}_K$-scheme

$$\operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}, \dots, t_{0,n_0}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

is called a *model polystable* $\mathcal{O}_K$-scheme.

Remark 2.18. We note that this definition allows $\pi_i = 0$. In particular, the generic fiber of an ALPT-polystable $\mathcal{O}_K$-scheme is not necessarily smooth.

Remark 2.19. This definition has an advantage over the notion of semi-stability is that any product of ALPT-polystable $\mathcal{O}_K$-schemes is ALPT-polystable. The analogous statement is false for semi-stable schemes.

Remark 2.20. Any ALPT-polystable $\mathcal{O}_K$-scheme is flat over $\mathcal{O}_K$. Indeed, it suffices to show that a model polystable $\mathcal{O}_K$-scheme is flat, but its function algebra clearly does not have $\mathcal{O}_K$-torsion hence $\mathcal{O}_K$-flat.

We will slightly modify the notion of poly-stability to make it more useful for our purposes later. We start with a few lemmas.

Lemma 2.21. Let $\mathcal{O}_K$ be a rank-1 valuation ring with algebraically closed residue field k, let X be an ALPT-polystable $\mathcal{O}_K$-scheme, and let $x \in X$ be a closed point in the closed fiber. Then there is an étale neighborhood $f\colon (U, u) \to (X, x)$ of x such that $f(u) = x$ and U admits an étale morphism

$$g\colon U \to \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}, \dots, t_{0,n_0}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

such that $t_{0,j}(g(u)) = 1$ for all $j \leq n_0$, and $t_{i,j}(g(u)) = 0$ for all $1 \leq i \leq l$, $j \leq n_l$.

Proof. Step 1. Reduce to the model case: As X is ALPT-polystable, we can find an étale neighborhood U of x with an étale morphism g to some model polystable $\mathcal{O}_K$-scheme Z. Pick $u \in U$ to be any point over x, and then it suffices to prove the claim for the pair $(Z, z) := (Z, g(u))$, i.e. we can assume that

$$Z = \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}, \dots, t_{0,n_0}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}.$$

Step 2. Reduce to the semi-stable model example: As the product of polystable models is polystable, and the product of étale morphisms is étale, it suffices to treat separately the cases of $Z = \operatorname{Spec} \mathcal{O}_K[t]$ and $Z = \operatorname{Spec} \mathcal{O}_K[t_0, \dots, t_n]/(t_0 \cdots t_n - \pi)$ for some element $\pi \in \mathfrak{m}_K$.

Step 3. Finish the proof in the semi-stable: The semi-stable case was already treated in [Tem17, Lemma 2.3.5], but the proof has a mathematical typo carried all over the proof. So we have decided to write a complete proof here.

The case $Z = \operatorname{Spec} \mathcal{O}_K[t]$ is easy. Consider $z \in Z$ as a closed point of the special fiber Z_s that is a variety over an algebraically closed field k. Thus it makes sense to evaluate t at z and get an element in k. Suppose that $t(z) = \overline{c} \in k$, lift it to some element $c \in \mathcal{O}_K$. Then the desired étale morphism is just the shift by $c-1$ on $Z = \operatorname{Spec} \mathcal{O}_K[t]$.

Now we deal with the case $Z = \operatorname{Spec} \mathcal{O}_K[t_0, \dots, t_n]/(t_0 \cdots t_n - \pi)$ for some $\pi \in \mathfrak{m}_K$. Then we rename $t_0, \dots, t_n$ so that $t_0(z), \dots, t_m(z) = 0 \in k$ but $t_{m+1}(z), \dots, t_n(z) \neq 0$. We consider $t_i(z) =: \overline{c_i} \in k$ and let c_i be any lift of $\overline{c_i}$ to $\mathcal{O}_K$. Finally, we define $t'_i := t_i$ for $0 \le i \le m-1$, $t'_m := t_m \cdots t_n$, and $t'_i = t_i - c_i + 1$ for $i \ge m+1$. This defines the natural morphism

$$\frac{\mathcal{O}_K[t'_0, \dots, t'_n]}{(t'_0 \cdots t'_m - \pi)} \to \frac{\mathcal{O}_K[t_0, \dots, t_n]}{(t_0 \cdots t_n - \pi)}$$

On the scheme level, it defines the morphism

$$g \colon Z \to \operatorname{Spec} \mathcal{O}_K[t'_0, \dots, t'_n]/(t'_0 \cdots t'_m - \pi)$$

that is easily seen to be étale at $z \in Z$. The target is ALPT-polystable, and $t'_i(g(z)) = 0 \in k$ for $i \le m$, $t'_i(g(z)) = 1 \in k$ for $i > m$. $\square$

Corollary 2.22. Let $\mathcal{O}_K$ be a rank-1 valuation ring with algebraically closed residue field k and fraction field K, let X be a K-smooth ALPT-polystable $\mathcal{O}_K$-scheme, and let $x \in X$ be any point. Then there is an étale neighborhood $f \colon (U, u) \to (X, x)$ of x such that $f(u) = x$ and U admits an étale morphism

$$g \colon U \to \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

for some *non-zero* $\pi_i \in \mathfrak{m}_K \setminus 0$.

Proof. If $x \in X$ lies in the generic fiber, the claim is easy. Indeed, $X \to \operatorname{Spec} \mathcal{O}_K$ is smooth at x, so there is a neighborhood $x \in U \subset X$ with an étale morphism $U \to \operatorname{Spec} \mathcal{O}_K[u_0, \dots, u_n]$. Then we can shift the image similar to what we done in Step 2 of Lemma 2.21 to assume that we get an étale map of the form $U \to \operatorname{Spec} \mathcal{O}_C[u_0^{\pm 1}, \dots, u_n^{\pm 1}]$.

The case of $x \in X$ lying in the special fiber essentially follows from Lemma 2.21. Namely, it suffices to prove the claim for closed points x since étale morphisms are open. In this case, we choose an étale neighborhood U of x with an étale morphism

$$g \colon U \to \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}, \dots, t_{0,n_0}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

such that $t_{0,j}(g(u)) = 1$ for all $j \le n_0$, and $t_{i,j}(g(u)) = 0$ for all $1 \le i \le l$, $j \le n_l$. The first condition ensures that the map factors as

$$g \colon U \to \operatorname{Spec} \frac{\mathcal{O}_K[u_1^{\pm 1}, \dots, u_m^{\pm 1}, t_{0,0}^{\pm 1}, \dots, t_{0,n_0}^{\pm 1}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}.$$

Now note that U is C-smooth as it is étale over X. Since g is étale, we conclude that the open subscheme $\operatorname{Im}(g)$ is K-smooth. Now we recall that $t_{i,j}(g(u)) = 0$ for all $1 \le i \le l$, $j \le n_l$, so the composition of g with the projection onto the last factors defines a map

$$h \colon U \to M := \operatorname{Spec} \frac{\mathcal{O}_K[t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

that is smooth, and $h(U)$ contains the origin O in the special fiber of M. Again, since h is smooth, we conclude that M is K-smooth at the image of h. In particular, $\mathcal{O}_{M,O} \otimes_{\mathcal{O}_K} K$ is K-(ind)smooth as $h(U)$ contains O. Now Remark 2.20 implies that each factor

$$M_i := \operatorname{Spec} \frac{\mathcal{O}_K[t_{i,0}, \dots, t_{i,n_i}]}{(t_{i,0} \cdots t_{i,n_i} - \pi_i)}$$

is $\mathcal{O}_K$-flat. Thus, [Sta21, Tag 02VL] implies that each factor the rings $\mathcal{O}_{M_i,O_i} \otimes_{\mathcal{O}_K} K$ is K-(ind)smooth, where O_i is the origin in the special fiber of M_i. Now we use the Jacobian criterion to ensure that the K-(ind)smoothness of $\mathcal{O}_{M_i,O_i} \otimes_{\mathcal{O}_K} K$ implies that $\pi_i \neq 0$. □

Definition 2.23. Let $\mathcal{O}_K$ be a rank-1 valuation ring. A flat, finitely presented, K-smooth $\mathcal{O}_K$-scheme X is called *polystable* if étale locally it admits an étale morphism

$$U \to \operatorname{Spec} \frac{\mathcal{O}_K[t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

for some $\pi_i \in \mathcal{O}_K \setminus \{0\}$. The $\mathcal{O}_K$-scheme

$$\operatorname{Spec} \frac{\mathcal{O}_K[t_{1,0}, \dots, t_{1,n_1}, \dots, t_{l,0}, \dots, t_{l,n_l}]}{(t_{1,0} \cdots t_{1,n_1} - \pi_1, \dots, t_{l,0} \cdots t_{l,n_l} - \pi_l)}$$

is called a *model polystable* $\mathcal{O}_K$-scheme.

Remark 2.24. Corollary 2.22 ensures that any K-smooth ALPT-polystable $\mathcal{O}_K$-scheme is polystable if the residue field k of $\mathcal{O}_K$ is algebraically closed. Indeed, it guarantees that we can choose non-zero $\pi_i \in \mathfrak{m}_K$ for all singular factors $\operatorname{Spec} \frac{\mathcal{O}_K[t_{i,0},\dots,t_{i,n_i}]}{(t_{i,0}\cdots t_{i,n_i} - \pi_i)}$, and the smooth factors $\operatorname{Spec} \mathcal{O}_K[u^{\pm 1}]$ can be rewritten as $\operatorname{Spec} \mathcal{O}_K[t_0, t_1]/(t_0 t_1 - 1)$.

One can show that these two notions of polystability are actually the same provided that X is K-smooth and k is algebraically closed. More generally, these two notions are the same if k is perfect. One can show this by passing to the strict henselization $\mathcal{O}_K^{\mathrm{sh}}$ and using [Sta21, Tag 0ASJ]. We will never use any of these results, so we do not give a proof. We do not know if these two notions are the same for a general $\mathcal{O}_K$.

3. Stable Modification Theorem for Curves Over Finite Rank Valuation Rings

In this section we prove a version of Temkin's Stable Modification Theorem over a finite rank valuation ring R. We fix some notations for this section: $S := \operatorname{Spec} R$, and $U := \eta$ be the generic point of $\operatorname{Spec} R$. There are different ways to define a rank of a valuation; we recall both of them below:

Definition 3.1. We say that a valuation ring R is of *finite rank*, if $\operatorname{Spec} R$ is a finite set. In this case, the rank r is defined as $|\operatorname{Spec} R| - 1$ (number of points in $\operatorname{Spec} R$ minus 1). It coincides with a rank (or height) of its valuation group Γ by [Bou98, Ch. VI, §4, n. 4, Proposition 5]. We denote rank of R by $\operatorname{rk} R$ or $\operatorname{rk} \Gamma$.

Definition 3.2. We say that a valuation ring R with a value group Γ has *rational rank* r, if $\dim_{\mathbf{Q}} \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} = r$ (it is infinite if dimension of $\Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$ is infinite). We denote it by $\operatorname{rk}_{\mathbf{Q}} \Gamma$ or $\operatorname{rk}_{\mathbf{Q}} R$.

Recall that for any valuation ring R we have inequality $\operatorname{rk}_{\mathbf{Q}} R \geq \operatorname{rk} R$ by [Bou98, Ch. 6, §10, n.2, Proposition 3]. Definition 3.2 is useful to verify that certain valuation rings have finite rank.

Now we recall the statement of Temkin's Stable Modification Theorem:

Theorem 3.3. [Tem10, Proposition 4.5.1] Let R be a finite rank valuation ring with a separably closed field of fractions K, and let $X \to \operatorname{Spec} R$ be a relative curve such that the generic fibre is semi-stable. There is a stable U-modification $f: X' \to X$ (in particular, X' is a semi-stable curve over $\operatorname{Spec} R$).

We want to show a (slightly) refined version of this theorem for an arbitrary finite rank valuation ring. Firstly, we recall some facts about valuation rings.

Lemma 3.4. Let R be a valuation ring with a field of fractions K, let K' be an algebraic extension of K and denote by R' the normalization of R in K'. Then the natural morphism $R' \otimes_R K \to K'$ is an isomorphism. Also, if R_1 is any valuation ring in K' that dominates R, then the natural map $R_1 \otimes_R K \to L$ is an isomorphism.

Proof. First if all, note that $R' \otimes_R K$ is a localization of the domain R'. Therefore, the natural map $R' \otimes_R K \to K'$ is injective, in particular it is a domain. In order to show that it is an isomorphism, it suffices to show that $R' \otimes_R K$ is a field. But $R' \otimes_R K$ is a domain integral over K, thus it must be a field.

As for the second claim, we note that R_1 is integrally closed in its fraction field K', so it contains R'. This easily implies that $R_1 \otimes_R K$ surjects onto K'. Injectivity follows easily from R-flatness of R_1. $\square$

Lemma 3.5. Let R be a valuation ring with a field of fractions K, and let K' be a finite Galois extension of K with Galois group G. Then the group G acts transitively on the set of maximal ideals in the normalization R' of R (in K'), and all these maximal ideals contract to the unique maximal ideal in R.

Proof. Note that the question is well-posed because G acts on a normalization of R, and therefore acts on its set of maximal ideals. Since the extension $R \subset R'$ is integral, we see that any maximal ideal in R' contracts to the maximal ideal in R by [Mat86, Section 9, Lemma 2], and [Mat86, Theorem 9.3] proves that the Galois group G acts transitively on these ideals. $\square$

Lemma 3.6. Let R be a valuation ring of finite rank with field of fractions K. Then the subset $\operatorname{Spec} K \subset R$ is open.

Proof. We note that $K = S^{-1}R$ is a localization of a ring R at the multiplicative set $S \coloneqq R \setminus \{0\}$. We have an equality $K = \operatorname{colim}_{x\in S} R_x$, so

$$\operatorname{Spec} K = \lim_{x\in S} \operatorname{Spec} R_x \cong \bigcap_{x\in S} \operatorname{Spec} R_x \subset \operatorname{Spec} R.$$

Since $\operatorname{Spec} R$ is has a finite underlying topological space, there are only finitely many options for $\operatorname{Spec} R_x \subset \operatorname{Spec} R$. Therefore, there exists some $x \in S$ such that $\operatorname{Spec} K = \operatorname{Spec} R_x$, so $\operatorname{Spec} K$ is open in $\operatorname{Spec} R$. $\square$

Theorem 3.7. Let $S = \operatorname{Spec} R$ be a spectrum of a valuation ring R with fraction field K, and let $f: X \to S$ be a relative curve such that the generic fibre is semi-stable. Then there is a finite Galois extension K'/K with R' the integral closure of R in K' such that $X_{S'} = X \times_S S'$ admits an S'-stable U-modification $g: X' \to X_{S'}$ with $S' = \operatorname{Spec} R'$. Moreover, an action of G on S' over S lifts to an action on X' over X.

Proof. Step 0: We use [Tem08, Corollary 2.1.3] to write R as a filtered colimit of its valuation subrings of finite height. We note that Lemma 2.5 implies that f is a semi-stable curve over some quasi-compact open subscheme of S. Then the standard approximation argument using Lemma A.3 and Lemma A.4 allows to assume that R is of finite rank.

Step 1: Choose a separable closure $K \subset K^{\mathrm{sep}}$, then [Mat86, Theorem 10.2] guarantees the existence of a valuation ring R' in K^{sep} that dominates R. Now by [Bou98, Ch. 6, §8, n.1, Corollary 1] we get that $\mathrm{rk}(R') = \mathrm{rk}(R)$, so R' is a valuation ring of finite rank. This means that we can apply Theorem 3.3 to get an S'-stable U-modification $g\colon X' \to X_{R'}$ with $S' = \operatorname{Spec} R'$ (note that $X_{R'} \times_S U = X_{K'}$ by Lemma 3.4).

Step 2: We write $K^{\mathrm{sep}} = \mathrm{colim}_{i\in I} K_i$ as a direct colimit of finite Galois extensions K_i of K. Note that each $R_i \coloneqq K_i \cap R'$ is again a valuation ring, $R' = \mathrm{colim}_{i\in I} R_i$, and for each i we have $R_i \otimes_R K \cong K_i$. We define $S_i \coloneqq \operatorname{Spec} R_i$. Then Lemma 3.6 says that U is open in $\operatorname{Spec} R$, so we can use the standard approximation techniques and Lemma A.4 to spread out the S'-stable U-modification $X' \to X_{S'}$ to an S_i-stable U-modification $g_i\colon X'_i \to X_{R_i}$. However, we are not done yet since R_i is generally not the integral closure of R in K_i (as this integral closure is usually just semi-local rather than local).

Step 3: Now we change notations and denote K_i by K', R_i by R_1, and the stable U-modification $g_i\colon X'_i \to X_{R_i}$ by $g\colon X'_1 \to X_{R_1}$. Also we denote by R' the integral closure of R in K'. Now note that [Bou98, Ch. 6, §8, n.6, Proposition 6] implies that R_1 is given by a localization of R' at some maximal ideal $\mathfrak{m}$, and Lemma 3.5 says that the Galois group $G \coloneqq \mathrm{Gal}(L/K)$ acts transitively on the set of maximal ideals in R'. For all $\sigma \in G$ the pullback $\sigma^*(g)\colon \sigma^*(X'_1) \to X_{(R_1)_{\sigma(\mathfrak{m})}}$ is an R_1-stable U-modification, so there exists an stable modification of $X_{R'}$ after a localization at each maximal ideal of R'. Then we use the standard approximation techniques once again to spread out this R_1-stable U-modification to a $V_{\mathfrak{n}}$-stable U-modification $g_{\mathfrak{n}}\colon X'_{\mathfrak{n}} \to X_{V_{\mathfrak{n}}}$ over some open $V_{\mathfrak{n}} \subset \operatorname{Spec} R'$ around each maximal ideal $\mathfrak{n} \subset R'$. Since $\operatorname{Spec} R'$ is normal, an R'-stable U-modification of $X_{R'}$ is unique up to a unique isomorphism by [Tem10, Corollary 1.3] and the same holds over all opens of $\operatorname{Spec} R'$. This uniqueness assertion allows us to glue various $X'_{\mathfrak{n}}$ to a unique R'-stable U-modification $g\colon X' \to X_{R'}$.

Step 4: The R'-stable U-modification $X' \to X_{R'}$ automatically admits a unique action of G extending that an action on its K'-fiber by [Tem10, Corollary 1.6] and the morphism $g\colon X' \to X_{R'}$ is G-equivariant. □

4. Stable Modification Theorem for Curves Over Finite Rank Semi-Valuation Rings

Let us recall a definition of a semi-valuation ring from [Tem11, §2].

Definition 4.1. A *valuation* on a ring A is commutative ordered group Γ with a multiplicative map $|\ |\colon A \to \Gamma \cup \{0\}$ which satisfied the inequality

$$|x+y| \leq \max(|x|, |y|)$$

and sends 1 to 1.

Definition 4.2. A *semi-valuation ring* is a ring $\mathcal{O}$ with a valuation $|\ |\colon \mathcal{O} \to \Gamma \cup \{0\}$ such that all zero-divisors of $\mathcal{O}$ lie in the prime kernel $\mathfrak{m} = \ker(|\ |)$ and for any pair $g, h \in \mathcal{O}$ with $|g| \le |h| \neq 0$ one has $h|g$ in $\mathcal{O}$. We say that $A \coloneqq \mathcal{O}_\mathfrak{m}$ is the *ring of semi-fractions* of $\mathcal{O}$.

In [Tem11, §2] Temkin gives a very useful description of all semi-valuation rings. Namely, they all come as a "composition" of a local ring A and a valuation ring R. Let us explain what this means. Given a semi-valuation ring $\mathcal{O}$ we see that the natural map $\mathcal{O} \to \mathcal{O}_\mathfrak{m} =: A$ is an injective morphism with a maximal ideal of A equal to $\mathfrak{m}A = \mathfrak{m}$ (exercise!), and the image $R = \mathcal{O}/\mathfrak{m}$ of $\mathcal{O}$ under the projection map $p\colon A \to A/\mathfrak{m}A$ is a valuation ring with fraction field $A/\mathfrak{m}A$. Indeed, the fraction field of R is naturally identified with $\mathcal{O}_\mathfrak{m}/\mathfrak{m}\mathcal{O}_\mathfrak{m} = A/\mathfrak{m}A$ and the valuation $|\ |\colon \mathcal{O} \to \Gamma \cup \{0\}$ induces a valuation on $A/\mathfrak{m}A$ with a ring of valuation being equal to R. Moreover one can easily show that $\mathfrak{m}A = \mathfrak{m}$ using that the ideal $\mathfrak{m}$ is $\mathcal{O} \setminus \mathfrak{m}$-divisible. This implies that $\mathcal{O} = p^{-1}(R)$, and gives an example of a "composition" of a local ring A and a valuation ring R:

Definition 4.3. The *composition* of a local ring $(A, \mathfrak{m}_A)$ and a valuation ring $R \subset A/\mathfrak{m}A$ (with fraction field $A/\mathfrak{m}A$) is the ring $\mathcal{O} \coloneqq p^{-1}(R)$, where $p\colon A \to A/\mathfrak{m}A$ is the natural projection map. It is easy to see that any such composition $\mathcal{O}$ is a semi-valuation ring (in the sense of Definition 4.2), and any semi-valuation ring $\mathcal{O}$ is a composition of $A = \mathcal{O}_\mathfrak{m}$ and $R = \mathcal{O}/\mathfrak{m} \subset K = A/\mathfrak{m}A$.

Actually this structure result will be more useful for us than the original definition, and from now on by a "semi-valuation ring" we will mean a pair $(\mathcal{O}, \mathfrak{m})$ consisting of a ring $\mathcal{O}$ and a prime ideal $\mathfrak{m}$ such that $\mathcal{O}$ injects into $A \coloneqq \mathcal{O}_\mathfrak{m}$ and the image of $\mathcal{O}/\mathfrak{m}$ in $A/\mathfrak{m}A$ is a valuation ring with fraction field $A/\mathfrak{m}A$. Given any semi-valuation ring $(\mathcal{O}, \mathfrak{m})$ we will always denote by A its localization $\mathcal{O}_\mathfrak{m}$, by K the residue field of $A/\mathfrak{m}A$ and by $R \subset K$ the valuation ring $\mathcal{O}/\mathfrak{m}$. Also, we reserve some notation for this section: we will always denote $\operatorname{Spec} \mathcal{O}$ by S, $\operatorname{Spec} A$ by U (so $U \to S$ is schematically dominant, monic and stable under generalization), $\operatorname{Spec} R$ by T, and $\operatorname{Spec} K$ by η (we follow notations from [Tem11]). Note that the natural commutative diagram

$$\begin{array}{ccc} \eta & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

is bi-Cartesian (easy to check from the very definition of semi-valuation ring).

Definition 4.4. We say that a semi-valuation ring $(\mathcal{O}, \mathfrak{m})$ is of *finite rank*, if the valuation ring R associated with this pair is of finite rank (in the sense of Definition 3.1).

Lemma 4.5. Let $\mathcal{O}$ be a finite rank semi-valuation ring with ring of semi-fractions A. Then the subset $\operatorname{Spec} A \subset \mathcal{O}$ is open.

Proof. We note that $A = S^{-1}\mathcal{O}$ is a localization of a ring $\mathcal{O}$ at a multiplicative set $S \coloneqq \mathcal{O} \setminus \mathfrak{m}$. So we have an equality $A = \operatorname{colim}_{x \in S} \mathcal{O}_x$, thus

$$\operatorname{Spec} A = \lim_{x \in S} \operatorname{Spec} \mathcal{O}_x \cong \bigcap_{x \in S} \operatorname{Spec} \mathcal{O}_x \subset \operatorname{Spec} \mathcal{O}.$$

Since the spectrum $\operatorname{Spec} R$ of the associated valuation ring R has a finite underlying topological space, there are only finitely many options for subsets of $\operatorname{Spec} \mathcal{O}$ containing $\operatorname{Spec} A$. For any $x \in S$ the subset $\operatorname{Spec} \mathcal{O}_x$ contains $\operatorname{Spec} A$, so we conclude that exists some $x \in S$ such that $\operatorname{Spec} A = \operatorname{Spec} \mathcal{O}_x$, so $\operatorname{Spec} A$ is open in $\operatorname{Spec} \mathcal{O}$. $\square$

Before going to the proof of Stable Modification Theorem over semi-valuation bases, we need to show two basic lemmas.

The following lemma is well-known and a form of it appears in [Gro63], but there is a connectedness assumption on X that we would like to avoid. So we write down a proof following the ideas from [Gro63].

Lemma 4.6. Let Y be a connected scheme and $f\colon X \to Y$ be a finite étale morphism. Then there is a finite group G (actually it can be chosen to be a symmetric group S_n) and a G-torsor $g\colon X' \to Y$ such that g dominates f, i.e. there is a factorization

$$X' \xrightarrow{h} X \xrightarrow{f} Y.$$

Proof. Note that the morphism $f\colon X \to Y$ is finitely presented (part of the definition of an étale morphism) and finite, so the degree function $\deg(X/Y)$ is locally constant on Y. Since Y is connected; f is of degree d for some integer d. Then we consider d-th fiber power

$$X^{d/Y} = X \times_Y X \times_Y \cdots \times_Y X.$$

The étaleness of the morphism f implies that the "big diagonal" $\Delta \subset X^{d/Y}$ is open (by the "big diagonal" we mean a union of all natural "diagonals" $X^{(d-1)/Y} \to X^{d/Y}$). But the separatedness of the morphism f implies that the "big diagonal" is also closed in $X^{d/Y}$. This means that $X' \coloneqq X^{d/Y} \setminus \Delta$ is open and closed in $X^{d/Y}$, so the natural morphism $g\colon X' \to Y$ is finite (as closed in a finite Y-scheme) and étale (as open inside an étale Y-scheme). Moreover, there is the natural action of the symmetric group S_d on X' because functorially it represents a functor of d fiberwise distinct points of X on the category of Y-schemes. So, in order to check that X' is an S_d-torsor, we need to check that the natural morphism of Y-schemes

$$p\colon G \times X' \to X' \times_Y X'$$

is an isomorphism. Since both schemes are finite étale over Y, we can check this on geometric fibers. But this claim is absolutely clear on geometric fibers from the functorial description of X'. Thus $g\colon X' \to Y$ is indeed an S_d-torsor.

Now we define a morphism $h\colon X' \to X$ as a map induced from the projection onto the first coordinate $p_1\colon X^{d/Y} \to X$. We need to check that h is surjective, it can be done on geometric fibers. And again on geometric fibers it is clear from the functorial description of X' and the definition of d. □

The next lemma is also well-known and we include it only for our convenience.

Lemma 4.7. Let $(A, \mathfrak{m}_A)$ be a local ring with a residue field K, and let K'/K be a finite separable extension. Then we can lift it to a finite étale morphism $A \to A'$, such that $A'/\mathfrak{m}_A A' = K'$. In particular, A' is local.

Proof. By the Primitive Element Theorem we know that $K' \cong K[T]/(\overline{P})$, where $\overline{P} \in K[T]$ is a monic separable polynomial. Lift this polynomial to an arbitrary *monic* polynomial $P \in A[T]$ and then $A' \coloneqq A[T]/(P)$ does the job. □

Theorem 4.8. Let $(\mathcal{O}, \mathfrak{m})$ be a semi-valuation ring of a finite rank and let $f\colon X \to S$ be a relative curve such that the base change $f_U\colon X_U \to U$ is a semi-stable relative curve. There is a finite group G and a faithfully flat finite and finitely presented morphism $\mathcal{O} \to \mathcal{O}'$ such that $\mathcal{O}'$ admits an action of G with the following properties:

- The natural morphism $f\colon S' \to S$ is G-invariant with $S' = \operatorname{Spec} \mathcal{O}'$.

- The restriction of f over U defines a G-torsor $f_U\colon S'_U \to U$ (in particular, it is finite etale).
- The base change $X_{S'}$ admits an S'-stable U-modification $g\colon X' \to X_{S'}$.

