LEFSCHETZ THEOREMS IN FLAT COHOMOLOGY AND APPLICATIONS

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ABSTRACT. We prove a version of the Lefschetz hyperplane theorem for fppf cohomology with coefficients in any finite flat commutative group scheme over the ground field. As consequences, we establish new Lefschetz results for the Picard scheme. We then use Godeaux-Serre varieties to give a number of examples of pathological behavior of families in positive characteristic.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** This paper is divided into two essentially disjoint parts. In the first part, we prove a version of the Lefschetz hyperplane theorem for finite flat commutative group scheme coefficients. Specifically, we have the following theorem.

Theorem 1.1.1. (Theorem 2.4.5) Let k be a field, Y a projective syntomic k-scheme, $X \subset Y$ a closed subscheme of dimension d, and G a finite flat commutative k-group scheme. Then the cone

$$\mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X, G\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z})$ if

(1) $Y \cong \mathbf{P}_k^N$ for some N and X is a global complete intersection, or

(2) $X \subset Y$ is a sufficiently ample Cartier divisor (see Definition 2.1.8 and Remark 2.1.9).

Remark 1.1.2. Example 2.4.6 shows that Theorem 1.1.1 may fail for ample (but not sufficiently ample) divisors $X \subset Y$. Example 2.4.7 shows that Theorem 1.1.1 may also fail for finite flat commutative Y-group schemes that are not defined over k. The assumption of syntomicity is similar to the assumptions on the Lefschetz theorems proved in [SGA2].

We expect that there is a version of Theorem 1.1.1 for non-commutative finite k-group schemes, but we cannot prove it.

Question 1.1.3. Let $X \subset \mathbf{P}_k^N$ be a complete intersection of dimension at least 2, and G a finite (not necessarily commutative) k-group scheme. Is the natural morphism $\mathrm{H}^1(\mathbf{P}_k^N, G) \to \mathrm{H}^1(X, G)$ a bijection? The same question may be asked for $X \subset Y$ a sufficiently ample Cartier divisor (with an appropriate definition of "sufficiently ample").

Remark 1.1.4. If both X and Y are smooth and the ground field k is algebraically closed, Question 1.1.3 has a positive answer. This follows from the Lefschetz type result for Nori's fundamental group (see [BH07, Theorem 1.1]).

By devissage, Theorem 1.1.1 is reduced to the cases $G = \mu_\ell$, α_p, and $\mathbf{Z}/p$, where ℓ is a prime number and p is the characteristic of k. The cases of α_p and $\mathbf{Z}/p$ are reduced to questions of coherent cohomology using standard exact sequences. For $\ell \neq p$, the case of μ_ℓ is settled using results in the theory of perverse sheaves.

The case of μ_p will give us the most difficulty. Here we will find it convenient to pivot to proving a Lefschetz hyperplane theorem for the cohomology of the Tate twists $\mathbf{Z}_p(i)$. Using the Nygaard filtration, this will ultimately be reduced to proving a Lefschetz hyperplane theorem for each filtered piece in the conjugate filtration on de Rham cohomology, which has been established in [ABM21]. In particular, we get a Lefschetz hyperplane theorem for the syntomic cohomology of the Tate twists $\mathbf{Z}_p(i)$ defined in [BMS19] (see also Section 1.2):

Theorem 1.1.5. (Corollary 2.2.2) Let k be a field of characteristic $p > 0$, Y a projective syntomic k-scheme, $X \subset Y$ a closed subscheme of dimension d, and G a finite flat commutative k-group scheme. Then the cone

$$C := \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{syn}}\left(Y, \mathbf{Z}_p\left(i\right)\right) \to \mathrm{R}\Gamma_{\mathrm{syn}}\left(X, \mathbf{Z}_p\left(i\right)\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z}_p)$ with $\mathrm{H}^d(C)$ torsion-free for $i \geq 0$ if

(1) $Y \cong \mathbf{P}_k^N$ for some N and X is a global complete intersection, or

(2) $X \subset Y$ is a sufficiently ample Cartier divisor.

As a consequence of Theorem 1.1.1, we prove a Lefschetz hyperplane theorem for $\mathbf{Pic}^\tau$ of projective syntomic k-schemes.

Theorem 1.1.6. (Theorem 2.5.2, Corollary 2.5.7) Let k be a field, and Y a projective syntomic k-scheme of pure dimension d, and $X \subset Y$ a sufficiently ample Cartier divisor. Then

(1) $\mathbf{Pic}^\tau_{Y/k} \to \mathbf{Pic}^\tau_{X/k}$ is an isomorphism if $d \geq 3$;

(2) $\mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$ is an isomorphism if $d \geq 4$.

Remark 1.1.7. In fact, our proof shows that the divisor $X \subset Y$ need only be a Hodge 2-equivalence (see Definition 2.1.1) with the property that $Y - X$ is affine. In particular, Theorem 2.1.4 implies that Theorem 1.1.6 holds for all ample Cartier divisors in either of the following two situations:

(1) Y is a smooth projective variety over a field of characteristic 0;

(2) Y is a smooth projective variety over a field of characteristic $p > 2$ such that Y admits a lift to $W_2(\overline{k})$.

However, Remark 2.5.8 gives some evidence that Theorem 1.1.6 is quite likely false for an ample (but not sufficiently ample) divisor on a more general Y.

Remark 1.1.8. A. Langer has informed us that an effective version of Theorem 1.1.6 for $\mathbf{Pic}^{\tau}_{\mathrm{red}}$ follows from his Lefschetz type theorem for the S-fundamental group when X and Y are smooth (see [Lan11, Theorem 10.2 and 10.4]).