Proof. We start by considering the bi-Cartesian square

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} \mathcal{O}. \end{array}$$

We know by Theorem 3.7 that we can find a finite Galois extension $K \subset K'$ with R' being the integral closure of R in K' such that the base change $X_{R'}$ admits a R'-stable U-modification $g\colon X' \to X_{R'}$. We use Lemma 4.7 to lift Galois extension $K \subset K'$ to a finite étale map $A \to A'$. And, moreover, we use Lemma 4.6 to find a finite étale A-algebra A'' such that $\operatorname{Spec} A'' \to \operatorname{Spec} A$ dominates $\operatorname{Spec} A' \to \operatorname{Spec} A$ and $\operatorname{Spec} A'' \to \operatorname{Spec} A$ is a torsor for some finite group G. Let K'' be $A''/\mathfrak{m}_A A''$; this is not necessarily a field, but still a finite étale K-algebra (so, it is a product of finite separable field extensions of K). Now let us denote by R'' the normalization of R in K''; note that the morphism

$$g_{R''}\colon X'_{R''} \to X_{R''}$$

is a R''-stable U-modification (because stability is a condition on geometric fibers).

Let us now denote K'' by K', R'' by R' and A'' by A'. So what we get from the previous discussion is that we have a G-torsor $\alpha\colon \operatorname{Spec} A' \to \operatorname{Spec} A$ for some finite group G such that $X_{R'}$ admits a R'-stable U-modification $g\colon X' \to X_{R'}$, where R' is the normalization of R in the special fibre of α. Note that by construction G acts on $\operatorname{Spec} K'$ and $\operatorname{Spec} R'$ over $\operatorname{Spec} R$.

Since K' is a finite étale K-algebra and $R' \otimes_R K = K'$ (apply Lemma 3.4 to factor fields of K'), we can find a finite number of elements $a_j \in R'$ such that $\{a_j\}$ is a basis of K' as a vector space over K. Write $R' = \operatorname{colim}_{i\in I} R_i$ as a filtered colimit of R-finite type subalgebras R_i of R' containing all the elements a_j. Since they are R-torsion free modules (as the subrings of R-torsion free R') they are actually flat over R. Then it is easy to see that $R_i \otimes_R K \to K'$ is an isomorphism (it is injective by R-flatness of R_i and it is surjective since R_i contains all K-basis elements a_j of K'). Moreover, since R_i are finite type and integral they are really finite. Then [Mat86, Theorem 7.10] implies that each R_i is actually finitely presented as an R-module, and therefore also finitely presented as an R-algebra by [Gro64, Proposition 1.4.7]. Now since R' has an action of G by R-algebra automorphisms, we can use Lemma A.1 (it is easy to see that all conditions of this Lemma are satisfied for a filtered system $\{R_i\}_{i\in I}$) to say that we can choose a subsystem R_j such that each R_j is G-stable subalgebra of R, and $R = \operatorname{colim}_{j\in J} R_j$. We note that Lemma 3.6 says that U is open under our assumptions, so we can use the standard approximation techniques and Lemma A.4 to spread out the R'-stable U-modification $X' \to X_{R'}$ to an R_j-stable U-modification $g_j\colon X'_j \to X_{R_j}$.

Now we use [CCO14, Proposition 1.4.4.1(ii), (iii)] to "glue" $\operatorname{Spec} R_j$ and $\operatorname{Spec} A'$ "along" $\operatorname{Spec} K'$ to achieve a faithfully flat, finite, finitely presented $\mathcal{O}$-algebra $\mathcal{O}'$ (explicitly, $\mathcal{O}'$ is equal to $A' \times_{K'} R_j$) with an $\mathcal{O}$-algebra action of G. Let $S' = \operatorname{Spec} \mathcal{O}'$ and we claim that $X_{S'}$ admits an S'-stable U-modification.

We know that X_{R_j} has a R_j-stable U-modification $g_j\colon X'_j \to X_{R_j}$, and X_U is a relative semi-stable U-curve by the assumption. Therefore, $X_{K'}$ is also a relative semi-stable K'-curve, and $(X'_j)_U$ is canonically isomorphic to $X_{K'}$. Now we can use [CCO14, Proposition 1.4.4.1(iii)] to "glue" X'_j and $X_{A'}$ to an S'-stable U-modification $X'' \to X_{\mathcal{O}'}$[5]. The last thing to observe is that the U-restriction of $\operatorname{Spec} \mathcal{O}' \to \operatorname{Spec} \mathcal{O}$ is a G-torsor because it coincides (by construction) with the G-torsor $\operatorname{Spec} A' \to \operatorname{Spec} A$.

□

5. Stable Modification Theorem for Curves Over General Bases

Let us set up some notations for this section.

Setup 5.1. We fix a quasi-compact quasi-separated scheme S, a stable under generalization quasi-compact subset U that admits a scheme structure such that $U \to S$ is schematically dominant, and a relative S-curve $f \colon X \to S$ (in the sense of Definition 2.1).

Remark 5.2. Any quasi-compact open subscheme U' containing the underlying topological space of U is schematically dense in S. Indeed, the natural map $U \to S$ clearly factors through the inclusion $U' \to S$, so the inclusion $U' \to S$ must be schematically dominant as $U \to S$ is.

Before we start the discussion of the Stable Modification Theorem over S, we need to recall some facts about relative Riemann-Zariski spaces from [Tem10] and [Tem11].

We consider the system of all U-modifications of S. This is a filtered system because any two U-modification S_i and S_j can be dominated by a U-modification obtained via taking a schematic closure of U in the product $S_i \times_S S_j$. So we take a limit over this filtered system in the category of locally ringed spaces; we call this limit as a *relative Riemann-Zariski space* $\mathrm{RZ}_U(S)$. We will usually denote it just by $\mathcal{S}$ (it will not cause any confusion in what follows). It has a natural "structure sheaf" obtained as

$$\mathcal{O}_{\mathcal{S}} = \operatorname{colim} \pi_i^{-1} \mathcal{O}_{S_i}, \text{ where } \pi_i \text{ is the natural projection } \pi_i \colon \mathcal{S} \to S_i.$$

Note that although it is proven in [Gro66, §8] that a filtered limit of schemes with affine transition maps exists as a scheme, it is extremely false without the affineness assumption. For example, one can show that the functor $\prod_{i=1}^{\infty} \mathbf{P}^1$ is not even an algebraic space. Thus there is no reason for $\mathcal{S}$ to be a scheme, but is rather just some abstract locally ringed space. It is difficult to work with this space, unless one proves results about the structure of this mysterious object. We summarize some of them below.

Lemma 5.3. [Tem10, Proposition 3.3.1][Tem11, Proposition 2.2.1] Fix $U, S, \mathcal{S}$ as in the Setup 5.1. Then

(1) The underlying topological space of $\mathcal{S}$ is quasi-compact, quasi-separated, and surjects onto S' for any U-modification $S' \to S$.
(2) For each point $\mathfrak{s} \in \mathcal{S}$ the local ring $\mathcal{O}_{\mathcal{S},\mathfrak{s}}$ has a natural structure of a semi-valuation ring with an isomorphism $\operatorname{Spec} A \cong U \times_S \operatorname{Spec} \mathcal{O}_{\mathcal{S},\mathfrak{s}}$, where A is the ring of semi-fractions of $\mathcal{O}_{\mathcal{S},\mathfrak{s}}$.

Proof. We set up dictionary between our notations and the notations in [Tem11]. What we call U (resp. S, resp. $\mathcal{S}$) is denoted by Y (resp. X, resp $\mathfrak{X}$) in [Tem11]. Then the first claim is shown in [Tem10, Prop. 3.3.1] and [Tem11, Prop. 2.2.1]. And the second claim is also implicitly obtained in *the proof* of [Tem11, Prop. 2.2.1] (and [Tem11, Prop. 2.2.2]).

We note that the proof of [Tem11, Proposition 2.2.1] works under the assumption that $f \colon Y \to X$ is decomposable (a composition of an affine morphism followed by a proper one). We can use

[5] Alternatively, one can use [Tem10, Theorem 1.1] and [Tem11, Lemma 2.3.1 (ii)].

[Tem11, Theorem 1.1.3] to see that our morphism $U \to S$ is automatically decomposable. But in the case of $U \to S$ being an open immersion (the only case we will really use in this paper[6]) it is much easier to show. Namely, we consider the complement $Z := S \setminus U$, a closed subset of S. Since S is quasi-compact quasi-separated and $U \to S$ is quasi-compact, [Con07, Lemma 1.3] says that there exists a finite type quasi-coherent ideal sheaf $\mathcal{I}$ with $\mathrm{V}(\mathcal{I}) = Z$. Then we define $S' := \mathbf{Proj}_S \bigoplus_{i \in \mathbf{N}} \mathcal{I}^n$ to be the blow-up of S along $\mathcal{I}$. Since $\mathcal{I}$ is finite type we conclude that $p\colon S' \to S$ is projective by [Gro61, Proposition (5.5.1)] (in particular, it is proper) . Since $\mathcal{I}|_U \cong \mathcal{O}_U$ we see that the U-restriction of $p\colon S' \to S$ is an isomorphism. Therefore, we have a commutative diagram:

$$\begin{array}{ccc} & & S' \\ & \overset{j'}{\nearrow} & \downarrow p \\ U & \xrightarrow{j} & S \end{array}$$

where p is proper and j' is an affine open immersion since its complement $S' \setminus j'(U)$ is a Cartier divisor. ☐

This Lemma explains the significance of semi-valuation rings for the purposes of proving the Stable Modification Theorem: we use various approximation techniques to reduce the case of a general S to the case of curves over local rings of a relative Riemann-Zariski spaces $\mathcal{S}$, which in turn are semi-valuation rings.

We will freely use the notion of "U-admissible blow-up" from [RG71, Section 5.1]. We now provide the reader with the definition and main properties of this construction

Definition 5.4. For an open immersion $U \to S$, a morphism $p\colon S' \to S$ is called a *U-admissible blow-up* if p is equal to the morphism $\mathbf{Proj}_S \bigoplus_n \mathcal{I}^n \to S$, where $\mathcal{I}$ is a finitely generated, quasi-coherent sheaf of ideals on S such that $\mathrm{V}(\mathcal{I}) \cap U = \varnothing$. In particular, p is of finite type.

We summarize the main properties of such blow-ups in the following lemma:

Lemma 5.5. Let S be a quasi-compact quasi-separated scheme, and let $U \subset S$ be a schematically dense quasi-compact open subscheme. Then any U-admissible blow-up $p\colon S' \to S$ is a projective admissible U-modification. Moreover, U-admissible blow-ups are cofinal among all U-modifications of S.

Proof. Projectivity follows the direct description of p as the Proj of a finitely generated quasi-coherent sheaf of ideals, and [Gro61, Proposition (5.5.1)]. The fact that p is an isomorphism can be easily seen from the fact that $\mathcal{I}|_U = \mathcal{O}_U$. Finally, admissible blow-ups are cofinal due to [RG71, Corollaire 5.7.12] (or [Sta21, Tag 081T] for a proof that does not mention algebraic spaces). ☐

Theorem 5.6. Let S, U be as in the Setup 5.1, and let $f\colon X \to S$ be an S-curve that is semi-stable over U. Then there exist

- A projective U-modification $h\colon S' \to S$ with a finite open Zariski covering $\cup_{i=1}^n V_i' = S'$ by quasi-compact opens $V_i' \subset S'$.
- A finite group G_i and a finite, finitely presented, faithfully flat, and U-étale G_i-invariant morphism $t_i\colon W_i' \to V_i'$ for each $i \le n$. In particular, the morphism $t\colon W' = \sqcup_{i=1}^n W_i' \to S$ is a U-étale covering.

[6]Even though we allow more general U in the formulation of Theorem 5.6, the first step there is reduction to the case of an open U.

satisfying the following properties:

(1) The induced morphisms $t_{i,U}\colon W'_{i,U} \to V'_{i,U}$ are G_i-torsors.

(2) Each $X_{W'_i}$ admits a W'_i-stable U-modification $g_i\colon X'_i \to X_{W'_i}$.

Remark 5.7. Note that the morphism

$$t\colon \bigsqcup_{i=1}^{n} W'_i \to S$$

from Theorem 5.6 is a quasi-projective U-étale covering. Indeed, it is a composition of the faithfully flat quasi-projective U-étale morphism

$$\bigsqcup W'_i \to S'$$

and the quasi-projective U-modification

$$S' \to S.$$

Both of those are U-étale coverings by Lemma 2.14, so their composition is also a quasi-projective U-étale covering.

Moreover, we recall that [Tem10, Theorem 1.1] shows that the W'_i-stable U-modifications $g_i\colon X'_i \to X_{W'_i}$ are also projective.

Proof. Step 0. Approximation to the case of finite type $\mathbf{Z}$*-schemes:* Note that [Tem10, Lemma 5.1] guarantees that any $x \in U$ has an open quasi-compact neighborhood $U_x \subset S$ such that X has semi-stable fibers over U_x. Since U is quasi-compact we can cover U by finitely many of those to find some quasi-compact schematically dense open subscheme $U' \subset S$ (see Remark 5.2) such that X has semi-stable fibers over U'. We can replace U by U' without loss of generality (any U'-modification is also a U-modification). So, from now on, we may (and do) assume that U is open in S.

Now we use Lemma A.2 and Lemma A.3 to show that the triple (S, U, X) is induced from some finite type $\mathbf{Z}$-scheme S', an open schematically dense subscheme U' and a relative curve $f'\colon X' \to S'$. Therefore, without loss of generality we can assume that S is finite type over $\operatorname{Spec}\mathbf{Z}$ and U is a schematically dense open subscheme of S.

Step 1. We solve the problem over local rings of $\mathcal{O}_{\mathcal{S}}$: Lemma 5.3 guarantees that all local rings of $\mathcal{O}_{\mathcal{S}}$ are semi-valuation rings. By Theorem 4.8 the only thing that we need to check is that under the assumption that S is finite type over $\mathbf{Z}$, all these semi-valuation rings are of finite rank. In order to do this we need to observe that the proof of [Tem11, Proposition 2.2.1] actually shows a bit more than stated there explicitly. Namely, choose an arbitrary point $\mathfrak{s} \in \mathcal{S}$, then Lemma 5.3 guarantees that there is a natural structure of a semi-valuation ring on a ring $\mathcal{O} \coloneqq \mathcal{O}_{\mathcal{S},\mathfrak{s}}$. Denote the kernel of corresponding valuation by $\mathfrak{m}$ (and let y be the corresponding point of $\operatorname{Spec}\mathcal{O}$), and the corresponding ring semi-fractions $\mathcal{O}_{\mathfrak{m}}$ by A. By the construction of relative Riemann-Zariski spaces we have a natural morphism of schemes $\pi_{\mathfrak{s}}\colon \operatorname{Spec}\mathcal{O} \to S$; consider the image $y' \coloneqq \pi_{\mathfrak{s}}(y) \in U$. Then the proof of [Tem11, Proposition 2.2.1] really proves that $A \cong \mathcal{O}_{U,y'}$. In particular, this is a ring essentially of finite type over $\mathbf{Z}$. Thus $K \coloneqq A/\mathfrak{m}A$ is a field of finite transcendence degree over the prime field $\mathbf{F}$. So we can use Abhyankar's inequality [Tem10, Lemma 2.1.2] to conclude that $R \subset K$ must be of finite rank.

Use Theorem 4.8 to get a finite group $G_{\mathfrak{s}}$ and a faithfully flat finite finitely presented $G_{\mathfrak{s}}$-invariant morphism $h_{\mathfrak{s}}\colon \operatorname{Spec}\mathcal{O}'_{\mathfrak{s}} \to \operatorname{Spec}\mathcal{O}_{\mathcal{S},\mathfrak{s}}$ such that $X_{\mathcal{O}'_{\mathfrak{s}}}$ admits a $\mathcal{O}'_{\mathfrak{s}}$-stable U-modification $g'_{\mathfrak{s}}\colon X'_{\mathfrak{s}} \to X_{\mathcal{O}'_{\mathfrak{s}}}$ and $(h_{\mathfrak{s}})_U\colon (\operatorname{Spec}\mathcal{O}'_{\mathfrak{s}})_U \to (\operatorname{Spec}\mathcal{O}_{\mathcal{S},\mathfrak{s}})_U$ becomes a $G_{\mathfrak{s}}$-torsor.

Step 2. We spread out this stable modification over a local ring of some U-modification of S: Let $\pi_i\colon \mathcal{S} \to S_i$ be the natural projection from a relative Riemann-Zariski space of S onto some U-modification S_i and denote by $s_i = \pi_i(\mathfrak{s})$ the image of $\mathfrak{s}$ in S_i. Then since filtered colimits commute with each other we see that

$$\mathcal{O}_{\mathcal{S},\mathfrak{s}} = \mathrm{colim}_i\, \mathcal{O}_{S_i,s_i}.$$

We have a faithfully flat finite finitely presented $G_{\mathfrak{s}}$-invariant morphism $\mathrm{Spec}\, \mathcal{O}'_{\mathfrak{s}} \to \mathrm{Spec}\, \mathcal{O}_{\mathcal{S},\mathfrak{s}}$ which is a $G_{\mathfrak{s}}$-torsor over U. Therefore, we can use the standard approximation techniques to find a faithfully flat finite finitely presented $G_{\mathfrak{s}}$-invariant morphism $\mathrm{Spec}\, \mathcal{O}'_{s_i} \to \mathrm{Spec}\, \mathcal{O}_{S_i,s_i}$ which becomes a $G_{\mathfrak{s}}$-torsor over U. Moreover, we can use standard approximation techniques and Lemma A.4 to get that (possibly after enlarging i) we can assume that $X_{\mathcal{O}'_{s_i}}$ admits a $\mathcal{O}'_{s_i}$-stable U-modification

Let us summarize what we have achieved so far. For each point $\mathfrak{s} \in \mathcal{S}$ there exists a big enough i such that for $s_i \coloneqq \pi_i(s)$ there is a U-modification $S_i \to S$ and a faithfully flat finite finitely presented $G_{\mathfrak{s}}$-invariant (for some finite group $G_{\mathfrak{s}}$) morphism $\mathrm{Spec}\, \mathcal{O}'_{s_i} \to \mathrm{Spec}\, \mathcal{O}_{S_i,s_i}$ such that it induces a $G_{\mathfrak{s}}$-torsor over U and $X_{\mathcal{O}'_{s_i}}$ admits a $\mathcal{O}'_{s_i}$-stable U-modification.

Step 3. We spread out this stable modification over some Zariski open subspace of the point s_i: Basically, we repeat the same argument as above, but now we use that

$$\mathcal{O}_{S_i,s_i} = \mathrm{colim}_{s_i\in V\text{-affine}}\, \mathcal{O}_{S_i}(V)$$

This means that we can again run the same approximation argument to find some open affine $V_i \subset S_i$ that contains s_i and a faithfully flat finite finitely presented $G_{\mathfrak{s}}$-invariant morphism $W_i \to V_i$ such that it induces $G_{\mathfrak{s}}$-torsor over U and X_{W_i} admits a W_i-stable U-modification. Note that by the very construction $V_{\mathfrak{s}} \coloneqq \pi_i^{-1}(V_i)$ is an open subset of $\mathcal{S}$ containing $\mathfrak{s}$.

Step 4. We use quasi-compactness of $\mathcal{S}$ to finish the argument: Since $\mathfrak{s}$ was an arbitrary point of $\mathcal{S}$, the previous construction provides us with a covering of $\mathcal{S}$ by open subsets $V_{\mathfrak{s}}$. Lemma 5.3 says that $\mathcal{S}$ has quasi-compact underlying topological space. Therefore we can choose a finite covering of $\mathcal{S}$ by means of such opens $V_{\mathfrak{s}}$. Denote the corresponding $\mathfrak{s}$'s just by $(\mathfrak{s}_j)_{j\in J}$ for some finite set J. Then after changing some notation, the previous discussion gives us a set of finite groups G_j, U-modifications $h_j\colon S_j \to S$, open affine subsets $V_j \subset S_j$ containing a point $\pi_j(\mathfrak{s}_j)$, and finite, finitely presented and finitely presented G_j-invariant morphisms $W_j \to V_j$ which become G_j-torsors over U, with the condition that X_{W_j} admits a W_j-stable U-modification for all j.

Recall that a system of all U-modifications is filtered, so we can find an U-modification $S' \to S$ that dominates all of the S_j. Then the data above induces open quasi-compact subsets

$$V'_j \coloneqq V_j \times_{S_j} S' \subset S'$$

and finite, finitely presented and faithfully flat G_j-invariants morphisms

$$W'_j \coloneqq W_j \times_{S_j} S' \to V'_j,$$

such that they induce G_j-torsors over U and $X_{W'_j}$ admits a W'_j-stable U-modification for all j.

We claim that open quasi-compact V'_j cover S'. To show this we note that under the natural projections $\pi_j\colon \mathcal{S} \to S_j$ and $\pi\colon \mathcal{S} \to S'$ we have an equality

$$V_{s_j} \coloneqq \pi_j^{-1}(V_j) = \pi^{-1}(V'_j).$$

This means that $\pi^{-1}(V'_j)$ form a covering $\mathcal{S}$. Therefore, we conclude that the V'_j cover S' because the projection $\mathcal{S} \to S'$ is surjective by Lemma 5.3.

The fact that we can choose $h\colon S' \to S$ to be a projective U-modification follows from Lemma 5.5. $\square$

6. Schematic Version of Local Uniformization

In what follows, we will freely use the following notations

Definition 6.1. We say that a morphism of S-schemes $f\colon X \to Y$ is *G-invariant* for some group G, if X admits an action of G by S-automorphisms, and $f \circ g = f$ for all $g \in G$.

Whenever we say that a morphism is G-invariant, we implicitly assume that there is a given action of G on X over S.

Definition 6.2. A morphism $f\colon X \to Y$ is called a *G-torsor over an open subset* $V \subset Y$, if f is a G-invariant morphism that factors through V, and

$$f_V\colon X \to V$$

is a G-torsor over V (i.e. it is flat finitely presented and the natural action morphism $X \times G \to X \times_V X$ is an isomorphism). In particular, V must be equal to the image $f(X) \subset Y$.

Lemma 6.3. Let G, H be two finite groups, and $f\colon X' \to X$ be a G-invariant map of schemes that is a G-torsor over its open image $V \coloneqq f(X')$ (in the sense of Definition 6.2). Let $g\colon X' \to Y$ be any G-invariant morphism, and let $h\colon Z \to Y$ be an H-invariant morphism that is an H-torsor over its open image $W \coloneqq h(Z) \subset Y$. Consider the base change morphism $h'\colon X'' \coloneqq Z \times_Y X' \to X'$ and the composition $t\colon X'' \to X$ in the commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow[f]{} & X' & \xleftarrow[h']{} & X'' \\ & & \downarrow{\scriptstyle g} & & \downarrow \\ & & Y & \xleftarrow[h]{} & Z \end{array}$$

The morphism t is $G \times H$-invariant, and it is a $G \times H$-torsor over the open $f(g^{-1}(W))$.

Proof. Firstly, we note that the subset $f(g^{-1}(W))$ is indeed open since f is flat and locally finitely presented by definition. Now we observe that since $g\colon X' \to Y$ is G-invariant, so $g^{-1}(W)$ is a G-stable open subset of X', the map $p\colon g^{-1}(W) \to X$ is a G-torsor over $f(g^{-1}(W))$. Therefore we can replace Y with W (so, we also replace X' with $X' \times_Y W$, Z with $Z \times_Y W$, and X'' with $X'' \times_Y W$) without affecting any assumptions of the lemma. Hence we can assume $h\colon Z \to Y$ is actually an H-torsor.

Now observe that we can replace X with $V' \coloneqq f(X')$ without affecting any assumptions of the lemma, so we can assume that $f\colon X' \to X$ is a G-torsor. Then we have a diagram

$$\begin{array}{ccccc} X & \xleftarrow[f]{} & X' & \xleftarrow[h']{} & X'' \\ & & \downarrow{\scriptstyle g} & & \downarrow \\ & & Y & \xleftarrow[h]{} & Z, \end{array}$$

where f is a G-torsor and h' is an H-torsor (as the base change of the H-torsor $h\colon Z \to Y$), and we need to show that $t = f \circ h'\colon X'' \to X'$ is a $G \times H$-torsor. We see that there is a natural action of the group $G \times H$ on X'' since g and h are, respectively, G and H-invariant morphisms. Moreover, the morphism h' is H-invariant and G-equivariant. Finally, we note that t is a finite étale morphism as a composition of two finite étale morphism.

It now suffices to check that the natural action $X'' \times (G \times H) \to X'' \times_X X''$ is an isomorphism. We can check this on geometric points (because t is finite étale). So we can reduce to the situation $X = \operatorname{Spec} k$ for an algebraically closed field k. Then X' is a disjoint union of $\operatorname{Spec} k$, and the G-invariance of $g\colon X' \to Y$ implies that g factors through a point in Y. This implies that we can also assume that Y is just a point $\operatorname{Spec} k$. In this case it is easy to see that X'' is just a disjoint union of $\#(G \times H)$ copies of $\operatorname{Spec} k$ with a simply transitive action of $G \times H$. Therefore $X'' \times (G \times H) \to X'' \times_X X''$ is an isomorphism in this case. □

We recall another technical definition that we will need to formulate the main theorem of this section.

Definition 6.4. A rank-1 valuation field K is *henselian* if its valuation ring $\mathcal{O}_K$ is a henselian local ring.

We summarize the main facts about henselian valuation rings in the following lemmas:

Lemma 6.5. Let K be a rank-1 valuation field with the valuation ring $\mathcal{O}_K$.

- Suppose that $\mathcal{O}_K$ is complete with respect to its valuation topology. Then K is a henselian field.
- Suppose K be henselian, and let L be an algebraic extension of K. Then L has a unique structure of a henselian field compatible with K. Moreover, the associated valuation ring $\mathcal{O}_L$ is equal to the normalization of $\mathcal{O}_K$ in L.

Proof. We start with the proof of the first statement. [Ber93, Proposition 2.4.3] implies that K is henselian if and only if K is quasi-complete in the sense of [Ber93, Definition 2.3.1]. Now quasi-completeness of $\mathcal{O}_K$ follows from [BGR84, Theorem 3.2.4/2].

Now we prove the second part. We note that [Ber93, Proposition 2.4.3] implies that K is quasi-complete. In particular, there is a unique extension of the norm on K to a norm on L. Now [Mat86, Theorem 10.4] implies that $\mathcal{O}_L$ is equal to the normalization of $\mathcal{O}_K$ in L. Finally, it is easy to see from [Ber93, Definition 2.3.1] that L is quasi-complete. Therefore, L is henselian by [Ber93, Proposition 2.4.3]. □

Our main goal of this section is to prove the following result.

Theorem 6.6. Let K be a henselian rank-1 valuation field with a valuation ring $\mathcal{O}_K$, and let $X \to \operatorname{Spec} \mathcal{O}_K$ be a flat, finitely presented morphism that is smooth over $\operatorname{Spec} K$. Then there is finite Galois extension $K \subset K'$, a finite extension $K' \subset K''$, and a finite set I of triples $\{g_i\colon X'_i \to X_{\mathcal{O}_{K'}}, h_i\colon X''_i \to X'_{i,\mathcal{O}_{K''}}, G_i, V_i\}_{i\in I}$ with the following properties:

(1) The morphism $g\colon \bigsqcup_{i\in I} X'_i \to X_{\mathcal{O}_{K'}}$ is a quasi-projective, K'-étale covering with each X'_i being a finitely presented, K'-admissible $\mathcal{O}_{K'}$-scheme.

(2) For all $i \in I$, G_i is a finite group, the morphism $g_i\colon X'_i \to X_{\mathcal{O}_{K'}}$ is G_i-invariant, and its base change $g_{i,K'}\colon X'_{i,K'} \to X_{K'}$ is a G_i-torsor over its image $V_i \coloneqq g_i(X'_{i,K'}) \subset X_{K'}$.