Theorem 1.1.6 appears to be new in every dimension, even for smooth X and Y. The main difficulty in deducing it from Theorem 1.1.1 is that the Picard schemes can be highly non-reduced in positive characteristic, so one cannot argue on the level of Picard groups, i.e., on the level of k-points. To overcome this issue, we need to use the structure theory of commutative group schemes over a field to obtain an isomorphism criterion (see Lemma 2.5.6), and to verify the hypotheses of this criterion we need to generalize Theorem 1.1.1 to more general base schemes, at least for $G = \mu_p$ (see Corollary 2.4.4).

In the case of complete intersections in projective space, we can show that the hypothesis of sufficient ampleness is not necessary, and we can make a more refined statement, recovering [CS21, Corollary 7.2.3].

Theorem 1.1.9. ([CS21, Corollary 7.2.3], Theorem 2.5.1) Let k be a field, and $X \subset \mathbf{P}^N_k$ be a complete intersection of dimension at least 2. Then

(1) $\mathrm{Pic}(X)_{\mathrm{tors}} = 0$;

(2) the group scheme $\mathbf{Pic}^{\tau}_{X/k}$ is trivial;

(3) the class of $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^N}(1)$ is a non-divisible element of $\mathrm{Pic}(X)$.

If $\dim X \geq 3$, then Theorem 1.1.9 was essentially settled by Grothendieck in [SGA2]. If $\dim X \geq 2$ and X is smooth, then this was settled by Deligne in [$\mathrm{SGA7_{II}}$, Exp. XI]. A version for weighted complete intersection surfaces with certain limited singularities can be found in [Lan84, §1]. The general case was established in [CS21, Corollary 7.2.3]. However, Theorem 1.1.6 does not seem to follow from their methods. When we started writing this paper, we were not aware that Theorem 1.1.9 was proven in [CS21].

Both proofs of Theorem 1.1.9 share a similar idea of using perfectoid techniques to reduce the study of flat cohomology of μ_p to studying the cohomology of certain coherent sheaves. However, the details of the proofs seem to be fairly different. Our proof is global and is based on the Lefschetz hyperplane theorem from [ABM21], while the proof in [CS21] is local; in their argument they relate the Picard group of X to the local Picard group of the vertex x of the affine cone over X, and then use local techniques to study that Picard group. Namely, if R is the local ring of x, $\mathrm{Pic}(X)/\mathbf{Z}[\mathcal{O}_X(1)]$ injects into $\mathrm{Pic}(\operatorname{Spec} R \setminus \{x\})$, and this is good enough to prove Theorem 1.1.9. However, the failure of the map $\mathrm{Pic}(X)/\mathbf{Z}[\mathcal{O}_X(1)] \to \mathrm{Pic}(\operatorname{Spec} R \setminus \{x\})$ to be surjective is the main reason why their methods do not seem to be sufficient to obtain a proof of Theorem 1.1.6. Both proofs are uniform in $\dim X$ and do not make any assumptions on singularities of X.

In the second part of this paper, we give a number of examples of the pathological behavior of various cohomology theories over fields of positive characteristic. Specifically, we give examples of the following types:

Theorem 1.1.10. Let R be a discrete valuation ring of equicharacteristic $p > 0$, and $d \geq 2$. Then there is a smooth, projective morphism $f\colon X \to S = \operatorname{Spec} R$ with geometrically connected fibers of pure dimension d such that

1. (Theorem 3.2.1 and Theorem 3.4.1) $\mathrm{rk}_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(X_s/k(s)) > \mathrm{rk}_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(X_\eta/k(\eta))$;
2. (Theorem 3.5.4) $\mathrm{R}^1 f_{\mathrm{crys},*}\mathcal{O}_{X/S}$ is not a crystal on $(S/S)_{\mathrm{crys}}$.
3. (Corollary 3.4.3) $\mathrm{H}^2_{\mathrm{dR}}(X/S)$ does not admit any stratification structure (in the sense of [BO78, Definition 2.10]). In particular, the Gauss-Manin connection on $\mathrm{H}^2_{\mathrm{dR}}(X/S)$ does not prolong to a stratification;

These examples seem not to have been previously established in the literature. Though it is well-known that de Rham numbers can jump in smooth families in characteristic p, it does not seem to be addressed anywhere besides a brief remark in [Ray79, 4.2.6(iii)]. The question of whether $\mathrm{R}^i f_{\mathrm{crys},*}\mathcal{O}_{X/S}$ are crystals in the equicharacteristic case (i.e., $p\mathcal{O}_S = 0$) is raised in [BO78, Remark 7.10]. Our second example negatively answers this question. As for the third example, Grothendieck constructed an example of a smooth projective family (see [Gro68a, §3.5] and [BO78, Example 2.18]) where the Gauss-Manin connection on $\mathrm{H}^1_{\mathrm{dR}}(X/S)$ does not admits any *functorial* stratification. Using p-curvature considerations and the relation between p-curvature and the Kodaira-Spencer map (see [Kat72, Theorem 3.2] for a precise statement), it is not hard to construct examples where the Gauss-Manin connection does not extend to a stratification in positive characteristic (see Remark 3.4.4). The novelty of our example is that $\mathrm{H}^2_{\mathrm{dR}}(X/S)$ has no stratification structure whatsoever.

All of the above examples are built using a general construction due to Godeaux and Serre, which we recall briefly in Theorem 3.1.1. We give an example which seems close to what Raynaud had in mind in Section 3.2, followed by another example in Section 3.4. The vague idea behind both examples is to find a sufficiently weird finite flat group scheme G over S such that the Godeaux-Serre construction applied to G gives the desired family $X \to S$.

Our first example in Section 3.2 is fairly elementary. It studies the Hodge numbers $h^{0,1}$ and $h^{1,0}$ of the fibers of $X \to S$ in an explicit way by relating them to the the tangent space of $\mathrm{Pic}^{\tau}_{X/S}$ and the global sections of differential forms respectively. To pass from Hodge numbers to de Rham numbers, we use the (partial) degeneration of the Hodge-to-de Rham spectral sequence for varieties which admit a lift to $W_2(k)$ (see [DI87]).

Our second example is based on the idea that the smooth, proper family of Artin stacks $BG \to S$ already has jumps in the first de Rham numbers of its fibers for some explicit G. To see this, we use a result of Mondal [Mon21] which relates the second crystalline cohomology group of BG to its Dieudonné module. This gives the desired family in the world of stacks. The second step is to use the Godeaux-Serre construction to approximate BG by a family of smooth, projective scheme without changing the first de Rham numbers of fibers. To do this, we use the notion of a Hodge d-equivalence from [ABM21].

1.2. **Terminology.** Throughout this paper, we extensively use the formalism of "derived" cohomology theories, so we briefly recall our conventions.

For a ring A, we denote by $D(A)$ its *triangulated derived category*, and by $\mathcal{D}(A)$ its ∞-enhancement. We denote by $\mathcal{D}F(A) = \mathrm{Fun}_\infty(\mathbf{N}, \mathcal{D}(A))$ a *filtered derived category of A*.

For a fixed prime p and an object $M \in \mathcal{D}(A)$, the *derived quotient* $[M/p] \coloneqq \mathrm{cone}(M \xrightarrow{p} M)$ is the cone of the multiplication by p map. We denote by $\widehat{\mathcal{D}}(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of p-adically derived complete objects (in the sense of [Sta21, Tag 091S]).

For an $\mathbf{F}_p$-algebra k, and a k-stack $\mathcal{X}$. We denote the *Frobenius twist* of $\mathcal{X}$ relative to k by $\mathcal{X}^{(1)}$.

For a ring k and a k-algebra R, we denote by $\wedge^i L_{R/k} \in \mathcal{D}(k)$ *derived i-th wedge power of the cotangent complex*. For a syntomic k-stack (in particular, syntomic k-scheme) $\mathcal{X}$, we define *(derived) Hodge cohomology*

$$\mathrm{R}\Gamma(\mathcal{X}, \wedge^i L_{\mathcal{X}/k}) \in \mathcal{D}(k)$$

by syntomic (hyper-)descent from the affine case (see [ABM21, Construction 2.7] for details).

Likewise, for an $\mathbf{F}_p$-algebra k and a k-algebra R, we denote by $\mathrm{dR}_{R/k} \in \mathcal{D}(k)$ its *derived de Rham complex*. This complex comes with an exhaustive conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet \mathrm{dR}_{R/k} \in \mathcal{D}F(k)$. For a syntomic k-stack $\mathcal{X}$, we define its *derived de Rham cohomology* and *its conjugate filtration*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/k) \in \mathcal{D}(k),\ \mathrm{Fil}^{\mathrm{conj}}_\bullet \mathrm{R}\Gamma_{\mathrm{dR}}(X/k) \in \mathcal{D}F(k)$$

by syntomic (hyper-)descent from the affine case. As explained in [ABM21, Definition 3.1(b')] (whose proof does not use perfectness of k), this filtration is exhaustive with associated graded pieces

$$\mathrm{gr}^{\mathrm{conj}}_i \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/k) \simeq \mathrm{R}\Gamma(\mathcal{X}, \wedge^i L_{\mathcal{X}^{(1)}/k})[-i].$$

Using [Bha12, Corollary 3.10] and Zariski (hyper-)descent, one sees that derived de Rham cohomology is canonically isomorphic to the classical de Rham cohomology for a k-smooth X, i.e.,

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/k) \simeq \mathrm{R}\Gamma(X, \Omega^\bullet_X).$$

For a perfect field k of characteristic $p > 0$ and a syntomic k-algebra R, we denote by $\mathrm{R}\Gamma_{\mathrm{crys}}(R/W(k)) \in \widehat{\mathcal{D}}(W(k))$ *crystalline cohomology*. For a syntomic k-stack $\mathcal{X}$, we define its *crystalline cohomology*

$$\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k)) \in \widehat{\mathcal{D}}(W(k))$$

by syntomic descent from the affine case. One can similarly check that it coincides with the usual crystalline cohomology for syntomic k-schemes. Using [Bha12, Theorem 3.27], [BdJ11, Corollary 3.10], and syntomic descent, we get a canonical isomorphism

$$\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k)) \otimes^L_{W(k)} k \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/k).$$

To define the *Nygaard filtration* on $\mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k))$ the crystalline cohomology of a syntomic k-scheme, we note that [BL22, Theorem 4.6.1 and Warning 4.6.2] give an isomorphism

$$F^*\mathrm{R}\Gamma_{\Delta}(X/W(k)) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k))$$

of the Frobenius twist of prismatic cohomology relative to the perfect prism $(W(k), (p))$ and crystalline cohomology relative to the standard pd-structure on $W(k)$. Thus, the relative Nygaard filtration defined on $F^*\mathrm{R}\Gamma_{\Delta}(X/W(k))$ in [BL22, §5.1] can be transported to the crystalline cohomology. Alternatively, one can define the Nygaard filtration as in [BMS19, §8], but this is less convenient for our purposes.

For a perfect field k of characteristic $p > 0$ and a syntomic k-scheme X, we define *syntomic complexes*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(i)) \in \widehat{\mathcal{D}}(\mathbf{Z}_p)$$

as in [BL22, Variant 7.4.12].