(3) For all $i \in I$, X'_i admits a structure of a successive semi-stable K'-smooth curve fibration over $\operatorname{Spec} \mathcal{O}_{K'}$.

(4) For all $i \in I$, $h_i\colon X''_i \to X'_{i,\mathcal{O}_{K''}}$ is a K''-modification with X''_i as K''-smooth, polystable $\mathcal{O}_{K''}$-scheme.

Remark 6.7. Lemma 6.5 ensures that K' (resp. K'') has the unique structure of a rank-1 valuation field compatible with that structure on K. In particular, $\mathcal{O}_{K'}$ (resp. $\mathcal{O}_{K''}$) makes sense as the valuation ring associated to that valuation.

Remark 6.8. Lemma 6.5 also implies that Theorem 6.6 holds for a complete rank-1 valuation field K.

The idea of our proof is rather easy in principle: we first reduce to the case when X has a structure of a successive curve fibration over a base S, then we "resolve" each curve fibration by a semi-stable one using Theorem 5.6. The last step is to apply [ALPT19, Theorem 5.2.16] to "resolve" each X'_i by a polystable scheme. There two main difficulties in this approach.

The first issue is to control the torsor condition on K-fibers. The problem already appears in the case of finite field extensions: a tower of two finite Galois extensions is not necessarily Galois. To resolve this issue we use (a trivial observation) that a composite of two Galois extension is Galois. So, if we try to resolve our tower of curve fibrations step by step from top to bottom, we can hope to preserve the G-torsor condition at each step of our inductive argument. This is exactly (with some extra technical complications) what we do in the proof below.

The other problem is to verify the assumptions of [ALPT19, Theorem 5.2.16], i.e. we need to construct a structure of a log smooth log variety on a successsive semi-stable K-smooth curve fibration over $\operatorname{Spec} \mathcal{O}_K$. This issue is resolved in Appendix C by a spreading out argument to the well-known case.

We prefer firstly to deal with the case of curve fibrations (over a general base) and then use some other techniques to reduce the general case to the case of a successive curve fibration.

The next proposition is rather lengthy to formulate; we apologize to the reader that we do not know how to avoid the heavy notation.

Lemma 6.9. Let R be a valuation ring with fraction field K. Let S be the affine scheme $\operatorname{Spec} R$. Suppose that $f\colon X \to S$ is a morphism that can be written as a composition

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = S$$

such that each morphism f_i is a relative curve (in the sense of Definition 2.1) that is K-smooth. Then there is a finite Galois extension $K \subset K'$ with R' the normalization of R in K', a finite set J, a set of finite groups $(G_j)_{j\in J}$ indexed by J, and diagrams:

$$\begin{array}{ccc}
X_{n,R'} & \xleftarrow{\;g_{n,j}\;} & X'_{n,j} \\
\big\downarrow {\scriptstyle f_{n,R'}} & & \big\downarrow {\scriptstyle h_{n,j}} \\
X_{n-1,R'} & \xleftarrow{\;g_{n-1,j}\;} & X'_{n-1,j} \\
\big\downarrow {\scriptstyle f_{n-1,R'}} & & \big\downarrow {\scriptstyle h_{n-1,j}} \\
\vdots & \xleftarrow{\;g_{k,j}\;} & \vdots \\
\big\downarrow {\scriptstyle f_{2,R'}} & & \big\downarrow {\scriptstyle h_{2,j}} \\
X_{1,R'} & \xleftarrow{\;g_{1,j}\;} & X'_{1,j} \\
\big\downarrow {\scriptstyle f_{1,R'}} & \swarrow {\scriptstyle h_{1,j}} & \\
\operatorname{Spec} R' & &
\end{array}$$

of flat, finitely presented $S' \coloneqq \operatorname{Spec} R'$-schemes such that

- $X'_{k,j}$ admits an R'-action of the group G_j for any $k \geq 1$ and $j \in J$, and $g_{k,j}$ is G_j-invariant and $h_{k,j}$ is G_j-equivariant for any $j \in J, k = 1, \dots, n$.

- The K'-restriction $(g_{k,j})_{K'}\colon (X'_{k,j})_{K'} \to (X_k)_{K'}$ is a G_j-torsor over its (open) image $V_{k,j}$.
- $h_{k,j}$ is a relative semi-stable K'-smooth curve for any $j \in J$ and any $k \geq 1$
- $g_{k,j}$ is quasi-projective for any $k \geq 1$ and $j \in J$ with the map $g_k\colon \sqcup_{j\in J} X'_{k,j} \to X_{k,R'}$ a K'-étale covering for any $k \geq 1$.

Proof. We only explain the main idea of the proof here and refer to Lemma 9.4 for a detailed proof of this Lemma.

The key idea is to argue by descending induction on n to construct the desired tower of semi-stable curve fibrations step by step. At each induction step we use Theorem 5.6 to "resolve" a new layer of our tower (and possibly increasing the set J). Then there are two issues: the first is to lift a group action on a semi-stable modification, and the second is to control the G-torsor condition. We deal with the first issue by invoking the Temkin's uniqueness result for *stable* modification over a normal base ([Tem10, Theorem 1.2]). And Lemma 6.3 allows us to effectively control the torsor condition. Details of this argument are carried over in Section 9. □

A delicate aspect of Theorem 6.6 is the finite presentation condition. Since the ring $\mathcal{O}_K$ is usually not noetherian, we need to be able to control that all our constructions preserve finite presentation condition. This can be done due to a (well-known) lemma below:

Lemma 6.10. Let R be a valuation ring with the fraction field K, and let X be a flat R-scheme locally of finite type. Then

- X is locally finitely presented over R.
- Any K-admissible blow-up X' of X is flat and locally finitely presented R-scheme. In particular, the blow-up morphism $p\colon X' \to X$ is locally of finite presentation.

Proof. The first claim follows from [RG71, Théorème (3.4.1)] (an alternative reference is [Sta21, Tag 053E]) and faithfully flat descent from the henselisation of R. As for the second claim we recall that X is R-flat if and only if $\mathcal{O}_X$ is R-torsion free. Then any K-admissible blow-up $X' := \mathbf{Proj}_X \bigoplus_{i\in\mathbf{N}} \mathcal{I}^n$ of X is again R-flat. Definition 5.4 ensures that $p\colon X' \to X$ is of finite type, thus X' is flat and locally of finite type as an R-scheme. So the first part implies that X' is locally of finite presentation. Therefore p is also locally of finite presentation. □

Remark 6.11. Before we start the proof of Theorem 6.6, we want to point out that it will be crucial for the argument that we do not make any properness or connectedness assumptions in Theorem 5.6. The proof starts by choosing an open cover of X_K by U_i with étale maps to $\mathbf{A}^n_K$. This defines relative curves $U_{i,K} \to \mathbf{A}^{n-1}_K$ and then after suitable modification we end up in the situation where Lemma 6.9 is applicable. But in order for this to work, we cannot impose any properness or geometric connectivity assumptions in Theorem 5.6. So this level of generality is actually important for proving Theorem 6.6.

Now we are finally ready to prove Theorem 6.6.

Proof of Theorem 6.6. We start the proof by noting that if X has a structure of a successive curve fibration over $\mathcal{O}_K$, then Lemma 6.9 constructs K'/K and $g_i\colon X'_i \to X_{\mathcal{O}_{K'}}$ satisfying properties (1), (2), and (3). So we want to reduce the question to the situation when X has a structure of a successive curve fibration over $\mathcal{O}_K$ so that Lemma 6.9 is applicable. Then we separately construct K''/K and $h_i\colon X''_i \to X'_{i,\mathcal{O}_{K''}}$ using [ALPT19, Theorem 5.2.16 and Corollary 5.1.14].

We note that Lemma 6.10 guarantees that for the purpose of the proof we can pass to a finite covering of X by quasi-compact open subsets and K-admissible blow-ups of those. Since the generic fibre X_K is smooth and quasi-compact over $\operatorname{Spec} K$, we can choose a finite covering of X_K by quasi-compact (affine) open subschemes U_i with étale K-morphisms $U_i \to \mathbf{A}_K^{n_i}$ for some n_i. Then [Tem17, Lemma 2.5.1] says that there is a K-admissible blow-up $\pi\colon X' \to X$ and a finite open quasi-compact covering $\cup_{i=1}^m X'_i = X'$ such that $X'_{i,K} \cap X_K = U_i$. We use the observation from the first sentence of this paragraph to replace X with X'_i, so we can assume that the generic fibre of X admits an étale morphism to some $\mathbf{A}_K^d$.

Now we are in the situation that X is finitely presented and flat over $\operatorname{Spec} \mathcal{O}_K$ and its generic fiber admits an étale morphism $f''\colon X_K \to \mathbf{A}_K^d$. Compose it with projection on the first $d-1$ coordinates to get a morphism $f''_d\colon X_K \to \mathbf{A}_K^{d-1}$ that is a smooth relative curve! For technical reasons it will be simpler to consider it as a morphism $f''_d\colon X_K \to (\mathbf{P}_K^1)^{d-1}$, which is also, certainly, a smooth relative curve. The reason is that we want to use [Con07, Theorem 2.4 and Remark 2.5] to find a K-admissible blow-up $\pi\colon X' \to X$ and a morphism $f'_d\colon X' \to (\mathbf{P}_{\mathcal{O}_K}^1)^{d-1}$ extending $f''_d \circ \pi_K$. And we can use the above observation to replace X with X', so we assume that X admits a map to $(\mathbf{P}_K^1)^{d-1}$ that is a smooth relative curve over the generic fibre.

Under the assumption as above, we almost have a structure of a successive K-smooth curve fibration on X. Indeed, we have a map $f'_d\colon X \to (\mathbf{P}_{\mathcal{O}_K}^1)^{d-1}$ over $\mathcal{O}_K$ that over K is a smooth relative curve, and also we have projection morphisms $p_i\colon (\mathbf{P}_{\mathcal{O}_K}^1)^i \to (\mathbf{P}_{\mathcal{O}_K}^1)^{i-1}$ that are K-smooth curve fibrations. The only issue is that f'_d is not necessary flat, but we deal with that issue by invoking [RG71, Corollaire 5.7.10] (or [Sta21, Tag 081R]) to find a K-admissible blow up $r_{d-1}\colon X_{d-1} \to (\mathbf{P}_{\mathcal{O}_K}^1)^{d-1}$ such that the strict transform $f_d\colon X' \to X_{d-1}$ is flat. Then we claim that f_d is automatically a relative curve. Indeed, it is flat and [Sta21, Tag 0D4H] implies that the relative dimension of all fibers is ≤ 1. Now [Sta21, Tag 02FZ] implies that the dimension of all non-empty fibers is ≥ 1 as X'_K is dense in X'. Thus, all non-empty fibers are of pure dimension 1.

Now we consider the map $X_{d-1} \to (\mathbf{P}_{\mathcal{O}_K}^1)^{d-2}$. It is not necessarily flat, but we can find those K-admissible blow-ups step by step (also affecting previous choices of f_i's via base change) to finally come up with a structure of a successive K-smooth curve fibration on a new X'. Then we just replace X with X' as we did above and use Lemma 6.9 to find a finite Galois extension $K \subset K'$ and morphisms $g_i\colon X'_i \to X_{\mathcal{O}_{K'}}$ with the source X'_i being a successive semi-stable K'-smooth curve fibration over $\operatorname{Spec} \mathcal{O}_{K'}$. Now we say that since K-admissible blow-ups are $\mathcal{O}_K$-projective, we get that the final morphisms $X'_i \to X_{\mathcal{O}_{K'}}$ are quasi-projective and they are finitely presented by Lemma 6.10.

The last thing we are left to show is how to construct a finite extension $K' \subset K''$ such that $X'_{i,\mathcal{O}_{K''}}$ admits a K''-modification $h_i\colon X''_i \to X'_{i,\mathcal{O}_{K''}}$ with a K''-smooth polystable X''_i. The usual approximation type argument allows to assume that K is algebraically closed. In this situation, the value group of $\mathcal{O}_K$ is divisible, so the results from [ALPT19] are applicable. Namely, [ALPT19, Theorem 5.2.16 and Corollary 5.1.14] imply that there is a K-modification $h_i\colon X''_i \to X'_i$ with the desired property once we know that there is a vertical[7] log structure on each X'_i making it into a log smooth log variety[8]. However, this structure is constructed Theorem C.3.1. This constructs the desired modification. $\square$

[7]This means that the log structure is trivial on the generic fiber, i.e. $\mathcal{M}_X|_{X_{K'}} \simeq \mathcal{O}_{X_{K'}}^\times$.

[8]Look at Definition C.1.30 and Definition C.2.4 for the precise meaning of these words.

7. Analytic Version of Local Uniformization

The idea for the proof of analytic version of the Local Uniformization result is to reduce the question to the schematic case. We do this by means of local algebraization. For what follows, we fix a complete rank-1 valuation ring K with a valuation ring $\mathcal{O}_K$.

Before starting the proof, we briefly discuss some notions related to formal schemes and their adic generic fibers. We refer the reader to Appendix B for a more detailed discussion. Here we mention only the main notions.

A *formal* $\mathcal{O}_K$*-scheme* will always mean an I-adic formal $\mathcal{O}_K$-scheme for a(ny) ideal of definition $I \subset \mathcal{O}_K$. It is easily seen to be independent of the choice of I. And by the *completion* $\widehat{X}$ of a finite type $\mathcal{O}_K$-scheme X, we always mean the I-adic completion. The same terminology is used in [Bos14]. This abuse of notation is common in p-adic geometry, but might be non-common in the other areas.

A *rigid-analytic space over* K will always mean a locally topologically finite type $(K, \mathcal{O}_K)$-adic space in the sense of [Hub94]. It is not necessary to use this approach to the foundations of non-archimedean geometry for our purposes, but we find it to be more convenient in our subsequent work. The reader, who is more familiar with the classical Tate approach to rigid-analytic geometry (as one used in [Bos14]), may safely use it in what follows. For a precise relation between those two notions, we refer the reader to [Hub94, Proposition 4.5].

Theorem 7.1. Let X be a quasi-compact and quasi-separated smooth rigid-analytic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ with a given admissible quasi-compact formal model $\mathfrak{X}$. Then there is a finite Galois extension $K \subset K'$, a finite extension $K' \subset K''$, a finite number of morphisms of admissible formal $\mathcal{O}_{K'}$-schemes (resp. $\mathcal{O}_{K''}$-schemes) $g_i \colon \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ and $h_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_{i,\mathcal{O}_{K''}}$, such that

- Each $\mathfrak{X}'_i$ admits an action of a finite group G_i such that $g_i \colon \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ is G_i-invariant for each i.
- The morphism $g \colon \mathfrak{X}' := \sqcup_i \mathfrak{X}'_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ is a rig-étale covering (in the sense of Definition B.5).
- On the generic fiber, each $\mathfrak{X}'_i$ becomes a G_i-torsor over its (quasi-compact) open image in the adic generic fiber $X_{K'} = \mathfrak{X}_{K'}$.
- Each $\mathfrak{X}'_i$ is formally quasi-projective over $\mathcal{O}_{K'}$ (in the sense of Definition B.14) and has a structure of a successive formal semi-stable rig-smooth curve fibration (in the sense of Definition B.9).
- Each $h_i \colon \mathfrak{X}''_i \to \mathfrak{X}'_{i,\mathcal{O}_{K''}}$ is a rig-isomorphism, and $\mathfrak{X}''_i$ is rig-smooth, polystable formal $\mathcal{O}_{K''}$-scheme (in the sense of Definition B.12).

Proof. We start by noting that the statement is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X}$ is an affine admissible rig-smooth formal model. We use [Tem17, Theorem 3.1.3] (it essentially boils down to [Elk73, Théoremèm 7 on page 582 and Remarque 2(c) on p.588] and [Tem08, Proposition 3.3.2]) that says that an affine rig-smooth formal scheme $\mathfrak{X}$ can be algebraized to an affine flat finitely presented $\mathcal{O}_K$-scheme Y with smooth generic fibre Y_K. We apply Theorem 6.6 to find a finite Galois extension K'/K, an extension K''/K, and morphisms $g'_i \colon Y'_i \to Y_{\mathcal{O}_{K'}}$, $h'_i \colon Y''_i \to Y'_{\mathcal{O}_{K''}}$ with all the properties from Theorem 6.6. Then we pass to the I-adic completions of those schemes $\widehat{g'_i} \colon \widehat{Y'_i} \to \widehat{Y_{\mathcal{O}_{K'}}}$, $\widehat{h'_i} \colon \widehat{Y''_i} \to \widehat{Y'_{\mathcal{O}_{K''}}}$ and apply [Tem17, Lemma 3.2.2], Lemma B.11, Lemma B.16 and an isomorphism $\widehat{Y_{\mathcal{O}_{K'}}} \cong \mathfrak{X}_{\mathcal{O}_{K'}}$ to get morphisms

$$g_i := \widehat{g'_i} \colon \widehat{Y'_i} \to \mathfrak{X}_{\mathcal{O}_{K'}} \text{ and } h_i := \widehat{h'_i} \colon \widehat{Y''_i} \to \widehat{Y'_{i,\mathcal{O}_{K''}}} = \mathfrak{X}_{\mathcal{O}_{K''}}$$

which satisfy all the properties from the formulation of the Theorem besides formal quasi-projectivity. But it is actually also formally quasi-projective over Spf $\mathcal{O}_{K'}$ because Theorem 6.6 ensures that Y_i is quasi-projective over an affine scheme Y for all i. Thus it is also quasi-projective over Spec $\mathcal{O}_{K'}$. And the completion of a finite type quasi-projective $\mathcal{O}_{K'}$-scheme is a quasi-projective topologically finite type formal $\mathcal{O}_{K'}$-scheme (since the completion along a finitely generated ideal in $\mathcal{O}_K$ preserves the special fibre). □

8. Uniformization by quotients of successive semi-stable curve fibrations

In the section we fix a complete non-archemidean field K. We show that any admissible quasi-compact and quasi-separated formal model of a smooth rigid-analytic space locally admits an uniformization (in a precise sense sense explained below) by a "good" quotient of a successive semi-stable curve fibration by an action of a finite group G. This is morally just a consequence of Theorem 7.1 (or really its proof); the basic idea is that we just pass from each model $\mathfrak{X}'_i$ in the formulation of Theorem 7.1 to a model $\mathfrak{X}'_i/G_i$. Existence of such quotients is systematically worked out in [Zav21b]. We now briefly review the definition of such quotients and the necessary existence results:

Definition 8.1. Let G be a finite group, S a ringed space (resp. topologically ringed space), and X an S-ringed (resp. topologically ringed) spaces with a right S-action of a finite group G. The *geometric quotient* $X/G = (|X/G|, \mathcal{O}_{X/G}, h)$ consists of:

- the topological space $|X/G| \coloneqq |X|/G$ with the quotient topology. We denote by $\pi : |X| \to |X/G|$ the natural projection,
- the sheaf of rings (resp. topological rings) $\mathcal{O}_{X/G} \coloneqq (\pi_*\mathcal{O}_X)^G$,
- the morphism $h : X/G \to S$ defined by the pair $(h, h^\#)$, where $h : |X|/G \to S$ is the unique morphism induced by $f : X \to S$, and $h^\#$ is the natural morphism
$$\mathcal{O}_S \to h_*\left(\mathcal{O}_{X/G}\right) = h_*\left((\pi_*\mathcal{O}_X)^G\right) = \left(h_*\left(\pi_*\mathcal{O}_X\right)\right)^G = \left(h_*\mathcal{O}_X\right)^G$$
that comes from G-invariance of f.

Now suppose that $\mathcal{O}_K$ is a complete rank-1 valuation ring. Then [Zav21b, Theorem 2.2.6] ensures that the geometric quotient of a quasi-projective, flat, locally finite type $\mathcal{O}_K$-scheme X exists as an admissible $\mathcal{O}_K$-scheme and the quotient map $X \to X/G$ is the universal G-invariant map in the category of $\mathcal{O}_K$-schemes. Similarly, [Zav21b, Theorem 3.3.4] guarantees that the geometric quotient of a formally projective, admissible formal $\mathcal{O}_K$-scheme exists as an admissible formal $\mathcal{O}_K$-scheme with the same universal property.

We now define precisely what we mean a "uniformization". Let $\mathfrak{X}$ be an locally topologically finite type formal $\mathcal{O}_K$-scheme, and let $\varphi_i : \mathfrak{X}_i \to \mathfrak{X}$ be a finite set of locally topologically finite type $\mathcal{O}_K$-morphisms.

Definition 8.2. We say that a set $(\mathfrak{X}_i, \varphi_i)_{i\in I}$ can be obtained as a *composition of open Zariski coverings and rig-isomorphisms of* $\mathfrak{X}$, if this set can be achieved in a finite number of steps using the following rules: we start with the set consisting of one morphism $(\mathfrak{X}, \mathrm{Id}_{\mathfrak{X}})$ and at each step we are allowed either to change one element $\varphi_i : \mathfrak{X}_i \to \mathfrak{X}$ by composing with a rig-isomorphism (in the sense of Definition B.4) $\varphi'_i : \mathfrak{X}'_i \to \mathfrak{X}$ or to replace $\varphi_i : X_i \to X$ (and keep other elements the same) by the compositions $\mathfrak{X}_{i,j} \to \mathfrak{X}_i \xrightarrow{\phi_i} \mathfrak{X}$ with $\mathfrak{X}_i = \cup_{j=1}^n \mathfrak{X}_{i,j}$ a Zariski cover.

It also has its schematic counterpart that will play an intermediate step in our proof of the uniformization result:

Definition 8.3. Let $\varphi_i\colon X_i \to X$ be a finite set of morphisms between $\mathcal{O}_K$-schemes. We say that a set $(X_i, \varphi_i)_{i\in I}$ can be obtained as a *composition of open Zariski coverings and K-modifications of X*, if this set can be achieved in a finite number of steps using the following rules: we start with the set consisting of one morphism (X, Id_X) and at each step we are allowed either to change one element $\varphi_i\colon X_i \to X$ by composing with a K-modification (in the sense of Definition 2.8) $\varphi_i'\colon X_i' \to X$ or to replace $\varphi_i\colon X_i \to X$ (and keep other elements the same) by the compositions $X_{i,j} \to X_i \xrightarrow{\phi_i} X$ with $X_i = \cup_{j=1}^n X_{i,j}$ a Zariski cover.

Lemma 8.4. Let $(X_i, \varphi_i)_{i\in I}$ be a finite set that can be obtained from X as a composition of open Zariski coverings and K-modifications. The set of I-adic completions $(\widehat{X_i}, \widehat{\varphi}_i)_{i\in I}$ is obtained from $\mathfrak{X}$ as a composition of open Zariski coverings and rig-isomorphisms of $\widehat{X}$.

Proof. This follows directly from the fact that the completion of a Zariski covering is a Zariski covering, and the completion of a K-modification is a rig-isomorphism (Lemma B.8). ☐

Theorem 8.5. Let $\mathfrak{X}$ be an admissible, quasi-compact and quasi-separated formal $\mathcal{O}_K$-scheme with the smooth generic fiber $\mathfrak{X}_K$. Then there is a finite Galois extension $K \subset K'$, and a finite extension $K' \subset K''$, a finite set $(\mathfrak{X}_i, \varphi_i)_{i\in I}$ of quasi-compact, quasi-separeted admissible formal $\mathcal{O}_{K'}$-schemes with morphisms $\varphi_i\colon \mathfrak{X}_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$ such that

- The set $(\mathfrak{X}_i, \varphi_i)$ can be obtained from $\mathfrak{X}_{\mathcal{O}_{K'}}$ as a composition of open Zariski coverings and rig-isomorphisms.
- Each $\mathfrak{X}_i$ is a geometric quotient of an admissible formal $\mathcal{O}_{K'}$-scheme $\mathfrak{X}_i'$ by an action of a finite group G_i such that the generic fiber of the quotient map $p_{i,K'}\colon \mathfrak{X}_{i,K'}' \to \mathfrak{X}_{i,K'}$ is a G_i-torsor.
- Each of $\mathfrak{X}_i'$ has a structure of a semi-stable, rig-smooth successive curve fibration over $\mathrm{Spf}\,\mathcal{O}_{K'}$.
- Each $\mathfrak{X}_{i,\mathcal{O}_{K''}}'$ admits a rig-isomorphism $h_i\colon \mathfrak{X}_i'' \to \mathfrak{X}_{i,\mathcal{O}_{K''}}'$ with a rig-smooth, polystable formal $\mathcal{O}_{K''}$-scheme $\mathfrak{X}_i''$.

Proof. The construction is very easy, we use Theorem 7.1 to get a finite Galois extension $K \subset K'$, finite extension $K' \subset K''$ (that can be chosen to be Galois if K is perfect), and a finite set of morphisms $g_i\colon \mathfrak{X}_i' \to \mathfrak{X}_{\mathcal{O}_{K'}}$, $h_i\colon \mathfrak{X}_i'' \to \mathfrak{X}_{i,\mathcal{O}_{K''}}'$ with all the properties from the Theorem 7.1. Formal quasi-projectivity of $\mathfrak{X}_i'$ over $\mathrm{Spf}\,\mathcal{O}_{K'}$ shows that we can use [Zav21b, Theorem 3.3.4] to see that the geometric quotient $p_i\colon \mathfrak{X}_i' \to \mathfrak{X}_i'/G_i$ exists as admissible formal $\mathcal{O}_{K'}$-scheme for any $i \in I$. We define

$$\mathfrak{X}_i \coloneqq \mathfrak{X}_i'/G_i$$

as the geometric quotient of $\mathfrak{X}_i'$ by an action of G_i, this naturally comes equipped with a map $\varphi_i\colon \mathfrak{X}_i \to \mathfrak{X}_{\mathcal{O}_{K'}}$.

Recall that [Zav21b, Proposition 5.2.1] implies that each $\mathfrak{X}_i$ is an admissible, quasi-compact and quasi-separated formal $\mathcal{O}_{K'}$-scheme. Moreover, [Zav21b, Theorem 4.4.1] shows that the geometric quotient of the generic fibre $\mathfrak{X}_{i,K'}'/G_i$ exists as adic space topologically finite type over $\mathrm{Spa}(K', \mathcal{O}_{K'})$, and we have a functorial isomorphism $\mathfrak{X}_{i,K'}'/G_i \cong (\mathfrak{X}_i'/G_i)_{K'}$. Theorem 7.1 says that the generic fibre $g_{i,K'}\colon \mathfrak{X}_{i,K'}' \to \mathfrak{X}_{K'}$ induces a G-torsor over some open $U_i \subset \mathfrak{X}_{K'}$. Therefore

$$(\mathfrak{X}_i'/G_i)_{K'} \cong \mathfrak{X}_{i,K'}'/G_i \cong U_i.$$

Thus the U_i-restriction of the map $p_{i,K'}\colon \mathfrak{X}_{i,K'}' \to \mathfrak{X}_{i,K'}$ is a G_i-torsor as well. Thus, the only thing we really need to show is that $(\mathfrak{X}_i, \varphi_i)_{i\in I}$ is obtained from $\mathfrak{X}_{\mathcal{O}_{K'}}$ as a composition of open

Zariski coverings and rig-isomorphisms. We do not know how to see this from the first principles, the only way to prove it that we are aware of is to use the explicit construction of each $\mathfrak{X}'_i$, as follows.

We briefly remind the reader the proof of Theorem 7.1. The first step was to reduce the situation to the affine case, then use Elkik's algebraization theorem to reduce to the schematic case. Since passing to affine coverings does no harm for our purpose (proving that $\mathfrak{X}_i$ can be obtained as a composition of open Zariski coverings and C-modifications) we can assume from the beginning that $\mathfrak{X}$ is affine. Moreover, Lemma 8.4 and Theorem [Zav21b, Theorem 3.4.1] show that it suffices to prove our claim in the schematic case. We change our notations here and denote an algebraization of $\mathfrak{X}$ by X[9].