Finally, if $S = \operatorname{Spec} R$ is a spectrum of a discrete valuation ring, we denote by $k(\eta) := \mathrm{Frac}(R)$ the quotient field, and $k(s) := R/\mathfrak{m}$ the residue field. Likewise, $\eta = \operatorname{Spec} k(\eta)$ denotes the generic point of S and $s = \operatorname{Spec} k(s)$ the closed point of S. The *geometric generic point* $\overline{\eta}$ is defined as $\operatorname{Spec} \overline{k(\eta)}$ for some choice of an algebraic closure $k(\eta) \subset \overline{k(\eta)}$, and the *geometric special point* $\overline{s}$ is defined as $\operatorname{Spec} \overline{k(s)}$ for some choice of an algebraic closure $k(s) \subset \overline{k(s)}$.

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2. Lefschetz hyperplane theorem for flat cohomology

2.1. Hodge d-equivalences.

This section recalls the main results from [ABM21, §5].

For the rest of the section, we fix a field k.

Definition 2.1.1. [ABM21, Definition 5.1] A k-morphism of syntomic k-stacks $\mathcal{X} \to \mathcal{Y}$ is a *Hodge d-equivalence* if, for every $s \geq 0$, we have

$$\operatorname{cone}\left(\mathrm{R}\Gamma\left(\mathcal{Y}, \wedge^s L_{\mathcal{Y}/k}\right) \to \mathrm{R}\Gamma\left(\mathcal{X}, \wedge^s L_{\mathcal{X}/k}\right)\right) \text{ lies in } D^{\geq d-s}(k).$$

Remark 2.1.2. Definition 2.1.1 may look a bit abstract, but really it just a formal way to say that f satisfies the conclusion of the "Lefschetz hyperplane theorem" for Hodge cohomology groups.

Using the conjugate filtration on derived de Rham cohomology, one can show that a Hodge d-equivalence f automatically induces an isomorphism on low degree de Rham (resp. crystalline) cohomology groups. This, together with the Nygaard filtration, will eventually allow us to get a Lefschetz hyperplane theorem for μ_p cohomology groups.

Lemma 2.1.3. [ABM21, Remark 5.2] Let k be a perfect field of characteristic $p > 0$, and $\mathcal{X} \to \mathcal{Y}$ be a Hodge d-equivalence of syntomic k-stacks. Then

(1) $\operatorname{cone}\left(\mathrm{Fil}_i^{\mathrm{conj}}\mathrm{R}\Gamma_{\mathrm{dR}}\left(\mathcal{Y}/k\right) \to \mathrm{Fil}_i^{\mathrm{conj}}\mathrm{R}\Gamma_{\mathrm{dR}}\left(\mathcal{X}/k\right)\right) \in D^{\geq d}(k)$ for every $i \geq 0$,

(2) $\operatorname{cone}\left(\mathrm{R}\Gamma_{\mathrm{dR}}\left(\mathcal{Y}/k\right) \to \mathrm{R}\Gamma_{\mathrm{dR}}\left(\mathcal{X}/k\right)\right) \in D^{\geq d}(k)$,

(3) $C \coloneqq \operatorname{cone}\left(\mathrm{R}\Gamma_{\mathrm{crys}}\left(\mathcal{Y}/W(k)\right) \to \mathrm{R}\Gamma_{\mathrm{crys}}\left(\mathcal{X}/W(k)\right)\right) \in D^{\geq d}(W(k))$ and $\mathrm{H}^d(C)$ is torsionfree.

Theorem 2.1.4. [ABM21] Let $i\colon X \hookrightarrow Y$ be a closed immersion of syntomic projective k-schemes.

(1) If $Y = \mathbf{P}^N_k$ and $X \subset Y$ is a d-dimensional (global) complete intersection over k, then i is a Hodge d-equivalence;

(2) If Y is a smooth variety of pure dimension $d+1$, $X \subset Y$ is an ample Cartier divisor, and k is a field of characteristic 0, then i is a Hodge d-equivalence;

(3) If Y is a smooth variety of pure dimension $d+1$ which lifts to $W_2(\overline{k})$, $X \subset Y$ is an ample Cartier divisor, and k is a field of characteristic $p > 0$ then i is a Hodge $(\inf(p, d+1) - 1)$-equivalence.

Proof. The first claim is [ABM21, Proposition 5.3]. The second claim follows from [ABM21, Example 5.6] and [ABM21, Proposition 5.7]. To prove the third claim, we can assume that $k = \overline{k}$ is algebraically closed. The proof of [ABM21, Proposition 5.7] shows that $X \to Y$ is a Hodge $(n-1)$-equivalence if

$$\mathrm{R}\Gamma(Y, \wedge^s L_Y \otimes \mathcal{O}_Y(-rX)) \simeq \mathrm{R}\Gamma(Y, \Omega^s_Y(-rX)) \in D^{\geq n-s}(k)$$

for any $s \geq 0$ and all $r > 0$. So we need to show that

$$\mathrm{H}^i\left(Y, \Omega_Y^s(-rX)\right) = 0$$

for $i + s < \inf(p, d+1)$ and all $r > 0$. This follows from [DI87, Corollary 2.8] since Y is smooth and admits a lift to $W_2(k)$ by assumption. □

Definition 2.1.5. [ABM21, Defintion 5.4] A Kodaira pair is an n-dimensional projective syntomic k-scheme Y and an ample line bundle $\mathcal{L}$ such that $\mathrm{R}\Gamma(Y, \wedge^s L_{Y/k} \otimes \mathcal{L}) \in D^{\geq n-s}(k)$ for all $s > 0$ and all $r > 0$.

Remark 2.1.6. If $\mathcal{L}$ is an ample line bundle on a projective syntomic k-scheme Y, then $(Y, \mathcal{L}^{\otimes d})$ is a Kodaira pair for a sufficiently large d. This follows from Serre vanishing and the fact that $\wedge^s L_{Y/k} \in D^{[-s,0]}(Y)$; see [Sta21, Tag 08SL] and [Lur18, Proposition 25.2.4.1, 25.2.4.2].

Theorem 2.1.7. [ABM21] Let Y be a syntomic k-scheme of pure dimension $d+1$, and let $\mathcal{L}$ be an ample line bundle on Y. Then there exists some integer n_0 such that for all $n \geq n_0$ and any effective Cartier divisor $X \subset Y$ defined by a section[1] of $\mathcal{L}^n$, the morphism $X \to Y$ is a Hodge d-equivalence.

Proof. This follows from [ABM21, Proposition 5.7] and Remark 2.1.6. □

Definition 2.1.8. An effective Cartier divisor $X \subset Y$ in a projective k-scheme of pure dimension $d+1$ is *sufficiently ample* if $X \to Y$ is a Hodge d-equivalence and $Y - X$ is an affine open in Y.

Remark 2.1.9. Theorem 2.1.7 implies that, for every ample line bundle $\mathcal{L}$ on Y, there is an integer n_0 such that for all $n \geq n_0$ and any effective Cartier divisor $X \subset Y$ defined by a section of $\mathcal{L}^n$, the morphism $X \to Y$ is sufficiently ample. This explains why we chose such a name.

By Theorem 2.1.4, any ample Cartier divisor in a smooth projective variety Y is sufficiently ample in either of the following situations:

(1) the ground field k is of characteristic 0;

(2) the ground field k is of characteristic $p > 0$, $\dim Y \leq p$, and Y admits a lift over $W_2\left(\overline{k}\right)$.

Theorem 2.1.10. [ABM21, Proposition 5.10] Let $f\colon X \to Y$ be a Hodge d-equivalence of syntomic k-schemes.

(1) For any syntomic k-scheme Z, $X \times_k Z \to Y \times_k Z$ is a Hodge d-equivalence.

(2) If an affine finite type k-group scheme G acts on X and Y making f into a G-equivariant morphism, then the natural morphism of syntomic k-stacks $[X/G] \to [Y/G]$ is a Hodge d-equivalence.

2.2. **$\mathbf{Z}_p(i)$ and μ_p coefficients.** In this section, we prove the Lefschetz hyperplane theorem for μ_p-cohomology groups for Hodge d-equivalences over a perfect field. More generally, we show it for $\mathbf{Z}_p(i)$-cohomology groups for all $i \geq 0$.

For the rest of the section, we fix a perfect field k of characteristic $p > 0$.

Theorem 2.2.1. Let $X \to Y$ be a Hodge d-equivalence of syntomic k-schemes. Then for every $i \geq 0$, the cone

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{syn}}\left(Y, \mathbf{Z}_p\left(i\right)\right) \to \mathrm{R}\Gamma_{\mathrm{syn}}\left(X, \mathbf{Z}_p\left(i\right)\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z}_p)$ and $\mathrm{H}^d(C)$ is torsion-free.

[1] i.e., an effective Cartier divisor such that $\mathcal{O}(X) \simeq \mathcal{L}^n$

We will give a proof shortly, but before doing so we discuss its main application for our purposes.

Corollary 2.2.2. Let Y a projective syntomic k-scheme, $X \subset Y$ a closed subscheme of dimension d, and G a finite flat commutative k-group scheme. Then the cone

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{syn}}\left(Y, \mathbf{Z}_p\left(i\right)\right) \to \mathrm{R}\Gamma_{\mathrm{syn}}\left(X, \mathbf{Z}_p\left(i\right)\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z}_p)$ with $\mathrm{H}^d(C)$ torsion-free for $i \geq 0$ if

(1) $Y \cong \mathbf{P}^N_k$ for some N and X is a global complete intersection, or
(2) $X \subset Y$ is a sufficiently ample Cartier divisor (see Definition 2.1.8).

Proof. In the first case, the map $X \to Y$ is a Hodge d-equivalence by Theorem 2.1.4, and in the second case this map is a Hodge d-equivalence by definition.

Thus the claim for the cohomologies of $\mathbf{Z}_p(i)$ follows from Theorem 2.2.1. The same result guarantees that

$$\mathrm{cone}\left(\left[\mathrm{R}\Gamma_{\mathrm{syn}}\left(Y, \mathbf{Z}_p\left(1\right)\right)/p\right] \to \left[\mathrm{R}\Gamma_{\mathrm{syn}}\left(X, \mathbf{Z}_p\left(1\right)\right)/p\right]\right) \in D^{\geq d}(\mathbf{F}_p).$$

Now [BL22, Proposition 8.4.13] and [Gro68b] imply that

$$\begin{aligned}\left[\mathrm{R}\Gamma_{\mathrm{syn}}\left(Y, \mathbf{Z}_p\left(1\right)\right)/p\right] &\simeq \left[\mathrm{R}\Gamma_{\text{ét}}(Y, \mathbf{G}_m)/p\right][-1] \\ &\simeq \left[\mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \mathbf{G}_m)/p\right][-1] \\ &\simeq \mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \mu_p)\end{aligned}$$

and similarly for X. Combining these observations, we conclude that

$$\mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \mu_p) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X, \mu_p)\right) \in D^{\geq d}(\mathbf{F}_p),$$

as desired. □

Now we go to the proof of Theorem 2.2.1. The main idea of the proof is to deduce it through a series of reduction from Theorem 2.1.4.

Lemma 2.2.3. Let $X \subset Y$ be a Hodge d-equivalence of syntomic k-schemes. Then, for any $i \geq 0$, the cone

$$\mathrm{cone}\left(\mathrm{Fil}^i_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}\left(Y/W(k)\right) \to \mathrm{Fil}^i_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}\left(X/W(k)\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z}_p)$ and $\mathrm{H}^d(C)$ is torsion-free.

Proof. We argue by induction on $i \geq 0$. The case of $i = 0$ is clear from Lemma 2.1.3 since $\mathrm{Fil}^0_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(-/W(k)) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(-/W(k))$.