The next step in the proof of Theorem 7.1 was to reduce the affine schematic version to the situation where X_K (is affine and) has a structure of a successive K-smooth curve fibration. This was done by means of projective K-modifications and Zariski open coverings, they also cause no harm for our purpose. So we can assume that X is an affine flat finitely presented $\mathcal{O}_K$-scheme with a structure of a successive K-smooth curve fibration

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = \operatorname{Spec} \mathcal{O}_K$$

such that each morphism f_i is quasi-projective. Then Lemma 9.4(5) finishes the proof. □

9. Proof of Lemma 6.9

We formulate here a more precise version of Lemma 6.9 and give a detailed proof. Even though the idea of the proof is rather easy, it turns out that it requires some patience to give a rigorous proof.

Proposition 9.1. Let R be a valuation ring with fraction field K. Let S be the affine scheme $\operatorname{Spec} R$. Suppose that $f\colon X \to S$ is a morphism that can be written as a composition

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = S$$

such that each morphism f_i is a relative curve (in the sense of Definition 2.1) that is K-smooth. There is a finite set J, a set of finite groups $(G_j)_{j\in J}$ indexed by J, and commutative diagrams:

$$\begin{array}{ccc}
X_n & \xleftarrow{\;g_{n,j}\;} & X_{n,j} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle h_{n,j}} \\
X_{n-1} & \xleftarrow{\;g_{n-1,j}\;} & X_{n-1,j} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle h_{n-1,j}} \\
\vdots & \xleftarrow{\;g_{k,j}\;} & \vdots \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle h_{2,j}} \\
X_1 & \xleftarrow{\;g_{1,j}\;} & X_{1,j} \\
\downarrow{\scriptstyle f_1} & & \\
\operatorname{Spec} R & &
\end{array}$$

of flat, finitely presented $S \coloneqq \operatorname{Spec} R$-schemes such that

[9]Since we will never use the adic generic fiber of $\mathfrak{X}$ in the rest of the proof, the possible confusion of X with the generic fiber $\mathfrak{X}_K$ will never occur.

(1) $X_{k,j}$ admits an R-action of the group G_j for any $k \geq 1$ and $j \in J$. Moreover, $g_{k,j}$ is G_j-invariant and $h_{k,j}$ is G_j-equivariant for any $j \in J, k = 1, \dots, n$.
(2) The K-restriction $(g_{k,j})_K \colon (X_{k,j})_K \to (X_k)_K$ is a G_j-torsor over its (open) image $V_{k,j}$.
(3) $h_{k,j}$ is a semi-stable K-smooth relative curve for any $j \in J$ and any $k \geq 2$
(4) $g_{k,j}$ is quasi-projective for any $k \geq 1$ and $j \in J$. Moreover, the map $g_k \colon \sqcup_{j\in J} X_{k,j} \to X_k$ is a K-étale covering[10] for any $k \geq 1$.

Proof. We prove the claim by descending induction on $m \leq n$. Namely, we show that for any $n \geq m \geq 1$ there is a finite set J_m, a set of finite groups $\left(G_j^{(m)}\right)_{j\in J_m}$ indexed by J_m, and commutative diagrams:

$$\begin{array}{ccc}
X_n & \xleftarrow{\;g_{n,j}^{(m)}\;} & X_{n,j}^{(m)} \\
\downarrow f_n & & \downarrow h_{n,j}^{(m)} \\
X_{n-1} & \xleftarrow{\;g_{n-1,j}^{(m)}\;} & X_{n-1,j}^{(m)} \\
\downarrow f_{n-1} & & \downarrow h_{n-1,j}^{(m)} \\
\vdots & \xleftarrow{\;g_{k+1,j}^{(m)}\;} & \vdots \\
\downarrow f_{m+1} & & \downarrow h_{m+1,j}^{(m)} \\
X_m & \xleftarrow{\;g_{m,j}^{(m)}\;} & X_{m,j}^{(m)}
\end{array}$$

of flat, finitely presented S-schemes such that

- $X_{k,j}^{(m)}$ admits an R-action of the group $G_j^{(m)}$ for any $k \geq m$ and $j \in J_m$. Moreover, $g_{k,j}^{(m)}$ is $G_j^{(m)}$-invariant and $h_{k,j}^{(m)}$ is $G_j^{(m)}$-equivariant for any $j \in J_m, k \geq m$.
- The K-restriction $(g_{k,j}^{(m)})_K \colon (X_{k,j}^{(m)})_K \to (X_k^{(m)})_K$ is a $G_j^{(m)}$-torsor over its (open) image $V_{k,j}^{(m)}$.
- $h_{k,j}^{(m)}$ is a semi-stable K-smooth relative curve for any $j \in J_m$ and any $k > m$.
- $g_{k,j}^{(m)}$ is quasi-projective for any $k \geq m$ and $j \in J_m$. Moreover, the map $g_k^{(m)} \colon \sqcup_{j\in J_m} X_{k,j}^{(m)} \to X_k$ is a K-étale covering for any $k \geq m$.

Remark 9.2. We use the superscript $^{(m)}$ to denote objects that come from the induction hypothesis. We do it to emphasize that the induction argument does change the objects constructed at the previous step. More precisely, given a tower $(X_{k,j}^{(m)})_{k\geq m}$ the resulting tower "$(X_{k,j}^{(m-1)})_{k\geq m-1}$" will not be an extension of $(X_{k,j}^{(m)})_{k\geq m}$ one layer lower, but rather some other tower built out of $(X_{k,j}^{(m)})_{k\geq m}$ using Theorem 5.6. In particular, the induction step does "enlarge" the sets J_m and groups $G_j^{(m)}$, but it shrinks the opens $V_{k,j}^{(m)}$.

The statement is trivial for $m = n$ (we can take a trivial group and the identity morphism $X_n \to X_n$). In particular, the case $n = 1$ is settled. Now suppose that $n \geq 2$ and we proved the

[10] In particular, each $g_{k,j}$ is finitely presented, see Definition 2.12.

claim for some $m \geq 2$, we deduce the claim for $m-1 \geq 1$ from that. We divide the proof into several steps for the convenience of the reader.

Step 1: We first use the induction hypothesis to find a commutative diagram

$$\begin{array}{ccc}
X_n & \xleftarrow{\quad g^{(m)}_{n,j} \quad} & X^{(m)}_{n,j} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle h^{(m)}_{n,j}} \\
X_{n-1} & \xleftarrow{\quad g^{(m)}_{n-1,j} \quad} & X^{(m)}_{n-1,j} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle h^{(m)}_{n-1,j}} \\
\vdots & \xleftarrow{\quad g^{(m)}_{k,j} \quad} & \vdots \\
\downarrow{\scriptstyle f_{m+1}} & & \downarrow{\scriptstyle h^{(m)}_{m+1,j}} \\
X_m & \xleftarrow{\quad g^{(m)}_{m,j} \quad} & X^{(m)}_{m,j} \\
\downarrow{\scriptstyle f_m} & & \\
X_{m-1} & &
\end{array}$$

with all the desired properties of $g^{(m)}_{k,j}$ and $h^{(m)}_{k,j}$. We now work separately with each $j \in J_m$, so we fix one such j for the rest of the induction argument.

We "modify" f_m. Consider the finitely presented morphism $\alpha_{m,j}\colon X^{(m)}_{m,j} \to X_{m-1}$ obtained as the composition $f_m \circ g^{(m)}_{m,j}$, this morphism is not necessarilly flat, but $(\alpha_{m,j})_K$ is a smooth relative curve. So [RG71, Corollaire 5.7.10] guarantees that there is an K-admissible blow-up $r_j\colon X'_{m-1,j} \to X_{m-1}$ such that the strict transform of $\alpha_{m,j}$ along the map r_j is an S-morphism $\alpha'_{m,j}\colon X'_{m,j} \to X'_{m-1,j}$ that is flat of pure relative dimension 1 and K-smooth. We note that $X'_{m-1,j}$ is flat and finitely presented over S by Lemma 6.10 (so $\alpha'_{m,j}$ is finitely presented), and the map r_j is a projective K-modification by Lemma 5.5. As a strict transform of a K-admissible blow-up is a K-admissible blow-up, we conclude that $r_{m,j}\colon X'_{m,j} \to X^{(m)}_{m,j}$ is a finitely presented, projective K-modification ($r_{m,j}$ is finitely presented since it is an S-map between finitely presented S-schemes).

We note that each $X^{(m)}_{m,j}$ admits a $G^{(m)}_j$-action over $X'_{m-1,j}$. Indeed, the morphism $\alpha_{m,j}$ is $G^{(m)}_j$-invariant for any $j \in J_m$ since $g^{(m)}_{m,j}$ is. Therefore, the strict transform $X'_{m,j}$ admits a $G^{(m)}_j$ such that $\alpha'_{m,j}$ is $G^{(m)}_j$-invariant and $r_{m,j}$ is $G^{(m)}_j$-equivariant for all $j \in J_m$.

We now "modify" f_k for $n \geq k \geq m+1$[11]. We consider the morphisms

$$\alpha_{k,j}\colon X^{(m)}_{k,j} \to X_{m-1} \text{ defined as } f_m \circ g^{(m)}_{m,j} \circ h^{(m)}_{m+1,j} \circ \cdots \circ h^{(m)}_{k,j}$$

and their strict transforms along the map $r_j\colon X'_{m-1,j} \to X_{m-1}$, denoted as $\alpha'_{k,j}\colon X'_{k,j} \to X'_{m-1,j}$. We need to note two things. Firstly, we recall that strict transform is defined as the schematic closure of the K-fiber

$$(X^{(m)}_{k,j} \times_{X_{m-1}} X'_{m-1,j})_K$$

[11]This case does not arise if $m = n$, but this case was already considered above.

in the total fibre product $X^{(m)}_{k,j} \times_{X_{m-1}} X'_{m-1,j}$, so the action of $G^{(m)}_j$ on $X^{(m)}_{k,j}$ defines the evident action on $X'_{k,j}$. Moreover, this shows that $X'_{k,j}$ is flat and finite type over R, so it is finitely presented by Lemma 6.10. Secondly, we note that we can consider $X'_{k,j}$ as the strict transform of $X^{(m)}_{k,j}$ along the K-admissible blow-up $X'_{m,j} \to X^{(m)}_{m,j}$. Thus all the S-maps $X'_{k,j} \to X^{(m)}_{k,j}$ between finitely presented S-schemes are K-admissible blow-ups; in particular, they are projective, finitely presented K-modifications. Moreover, by the assumption all the morphisms $X^{(m)}_{k,j} \to X^{(m)}_{m,j}$ are successive semi-stable curve fibrations, so they are flat.

Therefore, we have natural isomorphisms

$$X'_{k,j} \cong X^{(m)}_{k,j} \times_{X^{(m)}_{m,j}} X'_{m,j}$$

for any $k \geq m$. This implies that all the morphisms

$$h'_{k,j} \colon X'_{k,j} \to X'_{k-1,j}$$

arising from $h^{(m)}_{k,j}$ are semi-stable curve fibrations. As the fiber product, each $X'_{k,j}$ admits an action of $G^{(m)}_j$ such that $r_{k,j}$ and $h'_{k,j}$ are $G^{(m)}_j$-equivariant. In particular, the morphisms $g'_{k,j} \coloneqq g^{(m)}_{k,j} \circ r_{k,j}$ are $G^{(m)}_j$-invariant.

To sum up, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
X_n & \xleftarrow{\;g^{(m)}_{n,j}\;} & X^{(m)}_{n,j} & \xleftarrow{\;r_{n,j}\;} & X'_{n,j} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle h^{(m)}_{n,j}} & & \downarrow{\scriptstyle h'_{n,j}} \\
X_{n-1} & \xleftarrow{\;g^{(m)}_{n-1,j}\;} & X^{(m)}_{n-1,j} & \xleftarrow{\;r_{n-1,j}\;} & X'_{n-1,j} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle h^{(m)}_{n-1,j}} & & \downarrow{\scriptstyle h'_{n-1,j}} \\
\vdots & \xleftarrow{\;g^{(m)}_{k,j}\;} & \vdots & \xleftarrow{\;r_{k,j}\;} & \vdots \\
\downarrow{\scriptstyle f_{m+1}} & & \downarrow{\scriptstyle h^{(m)}_{m+1,j}} & & \downarrow{\scriptstyle h'_{m+1,j}} \\
X_m & \xleftarrow{\;g^{(m)}_{m,j}\;} & X^{(m)}_{m,j} & \xleftarrow{\;r_{m,j}\;} & X'_{m,j} \\
& \searrow{\scriptstyle f_m} & \downarrow{\scriptstyle \alpha_{m,j}} & & \downarrow{\scriptstyle \alpha'_{m,j}} \\
& & X_{m-1} & \xleftarrow{\;r_j\;} & X'_{m-1,j}
\end{array}$$

Here are the relevant properties of this construction:

- All the schemes in this diagram are flat and finitely presented over S.
- $X'_{k,j}$ admits an action of $G^{(m)}_j$ for any $k \geq m$. All the "horizontal" morphisms $r_{k,j}$ are $G^{(m)}_j$-equivariant. In particular, the morphisms $g'_{k,j} = g^{(m)}_{k,j} \circ r_{k,j}$ are $G^{(m)}_j$-invariant.
- for all $k \geq m+1$, $h'_{k,j}$ is $G^{(m)}_j$-equivariant and defines a semi-stable K-smooth curve fibration.

- The K-restriction $\left(g'_{k,j}\right)_K : \left(X'_{k,j}\right)_K \to (X_k)_K$ is a $G_j^{(m)}$-torsor over its image $V_{k,j}^{(m)}$ for any $k \geq m$. That follows from the fact that $r_{k,j}$ is a K-modification for any $k \geq m$.
- For each $k \geq m$, the morphism

$$\bigsqcup_{j \in J_m} g'_{k,j} \colon \bigsqcup_{j \in J_m} X'_{k,j} \to X_k$$

  is a quasi-projective K-étale covering (it is a composition of a K-étale covering and an K-admissible blow-up) for all $k \geq m$.
- The morphism $\alpha'_{m,j} \colon X'_{m,j} \to X'_{m-1,j}$ is a relative curve fibration that is also $G_j^{(m)}$-invariant.

Step 2: We "resolve" each $\alpha'_{m,j}$. We note that the K-restriction of $\alpha'_{m,j}$ is naturally identified with the K-restriction of $\alpha_{m,j}$ that is, in turn, the composition of $g_{m,j}^{(m)}$ and f_m. By assumption $g_{m,j}^{(m)}$ is K-étale and f_m is K-smooth, so $\alpha'_{m,j}$ is K-smooth as well. So we can apply Theorem 5.6 (together with Remark 5.7) to $\alpha'_{m,j}$ in the role of f there to obtain a non-empty finite set I, a set of finite groups $(H_i)_{i \in I}$ indexed by I, and H_i-invariant, quasi-projective K-étale morphisms

$$\psi_{m-1,j,i} \colon X_{m-1,j,i} \to X'_{m-1,j}$$ [12]

and $X_{m-1,j,i}$-stable (projective) K-modifications

$$\beta_{m,j,i} \colon X_{m,j,i} \to X'_{m,j,i} := X'_{m,j} \times_{X'_{m-1,j}} X_{m-1,j,i}$$

such that the diagram

$$\begin{array}{ccccccc}
X_{m,j}^{(m)} & \xleftarrow{r_{m,j}} & X'_{m,j} & \xleftarrow{\psi_{m,j,i}} & X'_{m,j,i} & \xleftarrow{\beta_{m,j,i}} & X_{m,j,i} \\
\downarrow{\scriptstyle \alpha_{m,j}} & & \downarrow{\scriptstyle \alpha'_{m,j}} & \square & \downarrow & \swarrow{\scriptstyle \text{semi-stable curve}} & \\
X_{m-1} & \xleftarrow{r_j} & X'_{m-1,j} & \xleftarrow{\psi_{m-1,j,i}} & X_{m-1,j,i} & &
\end{array} \tag{1}$$

is commutative and satisfies the following properties:

- For any $i \in I, j \in J_m$, $\psi_{m-1,j,i}$ is a composition $c \circ j \circ a$, where a is a projective K-modification, j an open immersion, and c a finite, finitely presented, faithfully flat H_i-invariant morphism, whose K-restriction becomes an H_i-torsor.

  Moreover, the map $\psi_{m-1,j} \colon \bigsqcup_{i \in I} X_{m-1,j,i} \to X'_{m-1,j}$ is a quasi-projective K-étale covering, so the natural morphism

$$\psi_{m-1} \colon \bigsqcup_{j \in J_m, i \in I} X_{m-1,j,i} \to X_{m-1}$$

  is also a quasi-projective K-étale covering.
- The K-restriction $(\psi_{m-1,j,i})_K : (X_{m-1,j,i})_K \to \left(X'_{m-1,j}\right)_K$ is an H_i-torsor over its (open) image.

[12] $X_{m-1,j,i}$ plays the role of W'_i in Theorem 5.6 and we suppress V'_i from there.

We note that all schemes in this diagram are flat and finite type over S, so they are finitely presented by Lemma 6.10.

Now observe that $X'_{m,j,i}$ admits an action of $G_j^{(m)} \times H_i$ via base change (since $\alpha'_{m,j}$ is $G_j^{(m)}$-invariant and $\psi_{m-1,j,i}$ is H_i-invariant), so the morphism $\psi_{m,j,i}$ is $G_j^{(m)}$-equivariant and the morphism $X'_{m,j,i} \to X_{m-1,j,i}$ is H_i-equivariant (see the Cartesian square in (1)). We now apply Lemma 6.3 to the morphisms

$$\begin{array}{ccccc} (X_m)_K & \xleftarrow{(g'_{m,j})_K} & \left(X'_{m,j}\right)_K & \xleftarrow{(\psi_{m,j,i})_K} & \left(X'_{m,j,i}\right)_K \\ & & \Big\downarrow {\scriptstyle (\alpha'_{m,j})_K} & & \Big\downarrow \\ & & \left(X'_{m-1,j}\right)_K & \xleftarrow{(\psi_{m-1,j,i})_K} & (X_{k-1,j_{k-1}})_K \end{array}$$

to obtain that $\left(X'_{m,j,i}\right)_K \to (X_m)_K$ is a $G_j^{(m)} \times H_i$-torsor over its open image $V_{m,j,i}$. Also, note that $\bigsqcup_{i\in I} X'_{m,j,i} \to X_{m,j}^{(m)}$ is a quasi-projective K-étale covering as a "base change over K" of a quasi-projective K-étale covering. Similarly, projectivity of the morphisms $\beta_{m,j,i}$ implies that the map $\bigsqcup_{i\in I} X_{m,j,i} \to X_{m,j}^{(m)}$ is a quasi-projective K-étale covering for any $j \in J_m$.

Now we do the key step: we want to lift the action of $G_j^{(m)} \times H_i$ on $X'_{m,j,i}$ to an action on $X_{m,j,i}$. A priori, there is no reason we should be able to lift this action. We solve this problem by *changing* $X_{m,j,i}$. Temkin proved the remarkable uniqueness statement [Tem10, Theorem 1.2] that over an irreducible *normal* base with generic point η, a *stable* $k(\eta)$-modification of a relative curve is unique. So the idea is to replace $X_{m-1,j,i}$ with the normalization $\widetilde{X}_{m,j,i}$ inside its generic fiber. However, there are two possible issues: the normalization may not be finitely presented over S, and it may not be integral. We resolve the first problem by an approximation argument, and we resolve the second problem by showing that $\widetilde{X}_{m-1,j,i}$ is a disjoint union of open integral, normal subschemes.

We note that the morphism $\widetilde{X}_{m-1,j,i} \to X_{m-1,j,i}$ an isomorphism over K, $\widetilde{X}_{m-1,j,i}$ is S-flat, and the action of H_i on $X_{m-1,j,i}$ lifts to an action on $\widetilde{X}_{m-1,j,i}$. Therefore, the going-down lemma [Mat86, Theorem 9.5] applied to an affine open covering of $\widetilde{X}_{m-1,j,i}$ implies that all generic points of $\widetilde{X}_{m-1,j,i}$ lie inside

$$\left(\widetilde{X}_{m-1,j,i}\right)_K \simeq (X_{m-1,j,i})_K$$

that is K-smooth and noetherian. In particular, $\widetilde{X}_{m-1,j,i}$ has finite number of irreducible components and normal. Therefore, [Sta21, Tag 0357] implies that $\widetilde{X}_{m-1,j,i}$ is a disjoint union of a finite number of normal integral (open) subschemes. Thus, we can apply [Tem10, Corollary 1.3] to all irreducible components of $\widetilde{X}_{m-1,j,i}$ to get that

$$\widetilde{X}_{m,j,i} \coloneqq X_{m,j,i} \times_{X_{m-1,j,i}} \widetilde{X}_{m-1,j,i}$$

is the unique *stable* modification of the relative curve

$$\widetilde{X'}_{m,j,i} \coloneqq X'_{m,j,i} \times_{X_{m-1,j,i}} \widetilde{X}_{m-1,j,i} \ (= X'_{m,j} \times_{X'_{m-1,j}} \widetilde{X}_{m-1,j,i}).$$

This allows us to lift the action of $G_j^{(m)} \times H_i$ on $\widetilde{X'}_{m,j,i}$ to an action on $\widetilde{X}_{m,j,i}$ for free[13]. Namely, the uniqueness result allows us to lift the action of each $g \in G_j^{(m)} \times H_i$ separately, and S-flatness of

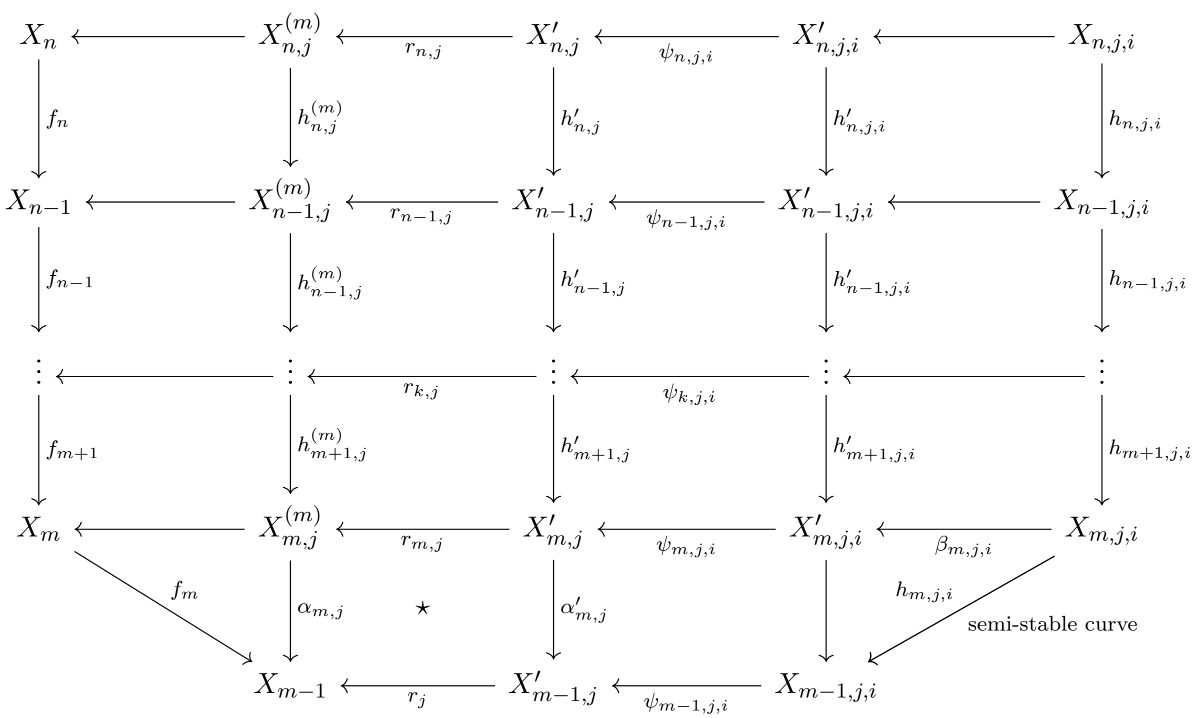
[13]We define action of $G_j^{(m)} \times H_i$ on $\widetilde{X'}_{m,j,i}$ via base change.

$\widetilde{X}_{m,j,i}$ and separatedness of $\widetilde{\beta}_{m,j,i} \colon \widetilde{X}_{m,j,i} \to \widetilde{X'}_{m,j,i}$ guarantee that this procedure defines an action of $G_j^{(m)} \times H_i$ and that $\widetilde{\beta}_{m,j,i}$ is automatically $G_j^{(m)} \times H_i$-equivariant. Clearly, its K-fiber is a torsor over the same

$$V_{m,j,i} \subset (X_{m-1,j,i})_K \simeq (\widetilde{X}_{m-1,j,i})_K.$$

Now we point out that in this non-noetherian situation, the normalization maps may not be finite, so $\widetilde{X}_{m-1,j,i}$ (and hence $\widetilde{X}_{m,j,i}$) may not be finite type over S. But we note that the morphism $\widetilde{X}_{m,j,i} \to X_{m,j,i}$ is at least integral, so it corresponds to a quasi-coherent $\mathcal{O}_{X_{m,j,i}}$-algebra $\mathcal{A}$ integral over $\mathcal{O}_{X_{m,j,i}}$. [GD71, Corollaire 6.9.15] gives that $\mathcal{A}$ is a filtered colimit of *finite* quasi-coherent $\mathcal{O}_{X_{m,j,i}}$-subalgebras $\mathcal{A}_i$. Note that all these $\mathcal{O}_{X_{m,j,i}}$-algebras are S-flat as they are torsion-free, so Lemma 6.10 implies that they are finite, finitely presented. In other words, $\widetilde{X}_{m,j,i}$ is a filtered colimit of finite, finitely presented, K-modifications $X^{\lambda}_{m,j,i}$ of $X_{m,j,i}$ with affine transition morphisms. Therefore, a standard approximation argument shows that the action $G_j^{(m)} \times H_i$ on $X'_{m,j,i}$ over S lifts to an action on some $X^{\lambda}_{m,j,i}$ that we may rename as $X_{m,j,i}$ for our purposes.

Step 3: The induction step. We extend our tower one layer lower. We roughly perform the base change of the initial tower $X_{k,j}^{(m)}$ along the maps $X_{m,j,i} \to X_{m,j}^{(m)}$ to get the desired result. Let us do everything carefully. We consider the following commutative diagram:



where all the squares except for the left column and the "starred" square are Cartesian[14]. We note all the schemes $X'_{k,j,i}$ and $X_{k,j,i}$ are flat, finitely presented schemes over S by construction (we use here that all f_k are flat). Moreover, we recall that all the morphisms $h'_{k,j}$ are $G_j^{(m)}$-equivariant, and the morphism $\psi_{m,j,i}$ is $G_j^{(m)}$-equivariant and H_i-invariant. This allows us to define via base change

[14]The maps $h_{k,j,i}$ along the right side are defined as the base change of $h'_{k,j,i}$ for $k > m$.

an action of $G_j^{(m)} \times H_i$ over S on each $X'_{k,j,i}$ such that all the maps $h'_{k,j,i}$ are $G_j^{(m)} \times H_i$-equivariant and the maps $\psi_{k,j,i}$ are $G_j^{(m)}$-equivariant and H_i-invariant.

The same construction defines an action of $G_j^{(m)} \times H_i$ over S on each $X_{k,j,i}$ (using that we already have an action on $X_{m,j,i}$) so that all the maps $\beta_{k,j,i}$ and $h_{k,j,i}$ are $G_j^{(m)} \times H_i$-equivariant. As the map $X_{k,j,i} \to X'_{k,j}$ is the base change of the quasi-projective K-étale map $X_{m,j,i} \to X'_{m,j}$, we conclude that the morphisms $X_{k,j,i} \to X'_{k,j}$ are quasi-projective K-étale morphisms. And the map

$$\bigsqcup_{j \in J_m, i \in I} X_{k,j,i} \to X_k$$

is a quasi-projective K-étale covering for any $k \geq m-1$. We also note that $h_{k,j,i}$ is a semi-stable, K-smooth curve fibration for any $k \geq m$. Finally, we observe that, for $k \geq m+1$, the K-restriction $(X_{k,j,i})_K \to (X_k)_K$ is a $G_j^{(m)} \times H_i$-torsor over its open image $V_{k,j,i}$ as it is a base change of the map $X_{m,j,i} \to X_m$ that shares the same property.