Now fix $i \geq 0$ and suppose we know the claim for i. Now we use [BL22, Theorem 4.6.1 and Warning 4.6.2] to get an isomorphism

$$F^*\mathrm{R}\Gamma_{\mathbb{\Delta}}(X/W(k)) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k))$$

of the frobenius twist of prismatic cohomology relative to the perfect prism $(W(k), (p))$ and crystalline cohomology relative to the standard pd-structure on $W(k)$. Using a canonical isomorphism

$$\mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k)) \otimes^L_{W(k)} k \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X/k),$$

[BL22, Proposition 5.1.1, Remark 5.1.2][2], and a $\mathbf{Z}_p$-linear (but not $W(k)$-linear) identification

$$(F^*)^{-1}\mathrm{Fil}^{\mathrm{conj}}_i\mathrm{R}\Gamma_{\mathrm{dR}}(X/k) \simeq \mathrm{Fil}^{\mathrm{conj}}_i\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$$

[2]We note that the Breuil-Kisin twists can be canonically trivialized for the prism $(W(k), (p))$

we get commutative diagram of exact triangles in $D(\mathbf{Z}_p)$:

$$\begin{array}{ccccc} \mathrm{Fil}^{i+1}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(Y/W(k)) & \longrightarrow & \mathrm{Fil}^{i}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(Y/W(k)) & \xrightarrow{\phi_i \bmod p} & \mathrm{Fil}^{\mathrm{conj}}_{i}\mathrm{R}\Gamma_{\mathrm{dR}}(Y/k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fil}^{i+1}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k)) & \longrightarrow & \mathrm{Fil}^{i}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k)) & \xrightarrow{\phi_i \bmod p} & \mathrm{Fil}^{\mathrm{conj}}_{i}\mathrm{R}\Gamma_{\mathrm{dR}}(X/k). \end{array}$$

Denote by C, C', and C'' cones of the left, middle, and right vertical maps respectively. Lemma 2.1.3 gives that $C'' \in D^{\geq d}(k)$ and the induction hypothesis gives that $C' \in D^{\geq d}(W(k))$ with $\mathrm{H}^d(C')$ torsion-free. This formally implies that $C \in D^{\geq d}(W(k))$ and that $\mathrm{H}^d(C)$ is torsion-free. □

Now we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Firstly, we note that [BL22, Theorem 5.6.2, Variant 7.4.12], and the considerations as in the proof of Lemma 2.2.3 imply that we have the following commutative diagram of exact triangles:

$$\begin{array}{ccccc} \mathrm{R}\Gamma_{\mathrm{syn}}(Y, \mathbf{Z}_p(i)) & \longrightarrow & \mathrm{Fil}^{i}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(Y/W(k)) & \xrightarrow{\phi_i - 1} & \mathrm{R}\Gamma_{\mathrm{crys}}(Y/W(k)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(i)) & \longrightarrow & \mathrm{Fil}^{i}_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k)) & \xrightarrow{\phi_i - 1} & \mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k)). \end{array}$$

Now, as in the proof of Lemma 2.2.3, we see that it suffices to prove the claim for $\mathrm{R}\Gamma_{\mathrm{crys}}(-/W(k))$ and $\mathrm{Fil}^i_{\mathrm{N}}\mathrm{R}\Gamma_{\mathrm{crys}}(-/W(k))$ separately. The first case is done in Lemma 2.1.3 and the second one in Lemma 2.2.3. □

2.3. μ_ℓ coefficients.

In this section, we give a proof of the Lefschetz hyperplane theorem for μ_ℓ-coefficients in the generality we will need later. The proof is probably well-known to the experts, but it seems hard to extract from the literature. The main difficulty is that we do not require the ambient space Y to be smooth, but only syntomic.

For the rest of the section, we fix a separably closed field k (possibly of characteristic 0) and a prime number ℓ not equal to the characteristic of k.

We recall that there is a well-behaved theory of perverse $\mathbf{F}_\ell$-sheaves on finite type k-schemes; see [BBD82, Intro to Ch. 4] or [BH21, §4][3] for a more detailed discussion. We only mention two main results that we will need in this section.

Lemma 2.3.1. Let X a finite type k-scheme of pure dimension d. Then

(1) the sheaf $\mu_\ell[d]$ is a perverse sheaf on X if X is k-syntomic;
(2) for a perverse $\mathbf{F}_\ell$-sheaf $\mathcal{L}$, the complex $\mathrm{R}\Gamma_c(X_{\text{ét}}, \mathcal{L})$ lies in $D^{\geq 0}(\mathbf{F}_\ell)$ if X is affine.

Proof. The first claim is [Ill03, Corollaire 1.4]. The second statement is [Ill03, Théorème 2.4] or [BBD82, Théorème 4.1.1]. □

Theorem 2.3.2. Let Y be a syntomic projective k-scheme of pure dimension $d+1$, and let $X \subset Y$ be an ample Cartier divisor. Then the cone

$$\mathrm{cone}\,(\mathrm{R}\Gamma_{\text{ét}}(Y, \mu_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mu_\ell))$$

lies in $D^{\geq d}(\mathbf{F}_\ell)$.

[3]This is written for rigid-analytic spaces, but similar (and, in fact, easier) proofs work in the algebraic situation.

Proof. Denote the complement of X in Y by U. Then we have an exact triangle

$$\mathrm{R}\Gamma_c(U_{\text{ét}}, \mu_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(Y, \mu_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mu_\ell).$$

By Lemma 2.3.1, $\mu_\ell[d+1]$ is a perverse sheaf on U. Therefore the same lemma implies that $\mathrm{R}\Gamma_c(U_{\text{ét}}, \mu_\ell) \in D^{\geq d+1}(\mathbf{F}_\ell)$ finishing the proof. □

2.4. **Finite flat commutative group scheme coefficients.** In this section, we prove the general version of the Lefschetz hyperplane theorem. The strategy is to reduce the general case to the cases of finite flat group schemes $G = \mu_\ell$, μ_p, α_p, and $\mathbf{Z}/p$, and deal with each case separately.

Lemma 2.4.1. Let k be a perfect field of characteristic $p > 0$, $X \to Y$ a Hodge d-equivalence of syntomic k-schemes, and G a commutative finite flat k-group scheme with a finite filtration $\mathrm{Fil}^\bullet G$ such that all $\mathrm{gr}^i G$ are isomorphic to μ_p, α_p, or $\mathbf{Z}/p$. Then

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X, G\right)\right) \in D^{\geq d}(\mathbf{Z}).$$

Proof of Lemma 2.4.1. One easily reduces to the case $G = \mu_p$, $G = \alpha_p$, or $G = \mathbf{Z}/p$. The first case is just Theorem 2.2.1. In the second case, one uses the short exact sequence

$$0 \to \alpha_p \to \mathbf{G}_a \xrightarrow{f \mapsto f^p} \mathbf{G}_a \to 0$$

to reduce the claim to Theorem 2.1.4. In the last case, one uses the sequence

$$0 \to \mathbf{Z}/p \to \mathbf{G}_a \xrightarrow{f \mapsto f^p - f} \mathbf{G}_a \to 0$$

to reduce to Theorem 2.1.4 again. □

Corollary 2.4.2. Let k be a perfect field of characteristic $p > 0$, $f\colon X \to Y$ be a morphism of quasi-compact, quasi-separated k-schemes, and G a commutative finite flat k-group scheme such that

(1) $Y = \lim_I Y_i$ is a cofiltered limit of syntomic quasi-compact quasi-separated k-schemes with affine transition maps $Y_i \to Y_j$ for $i > j$;
(2) there is $i_0 \in I$, a syntomic quasi-compact quasi-separated k-scheme X_{i_0}, and a morphism $f_{i_0}\colon X_{i_0} \to Y_{i_0}$ such that $f_{i_0} \times_{Y_{i_0}} Y\colon X_{i_0} \times_{Y_{i_0}} Y \to Y$ is isomorphic to $f\colon X \to Y$ (in particular, $X_{i_0} \times_{Y_{i,0}} Y \simeq X$);
(3) for each $i \geq i_0$, the fiber product $f_i\colon f_{i_0} \times_{Y_{i_0}} Y_i\colon X_{i_0} \times_{Y_{i,0}} Y_i \to Y_i$ is a Hodge d-equivalence of syntomic k-schemes;
(4) there is a finite filtration $\mathrm{Fil}^\bullet G$ such that all $\mathrm{gr}^i G$ are isomorphic to μ_p, α_p, or $\mathbf{Z}/p$.

Then

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X, G\right)\right) \in D^{\geq d}(\mathbf{Z}).$$

Proof. For brevity, we denote the fiber product $X_{i_0} \times_{Y_{i_0}} Y_i$ by X_i. Then $X = \lim X_i$, so a standard approximation result (similar to [Fu11, Proposition 5.9.2])) implies that the natural morphism

$$\mathrm{hocolim}_{i \geq i_0} \mathrm{R}\Gamma_{\mathrm{fppf}}(X_i, G) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X, G)$$

is an equivalence (and the same for Y_i and Y). Since $\mathcal{D}(\mathbf{Z})$ is closed under (homotopy) colimits, it suffices to show that

$$\mathrm{R}\Gamma_{\mathrm{fppf}}(Y_i, G) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X_i, G)$$

has cone in $D^{\geq d}(\mathbf{Z})$. This follows from Lemma 2.4.1. □

Corollary 2.4.3. Let k be any field of characteristic $p > 0$, $X \to Y$ a Hodge d-equivalence of syntomic k-schemes, and G a commutative finite flat k-group scheme of order p^m for some m. Then

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X, G\right)\right) \in D^{\geq d}(\mathbf{Z}).$$

Proof of Corollary 2.4.3. Theorem 2.1.10 guarantees that all morphisms

$$X_n \coloneqq X_{\overline{k}^{\otimes_k n}} \to Y_n \coloneqq Y_{\overline{k}^{\otimes_k n}}$$

fit into the assumption of Corollary 2.4.2. Therefore, we see that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{fppf}}(Y, G) & \longrightarrow & \mathrm{R}\lim_{n\in\Delta}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}(Y_n, G)\right) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{fppf}}(X, G) & \longrightarrow & \mathrm{R}\lim_{n\in\Delta}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}(X_n, G)\right) \end{array}$$

whose horizontal arrows are isomorphisms by Lemma B.1.

Note that $G_{\overline{k}}$ has a filtration with associated graded pieces being equal to μ_p, α_p, or $\underline{\mathbf{Z}/p}$, by the classification of commutative finite group schemes of p-power order over an algebraically closed field. Therefore, we conclude that each map

$$\mathrm{R}\Gamma_{\mathrm{fppf}}(Y_n, G) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X_n, G)$$

has cone $C_n \in D^{\geq d}(\mathbf{Z})$ by Corollary 2.4.2. Therefore, the cone of the map

$$\mathrm{R}\Gamma_{\mathrm{fppf}}(Y, G) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X, G)$$

is equal to $\mathrm{R}\lim_{n\in\Delta}(C_n) \in \mathcal{D}^{\geq d}(\mathbf{Z})$. □

Corollary 2.4.4. Let k be a field of characteristic $p > 0$, Y a k-syntomic scheme of dimension $d+1$, $X \subset Y$ be a sufficiently ample Cartier divisor, and G a commutative finite flat k-group scheme of order p^m for some m. Then

$$C \coloneqq \mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y_S, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X_S, G\right)\right) \in D^{\geq d}(\mathbf{Z})$$

for any syntomic k-scheme S.