We are now ready to define the set J_{m-1} and all other corresponding objects that we need in the inductive step for $m-1$. Namely, we define J_{m-1} to be the direct product $J_m \times I$, and the set of finite groups $\left\{G_{j'}^{(m-1)}\right\}_{j' \in J_{m-1}}$ to be the set of products $\left\{G_j^{(m)} \times H_i\right\}_{j \in J_m, i \in I}$. We also define a flat, finitely presented S-scheme $X_{k,j'}^{(m-1)} := X_{k,j,i}$ where $j' = (j,i)$ for any $k \geq m-1$ and $j' \in J_{m-1}$. The maps $g_{k,j'}^{(m-1)} \colon X_{k,j'}^{(m-1)} \to X_k$ are the maps $X_{k,j,i} \to X_k$ in the diagram above, and the maps $h_{k,j'}^{(m-1)} \colon X_{k,j'}^{(m-1)} \to X_{k-1,j'}^{(m-1)}$ are defined as corresponding maps $h_{k,j,i}$ for $j' = (j,i)$ and any $k \geq m$. The discussion above shows that the diagrams (indexed by $j' \in J_{m-1}$):

$$\begin{array}{ccc}
X_n & \xleftarrow[g_{n,j'}^{(m-1)}]{} & X_{n,j'}^{(m-1)} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle h_{n,j'}^{(m-1)}} \\
X_{n-1} & \xleftarrow[g_{n-1,j'}^{(m-1)}]{} & X_{n-1,j'}^{(m-1)} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle h_{n-1,j'}^{(m-1)}} \\
\vdots & \xleftarrow[g_{k+1,j'}^{(m-1)}]{} & \vdots \\
\downarrow{\scriptstyle f_{m+1}} & & \downarrow{\scriptstyle h_{m+1,j'}^{(m-1)}} \\
X_m & \xleftarrow[g_{m,j'}^{(m-1)}]{} & X_{m,j'}^{(m-1)} \\
\downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle h_{m,j'}^{(m-1)}} \\
X_{m-1} & \xleftarrow[g_{m-1,j'}^{(m-1)}]{} & X_{m-1,j'}^{(m-1)}
\end{array}$$

satisfy all the requirements. This finishes the induction argument. □

Corollary 9.3. Under the notation of Proposition 9.1, suppose that each f_i is quasi-projective (so each $X_{n,j}$ is quasi-projective over R, too). Then the geometric quotient $X_{n,j}/G_j$ exists as a flat, finitely presented S-scheme for any $j \in J$. We define $\overline{g_{n,j}} \colon X_{n,j}/G_j \to X_n$ as the map induced by $g_{n,j}$ for any $j \in J$. Then we can choose $J, G_j, X_{n,j}, \ldots$ in a way that the set $(X_{n,j}/G_j,\ \overline{g_{n,j}})_{j \in J}$

can be obtained from X_n as a composition of Zariski open coverings and K-modifications (in the sense of Definition 8.3).

Proof. Since $X_{n,j}$ is quasi-projective over R, we see by [Zav21b, Theorem 2.2.6 and Proposition 5.1.1] that $X_{n,j}/G_j$ exists as a flat and finitely presented S-scheme for any $j \in J$.

Now we show that the schemes $X_{n,j}$ constructed in the *proof* of Proposition 9.1 satisfy the property that $(X_{n,j}/G_j, \overline{g_{n,j}})_{j\in J}$ can be obtained from X_n as a composition of Zariski open coverings and K-modifications. Throughout the proof we use the same notation as in the proof of Proposition 9.1; we refer to Steps 2 and 3 therein for the construction of $X^{(m)}_{k,j}$.

We show the claim by descending induction on $n \geq m \geq 1$. Namely, we start the induction by declaring $J_n \coloneqq \{\varnothing\}, G_\varnothing \coloneqq \{e\}$ and the map $X^{(n)}_{n,\varnothing} \to X_n$ to be the identity map $X_n \to X_n$. This clearly satisfies the condition that $X^{(n)}_{n,\varnothing}/G_\varnothing$ can be obtained from X_n as a composition of Zariski open coverings and K-modifications. The claim now is that the induction argument from the proof of Proposition 9.1 that builds up $\left\{X^{(m-1)}_{n,j'}\right\}$ out of $\left\{X^{(m)}_{n,j}\right\}$ preserves this property. We suppose that this property holds for some $n \geq m > 1$, and show it for $m-1$.

The induction argument constructs the schemes $X^{(m-1)}_{n,j'}$ for $j' \in J_{m-1} = J_m \times I$ in a rather specific way. We are going to use this explicit construction. First of all, we note that each $X^{(m-1)}_{n,j'}$ is a quasi-projective, finitely presented, flat S-scheme. Thus, [Zav21b, Theorem 2.2.6 and Proposition 5.1.1] imply that $X^{(m-1)}_{n,j'}/G^{(m-1)}_{j'}$ exists as a flat and finitely presented S-scheme. Moreover, the $G^{(m-1)}_{j'}$-invariant map $g^{(m-1)}_{n,j'}\colon X^{(m-1)}_{n,j'} \to X_n$ induces a map

$$\overline{g^{(m-1)}_{n,j'}}\colon X^{(m-1)}_{n,j'}/G^{(m-1)}_{j'} \to X_n$$

We claim that that the set $\left(X^{(m-1)}_{n,j'}/G^{(m-1)}_{j'}, \overline{g^{(m-1)}_{n,j'}}\right)_{j'\in J_{m-1}}$ can be obtained from X_n as a composition of K-modifications and open Zariski coverings (in the sense of Definition 8.3). We take a closer look at the construction of this morphism in Steps 2 and 3.

Firstly, we recall that $G^{(m-1)}_j = G^{(m)}_j \times H_i$, and the morphism $\psi_{m-1,j,i}\colon X_{m-1,j,i} \to X'_{m-1,j}$ was obtained as a composition of a (projective) K-modification followed by an open immersion, followed by a finite, finitely presented H_i-invariant morphism that becomes an H_i-torsor on generic fibers[15]. So we decompose the morphism $\psi_{m-1,j,i}$ as follows:

$$X'_{m-1,j} \xleftarrow{a} X''_{m-1,j} \xleftarrow{j} U_{m-1,j,i} \xleftarrow{c} X_{m-1,j,i}$$

where a is a projective K-modification, j is an open immersion, and c is a finite, finitely presented H_i-invariant morphism c that becomes an H_i-torsor over the generic fiber $(U_{m-1,j,i})_K$. Now we draw the diagram that defines the scheme $X^{(m-1)}_{n,j'} = X_{n,j,i}$ for $j' = (j,i)$:

[15]We note that last map was obtained as a composition of a faithfully flat, finitely presented morphism and a finitely presented approximation of the normalization in the generic fiber.

$$
\begin{array}{ccccccccccc}
X_{n,j}^{(m)} & \xleftarrow{r_{n,j}} & X'_{n,j} & \xleftarrow{a_n} & X''_{n,j,i} & \xleftarrow{j_n} & U_{n,j,i} & \xleftarrow{c_n} & X'_{n,j,i} & \xleftarrow{\beta_{n,j,i}} & X_{n,j,i} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{m,j}^{(m)} & \xleftarrow{r_{m,j}} & X'_{m,j} & \xleftarrow{a_m} & X''_{m,j,i} & \xleftarrow{j_m} & U_{m,j,i} & \xleftarrow{c_m} & X'_{m,j,i} & \xleftarrow{\beta_{m,j,i}} & X_{m,j,i} \\
\downarrow{\scriptstyle \alpha_{m,j}} & \star & \downarrow{\scriptstyle \alpha'_{m,j}} & & \downarrow & & \downarrow & & \downarrow & \swarrow{\scriptstyle \text{semi-stable curve}} & \\
X_{m-1} & \xleftarrow{r} & X'_{m-1,j} & \xleftarrow{a} & X''_{m-1,j} & \xleftarrow{j} & U_{m-1,j,i} & \xleftarrow{c} & X_{m-1,j,i} & &
\end{array}
$$

where all the squares except for the "starred" one are Cartesian. All the schemes in this diagram are quasi-projective, flat and finitely presented over S. Thus, [Zav21b, Theorem 2.2.6 and Proposition 5.1.1] guarantees that the geometric quotient of any of those schemes for the action over S by a finite group exists as a flat, finitely presented S-scheme. The strategy now is to use this diagram to decompose the map $X_{n,j,i}/\left(G_j^{(m)} \times H_i\right) \to X_n$ into a composition of certain maps that we can understand.

We recall that $\alpha'_{m,j}$ is $G_j^{(m)}$-invariant, so $X'_{m,j} \to X'_{m-1,j}$ is $G_j^{(m)}$-invariant as well. Thus, $X''_{n,j,i}$ and $U_{n,j,i}$ admit an action of $G_j^{(m)}$ over S via base change so that a_n and j_n are $G_j^{(m)}$-equivariant. Now $X_{m-1,j,i}$ admits an action of H_i so that c is H_i-invariant (and an H_i-torsor over K). Hence, $X'_{n,j,i}$ similarly admits an action of $G_j^{(m)} \times H_i$ over S via base change. The morphism c_n is $G_j^{(m)}$-equivariant and H_i-invariant. The action of $G_j^{(m)} \times H_i$ on $X'_{n,j,i}$ lifts to an action on $X_{n,j,i}$ over S so that $\beta_{n,j,i}$ is equivariant. This is shown in the last paragraph of the proof of Step 2 of Proposition 9.1[16].

The map j_n is a $G_j^{(m)}$-stable open immersion; moreover, $\{U_{n,j,i}\}_{i \in I}$ forms a covering of $X''_{n,j}$ for any $j \in J_m$. Finally, the maps c_n and $\beta_{n,j,i}$ are both $G_j^{(m)} \times H_i$-equivariant K-modifications. The induction hypothesis implies that the set

$$\left(X_{n,j}^{(m)}/G_j^{(m)}, \overline{g_j^{(m)}}\right)_{j \in J_m} \tag{2}$$

is obtained from X_n as a composition of K-modification and Zariski open coverings.

We note that [Zav21b, Proposition 5.1.2] shows that $X''_{n,j}/G_j^{(m)} \to X_{n,j}^{(m)}/G_j^{(m)}$ is a K-modification. The construction of the geometric quotients in [Zav21b, Definition 2.1.1] implies that $U_{n,j,i}/G_j^{(m)} \subset X''_{n,j}/G_j^{(m)}$ is a Zariski open subscheme, and the union (indexed by I) of those open subsets is a covering of $X_{n,j}^{(m)}/G_j^{(m)}$. Now we note that

$$X'_{n,j,i}/\left(G_j^{(m)} \times H_i\right) = \left(X'_{n,j,i}/H_i\right)/G_j^{(m)}$$

[16]It is shown there for $m = n$, one can then define the action on $X_{n,j,i}$ via base change.

Therefore, [Zav21b, Proposition 5.1.2] shows that the map $X'_{n,j,i}/H_i \to U_{n,j,i}$ is a K-modification (as finite morphisms are proper), and *loc. cit* shows that the map

$$X'_{n,j,i}/\left(G_j^{(m)} \times H_i\right) = \left(X'_{n,j,i}/H_i\right)/G_j^{(m)} \to U_{n,j,i}/G_j^{(m)}$$

is a K-modification. Finally, we use *loc. cit.* once again to conclude that the map $X_{n,j,i}/\left(G_j^{(m)} \times H_i\right) \to X'_{n,j,i}/\left(G_j^{(m)} \times H_i\right)$ is a K-modification. Combining all these observations and the descending inductive hypothesis with (2), we conclude that the set

$$\left(X_{n,j'}^{(m-1)}/G_{j'}^{(m-1)}, \overline{g_{n,j'}^{(m-1)}}\right)_{j' \in J_{m-1}} = \left(X_{n,j,i}/\left(G_j^{(m)} \times H_i\right), \overline{g_{n,j,i}}\right)_{j \in J_m, i \in I}$$

can be obtained from X_n as a composition of open Zariski coverings and K-modifications. □

Lemma 9.4. Let R be a valuation ring with fraction field K. Let S be the affine scheme $\operatorname{Spec} R$. Suppose that $f\colon X \to S$ is a morphism that can be written as a composition

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = S$$

such that each morphism f_i is a relative curve (in the sense of Definition 2.1) that is K-smooth. Then there is a finite Galois extension $K \subset K'$ with R' the normalization of R in K', a finite non-empty set J, a set of finite groups $(G_j)_{j\in J}$ indexed by J, and commutative diagrams:

$$\begin{array}{ccc}
X_{n,R'} & \xleftarrow{\;g_{n,j}\;} & X'_{n,j} \\
\downarrow{\scriptstyle f_{n,R'}} & & \downarrow{\scriptstyle h_{n,j}} \\
X_{n-1,R'} & \xleftarrow{\;g_{n-1,j}\;} & X'_{n-1,j} \\
\downarrow{\scriptstyle f_{n-1,R'}} & & \downarrow{\scriptstyle h_{n-1,j}} \\
\vdots & \xleftarrow{\;g_{k,j}\;} & \vdots \\
\downarrow{\scriptstyle f_{2,R'}} & & \downarrow{\scriptstyle h_{2,j}} \\
X_{1,R'} & \xleftarrow{\;g_{1,j}\;} & X'_{1,j} \\
\downarrow{\scriptstyle f_{1,R'}} & \swarrow{\scriptstyle h_{1,j}} & \\
\operatorname{Spec} R' & &
\end{array}$$

of flat, finitely presented $S' := \operatorname{Spec} R'$-schemes such that

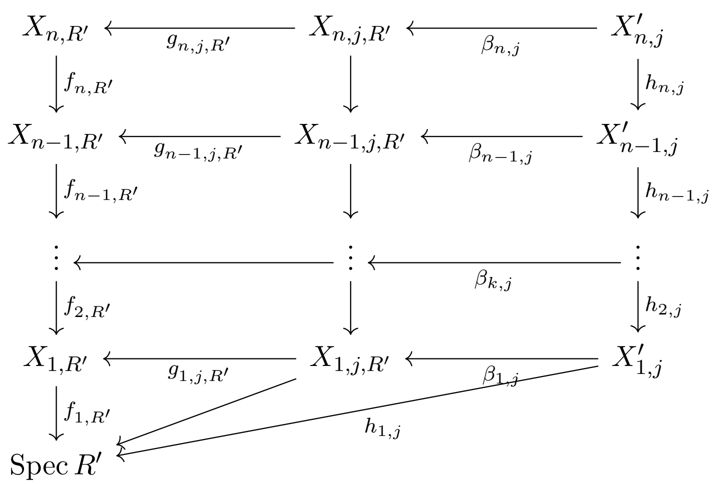
(1) $X'_{k,j}$ admits an R'-action of the group G_j for any $k \geq 1$ and $j \in J$. Moreover, $g_{k,j}$ is G_j-invariant and $h_{k,j}$ is G_j-equivariant for any $j \in J, k = 1, \dots, n$.
(2) The K'-restriction $(g_{k,j})_{K'}\colon (X'_{k,j})_{K'} \to (X_k)_{K'}$ is a G_j-torsor over its (open) image $V_{k,j}$ for $k \geq 1$.
(3) $h_{k,j}$ is a semi-stable, K'-smooth relative curve for any $j \in J$ and any $k \geq 1$
(4) $g_{k,j}$ is quasi-projective for any $k \geq 1$ and $j \in J$. Moreover, the map $g_k\colon \sqcup_{j\in J} X'_{k,j} \to X_{k,R'}$ is a K'-étale covering for any $k \geq 1$.

(5) Assume that K is a henselian rank-1 valued field[17] (in the sense of Definition 6.4) and f_i is quasi-projective for every $i \geq 1$. Then the geometric quotient $X'_{n,j}/G_j$ exists as a flat, finitely presented S'-scheme for any $j \in J$. Moreover, we can choose $X'_{n,j}$ so that the set $\left(X'_{n,j}/G_j,\ \overline{g_{n,j}}\right)_{j\in J}$ can be obtained from $X_{n,R'}$ as a composition of Zariski open coverings and K-modifications (in the sense of Definition 8.3).

Proof. The proof is similar to Proposition 9.1, but one needs to be careful as the extension $R \to R'$ may not be finite. Instead we pick $J,\ G_j,\ X_{j,k} \xrightarrow{g_{j,k}} X_j$ as in Proposition 9.1 (or as in Corollary 9.3 in the case of henselian K and quasi-projective f_i). In particular, $X_{k,j} \to X_{k-1,j}$ is a semi-stable K-smooth relative curve for $k \geq 2$, $j \in J$.

We note that $X_{1,j}$ is automatically a K-smooth S-curve as it is S-flat and finitely presented by construction. Thus, we can use Theorem 3.7 to get a finite Galois extension $K \subset K'$ with R' the integral closure of R in K' such that the R'-scheme $X_{1,j,R'}$ admits a R'-stable K-modification $\beta_{1,j}\colon X'_{1,j} \to X_{1,j,R'}$ for any $j \in J$. We note that Theorem 3.7 a priori only gives us such an extension K'_j separately for each $j \in J$, but then we can find some finite Galois extension $K \subset K'$ that dominates all K'_j. So this field can be chosen independently of $j \in J$.

So we have commutative diagrams (indexed by J):



where each right square is Cartesian[18]. We note that $h_{k,j}$ is a relative semi-stable curve for each $k \geq 1$. Moreover, [Tem10, Theorem 1.1] implies that $\beta_{1,j}$ is a projective K'-modification, so the same holds for $\beta_{k,j}$ for any $k \geq 1$. So if we define $g_{k,j}\colon X'_{k,j} \to X_{k,R'}$ as the composition $g_{k,j,R'} \circ \beta_{k,j}$, then we see that each of those $g_{k,j}$ is quasi-projective and the total map

$$g_k\colon \bigsqcup_{j\in J} X'_{k,j} \to X_{k,R'}$$

is a quasi-projective K'-étale covering for any $k \geq 1$. As for the group action, we use the uniqueness result [Tem10, Theorem 1.2] to lift the R'-action of G_j on $X_{1,j,R'}$ to an R'-action of G_j on $X'_{1,j}$.

[17] The results holds true if one only assumes that R is a henselian valuation ring. The rank-1 assumption is only used to show that the integral closure of R in an algebraic field extension L/K is a valuation ring. This can be proven without the rank-1 assumption using [Bou98, Ch. 6, §8, n.6, Proposition 6] and [Sta21, Tag 09XI].

[18] We define $h_{k,j}$ as the base change of $X_{k,j,R'} \to X_{k-1,j,R'}$ for $k \geq 2$. Similarly, for $k \geq 2$, $\beta_{k,j}$ are defined as the base change of $\beta_{1,j}$.

Since $\beta_{1,j}$ is G_j-equivariant and all the vertical maps in the middle column are G_j-equivariant, we obtain a canonical R'-action of G_j on each $X'_{k,j}$ making morphisms $\beta_{k,j}$ and $h_{k,j}$ be equivariant for any $k \geq 1$. Then it is easy to see that each $g_{k,j}$ is G_j-invariant and its K'-restriction $(g_{k,j})_{K'}$ becomes a G_j torsor over its open image.

Now we go to the situation of a henselian rank-1 valued field K and quasi-projective f_i. We note that Lemma 6.5 implies that the normalization R' is a valuation ring. So we can use [Zav21b, Theorem 2.2.7 and Proposition 5.1.1] to conclude that $X'_{n,j}/G_j$ exists as a flat, finitely presented R'-scheme. Since the formation of the geometric quotient commutes with flat base change [Zav21b, Theorem 2.1.16], we use Corollary 9.3 to see that the set $\left(X_{n,j,R'}/G_j, \overline{g_{n,j,R'}}\right)$ can be obtained from $X_{n,R'}$ as a composition of Zariski open coverings and K'-modifications.

Now we note that [Zav21b, Proposition 5.1.2] gives that the map $X'_{n,j}/G_j \to X_{n,j,R'}/G_j$ is a K'-modification for any $j \in J$. This, in turn, implies that the set $\left(X'_{n,j}/G_j, \overline{g_{n,j}}\right)$ can be obtained from $X_{n,R'}$ as a composition of Zariski open coverings and K'-modifications. This finishes the proof. □

Appendix

Appendix A. Approximation Techniques

Lemma A.1. Let R be a ring, let G be a finite group, and let A be an R-algebra with an R-algebra action of G. Suppose that an R-algebra $A = \mathrm{colim}_{i \in I} A_i$ if a filtered colimit of R-algebras A_i with the following properties

- Each A_i is a finite type R-subalgebra of A.
- Each finite type R-subalgebra $B \subset A$, which contains A_i for some i, is equal to A_j for some other j.

The subsystem $J = \{i \in I \mid A_i \text{ is } G\text{-stable subalgebra of } A\}$ is filtered and $A = \mathrm{colim}_{j \in J} A_j$.

Proof. It suffices to show that for any $i \in I$ there is $j \in I$ such that $A_i \subset A_j$, and A_j is G-stable subalgebra of A. Pick some R-algebra generators $\{h_\alpha\}_{\alpha \in T}$ of A_i for some finite set T. And now define B as an R-subalgebra of A generated by $g(h_\alpha)$ for all $g \in G$ and $\alpha \in T$. Since G acts on A by R-algebra automorphisms, we conclude that B is a G-stable R-algebra containing A_i. Moreover, this is still a finite type R-algebra because the group G is finite. Thus our assumption implies that $B = A_j$ for some $j \in J$. This proves the claim. □

Lemma A.2. Let S be a quasi-compact quasi-separated scheme and $U \subset S$ be a schematically dense quasi-compact open subscheme of S. Then there is a filtered system of finite type $\mathbf{Z}$-schemes S_i with affine transition maps $v_{i,j} \colon S_i \to S_j$, and schematically dense open subschemes $U_i \subset S_i$ such that $S \cong \lim_{i \in I} S_i$ and $U \cong \lim_{i \in I} U_i$. Moreover, if S is integral and normal, then we can assume that all S_i are integral and normal as well.

Proof. We use [TT90, Theorem C.9] to write $S = \lim_{i \in I} S_i$ for some filtered limit of finitely presented $\mathbf{Z}$-scheme S_i with affine schematically dominant transition maps $v_{i,j} \colon S_i \to S_j$. Note that this formally implies that $v_i \colon S \to S_i$ is schematically dominant. Now [Gro66, Proposition 8.6.3] guarantees that U comes as a base change of some (quasi-compact) open $U_i \subset S_i$ for some large i. Define $U_j \coloneqq v_{ji}^{-1}(U_i)$ and consider a filtered subsystem I' of indices at least i. Then $S = \lim_{i \in I'} S_i$ and $U = \lim_{i \in I'} U_i$ (in the last equality we use that fiber product commutes with filtered limits).

The only thing we are left to check is that morphisms $U_i \to S_i$ are schematically dense for all $i \in I'$. Since the morphism $U_i \to S_i$ is quasi-compact and quasi-separated (and the same for $U \to S$), the notion of schematic density is Zariski-local on S_i. So we can assume that each S_i is affine; this also implies that S is affine as well. In this situation, the schematic density of U_i is equivalent to injectivity of the map $\gamma_i \colon \mathcal{O}_{S_i}(S_i) \to \mathcal{O}_{U_i}(U_i)$. Recall that we have chosen S_i such that $S \to S_i$ is schematically dominant for all i. This implies that we have the following commutative square

$$\begin{array}{ccc} \mathcal{O}_{S_i}(S_i) & \xrightarrow{\gamma_i} & \mathcal{O}_{U_i}(U_i) \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow \\ \mathcal{O}_S(S) & \xrightarrow{\beta} & \mathcal{O}_U(U), \end{array}$$

where morphisms α_i and β are injective. This implies that γ_i is also injective.

The last thing we are left to address is the fact that we can choose all S_i to be integral and normal if S is integral and normal. Note that since $S \to S_i$ is schematically dominant, we can conclude that each S_i must be integral if S is. As for normality we just recall that any finite type $\mathbf{Z}$-scheme is excellent by [Gro65, Scholie 7.8.3 (iii)] and [Gro65, Proposition 7.8.6(i)], and a normalization of an excellent scheme is finite by [Gro65, Proposition 7.8.6(ii)]. Thus the normalization S_i' of S_i is a finite type, normal $\mathbf{Z}$-scheme for all i. The schematic dominance of the morphisms $S \to S_i$ implies that the map $S \to S_i$ factors as $S \to S_i' \to S_i$ with S_i' being the normalization of S_i. This defines the natural map $S \to \lim_{i\in I} S_i'$. This map is an isomorphism as this can be checked affine locally, where it follows from normality of S. Thus, S is a limit of finite type integral normal $\mathbf{Z}$-schemes with affine transition maps (and we set $U_i' := U_i \times_{S_i} S_i'$). □

Lemma A.3. Let $\{S_i\}_{i\in I}$ be a filtered system of quasi-compact quasi-separated schemes with affine transition maps $v_{i,j} \colon S_i \to S_j$, and let $f_i \colon X_i \to S_i$ be a system of finitely presented S_i-schemes so that compatibly $X_i \cong X_j \times_{S_j} S_i$ for all $i \geq j$. Denote $\lim_{i\in I} S_i$ by S and $\lim_{i\in I} X_i$ by X. Suppose that X is a relative S-curve (resp. semi-stable S-curve, resp. smooth S-curve). Then there is $i \in I$ such that for all $j \geq i$, X_j is also a relative S-curve (resp. semi-stable S-curve, resp. smooth S-curve).

Proof. We firstly use [Gro66, Théorème 11.2.6] to achieve that X_i is flat over S_i for all large i. Now we want to show that that X_i is of pure relative dimension 1 for all large i. In order to achieve this we use [Gro66, Proposition 9.9.1(i)] to say that the locus

$$x \in X, \text{ such that} \dim_x f^{-1}(f(x)) = 1$$

is constructible for any finitely presented morphism $f \colon X \to Y$ with quasi-compact and quasi-separated scheme Y. We apply it to $f_i \colon X_i \to S_i$ to see that the locus of relative dimension 1 is constructible in X_i for all i. Denote this locus by $E_i \subset X_i$ and denote the same locus in X by E, then it is clear that $E_i = u_{i,j}^{-1}(E_j)$ and $E = u_i^{-1}(E_i)$, where $u_{i,j} \colon X_i \to X_j$ are transition morphisms and $u_i \colon X \to X_i$ is natural projections. By assumption we have that $E = X$, so [Gro66, Corollaire 8.3.4] implies that for all large i we have $E_i = X_i$. This means that $f_i \colon X_i \to S_i$ are actually relative curves for all large i.

The last part of the lemma is to show that if X is a semi-stable S-curve (resp. smooth S-curve), then so is X_i for all large i. We have already seen that X_i is a relative curve for all large i. So, the only thing we need to check is that it becomes semi-stable (resp. smooth) for all large i. The rest of the argument is basically the same (instead of [Gro66, Proposition 9.9.1(i)] we use Lemma 2.5 to get the constructibility result)(resp. [Gro67, Proposition 17.7.11]) once we noticed that the formation of semi-stable (smooth) locus commutes with arbitrary base change. The latter claim is clear from definition. □

In the following two lemmas we put an assumption on schematic density of U_i just to guarantees that Definition 2.8 of (stable) U-modification makes sense. Really one can prove a version of these lemmas without this assumption, if one is willing to give a more general definition of (stable) U-modification (without any restrictions on U).

Lemma A.4. Let $\{(S_i, U_i)\}_{i\in I}$ be a filtered system of quasi-compact quasi-separated schemes S_i, and open schematically dense quasi-compact subschemes $U_i \subset S_i$ with affine transition maps $v_{i,j}\colon S_i \to S_j$ such that $v_{i,j}^{-1}(U_j) = U_i$. And let $g_i\colon X_i' \to X_i$ be a system of S_i-morphisms between finitely presented S_i-schemes such that $X_j \times_{S_j} S_i \cong X_i$, $X_j' \times_{S_j} S_i \cong X_i'$ and under these morphisms we have $(g_j)_{S_i} = g_i$ for all $i \geq j$.

Denote $\lim_{i\in I} S_i$ by S (resp. $\lim_{i\in I} U_i$ by U), $\lim_{i\in I} X_i$ by X (resp. $\lim_{i\in I} X_i'$ by X'), and $\lim_{i\in I} g_i$ by $g\colon X' \to X$. Suppose that X_i is a relative S_i-curve for all i, and that $g\colon X' \to X$ is a stable U-modification (so X_U' is schematically dense in X'). Then, for all large i, the morphism $g_i\colon X_i' \to X_i$ is a stable U_i-modification.

Proof. The proof is similar to the proof of previous Lemma. We prove various desired properties of g_i step by step. First of all, we see that [Gro66, Théorème 8.10.5 (i), (xii)] implies that g_i is proper and an isomorphism over U_i for all large i (here we use that fiber product commutes with filtered limits). Then we use Lemma A.3 to confirm that X_i' is a semi-stable S_i-curve for all large i. So the last step is to show that stability of this modification is also preserved for all large i. By [Tem10, Lemma 5.1(ii)] the locus $\{s \in S_i \mid X_{i,s}' \to X_{i,s} \text{ is not stable}\}$ is constructible in S_i and the same locus in S is empty. Then the same argument as above (using [Gro66, Corollaire 8.3.3] instead of [Gro66, Corollaire 8.3.4]) implies that this locus is empty for all large i. Also, $(X_i')_{U_i} \subset X_i'$ is schematically dense since $X_i' \to S_i$ is flat and U_i is schematically dense in S_i. All these statements together imply that for all large i the morphism $g_i\colon X_i' \to X_i$ is a stable U_i-modification. The only non-automatic part is to check $(X_i')_{U_i}$ □

Definition A.5. Let X be a regular scheme. A *strict normal crossing divisor* (or an *snc divisor*) in X is a closed subscheme Y such that the intersection of every set of irreducible components of Y is also regular.

Lemma A.6. Let $(\mathcal{O}, \pi)$ be a pair of a rank-1 valuation ring $\mathcal{O}$ with algebraically closed fraction field K and a pseudo-uniformizer π. Then it can be written as a filtered colimit $(\mathcal{O}, \pi) \simeq \mathrm{colim}(A_i, t_i)$, where each A_i is a regular noetherian subring of $\mathcal{O}$ and $\mathrm{V}(t_i)_{\mathrm{red}}$ is an snc divisor in $\mathrm{Spec}\, A_i$.

Proof. We consider 3 cases separately: $\mathcal{O}$ is of equal characteristic 0, $\mathcal{O}$ is of equal characteristic $p > 0$, or $\mathcal{O}$ of mixed characteristic $(0, p)$. Depending on the case, we can we can write $\mathcal{O}$ as a filtered colimit of finite type $\mathbf{Q}$ (resp. $\mathbf{F}_p$, resp. $\mathbf{Z}_{(p)}$) sub-algebras A_i containing π. Now it suffices to show that for any $A_i \to \mathcal{O}_C$ there is a factorization $A_i \to A_j \to \mathcal{O}$ such that A_j is regular, finite type over $\mathbf{Q}$ (resp. $\mathbf{F}_p$, resp. $\mathbf{Z}_{(p)}$), and $\mathrm{V}(\pi)_{\mathrm{red}}$ is snc in $\mathrm{Spec}\, A_j$.

We use de Jong alternation theorem [Jon96, Theorem 4.1] in the equal characteristic case, and [Jon96, Theorem 6.5] in the mixed characteristic case, to say that there is an alteration $f\colon X \to \mathrm{Spec}\, A_i$ such that X is regular and $f^{-1}(\mathrm{V}(\pi))_{\mathrm{red}}$ is snc. Since alterations are generically finite and K is algebraically closed, we can lift a map $\mathrm{Frac}(A_i) \to K$ to the map $K(X) \to K$. Then since X is proper over A_i, we can lift a map $\mathrm{Spec}\, \mathcal{O} \to \mathrm{Spec}\, A_i$ to the map $\mathrm{Spec}\, \mathcal{O} \to X$. We now choose some open affine neighborhood of its image and denote it by $\operatorname{Spec} A_j$. Then A_j is regular by construction, and there is a factorization $A_i \to A_j \to \mathcal{O}$. The second map is injective as it is compatible with the injection $\operatorname{Frac}(A_j) = K(X) \to K$. Finally, consider $g\colon \operatorname{Spec} A_j \to \operatorname{Spec} A_i$. The construction said that $g^{-1}(\mathrm{V}(\pi)_{\mathrm{red}})$ is snc, but it is the same as $\mathrm{V}(g^*(\pi))_{\mathrm{red}} = \mathrm{V}(\pi)_{\mathrm{red}}$, where we consider π as an element of A_j. This finishes the proof. □

Appendix B. Formal And Rigid-Analytic Geometry

The two main goals of this section are to spell out precisely some notions related to formal algebraic and rigid-analytic geometry that we use in the paper, and to compare these notions to their algebraic counterparts. Since terminology in these areas slightly vary from reference to reference, we decide to make the definitions as precise as we can. But we do not really discuss terminology that seem to have only one interpretation.

B.1. **Formal Geometry.** We fix a complete rank-1 valuation field $\mathcal{O}_K$ with fraction field K. We recall that a formal $\mathcal{O}_K$-scheme always means for us an I-adic formal $\mathcal{O}_K$-scheme for a(ny) ideal of definition $I \subset \mathcal{O}_K$. And by a completion $\widehat{X}$ of finitely type $\mathcal{O}_K$-scheme X, we always mean the I-adic completion.

We refer to [Bos14, §7.4] for the discussion of the notions of morphisms of topologically finite type and of topologically finite presentation in the context of formal $\mathcal{O}_K$-schemes. We now recall the notion of admissible formal $\mathcal{O}_K$-schemes from [Bos14, §7.4] and relate it to its algebraic counterpart.

Definition B.1. A formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ is *admissible*, if it is $\mathcal{O}_K$-flat and locally of topologically finite type.

Remark B.2. We note that admissible formal $\mathcal{O}_K$-schemes are always topologically finitely presented by [Bos14, Corollary 7.3/5]. Thus our definition coincides with [Bos14, Definition 7.4/1]

Sometimes it is required that an admissible formal scheme should be also quasi-compact and quasi-separated. However, we follow the terminology of [Bos14] and [BL93] and do not put those conditions into the definition of admissible formal schemes. We also note that in our situation quasi-compactness of $\mathfrak{X}$ automatically implies quasi-separatedness of $\mathfrak{X}$. Indeed, $|\mathfrak{X}| = |\overline{\mathfrak{X}}|$ and the latter is a noetherian topological space.

Lemma B.3. Let X be an admissible $\mathcal{O}_K$-scheme (in the sense of Definition 2.6), then its completion $\widehat{X}$ is an admissible formal $\mathcal{O}_K$-scheme.

Proof. We note that X_K is schematically dense in X if and only if $\mathcal{O}_X$ is a $\mathcal{O}_K$-torsion free sheaf. We claim that this implies $\mathcal{O}_K$-flatness of $\mathfrak{X}$. The statement is Zariski local on $\mathfrak{X}$, so we may and do assume that X is affine. Now the completion of a flat $\mathcal{O}_K$-module is $\mathcal{O}_K$-flat by [Bos14, Proposition 7.3/11 and Remark 7.1/6]. Thus $\widehat{X}$ is admissible as a formal $\mathcal{O}_K$-scheme. □

We introduce the notion of a K-modification in the set-up of formal geometry and compare it to its algebraic analogue.

Definition B.4. We say that a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of admissible, quasi-compact, quasi-separated $\mathcal{O}_K$-formal schemes is a *rig-isomorphism* (resp. *rig-étale*, resp. *rig-smooth*, resp. *rig-surjective*), if the generic fibre $f_K\colon \mathfrak{X}_K \to \mathfrak{Y}_K$ is an isomorphism (resp. étale, resp. smooth, resp. surjective morphism) of adic spaces.

Definition B.5. We say that a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of admissible, quasi-compact, quasi-separated $\mathcal{O}_K$-formal schemes is a *rig-étale covering* if it is rig-étale and rig-surjective.

Remark B.6. We note that Definition B.4 implies that any rig-isomorphism f is proper (in the sense of [FK18, Definition 4.7.1]). This follows from [Lüt90, Theorem 3.1] in the case $\mathcal{O}_K$ is discretely valued and from [Tem00, Corollary 4.4. and Corollary 4.5] in general.

Now we need to the following lemma that was proven in [RG71, Corollaire 5.7.12]. However, the statement there is formulated for algebraic spaces, so to make things more accessible, we provide a short argument that extracts the result in the scheme case.

Lemma B.7. Let $U \subset S$ be a schematically dense quasi-compact open subscheme of a quasi-compact, quasi-separated scheme S. Let $f\colon X \to S$ be a proper, finitely presented morphism that is an isomorphism over U. Then there is an $f^{-1}(U)$-admissible blow-up $g\colon X' \to X$ such that the composition $h\colon X' \to X$ is also a U-admissible blow-up.

Proof. We note that [Sta21, Tag 081R] implies that there is a U-admissible blow-up $S' \to S$ such that the strict transform $X' \to S'$ is an open immersion. It is also proper, so it must be an isomorphism by schematic density of U. Now [Sta21, Tag 080E] implies that $X' \to X$ is an $f^{-1}(U)$-admissible blow-up. Thus we get a factorization $X' \xrightarrow{g} X \xrightarrow{f} S$ such that g and $f \circ g$ are admissible blow-ups. ☐

Lemma B.8. Let $f\colon X \to Y$ be a K-modification (in the sense of Definition 2.8) of flat, finitely presented $\mathcal{O}_K$-schemes. Then its completion $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ is a rig-isomorphism (in the sense of Definition B.4).

Proof. Lemma B.3 says that $\widehat{X}$ is an admissible formal $\mathcal{O}_K$-scheme. Moreover, the same argument (using [Bos14, Proposition 7.3/10] and [FK18, Proposition 4.7.3] instead [Bos14, Proposition 7.3/11]) shows that $\widehat{f}$ is proper since f is proper.

Now we use Lemma B.7 to find a morphism $g\colon X' \to X$ such that g and $f \circ g$ are both admissible K-blow ups. Therefore, the completions $\widehat{g}\colon \widehat{X'} \to \widehat{X}$ and $\widehat{f \circ g}\colon \widehat{X'} \to \widehat{Y}$ are both rig-isomorphisms by [Bos14, Proposition 8.4/2]. This implies that $\widehat{f}$ is a K-modification as well. ☐

Now we are ready to give a definition of a semi-stable formal curve fibration and relate it to the algebraic version of it.

Definition B.9. We say that a morphism of locally topologically finitely presented formal $\mathcal{O}_K$-schemes $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a *semi-stable formal relative curve* (or *semi-stable formal curve fibration*), if for any point $x \in \mathfrak{X}$ either f is smooth at x, or there is an open affine neighborhood $\mathrm{Spf}\, B \subset \mathfrak{X}$ containing x and an open affine neighborhood $\mathrm{Spf}\, A \subset \mathfrak{Y}$ containing $y = f(x)$, such that there exists a diagram of pointed schemes

$$\begin{array}{ccccc} & & (\mathrm{Spf}\, C, s) & & \\ & \overset{g}{\swarrow} & & \overset{h}{\searrow} & \\ (\mathrm{Spf}\, B, x) & & & & \left(\mathrm{Spf}\, \frac{A\langle U,V\rangle}{(UV-a)}, \{y,0,0\}\right), \end{array}$$

where g and h are étale and a lies in the ideal of y.

Remark B.10. We note that any semi-stable formal curve fibration is flat. Indeed, [FK18, Corollary I.4.8.2] implies that it suffices to check the claim modulo any power of a pseudo-uniformizer π. So the claim boils down to the fact that a (schematic) semi-stable curve fibration is flat.

Lemma B.11. Let $f\colon X\to Y$ be a morphism of flat, finitely presented $\mathcal{O}_K$-schemes that is a relative semi-stable curve (in the sense of Definition 2.4). Then its completion $\widehat{f}\colon \widehat{X}\to\widehat{Y}$ is a semi-stable formal relative curve (in the sense of Definition B.9).

Proof. Using Lemma 2.5 the question boils down to the fact that a completion of an étale morphism of algebras is formally étale. This condition can be checked modulo all powers of a pseudo-uniformizer, but since $\widehat{A}/I^n\cong A/I^n$ holds for any completion along a *finitely generated* ideal, we conclude the statement. □

Definition B.12. Let $\mathcal{O}_K$ be a complete rank-1 valuation ring. An admissible rig-smooth formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ is called *polystable* if étale locally it admits an étale morphism

$$\mathfrak{U}\to \operatorname{Spf}\frac{\mathcal{O}_K\langle t_{1,0},\dots,t_{1,n_0},\dots,t_{l,0},\dots,t_{l,n_l}\rangle}{(t_{1,0}\cdots t_{1,n_1}-\pi_1,\dots,t_{l,0}\cdots t_{l,n_l}-\pi_l)}$$

for some $\pi_i\in\mathcal{O}_K\setminus\{0\}$.

Lemma B.13. Let X be a morphism of flat, finitely presented $\mathcal{O}_K$-schemes that is K-smooth and polystable (in the sense of Definition 2.23). Then its completion $\widehat{X}$ is a rig-smooth polystable formal $\mathcal{O}_K$-scheme (in the sense of Definition B.12).

Proof. The proof is identical to that of Lemma B.11. □

We need to introduce another definition that is not standard:

Definition B.14. We say that a morphism $f\colon\mathfrak{X}\to\mathfrak{Y}$ of topologically finite type formal $\mathcal{O}_K$-schemes is *formally quasi-projective*, if its special fibre

$$\overline{f}\colon\overline{\mathfrak{X}}\to\overline{\mathfrak{Y}}$$

is a quasi-projective morphism of $\mathcal{O}_K/\mathfrak{m}_K$-schemes (in the sense of definition [Gro61, Definition 5.3.1]). It is clear that a composition of formally quasi-projective morphisms is formally quasi-projective.

B.2. **Rigid-Analytic Geometry.** We remind the reader that we use Huber's foundation for rigid-analytic geometry in this paper. So, a rigid-analytic space over a complete rank-1 valuation field K always means here an adic space locally topologically finite type over $\operatorname{Spa}(K,\mathcal{O}_K)$. When we need to use classical rigid-analytic spaces, we refer to them as Tate rigid-analytic spaces.

We recall that there is a fully faithfull functor

$$r_K\colon\{\text{Tate Rigid-Analytic Spaces over }K\}\to\left\{\begin{array}{l}\text{Adic Spaces locally of topologically}\\ \text{finite type over }\operatorname{Spa}(K,\mathcal{O}_K)\end{array}\right\}$$

that becomes an equivalence on categories when it is restricted to quasi-compact and quasi-separated objects on both sides. We do not define this functor here, instead we only mention that it sends an affinoid rigid-analytic space $\operatorname{Sp}(A)$ to the adic space $\operatorname{Spa}(A,A^\circ)$. The main difficulty then is to show that we can "glue" to define r_K on non-affinoids. We refer to [Hub94, §4] and [Sem15, Lecture 16] for the full construction and discussion of some properties of this functor.

We also need the functor of adic generic fiber:

$$(-)_K\colon\left\{\begin{array}{l}\text{locally topologically finite}\\ \text{type formal }\mathcal{O}_K\text{-schemes}\end{array}\right\}\to\left\{\begin{array}{l}\text{Adic Spaces locally of topologically}\\ \text{finite type over }\operatorname{Spa}(K,\mathcal{O}_K)\end{array}\right\}$$

that is defined in [Hub96, §1.9] (it is denoted by d there). Given an affine formal topologically finite type $\mathcal{O}_K$-scheme $\mathrm{Spf}(A)$, this functor assigns the affinoid adic space $\mathrm{Spa}(A\otimes_{\mathcal{O}_K} K, A^+)$ where A^+ is the integral closure of A in $A\otimes_{\mathcal{O}_K} K$. One of the main features of this functor is that there is a natural isomorphism of functors:

$$(-)_K \cong r_K \circ (-)_{\mathrm{rig}}$$

where $(-)_{\mathrm{rig}}$ is the "classical" Raynaud generic fiber (as defined in [Bos14, §7.4]).

The last functor of interest is the analytification functor:

$$(-)^{\mathrm{an}}\colon \left\{ \begin{array}{l} \text{locally finite type} \\ K\text{-schemes} \end{array} \right\} \to \left\{ \begin{array}{l} \text{Adic Spaces locally of topologically} \\ \text{finite type over } \mathrm{Spa}(K,\mathcal{O}_K) \end{array} \right\}$$

that is defined as a composition

$$(-)^{\mathrm{an}} = r_K \circ (-)^{\mathrm{rig}}$$

of the classical analytification functor $(-)^{\mathrm{rig}}$ (as it is defined in [Bos14, §5.4][19]) and the functor r_K.

The last thing we want to remind the reader is that given any morphism $\varphi\colon X\to Y$ of finite type $\mathcal{O}_K$-schemes, there are two ways to take the "generic fibre" of φ. The first one is to form the map between the usual generic fibres in the category of schemes and then consider its analytification. We denote the resulting morphism by $\varphi^{\mathrm{an}}_K\colon X^{\mathrm{an}}_K \to Y^{\mathrm{an}}_K$. Another way is first to complete φ and then take its adic generic fibre in the sense of Raynaud; the result of this approach we denote by $\widehat{\varphi}_K\colon \widehat{X}_K\to\widehat{Y}_K$. In general, these two approaches are very different, i.e. $\mathbf{A}^{1,\mathrm{an}}_K$ is an analytic affine line, but $(\widehat{\mathbf{A}^1_{\mathcal{O}_K}})_K \cong \mathbf{D}_K$ is the closed unit disc.

Lemma B.15. Let K be a complete rank-1 valuation field, and let X be a locally finitely presented $\mathcal{O}_K$-scheme. Then there is a functorial morphism

$$j_X\colon \widehat{X}_K \to X^{\mathrm{an}}_K$$

such that j_X is a quasi-compact open immersion if X is separated and admits a locally finite affine covering.

Proof. This is an easy reformulation of [Con99, Theorem 5.3.1] in the language of adic spaces. Namely, Conrad constructs[20] a functorial morphism

$$i_X\colon \widehat{X}_{\mathrm{rig}} \to X^{\mathrm{rig}}_K$$

that is a quasi-compact open immersion if X is separated and admits a locally finite affine covering. We apply the functor r_K to j_X to get the canonical morphism

$$j_X \coloneqq r_K(i_X)\colon \widehat{X}_K \to X^{\mathrm{an}}_K.$$

We only need to show that r_K sends quasi-compact open immersions to quasi-compact open immersions. We check it separately.

The fact that r_K carries open immersions to open immersions is explained on page 2 of [Sem15].

Now we show that r_K preserves quasi-compactness of morphisms. Suppose $f\colon Y\to Z$ is a quasi-compact morphism of rigid-analytic spaces over K. We choose an admissible covering of Z by open admissible affinoid subspaces U_i. In order to prove that $r_K(f)$ is quasi-compact, it suffices to verify that

$$r_K(f)^{-1}(r_K(U_i))$$

[19]We want to emphasize that the functors $(-)^{\mathrm{rig}}$ and $(-)_{\mathrm{rig}}$ are different. In particular, they have different source categories. The target category for both functors is the category of Tate Rigid-Analytic Spaces over K.

[20]Slightly different notations are used in that paper.

is quasi-compact for any i. Since

$$r_K(f)^{-1}(r_K(U_i)) = r_K(f^{-1}(U))$$

we conclude that it is enough to show that r_K preserves quasi-compactness. This is easy to see using the fact that any affinoid adic space is quasi-compact. $\square$

Lemma B.16. Let $\varphi\colon X \to Y$ be a morphism of flat finitely presented separated $\mathcal{O}_K$-schemes, and suppose that $\varphi_K\colon X_K \to Y_K$ factors as a G-torsor over its open quasi-compact image $U \subset Y_K$ for a finite group G. Then the morphism $\widehat{\varphi}_K\colon \widehat{X}_K \to \widehat{Y}_K$ factors as a G-torsor over its open quasi-compact image $V \subset (\widehat{Y})_K$.

Proof. We start by observing that $\varphi_K^{\mathrm{an}}\colon X_K^{\mathrm{an}} \to Y_K^{\mathrm{an}}$ factors as a G-torsor over an open U^{an}. We want to use it to deduce the result for $\widehat{\varphi}_K$ via the commutative square

$$\begin{array}{ccc} \widehat{X}_K & \xrightarrow{j_X} & X_K^{\mathrm{an}} \\ \downarrow{\scriptstyle \widehat{\varphi}_K} & & \downarrow{\scriptstyle \varphi_K^{\mathrm{an}}} \\ \widehat{Y}_K & \xrightarrow{j_Y} & Y_K^{\mathrm{an}} \end{array}$$

where j_X the G-equivariant from Lemma B.15. Clearly, both morphisms $\widehat{\varphi}_K$ and φ_K^{an} are G-invariant. The morphism φ_K^{an} is étale as the analytification of the étale morphism φ_K. Lemma B.15 gurantees that j_X and j_Y are both open immersions, so $\widehat{\varphi}_K$ is also étale. In particular, its image $V := \widehat{\varphi}_K(\widehat{X}_K)$ is open and quasi-compact (as $\widehat{X}_K$ is quasi-compact).

We claim that $\widehat{\varphi}_K$ is a G-torsor over V. It suffices to prove that the natural map $\widehat{X}_K \to X_K^{\mathrm{an}} \times_{Y_K^{\mathrm{an}}} V$ is an isomorphism. Since j_X and j_Y are quasi-compact open immersions by Lemma B.15, we conclude that it is equivalent to show that $(\varphi_K^{\mathrm{an}})^{-1}(V) = \widehat{X}_K$.

We know that $\widehat{X}_K \subset (\varphi_K^{\mathrm{an}})^{-1}(V)$ by the construction of V as an image of $\widehat{X}_K$. So it is enough to show the reverse inclusion. Note that since $V \subset U^{\mathrm{an}}$, we see that $(\varphi_K^{\mathrm{an}})^{-1}(V) \to V$ is a G-torsor as a pullback of a G-torsor. Moreover, note that a fibre over any point $v \in V$ contains at least one point from $\widehat{X}_K$. This implies that $(\varphi_K^{\mathrm{an}})^{-1}(V) = G.\widehat{X}_K$ (the orbit of $\widehat{X}_K$ under the action of G), thus $G.\widehat{X}_K = \widehat{X}_K$ as $\widehat{X}_K$ is G-stable. We finally deduce that $\widehat{X}_K = (\varphi_K^{\mathrm{an}})^{-1}(V)$, so $\widehat{\varphi}_K$ is a G-torsor over V. $\square$

Appendix C. Log Geometry

The main goal of this section is to construct a structure of log-smooth, log-variety on a successive semi-stable C-smooth fibration over a rank-1 valuation ring $\mathcal{O}_C$ with algebraically closed fraction field C (see Theorem C.3.1); this is used at the end of the proof of Theorem 6.6. The natural log structure on $\operatorname{Spec} \mathcal{O}_C$ is not fine, so one needs to be careful while working log schemes over $\operatorname{Spec} \mathcal{O}_C$. We handle this issue by using Lemma A.6.

For the readers convenience, we recall the main definitions of log geometry in this appendix. In particular, we recall the definition of a log $\mathcal{O}_C$-variety from [ALPT19]. We found [Kat89], [Ogu18] and [Niz06] to be especially useful sources on this subject; we refer there for a more comprehensive treatment of the theory.

Throughout this appendix, all monoids are meant to be commutative monoids. Also, we work only with étale log structures. In particular, all sheaves in this sections are considered as sheaves on a small étale site of X.

C.1. **Basic Definitions.**

Definition C.1.1. A *pre-logarithmic structure* (or just *pre-log structure*) on a scheme X is a homomorphism of sheaves of monoids $\alpha\colon \mathcal{M}_X \to \mathcal{O}_X$ on $X_{\text{ét}}$. A *logarithmic structure* (or just *log structure*) is a pre-logarithmic structure such that the induced homomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. Morphisms are defined in the evident way.

We denote the category of prelog structures on X as $\mathbf{Plog}_X$ and the category of log structures on X by $\mathbf{log}_X$. There is the natural forgetful functor $r_X\colon \mathbf{log}_X \to \mathbf{Plog}_X$ denoted also as r if it does not cause any confusion.

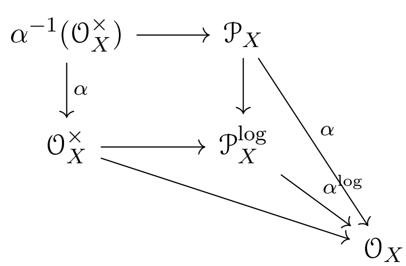
Definition C.1.2. A *log scheme* is a scheme X endowed with a log structure $\alpha\colon \mathcal{M} \to \mathcal{O}_X$.

Example C.1.3. Let $Z \subset X$ be a closed subscheme with the complement $j\colon U \to X$. We define the log structure $(\mathcal{M}_X, \alpha)$ *associated to the pair* (X, Z) as follows. We define the sheaf of monoids $\mathcal{M}_X$ as $\mathcal{M}_X = \mathcal{O}_X \times_{j_*\mathcal{O}_U} j_*\mathcal{O}_U^\times$ that is sometimes informally denoted by $\mathcal{O}_X \cap j_*\mathcal{O}_U^\times$. We define the map $\alpha\colon \mathcal{M}_X \to \mathcal{O}_X$ to be the natural inclusion map. Then it is straightforward to check that this defines a logarithmic structure on X, or that $(X, \mathcal{M}_X, \alpha)$ is a log scheme.

Example C.1.4. Let $\mathcal{O}_K$ be a rank-1 valuation ring with a uniformizer π and the residue field k. We call the log structure associated with the pair $(\operatorname{Spec} \mathcal{O}_K, \operatorname{Spec} k)$ the *standard* log structure on $\operatorname{Spec} \mathcal{O}_K$. Explicitly, $\mathcal{M}(\operatorname{Spec} A) = A \cap A[\frac{1}{\pi}]^\times$ for any affine étale map $\operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_K$.

The notion of a log scheme is too general. The sheaf of monoids $\mathcal{M}_X$ can be pretty much anything. We need to define the notion of quasi-coherent log structures. Before doing this, we need to discuss the way to associate log structures to monoids or, more generally, to any pre-log structure.

Definition C.1.5. Let $\alpha\colon \mathcal{P}_X \to \mathcal{O}_X$ be a pre-log structure on X. Then we define the *associated log structure* $(\mathcal{P}_X^{\log}, \alpha^{\log})$ as the pushout in the following diagram:



Lemma C.1.6. The functor $(-)^{\log}\colon \mathbf{Plog}_X \to \mathbf{log}_X$ defines a left adjoint to the forgetful functor $r_X\colon \mathbf{log}_X \to \mathbf{Plog}_X$. More precisely, the natural map $\mathcal{P}_X \to \mathcal{P}_X^{\log}$ induces an isomorphism $\operatorname{Hom}_{\mathbf{log_X}}(\mathcal{P}_X^{\log}, \mathcal{M}) \simeq \operatorname{Hom}_{\mathbf{Plog_X}}(\mathcal{P}_X, r_X(\mathcal{M}))$ for any log-structure $\mathcal{M}$.

Proof. The proof is essentially trivial and left as an exercise. □

Definition C.1.7. Let X be a scheme and P a monoid. A sheaf of monoids $\underline{P}_X$ is defined as the constant sheaf associated to P.

Example C.1.8. Let R be a ring, P be a monoid, and $R[P]$ be the monoid ring of P. Then $\operatorname{Spec} R[P]$ has a canonical log structure. Namely, it is defined as the log structure associated to the pre-log structure given by a morphism of sheaves of monoid $\underline{P}_X \to \mathcal{O}_X$ on $X = \operatorname{Spec} R[P]$. That morphism, in turn, comes from the natural morphism of monoids $P \to R[P]$.

We recall that the category of monoids admits all small colimits [Ogu18, page 2]. For example, if $P \underset{u}{\overset{v}{\rightrightarrows}} Q$ are homomorphisms of monoids, the *coequalizer* $\operatorname{coeq}(P \underset{u}{\overset{v}{\rightrightarrows}} Q)$ is constructed as the quotient of Q by the minimal congruence relation[21] $R \subset Q \oplus Q$ containing elements $(u(p), v(p)) \subset Q \oplus Q$ for all $p \in P$. One can construct the *cokernel* of $u\colon P \to Q$ as $\operatorname{coker}(u) = \operatorname{coeq}(P \underset{u}{\overset{1}{\rightrightarrows}} Q)$

Definition C.1.9. The *Grothendieck group* (P^{gp}, p) of a monoid P is defined as a group P^{gp} with a morphism of monoids $p\colon P \to P^{\mathrm{gp}}$ that is universal among maps to commutative groups.

The *group of units* $P^\times$ of a monoid P is a group of invertible elements in P.

The *sharpening* of a monoid P is the monoid $\overline{P} \coloneqq P/P^\times$.

Remark C.1.10. Existence of (P^{gp}, p) is discussed at [Ogu18, Beginning of Section I.1.3]. Namely, P^{gp} can be identified with the cokernel of the diagonal map $P \to P \oplus P$.

We use the terminology introduced in [ALPT19] and define the monoid $M[t_1, \dots, t_n]$ to be the monoid $M \oplus \mathbf{N}^{\oplus n}$. We define the monoid $M[t_1, \dots, t_n]/(f_i = g_i)$ for $f_i, g_i \in M[t_1, \dots, t_n]$ $(i = 1, \dots, m)$ to be the coequalizer

$$\frac{M[t_1, \dots, t_n]}{(f_i = g_i)} \coloneqq \operatorname{coeq}(\mathbf{N}e_1 \oplus \dots \oplus \mathbf{N}e_m \underset{u}{\overset{v}{\rightrightarrows}} M[t_1, \dots, t_n])$$

where $u(e_i) = f_i$ and $v(e_i) = g_i$.

Definition C.1.11. A monoid P is called *integral* if the cancellation low holds in P, i.e. $a \cdot b = a \cdot c$ implies $b = c$ in P.

A monoid P is called *u-integral* if $a \in P$, $u' \in P^\times$ and $a \cdot u' = a$ implies that $u' = 1$.

A monoid P is called *saturated* if it is integral and if whenever $p \in P^{\mathrm{gp}}$ is such that $np \in P$ for some $n \in \mathbf{Z}_{\geq 1}$, then $p \in P$.

A map of monoids $\phi\colon N \to M$ is called *finitely generated* if it can be extended to a surjective map of monoids $\psi\colon N[t_1, \dots, t_n] \to M$.

If, in addition, one can choose ϕ defined by an equivalence relation generated by finitely many relations $f_i = g_i$ with $f_i, g_i \in N[t_1, \dots, t_n]$ (i.e. $M = N[t_1, \dots, t_n]/(f_1 = g_1, \dots, g_m = f_m)$), then we say that ϕ is *finitely presented.*

A monoid P is called *finitely generated* if the natural map $e \to P$ from the trivial monoid is finitely generated.

A monoid P is called *fine* if it is finitely generated and integral.

Example C.1.12. Let $\mathcal{O}_K$ be a valuation ring of rank-1, and let $\Gamma_{\leq 1} \subset \Gamma \subset (\mathbf{R}_{>0}, \times)$ be the monoid of elements of norm less or equal than 1. Then $\Gamma_{\leq 1}$ is integral and saturated, but it is usually not finitely generated.

[21] Equivalence relation that is also a monoid

Definition C.1.13. A (global) *chart*[22] for the log structure $\mathcal{M}_X$ consists of a monoid P and a homomorphism $P \to \Gamma(X, \mathcal{O}_X)$, equivalently a morphism $\underline{P}_X \to \mathcal{O}_X$, such that the associated log structure $\underline{P}_X^{\log}$ is isomorphic to $\mathcal{M}_X$. We will usually denote charts by $(P \to \mathcal{M}_X)$.

Remark C.1.14. Suppose $(X, \mathcal{M}_X)$ is a log scheme with a chart $(P \to \Gamma(X, \mathcal{O}_X))$ and $Y \to X$ is an étale morphism with the log structure $(Y, \mathcal{M}_X|_Y)$. Then the associated morphism $(P \to \Gamma(Y, \mathcal{O}_Y))$ defines a chart for $(Y, \mathcal{M}_X|_Y)$.

Lemma C.1.15. Let $(P \to \mathcal{M}_X)$ be a chart for a log scheme $(X, \mathcal{M}_X)$. Then the natural map $\overline{P} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is surjective for any $x \in X$.

Proof. The proof is essentially trivial and follows from the definitions. □

Definition C.1.16. A log scheme X is called *quasi-coherent* if its log structure possesses charts étale locally.

A quasi-coherent log scheme X is called *integral* (resp. saturated, resp. coherent) if every point $x \in X$ étale locally admits a chart with an integral (resp. saturated, resp. finitely generated) monoid P.

A quasi-coherent log scheme X is called *fine* if it is integral and coherent.

We give another characterization of integral (resp. saturated) log schemes that depends less on a choice of a chart.

Lemma C.1.17. Let $(X, \mathcal{M}_X)$ be a quasi-coherent log scheme. Then it is integral (resp. saturated) if and only if $\mathcal{M}_X$ is a sheaf of integral (resp. saturated) monoids, i.e. $\mathcal{M}_X(U)$ is integral (resp. separated) for any étale $U \to X$.

Proof. Suppose that $(X, \mathcal{M}_X)$ is integral (resp. saturated). Then we can show that $\mathcal{M}_X$ is a sheaf of integral (resp. saturated) monoids étale locally on X ([Ogu18, Proposition II.1.1.3]). So we can assume that X admits a chart $(P \to \mathcal{M}_X)$. Then [Ogu18, Proposition II.1.1.8(2)] ensures that $\underline{P}_X^{\log} \simeq \mathcal{M}_X$ is a sheaf of integral (resp. saturated) monoids. This shows the "only if" direction.

Now suppose that $(X, \mathcal{M}_X)$ is quasi-coherent and $\mathcal{M}_X$ is a sheaf of integral monoids. We need to construct charts $(P_x \to \mathcal{M}_X)$ étale locally on X with *integral* monoids P_x. Since $\mathcal{M}_X$ is quasi-coherent, we may and do assume that $\mathcal{M}_X$ admits a chart $(P \to \mathcal{M}_X)$.

Consider the monoid $P^{\mathrm{int}} := \mathrm{Im}(P \to P^{\mathrm{gp}})$. It is clear that P^{int} is an integral monoid, and that the chart map $\alpha \colon \underline{P}_X \to \mathcal{M}_X$ factors through $\beta \colon \underline{P}_X^{\mathrm{int}} \to \mathcal{M}_X$ as $\mathcal{M}_X$ is a sheaf of integral monoids. Unravelling the definition of a chart, we need to show that the diagram

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \underline{P}_X^{\mathrm{gp}} \\ \downarrow & & \downarrow \beta \\ \mathcal{O}_X^\times & \longrightarrow & \mathcal{M}_X \end{array}$$

is co-cartesian. This formally follows from the fact the diagram

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \underline{P}_X \\ \downarrow & & \downarrow \alpha \\ \mathcal{O}_X^\times & \longrightarrow & \mathcal{M}_X \end{array}$$

[22] It is called by a (global) affine chart in [ALPT19].

is co-cartesian and the maps $\underline{P}_X \to \underline{P}^{\mathrm{int}}_X$ and $\alpha^{-1}(\mathcal{O}_X^\times) \to \beta^{-1}(\mathcal{O}_X^\times)$ are surjective. This shows that $(P^{\mathrm{int}} \to \mathcal{M}_X)$ defines an integral chart on X.

Finally, suppose that the log scheme $(X, \mathcal{M}_X)$ is saturated (so it is automatically integral). We already know that étale locally we can choose a chart $(P \to \mathcal{M}_X)$ with integral P. Now we consider monoid

$$P^{\mathrm{sat}} := \{x \in P^{\mathrm{gp}} \mid x^n \in P \text{ for some integer } n \geq 1\}.$$

The chart map $\alpha\colon \underline{P}_X \to \mathcal{M}_X$ extends uniquely to the map $\beta\colon \underline{P}^{\mathrm{sat}}_X \to \mathcal{M}_X$ since $\mathcal{M}_X$ is a sheaf of saturated monoids. Since any integral monoid is u-integral, [Ogu18, Proposition II.2.1.4] ensures that a map $\gamma\colon \underline{Q}_X \to \mathcal{M}_X$ is a chart if and only if the natural homomorphism

$$Q/\gamma_{\overline{x}}^{-1}(\mathcal{M}^\times_{X,\overline{x}}) \to \overline{\mathcal{M}}_{X,\overline{x}}$$

is an isomorphism for all $x \in X$. But clearly α satisfies this property if and only if β does. So $(P^{\mathrm{sat}} \to \mathcal{M}_X)$ is indeed a chart with a saturated monoid P^{sat}. □

Corollary C.1.18. Let $(X, \mathcal{M}_X)$ be a fine (resp. fine and saturated) log scheme. Then, étale locally on X, $(X, \mathcal{M}_X)$ admits a chart $(P \to \mathcal{M}_X)$ with a fine (resp. fine and saturated) monoid P.

Proof. This follows from Lemma C.1.17 and [Ogu18, Corollary II.2.3.6]. Alternatively, one can repeat the proof of Lemma C.1.17 noting that P^{int} is finitely generated for any finitely generated monoid P, and P^{sat} is finitely generated for any integral, finitely generated monoid P (It can be deduced from [Ogu18, Proposition I.2.1.1] and [Ogu18, Theorem I.2.1.7]). □

Remark C.1.19. We warn the reader that the notion of a fine log scheme is not independent of a choice of a chart P. For instance, let $\mathcal{O}_K$ be a dvr and $(X = \operatorname{Spec} \mathcal{O}_K, \mathcal{M}_X)$ the associated log scheme with the standard log structure. Then $(\mathcal{O}_K \setminus \{0\} \to \mathcal{M}_X)$ is a chart and the monoid $\mathcal{O}_K \setminus \{0\}$ is usually not finitely generated. However, if we pick a uniformizer π then the induced map $(\mathbf{N}\pi \to \mathcal{M}_X)$ is also a chart with finitely generated, integral monoid $\mathbf{N}\pi$.

Remark C.1.20. We also warn the reader that the notion of quasi-coherent log scheme is quite restrictive. Unlike the case of quasi-coherent sheaves in algebraic geometry, many natural constructions of log schemes are not quasi-coherent. For instance, the log structures associated with pairs (X, Z) as in the Example C.1.3 are often not quasi-coherent unless Z is a strict normal crossing divisor in a regular scheme X (see Definition A.5).

In the case of a strict normal crossing divisor $Z \subset X$ and the associated log structure $(X, \mathcal{M}_X)$, the key observation is that any point $x \in X$ admits a regular sequence $(t_1, \dots, t_m)$ generating the maximal ideal of $\mathcal{O}_{X,x}$ and a natural number r such that the product $t_1 \cdot \dots \cdot t_r$ generates the ideal of Z in $\mathcal{O}_{X,x}$. One shows that the natural map $\oplus_{i=1}^r \mathbf{N} t_i \to \mathcal{M}_U$ is a fine chart in some neighborhood U of a point x. See [Ogu18, Proposition III.1.8.2 and III.1.7.3] for a more detailed discussion.

Example C.1.21. The standard log structure on $\operatorname{Spec} \mathcal{O}_K$ for a rank-1 valuation ring $\mathcal{O}_K$ is integral and quasi-coherent similarly to example in Remark C.1.19. However, it is not fine unless the associated valuation monoid $\Gamma_{\leq 1}$ is finitely generated. For example, it never happens if K is algebraically closed.

Remark C.1.22. We will be mostly concerned with integral quasi-coherent log schemes. However, the fineness assumption is too strong for our purposes since $\operatorname{Spec} \mathcal{O}_K$ with its standard log structure is usually not fine (Example C.1.21).

Definition C.1.23. Let $f\colon X' \to X$ be a morphism of schemes.

- Let $(\mathcal{M}_{X'}, \alpha)$ be a pre-log structure on X'. We define its pushforward $f_*^{\mathrm{plog}}(\mathcal{M}_{X'}, \alpha)$, or just $f_*^{\mathrm{plog}}(\mathcal{M}_{X'})$, as the fiber product

$$\begin{array}{ccc} f_*^{\mathrm{plog}}(\mathcal{M}_{X'}) & \xrightarrow{f_*^{\mathrm{plog}}(\alpha)} & \mathcal{O}_X \\ \downarrow & & \downarrow \\ f_*(\mathcal{M}_{X'}) & \xrightarrow{f_*(\alpha)} & f_*(\mathcal{O}_{X'}) \end{array}$$

- Let $(\mathcal{M}_X, \alpha)$ be a pre-log structure on X. We define its pullback $f^*_{\mathrm{plog}}(\mathcal{M}_X, \alpha)$, or just $f^*_{\mathrm{plog}}(\mathcal{M}_X)$, as the pre-log structure given by $f^{-1}(\mathcal{M}_X) \to f^{-1}(\mathcal{O}_X) \to \mathcal{O}_{X'}$.

Definition C.1.24. Let $f \colon X' \to X$ be a morphism of schemes.

- Let $(\mathcal{M}_{X'}, \alpha)$ be a log structure on X'. We define its pushforward $f_*^{\log}(\mathcal{M}_{X'}, \alpha)$, or just $f_*^{\log}(\mathcal{M}_{X'})$, as the fiber product

$$\begin{array}{ccc} f_*^{\log}(\mathcal{M}_{X'}) & \xrightarrow{f_*^{\log}(\alpha)} & \mathcal{O}_X \\ \downarrow & & \downarrow f^{\#} \\ f_*(\mathcal{M}_{X'}) & \xrightarrow{f_*(\alpha)} & f_*(\mathcal{O}_{X'}) \end{array}$$

This is already a log structure as can be easily checked.

- Let $(\mathcal{M}_X, \alpha)$ be a log structure on X. We define its pullback $f^*_{\log}(\mathcal{M}_X, \alpha)$, or just $f^*_{\log}(\mathcal{M}_X)$, as the log structure associated to the pre-log structure given by

$$f^{-1}(\mathcal{M}_X) \xrightarrow{f^{-1}(\alpha)} f^{-1}(\mathcal{O}_X) \xrightarrow{f^{\#}} \mathcal{O}_{X'} \ .$$

Lemma C.1.25. Let $f \colon X' \to X$ be a morphism of schemes. The functor $f^*_{\mathrm{plog}} \colon \mathbf{Plog}_{X'} \to \mathbf{Plog}_X$ is a left adjoint to the functor $f_*^{\mathrm{plog}} \colon \mathbf{Plog}_X \to \mathbf{Plog}_{X'}$. Similarly, $f^*_{\log}$ is left adjoint to $f_*^{\log}$.

Proof. Straightforward and left as an exercise. □

Corollary C.1.26. Let $f \colon X' \to X$ be a morphism of schemes and $\mathcal{M}_X$ a pre-log structure on X. Then $\left(f^*_{\mathrm{plog}}\left(\mathcal{M}_X\right)\right)^{\log} \simeq f^*_{\log}\left(\mathcal{M}_X^{\log}\right)$.

Proof. Note that the functors $f_*^{\mathrm{plog}} \circ r_{X'} \colon \mathbf{log}_{X'} \to \mathbf{Plog}_X$ and $r_X \circ f_*^{\log}$ are canonically identified simply by the definitions of f_*^{plog} and $f_*^{\log}$. This means that their left adjoints are canonically isomorphic. The first functor has a left adjoint $(-)^{\log} \circ f^*_{\mathrm{plog}}$ and the second functor has a left adjoint $f^*_{\log} \circ (-)^{\log}$. This says that $(f^*_{\mathrm{plog}}(\mathcal{M}_X))^{\log}$ and $f^*_{\log}(\mathcal{M}_X^{\log})$ are canonically isomorphic for any pre-log structure $\mathcal{M}_X$. □

Definition C.1.27. A morphism of log schemes $f \colon (X, \mathcal{M}_X) \to (X', \mathcal{M}_{X'})$ is called *strict* if the natural map $f^*_{\log}(\mathcal{M}_{X'}) \to \mathcal{M}_X$ is an isomorphism.

Remark C.1.28. A chart for a log structure on an R-scheme X can be understood as a strict morphism $X \to \operatorname{Spec} R[P]$ where the target is endowed with the natural log structure from Example C.1.8.

Example C.1.29. Let $\mathcal{O}_K$ be a rank-1 valuation ring with the value group Γ and the associated monoid $V \coloneqq \mathcal{O}_K \setminus \{0\}$[23]. Then its sharpening $\overline{V}$ is isomorphic to $\Gamma_{\leq 1}$.

[23] It is denoted as R is [ALPT19]. However, we prefer this notation since R can be easily confused with the base ring.

In what follows, V denotes the monoid $\mathcal{O}_K \setminus \{0\}$ associated to the rank-1 valuation ring $\mathcal{O}_K$.

Definition C.1.30. A *log variety* over a rank-1 valuation ring $\mathcal{O}_K$ is an integral quasi-coherent log scheme $(X, \mathcal{M}_X)$ such that the underlying morphism of schemes $X \to \operatorname{Spec} \mathcal{O}_K$ is flat of finite presentation and each homomorphism of monoids $\overline{V} = \Gamma_{\leq 1} \to \overline{\mathcal{M}}_{X,\overline{x}}$ is finitely generated.

Remark C.1.31. The last condition is automatic if an $\mathcal{O}_K$-log scheme $(X, \mathcal{M}_X)$ admits charts that are finitely generated over V. Indeed, if $x \in X$ étale locally admits a chart given by a finitely generated R-monoid M, then the associated log structure is given by $\underline{M}^{\log}$. Therefore, $\overline{M} \to \overline{\underline{M}^{\log}_{X,\overline{x}}} \simeq \overline{\mathcal{M}}_{X,\overline{x}}$ is surjective by Lemma C.1.15. Therefore, $\overline{\mathcal{M}}_{X,\overline{x}}$ is finitely generated over $\overline{V} \simeq \Gamma_{\leq 1}$.

C.2. Log Smoothness.

Definition C.2.1. Let $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of fine quasi-coherent log schemes with a choice of a fine chart $(Q \to \mathcal{M}_Y)$. Then f is called *log smooth* if, for each $x \in X$,

- There is a finitely presented morphism of integral monoids $Q \to P_x$ such that the kernel and the torsion part of the cokernel $Q^{\mathrm{gp}} \to P_x^{\mathrm{gp}}$ are finite groups of orders invertible on X
- There is an étale neighborhood $U \to X$ of x, such that the natural map $U \to Y$ factors as $U \xrightarrow{g} Y \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P_x] \to Y$ and the map g is strict and étale (in the usual sense).

Remark C.2.2. We note that [Kat89, Theorem 3.5] shows that this definition does recover the standard notion of log smooth morphism [Kat89, Definition at the beginning of (3.3)] defined in terms of the log analog of the infinitesimal criterion of smoothness. In particular, this definition does not depend on a choice of a chart Q.

Example C.2.3. A non-trivial example of a log smooth morphism of log varieties is given by a semi-stable degeneration over a dvr $\mathcal{O}_K$. More precisely, let $X \to \operatorname{Spec} \mathcal{O}_K$ be a flat, finite type morphism with a regular total space X and the closed fiber X_s a strict normal crossing divisor in X. Consider the log scheme $(X, \mathcal{M}_X)$ with the log structure associated with the closed fiber X_s, and the log scheme $(\operatorname{Spec} \mathcal{O}_K, \mathcal{M}_K)$ with the classical log structure. Then the morphism $(X, \mathcal{M}_X) \to (\operatorname{Spec} \mathcal{O}_K, \mathcal{M}_K)$ is log smooth by [Ogu18, Corollary IV.3.1.18].

Following [ALPT19], we give the following ad-hoc definition of log smooth log variety:

Definition C.2.4. Let $\mathcal{O}_K$ be a rank-1 valuation ring with a divisible value group Γ. Then we say that a log $\mathcal{O}_K$-variety X is *log smooth* if for every $x \in X$ there is an étale neighborhood U of x with a strict étale morphism

$$U \to \operatorname{Spec} \mathcal{O}_K \times_{\operatorname{Spec} \mathbf{Z}[V]} \operatorname{Spec} \mathbf{Z}[P]$$

for some integral monoid P with a finitely presented, injective morphism $V \hookrightarrow P$.

The main goal of this section is to show a successive semi-stable C-smooth curve fibration over a rank-1 valuation ring with algebraically closed fraction field is indeed a log smooth log variety. Moreover, we will actually show that the desired log structure is the log structure associated to its special fiber. However, before proving this claim we need to deal with a noetherian situation. In order to do so, we need to use the notion of log regular log structures.

Unfortunately, it is quite difficult to define log regular log structure, so instead we only mention the main properties and refer to [Tsu19, Definition 4.4.5] or [Ogu18, Theorem III.1.11.1] for a precise definition.

Facts C.2.5. (1) [Ogu18, Ex. III.1.11.9] Suppose that X is a regular noetherian scheme with a strict normal crossing divisor D. Then the log structure associated to D is log regular.

(2) Any log regular log scheme $(X, \mathcal{M}_X)$ is fine and saturated (see Corollary C.1.18).

(3) [Niz06, Prop. 2.6] Suppose that $(X, \mathcal{M}_X)$ is a log regular log scheme. Then the locus of triviality $X_{\mathrm{tr}} \coloneqq \{x \in X \mid \mathcal{M}_{X,\overline{X}} = \mathcal{O}^{\times}_{X,\overline{x}}\}$ is a dense open subset of X, and $\mathcal{M}_X \simeq \mathcal{O}_X \cap j_*\mathcal{O}^{\times}_{X_{\mathrm{tr}}}$, where $j\colon X_{\mathrm{tr}} \to X$ is the natural open immersion.

(4) [Ogu18, Prop. IV.3.5.3] Let $(X', \mathcal{M}_{X'}) \to (X, \mathcal{M}_X)$ be a log smooth morphism of fine saturated log schemes whose underlying schemes are locally noetherian. If $(X, \mathcal{M}_X)$ is log regular then so is $(X', \mathcal{M}_{X'})$.

Definition C.2.6. We say that a pair (X, Z) of a locally noetherian scheme and closed subscheme Z is *log regular* if the log structure associated to the pair (X, Z) is log regular.

Proposition C.2.7. [ILO14, Exp. VI, Prop. 1.9] Let (Y, T) be a log regular pair. And let $f\colon X \to Y$ be a semi-stable curve fibration that is smooth over $Y \setminus T$. Then the pair $(X, f^{-1}(T))$ is also log regular and the canonical morphism of log schemes $X \to Y$ is log smooth.

Proof. We denote the complement of T in Y as $j\colon U \to Y$ and its pre-image in X as V. Pick a point $x \in X$ with $y \coloneqq f(x) \in Y$. The statement is étale local on X and Y, so we can replace X and Y with étale neighborhoods of x and y to assume that $Y = \operatorname{Spec} A$ is affine, admits a global fine and saturated chart $P \to \mathcal{M}_Y$ (Corollary C.1.18), and $f\colon X \to Y$ factors as

$$\begin{array}{ccc} X & \xrightarrow{h} & \operatorname{Spec} A[x,y]/(xy-a) \\ \big\downarrow f & \swarrow g & \\ Y & & \end{array}$$

where h is étale and $a \in A \cap \mathcal{O}(U)^{\times}$[24] ($U$ is dense in Y by Fact C.2.5(3)). The condition that a is invertible in $\mathcal{O}(U)$ comes from the fact that $X \to Y$ is assumed to be smooth over U.

Since P is a chart for $\mathcal{M}_Y = \mathcal{O}_Y \cap j_*\mathcal{O}^{\times}_U$, we can pass to some étale neighborhood of y to assume that $a = um$ for some $m \in P$ and $u \in \mathcal{O}_Y(Y)^{\times}$. Then we use the isomorphism $\operatorname{Spec} A[x,y]/(xy - a) \cong \operatorname{Spec} A[x,y]/(xy-m)$ to assume that a lies in (the image of) P.

Now we define a log structure on $\operatorname{Spec} A[x,y]/(xy-a)$ associated to the monoid $Q \coloneqq P[x,y]/(xy = a)$ with the obvious map $\beta\colon Q \to A[x,y]/(xy-a)$. This is clearly a fine saturated monoid, and the homomorphism $\phi\colon P \to Q$ defines the homomorphism $\phi^{\mathrm{gp}}\colon P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ that can be identified with the natural inclusion $P^{\mathrm{gp}} \to P^{\mathrm{gp}} \oplus \mathbf{Z}$. Therefore, $\ker(\phi^{\mathrm{gp}}) = \{e\}$ and $\operatorname{coker}(\phi^{\mathrm{gp}}) \simeq \mathbf{Z}$, in particular, it is torsion-free. Finally, we note that the natural map

$$\operatorname{Spec} A[x,y]/(xy-a) \to \operatorname{Spec} A \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$$

is an isomorphism since $\mathbf{Z}[Q] \simeq \mathbf{Z}[P][x,y]/(xy-a)$[25]. This implies that the map

$$(\operatorname{Spec} A[x,y]/(xy-a), Q^{\log}) \to (\operatorname{Spec} A, \mathcal{M}_Y)$$

is a log smooth morphism of fine, saturated log schemes. In particular, Fact C.2.5(4) ensures that $(\operatorname{Spec} A[x,y]/(xy-a), Q^{\log})$ is log regular. And now Fact C.2.5(3), in turn, guarantees that the log structure on $\operatorname{Spec} A[x,y]/(xy-a)$ coincides with the log structure associated to $g^{-1}(T)$.

Finally, we come back to showing that $f\colon X \to Y$ is log smooth. We reduced the situation to the case that X admits an étale map to $\operatorname{Spec} A[x,y]/(xy-a)$. Given all the work above, it is suffices to

[24]If f is smooth at $x \in X$, we can choose $a = 1$, and otherwise we use Lemma 2.5.

[25]There is a natural map $\mathbf{Z}[Q] \to \mathbf{Z}[P][x,y]/(xy-a)$ that can be checked to be an isomorphism "by hands". Alternatively, see Remark C.2.11.

show that this map is strict when X is provided with its log structure associated with $f^{-1}(T)$ and $\operatorname{Spec} A[x,y]/(xy-a)$ is provided with the log structure $Q^{\log}$ constructed above. However, we have already shown the latter log structure coincides with the log structure associated to $g^{-1}(T)$. And it is clear that the restriction of this log structure on X exactly coincides with the log structure associated with $f^{-1}(T)$ as h is étale. In other words, the map $h\colon X \to \operatorname{Spec} A[x,y]/(xy-a)$ is a strict étale morphism. This ensures that f is log smooth according to Definition C.2.1. Finally, $(X, f^{-1}(T))$ is log regular by Fact C.2.5(4). □

Remark C.2.8. The proof actually shows more. If Y has a global chart P, then étale locally X has a chart of the form $P[x,y]/(xy=a)$. Strictly speaking, we constructed the global chart only in the model example $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} A[x,y]/(xy-a)$. However, it defines a chart for X étale over $\operatorname{Spec} A[x,y]/(xy-a)$ by Remark C.1.14.

Corollary C.2.9. Let $g\colon (Y',T') \to (Y,T)$ be a morphism of log regular pairs, i.e. a morphism of schemes $g\colon Y' \to Y$ such that $g^{-1}(|T|) = |T'|$. And let $f\colon X \to Y$ be a semi-stable curve fibration that is smooth over $Y \setminus T$, and let $f'\colon X' \to Y'$ be its base-change. Then the natural morphism of log schemes

$$(X', \mathcal{O}_{X'} \cap j'_*\mathcal{O}^\times_{f'^{-1}(Y'\setminus T')}) \to (X, \mathcal{O}_X \cap j_*\mathcal{O}^\times_{f^{-1}(Y\setminus T)}) \times_{(Y, \mathcal{O}_Y \cap i_*\mathcal{O}^\times_{Y\setminus T})} (Y', \mathcal{O}_{Y'} \cap i'_*\mathcal{O}^\times_{Y'\setminus T'})$$

is an isomorphism.

Proof. The claim is étale local on X, Y and Y'. So we can reduce to the case X, $Y = \operatorname{Spec} A$, and $Y' = \operatorname{Spec} A'$ are affine with $(Y, \mathcal{O}_Y \cap i_*\mathcal{O}^\times_{Y\setminus T})$ and $(Y', \mathcal{O}_{Y'} \cap i'_*\mathcal{O}^\times_{Y'\setminus T'})$ having global compatible charts P and P', and X having a strict étale morphism to

$$\operatorname{Spec} A[x,y]/(xy-a) = \operatorname{Spec} A \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}\left[\frac{P[x,y]}{(xy=a)}\right]$$

for some $a \in P$. This follows from Remark C.2.8.

So it suffices to prove the claim in the case $X = \operatorname{Spec} A[x,y]/(xy-a)$ with the chart $P[x,y]/(xy = a) \to \mathcal{M}_X$. But then its pullback along the log map $Y' \to Y$ is given by $\operatorname{Spec} A'[x,y]/(xy-a)$ with the chart $P'[x,y]/(xy=a)$. The proof of Proposition C.2.7 (and Remark C.2.8) ensures that this is exactly the log structure associated with the closed subscheme $(f')^{-1}(T')$. □

We are almost ready to show that any successive semi-stable C-smooth curve fibration has a structure of a log smooth log $\mathcal{O}_C$-variety. We introduce the following definition that will be useful in the proof.

Definition C.2.10. If M is a monoid and $a \in M$ an element, we define the *semi-stable model* monoid over M as $F_a(M) \coloneqq M[x,y]/(xy=a)$.

A *good sequence for a monoid* M $(a_n, \dots, a_1)$ is a sequence of elements $a_1 \in M$, $a_2 \in F_{a_1}(M)$, $a_3 \in F_{a_2}(F_{a_1}(M)), \dots$.

If $\underline{a} = (a_n, \dots, a_1)$ is a good sequence for a monoid M, we define *successive semi-stable* monoid over it as $F_{\underline{a}}(M) \coloneqq F_{a_n}(F_{a_{n-1}}(\dots F_{a_1}(M)))$.

Similarly, if A is an algebra with an element $a \in A$, we define the *semi-stable model* algebra over A as $G_a(A) \coloneqq A[x,y]/(xy-a)$.

A *good sequence for an algebra* A $(a_n, \dots, a_1)$ is a sequence of elements $a_1 \in A$, $a_2 \in G_{a_1}(A)$, $a_3 \in G_{a_2}(G_{a_1}(A)), \dots$.

And, if $\underline{a} = (a_n, \dots, a_1)$ is a good sequence for an algebra A, we define *successive semi-stable* algebra over it as $G_{\underline{a}}(A) \coloneqq G_{a_n}(G_{a_{n-1}}(\dots G_{a_1}(A)))$

Remark C.2.11. There is a natural isomorphism $\mathbf{Z}[F_{\underline{a}}(M)] \simeq G_{\underline{a}}(\mathbf{Z}[M])$. Indeed, it suffices to prove the claim for one element $a \in M$. Then we have a natural map

$$\mathbf{Z}\left[F_a\left(M\right)\right] = \mathbf{Z}\left[\frac{M\left[x,y\right]}{(xy=a)}\right] \to G_a\left(\mathbf{Z}\left[M\right]\right) = \frac{\mathbf{Z}\left[M\right]\left[x,y\right]}{(xy-a)}.$$

Firstly, we note that $\mathbf{Z}[M[x,y]] \simeq \mathbf{Z}[M][x,y]$, so it suffices to show that $\mathbf{Z}[-]$ commutes with coequalizers. However, it is a left adjoint functor to the forget functor from (commutative) algebras to (commutative) monoids. Therefore, it does commute with coequalizers.

Remark C.2.12. One can check "by hand" that $F_{\underline{a}}(M)$ is always finitely presented over M, and it is integral and saturated provided that so is M.

C.3. **Main Result.**

Theorem C.3.1. Let $\mathcal{O}_C$ be a rank-1 valuation ring with the algebraically closed fraction field C and the residue field k. Let $f\colon X \to \operatorname{Spec} \mathcal{O}_C$ be a successive semi-stable C-smooth fibration. Then the log structure associated with the closed fiber X_k defines the structure of a log smooth log $\mathcal{O}_C$-variety.

Proof. We start by using Lemma A.6 to write $(\mathcal{O}_C, \pi)$ as a filtered colimit of noetherian, regular subrings $(A_i, t_i)_{i\in I}$ with $\mathrm{V}(t_i)_{\mathrm{red}}$ an snc divisor. We may replace each A_i with the localization $(A_i)_{\mathfrak{m}_C \cap A_i}$ to assume that all A_i are local and the morphisms $A_i \to \mathcal{O}_C$ are local as well. In what follows, we denote the standard log structure on $\operatorname{Spec} \mathcal{O}_C$ by $\mathcal{M}_C$.

Step 1. Spread X over some A_i: We present X as a successive semi-stable curve fibration

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \to X_1 \xrightarrow{f_1} X_0 = \operatorname{Spec} \mathcal{O}_C$$

such that each f_i is a relative semi-stable curve smooth over C-fibers. Now we use [Gro66, Théorème 8.8.2] to successively spread this tower over some A_i to get a tower

$$X^{(i)} \coloneqq X_{n,i} \xrightarrow{f_{n,i}} X_{n-1,i} \xrightarrow{f_{n-1,i}} \cdots \to X_{1,i} \xrightarrow{f_{1,i}} X_{0,i} = \operatorname{Spec} A_i$$

of A_i-schemes. Lemma A.3 ensures that, after possibly enlarging i, we can assume that each $f_{n,i}$ is a relative semi-stable curve whose restriction over $\operatorname{Spec} A_i[1/t_i]$ is smooth. Now we replace the filtered set I with $I_{\geq i}$, and define $X_{k,j} \coloneqq X_{i,j} \times_{\operatorname{Spec} A_i} \operatorname{Spec} A_j$ for any $j \geq i$. Clearly, the tower

$$X^{(j)} \coloneqq X_{n,j} \xrightarrow{f_{n,j}} X_{n-1,j} \xrightarrow{f_{n-1,j}} \cdots \to X_{1,j} \xrightarrow{f_{1,j}} X_{0,j} = \operatorname{Spec} A_j$$

is a successive semi-stable curve fibration whose restriction over $\operatorname{Spec} A_j[1/t_j]$ is smooth.

Step 2. Construct the log structure over the finite layer A_i: Now we note that $\operatorname{Spec} A_i$ has a canonical log structure associated to the closed subset $\mathrm{V}(t_i)_{\mathrm{red}}$. Moreover, this log structure is log regular by Facts C.2.5(1), and it admits a global chart with $P_i = \mathbf{N}^d$ where d is the number of irreducible components of $\mathrm{V}(t_i)_{\mathrm{red}}$. Only the latter claim requires a justification, in fact, it follows from the discussion before [Ogu18, Proposition III.1.7.3] and the fact that, for each irreducible component $D_i \subset \mathrm{V}(t_i)_{\mathrm{red}}$, the ideal sheaf $\mathcal{O}(-D_i)$ has a global generator as A_i is a local regular ring. We denote this log structure on $\operatorname{Spec} A_i$ by $\mathcal{M}_i$.

Now we endow each $X_{k,i}$ with the log structure $\mathcal{M}_{X_{k,i}}$ associated with the closed subscheme $X_{k,i} \times_{\operatorname{Spec} A_i} \operatorname{Spec}(A_i/t_i)$. We use (the proof of) Proposition C.2.7 and Remark C.2.8 successively to show that $X^{(i)} = X_{n,i}$ étale locally admits a chart of the form $F_{\underline{a}}(P_i)$ for some good sequence $\underline{a}$ for the monoid P_i (see Definition C.2.10). And, moreover, it admits a strict étale morphism

$$X_i \to \operatorname{Spec} A_i \times_{\operatorname{Spec} \mathbf{Z}[P_i]} \operatorname{Spec} \mathbf{Z}[F_{\underline{a}}(P_i)] \simeq \operatorname{Spec} A_i \times_{\operatorname{Spec} \mathbf{Z}[P_i]} \operatorname{Spec} G_{\underline{a}}(\mathbf{Z}[P_i]) \simeq \operatorname{Spec} G_{\underline{a}}(A_i)$$

where the target has the log structure associated with the chart $F_{\underline{a}}(P_i) \to G_{\underline{a}}(A_i) \cap \left(G_{\underline{a}}(A_i)[\frac{1}{t_i}]\right)^{\times}$.

Step 3. Construct some structure of a log smooth log $\mathcal{O}_C$-variety on X: Now we simply define the log structure on X to be the fiber product $(X^{(i)}, \mathcal{M}_{X^{(i)}}) \times_{(\operatorname{Spec} A_i, \mathcal{M}_i)} (\operatorname{Spec} \mathcal{O}_C, \mathcal{M}_C)$ in the category of log schemes. Recall that [Ogu18, Proposition III.2.1.2] ensures that the functor sending a log scheme to underlying scheme commutes with fiber products. Thus the underlying scheme of the fiber product $(X^{(i)}, \mathcal{M}_{X^{(i)}}) \times_{(\operatorname{Spec} A_i, \mathcal{M}_i)} (\operatorname{Spec} \mathcal{O}_C, \mathcal{M}_C)$ is exactly $X^{(i)} \times_{\operatorname{Spec} A_i} \operatorname{Spec} \mathcal{O}_C = X$. Thus, this does define some log structure $(X, \mathcal{M}_{X,i})$. We prefer to use this notation to emphasize that we do not know at this point if this structure is independent of a choice of i.

Moreover, X with this log structure étale locally admits a strict étale morphism[26]

$$\begin{aligned} X \to & \operatorname{Spec} \mathcal{O}_C \times_{\operatorname{Spec} A_i} \operatorname{Spec} G_{\overline{a}}(A_i) \\ & \simeq \operatorname{Spec} G_{\overline{a}}(\mathcal{O}_C) \\ & \simeq \operatorname{Spec} \mathcal{O}_C \times_{\operatorname{Spec} \mathbf{Z}[V]} \operatorname{Spec} G_{\overline{a}}(\mathbf{Z}[V]) \\ & \simeq \operatorname{Spec} \mathcal{O}_C \times_{\operatorname{Spec} \mathbf{Z}[V]} \operatorname{Spec} (\mathbf{Z}[F_{\overline{a}}(V)]) \end{aligned}$$

and the target has a chart given by $F_{\overline{a}}(V) \to G_{\overline{a}}(\mathcal{O}_C) \cap \left(G_{\overline{a}}(\mathcal{O}_C)[\frac{1}{\pi}]\right)^{\times}$. In particular, the log scheme $(X, \mathcal{M}_{X,i})$ admits charts étale locally with the associated monoids $F_{\overline{a}}(V)$. This already implies that the constructed log structure is quasi-coherent. Now we need to study properties of these monoids to make sure that $(X, \mathcal{M}_{X,i})$ is actually a log smooth log $\mathcal{O}_C$-variety.

We recall that Remark C.2.12 guarantees that $F_{\overline{a}}(V)$ is V-finitely presented, integral and the natural map $V \to F_{\overline{a}}(V)$ is injective. In particular, X becomes an integral quasi-coherent log scheme with V-finitely presented charts. This implies X is a log $\mathcal{O}_C$-variety by Remark C.1.31. Furthermore, X is log smooth as it admits strict étale morphisms $X \to \operatorname{Spec} \mathcal{O}_C \times_{\operatorname{Spec} \mathbf{Z}[V]} \operatorname{Spec} (\mathbf{Z}[F_{\overline{a}}(V)])$ étale locally on X.

Step 4. Show that the log structure $\mathcal{M}_{X,i}$ is independent on i and coincides with $\mathcal{O}_X \cap j_* \mathcal{O}_{X_C}^{\times}$: Similarly to what we did in Step 3, we can define the log structure $\mathcal{M}_{X,j}$ on X as the pullback $(X^{(j)}, \mathcal{M}_{X^{(j)}}) \times_{(\operatorname{Spec} A_j, \mathcal{M}_j)} (\operatorname{Spec} \mathcal{O}_C, \mathcal{M}_C)$ for any $j \geq i$. However, Corollary C.2.9 guarantees that $(X, \mathcal{M}_{X,j}) \simeq (X, \mathcal{M}_{X,i})$ as $(X^{(j)}, \mathcal{M}_{X^{(j)}}) \simeq (X^{(i)}, \mathcal{M}_{X^{(i)}}) \times_{(\operatorname{Spec} A_i, \mathcal{M}_i)} (\operatorname{Spec} A_j, \mathcal{M}_j)$.

Now we consider the fiber square

$$\begin{array}{ccc} X & \xrightarrow{g'^{(j)}} & X^{(j)} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{(j)}} \\ \operatorname{Spec} \mathcal{O}_C & \xrightarrow{g^{(j)}} & \operatorname{Spec} A_j \end{array}$$

and denote the composition $h^{(j)} := f^{(j)} \circ g'^{(j)} = g^{(j)} \circ f$. Then the proof of [Ogu18, Proposition III.2.1.2] shows that the log structure on X is given by the the coproduct of morphisms of log

[26]In the formula below, we slightly abuse notation and consider $\underline{a}$ as a sequence for V using the natural morphism $P_i \to A_i \cap A_i[\frac{1}{t_i}]^{\times} \to \mathcal{O}_C \setminus \{0\} = V$.

structures $h_{\log}^{(j),*}(\mathcal{M}_j) \to f_{\log}^*(\mathcal{M}_C)$ and $h_{\log}^{(j),*}(\mathcal{M}_j) \to g_{\log}'^{(j),*}(\mathcal{M}_{X^{(j)}})$. We denote this coproduct as $f_{\log}^*(\mathcal{M}_C) \oplus^{\log}_{h_{\log}^{(j),*}(\mathcal{M}_j)} g_{\log}'^{(j),*}(\mathcal{M}_{X^{(j)}})$. Now we write $\mathcal{M}_{X,i} \simeq \operatorname{colim} \mathcal{M}_{X,j}$ as this system is just constant. We claim there are isomorphisms

$$
\begin{aligned}
\operatorname{colim} \mathcal{M}_{X,j} &\simeq \operatorname{colim} \left(f_{\log}^*(\mathcal{M}_C) \bigoplus_{h_{\log}^{(j),*}(\mathcal{M}_j)}^{\log} g_{\log}'^{(j),*}\left(\mathcal{M}_{X^{(j)}}\right) \right) \\
&\simeq \left(\operatorname{colim} f_{\log}^*\left(\mathcal{M}_C\right)\right) \bigoplus_{\operatorname{colim} h_{\log}^{(j),*}(\mathcal{M}_j)}^{\log} \left(\operatorname{colim} g_{\log}'^{(j),*}\left(\mathcal{M}_{X^{(j)}}\right)\right) \\
&\simeq f_{\log}^*(\mathcal{M}_C) \bigoplus_{\operatorname{colim} f_{\log}^*\left(g_{\log}^{(j),*}\mathcal{M}_i\right)}^{\log} \left(\operatorname{colim} g_{\log}'^{(j),*}\left(\mathcal{M}_{X^{(j)}}\right)\right) \\
&\simeq f_{\log}^*(\mathcal{M}_C) \bigoplus_{f_{\log}^*\left(\operatorname{colim}\left(\left(g_{\operatorname{plog}}^{(j),*}\mathcal{M}_j\right)^{\log}\right)\right)}^{\log} \operatorname{colim}\left(\left(g_{\operatorname{plog}}'^{(j),*}\mathcal{M}_{X^{(j)}}\right)^{\log}\right) \\
&\simeq f_{\log}^*(\mathcal{M}_C) \bigoplus_{f_{\log}^*\left(\left(\operatorname{colim} g_{\operatorname{plog}}^{(j),*}\mathcal{M}_j\right)^{\log}\right)}^{\log} \left(\operatorname{colim} g_{\operatorname{plog}}'^{(j),*}\mathcal{M}_{X^{(j)}}\right)^{\log} \\
&\simeq f_{\log}^*(\mathcal{M}_C) \bigoplus_{f_{\log}^*(\mathcal{M}_C)}^{\log} \left(\mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times\right) \\
&\simeq \mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times \ .
\end{aligned}
$$

Now we explain each isomorphism. The first is just the definition of $\mathcal{M}_{X,j}$. The second comes from the fact that filtered colimits commute with push-outs. The third just uses the fact that $h^{(j)} = g^{(j)} \circ f$ and that the colimit of a constant system is isomorphic to that constant term. The fourth uses that $f_{\log}^*$ is left adjoint (so it commutes with all colimits) and Corollary C.1.26. The fifth uses that $(-)^{\log}$ is left adjoint, thus it commutes with arbitrary colimits, where colimits are understood in the category of *pre-log structures*. The sixth uses that $\operatorname{colim} {g_{\operatorname{plog}}^{(j)}}^{*}\mathcal{M}_i$ is already isomorphic to $\mathcal{M}_C$ (as we will soon justify), and so it stays the same after applying $(-)^{\log}$. Similarly, $\operatorname{colim} g_{\operatorname{plog}}'^{(j),*}\mathcal{M}_{X^{(j)}}$ is already isomorphic to $\mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times$ (as we will soon justify), so so it stays the same after applying $(-)^{\log}$. The last isomorphism is trivial.

So, overall, the only thing we are left to show is that $\operatorname{colim} g_{\operatorname{plog}}^{(j),*}\mathcal{M}_j \simeq \mathcal{M}_C$ and $\operatorname{colim} g_{\operatorname{plog}}'^{(j),*}\mathcal{M}_{X^{(j)}} \simeq \mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times$. We show the second, and the proof of the first is similar. The pre-log pullback $g_{\operatorname{plog}}'^{(j),*}\mathcal{M}_{X^{(j)}}$ is given by

$$
g'^{(j),-1}\mathcal{M}_{X^{(j)}} = g'^{(j),-1}(\mathcal{O}_{X^{(j)}} \cap \iota_*^{(j)}\mathcal{O}_{U^{(j)}}^\times) \to g'^{(j),-1}\mathcal{O}_{X^{(j)}} \to \mathcal{O}_X \ .
$$

where $\iota^{(j)}\colon U^{(j)} \to X^{(j)}$ is the complement of $X^{(j)} \times_{\operatorname{Spec} A_j} \operatorname{Spec} A_j/t_j$. So the question boils down to showing that the natural morphism

$$\operatorname{colim} g'^{(j),-1}(\mathcal{O}_{X^{(j)}} \cap \iota_*^{(j)}\mathcal{O}_{U^{(j)}}^\times)) \to \mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times$$

is an isomorphism, where colim is understood as the colimit in the category of sheaves of monoids. Since $g'^{(j),-1}$ is exact, it commutes with intersection (or, actually, any fiber product). Moreover, filtered colimits commute with finite limits, so we can rewrite the map as:

$$\beta\colon \operatorname{colim} g'^{(j),-1}\mathcal{O}_{X^{(j)}} \cap \operatorname{colim} g^{(j),-1}\iota_*^{(j)}\mathcal{O}_{U^{(j)}}^\times \to \mathcal{O}_X \cap j_*\mathcal{O}_{X_C}^\times$$

that we want to be an isomorphism. Now [Gro66, Corollary 8.2.12] implies that $\operatorname{colim} g'^{(j),-1}\mathcal{O}_{X^{(j)}} \to \mathcal{O}_X$ is an isomorphism. In order to establish that the morphism β is an isomorphism, it suffices to show a local section f of $\mathcal{O}_X$ is invertible on X_C if and only if it comes from some local section of $\mathcal{O}_{X^{(j)}}$ invertible on $U^{(j)}$ for some large j. We have already shown that f comes from some finite level i, and so we only have to show that it becomes invertible on $U^{(j)}$ for some large $j \geq i$. This again follows from [Gro66, Corollary 8.2.12] as $X_C = \lim U^{(j)}$. This finishes the proof that the structure of the log smooth log $\mathcal{O}_C$-variety on X constructed above coincides with the log structure associated to the special fiber. □

References

[AK00] D. Abramovich and K. Karu. "Weak semistable reduction in characteristic 0". In: *Invent. Math.* 139.2 (2000), pp. 241–273.

[ALPT19] K. Adiprasito, G. Liu, I. Pak, and M. Temkin. *Log smoothness and polystability over valuation rings*. https://arxiv.org/pdf/1806.09168.pdf. 2019.

[ALT19] K. Adiprasito, G. Liu, and M. Temkin. *Semistable reduction in characteristic* 0. https://arxiv.org/abs/1810.03131. 2019.

[Ber93] V. Berkovich. "Étale cohomology for non-Archimedean analytic spaces". In: *Inst. Hautes Études Sci. Publ. Math.* 78 (1993), 5–161 (1994).

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436.

[BL93] Siegfried Bosch and Werner Lutkebohmert. "Formal and rigid geometry. I. Rigid spaces." In: *Mathematische Annalen* 295 (1993), pp. 291–318.

[Bos14] S. Bosch. *Lectures on formal and rigid geometry*. Vol. 2105. Lecture Notes in Mathematics. Springer, Cham, 2014, pp. viii+254.

[Bou98] N. Bourbaki. *Commutative algebra. Chapters 1–7*. Translated from the French, Reprint of the 1989 English translation. Springer-Verlag, Berlin, 1998, pp. xxiv+625.

[CCO14] C. Chai, B. Conrad, and F. Oort. *Complex multiplication and lifting problems*. Vol. 195. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2014, pp. x+387.

[Con07] B. Conrad. "Deligne's notes on Nagata compactifications". In: *J. Ramanujan Math. Soc.* 22.3 (2007), pp. 205–257.

[Con99] B. Conrad. "Irreducible components of rigid spaces". In: *Ann. Inst. Fourier (Grenoble)* 49.2 (1999), pp. 473–541.

[CS86] G. Cornell and J. H. Silverman, eds. *Arithmetic geometry*. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984. Springer-Verlag, New York, 1986, pp. xvi+353.

[Del85] P. Deligne. "Le lemme de Gabber". In: *Astérisque* 127 (1985). Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), pp. 131–150.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 75–109.

[Elk73] Renée Elkik. "Solutions d'équations à coefficients dans un anneau hensélien". In: *Ann. Sci. École Norm. Sup. (4)* 6 (1973), 553–603 (1974).

[FK18] K. Fujiwara and F. Kato. *Foundations of rigid geometry. I.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2018, pp. xxxiv+829.

[FK88] E. Freitag and R. Kiehl. *Étale cohomology and the Weil conjecture.* Springer-Verlag, Berlin, 1988, pp. xviii+317.

[GD71] A. Grothendieck and J. A. Dieudonné. *Eléments de géométrie algébrique. I.* Vol. 166. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1971, pp. ix+466.

[Gro61] A. Grothendieck. "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes". In: *Inst. Hautes Études Sci. Publ. Math.* 8 (1961), p. 222.

[Gro63] A. Grothendieck. *Revêtements étales et groupe fondamental.* Vol. 1960/61. Séminaire de Géométrie Algébrique. Institut des Hautes Études Scientifiques, Paris, 1963, iv+143 pp. (not consecutively paged) (loose errata).

[Gro64] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I". In: *Inst. Hautes Études Sci. Publ. Math.* 20 (1964), p. 259.

[Gro65] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II". In: *Inst. Hautes Études Sci. Publ. Math.* 24 (1965), p. 231.

[Gro66] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III". In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255.

[Gro67] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV". In: *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), p. 361.

[Har03] Urs T. Hartl. "Semi-stable models for rigid-analytic spaces". In: *Manuscripta Math.* 110.3 (2003), pp. 365–380.

[Hub94] R. Huber. "A generalization of formal schemes and rigid analytic varieties". In: *Math. Z.* 217.4 (1994), pp. 513–551.

[Hub96] R. Huber. *Étale cohomology of rigid analytic varieties and adic spaces.* Friedr. Vieweg & Sohn, Braunschweig, 1996, pp. x+450.

[ILO14] L. Illusie, Y. Laszlo, and F. Orgogozo, eds. *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents.* Séminaire à l'École Polytechnique 2006–2008. [Seminar of the Polytechnic School 2006–2008], With the collaboration of Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin and Weizhe Zheng, Astérisque No. 363-364 (2014) (2014). Société Mathématique de France, Paris, 2014, i–xxiv and 1–625.

[Jon96] A. J. de Jong. "Smoothness, semi-stability and alterations". In: *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), pp. 51–93.

[Jon97] A. J. de Jong. "Families of curves and alterations". In: *Ann. Inst. Fourier (Grenoble)* 47.2 (1997), pp. 599–621.

[Kat89] K. Kato. "Logarithmic structures of Fontaine-Illusie". In: *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*. Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.

[KKMS73] G. Kempf, F. Knudsen, D. Mumford, and G Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973, pp. viii+209.

[Lüt90] Werner Lütkebohmert. "Formal-algebraic and rigid-analytic geometry". In: *Math. Ann.* 286.1-3 (1990), pp. 341–371.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1986.

[Niz06] W. Nizioł. "Toric singularities: log-blow-ups and global resolutions". In: *J. Algebraic Geom.* 15.1 (2006), pp. 1–29.

[Ogu18] A. Ogus. *Lectures on logarithmic algebraic geometry*. Vol. 178. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2018, pp. xviii+539.

[RG71] M. Raynaud and L. Gruson. "Critères de platitude et de projectivité. Techniques de "platification" d'un module". In: *Invent. Math.* 13 (1971), pp. 1–89.

[Sem15] The Learning Seminar authors. *Stanford Learning Seminar*. http://virtualmath1.stanford.edu/~conrad/Perfseminar/. 2014-2015.

[Sta21] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2021.

[Tem00] M. Temkin. "On local properties of non-Archimedean analytic spaces". In: *Math. Ann* 318(3) (2000), pp. 585–607.

[Tem08] M. Temkin. "Desingularization of quasi-excellent schemes in characteristic zero". In: *Adv. Math.* 219.2 (2008), pp. 488–522.

[Tem10] M. Temkin. "Stable modification of relative curves". In: *J. Algebraic Geom.* 19.4 (2010), pp. 603–677.

[Tem11] M. Temkin. "Relative Riemann-Zariski spaces". In: *Israel J. Math.* 185 (2011), pp. 1–42.

[Tem17] M. Temkin. "Altered local uniformization of Berkovich spaces". In: *Israel J. Math.* 221.2 (2017), pp. 585–603.

[Tsu19] Takeshi Tsuji. "Saturated morphisms of logarithmic schemes". In: *Tunis. J. Math.* 1.2 (2019), pp. 185–220.

[TT90] R. W. Thomason and Thomas Trobaugh. "Higher algebraic K-theory of schemes and of derived categories". In: *The Grothendieck Festschrift, Vol. III*. Vol. 88. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[Zav21a] B. Zavyalov. *Mod-p Poincaré Duality in p-adic Analytic Geometry*. http://stanford.edu/~bogdzav/refs/Poincare_Duality.pdf. 2021.

[Zav21b] B. Zavyalov. *Quotients Of Admissible Formal Schemes and Adic Space by Finite Groups*. https://arxiv.org/abs/2102.02762. 2021.