Proof. The closed embedding $X \to Y$ is a Hodge d-equivalence by definition. Then $X_S \to Y_S$ is a Hodge d-equivalence by Theorem 2.1.10. Moreover, both X_S and Y_S are syntomic over k because syntomic morphisms are closed under pullbacks and compositions. Therefore, Corollary 2.4.3 implies the claim. □

Theorem 2.4.5. Let k be a field, Y a projective syntomic k-scheme, $X \subset Y$ a closed subscheme of dimension d, and G a finite flat commutative k-group scheme. Then the cone

$$\mathrm{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(Y, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(X, G\right)\right)$$

lies in $D^{\geq d}(\mathbf{Z})$ if

(1) $Y \cong \mathbf{P}^N_k$ for some N and X is a global complete intersection, or
(2) $X \subset Y$ is a sufficiently ample Cartier divisor.

Proof. Suppose that k is a field of characteristic $p \geq 0$ and consider the short exact sequence

$$0 \to G[p^\infty] \to G \to G/G[p^\infty] \to 0.$$

The group $G' \coloneqq G/G[p^\infty]$ is a p-torsion-free finite étale commutative k-group scheme. Therefore, it suffices to prove the claim separately for a p-power torsion $G[p^\infty]$ and for a p-torsion-free étale G'.

In the case of a p-torsion-free étale group scheme, the claim follows from Theorem 2.3.2. Firstly, we reduce to the case of a separably closed field k by writing

$$\mathrm{R}\Gamma_{\mathrm{fppf}}\left(-, G'\right) \simeq \mathrm{R}\Gamma_{\text{ét}}\left(-, G'\right) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_k, \mathrm{R}\Gamma_{\text{ét}}((-)_{k^{\mathrm{sep}}}, G')).$$

In this case, we can find a filtration of G' by finite étale group subschemes with associated graded pieces equal to μ_ℓ. Therefore, it suffices to prove the claim for $G = G' = \mu_\ell$ and k a separably closed field. Now if $X \subset Y$ is a sufficiently ample Cartier divisor, the claim follows from Theorem 2.3.2. If $X \subset Y = \mathbf{P}^N_k$ is a global complete intersection, then write $X \to Y$ as a composition of immersions of ample Cartier divisors and apply Theorem 2.3.2 to each of those immersions.

Now we assume that $G = G[p^\infty]$ is a p-power torsion flat commutative k-group scheme, and we may assume $p > 0$. Then $X \to Y$ is a Hodge d-equivalence if it is a sufficiently ample Cartier divisor (by definition), and it is a Hodge d-equivalence in case of a global complete intersection due to Theorem 2.1.4. Therefore, the claim follows from Corollary 2.4.3. □

Example 2.4.6. ([BH07, §2], [Lan11, Ex. 10.1]) Let k be a perfect field of characteristic $p > 0$, and $X \subset Y$ an ample Cartier divisor such that

(1) X and Y are smooth and connected;
(2) Y is of pure dimension $d \geq 2$;
(3) $\mathrm{H}^1(Y, \mathcal{O}_Y(-X)) \neq 0$. (For examples of such pairs with $d = 2$, see [Eke88, Proposition 2.14][4].)

Then $\mathrm{H}^1_{\mathrm{fppf}}(Y, \alpha_p) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \alpha_p)$ is not injective. In particular,

$$\mathrm{cone}(\mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \alpha_p) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \alpha_p))$$

does not lie in $D^{\geq d}(\mathbf{Z})$.

Proof. Our assumptions on X and Y imply that

$$\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(Y, \mathcal{O}_Y) \simeq k.$$

Therefore, the map $\mathrm{H}^1(Y, \mathcal{O}_Y(-X)) \to \mathrm{H}^1(Y, \mathcal{O}_Y)$ is injective. So any non-trivial class in $\mathrm{H}^1(Y, \mathcal{O}_Y(-X))$ defines a non-trivial class $x \in \mathrm{H}^1(Y, \mathcal{O}_Y)$ such that $x|_X = 0 \in \mathrm{H}^1(X, \mathcal{O}_X)$. We claim that $(F^n_Y)^*(x) = 0$ for some $n \geq 0$. Indeed, by functoriality, $(F^n_Y)^*(x)$ lies in

$$\mathrm{H}^1(Y, \mathcal{O}_Y(-((F^n_Y)^*X))) = \mathrm{H}^1(Y, \mathcal{O}_Y(-p^n X)) = 0$$

for large $n \gg 0$.

We replace x with $(F^{n-1}_Y)^*(x)$ to assume that $F^*_Y(x) = 0$. Since F^*_Y and F^*_X are bijective on $\mathrm{H}^0(Y, \mathcal{O}_Y)$ and $\mathrm{H}^0(X, \mathcal{O}_X)$ respectively, we may use the short exact sequence

$$0 \to \alpha_p \to \mathbf{G}_a \xrightarrow{F^*} \mathbf{G}_a \to 0,$$

to conclude that

$$\mathrm{H}^1_{\mathrm{fppf}}(Y, \alpha_p) \simeq \ker(F^*_Y \colon \mathrm{H}^1(Y, \mathcal{O}_Y) \to \mathrm{H}^1(Y, \mathcal{O}_Y))$$

and the same for X. In particular, $\mathrm{H}^1_{\mathrm{fppf}}(Y, \alpha_p) \to \mathrm{H}^1(Y, \mathcal{O}_X)$ is injective (and the same for X). Therefore, x (uniquely) defines a non-trivial class in $\mathrm{H}^1_{\mathrm{fppf}}(Y, \alpha_p)$ that lies in the kernel of the restriction map

$$\mathrm{H}^1_{\mathrm{fppf}}(Y, \alpha_p) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \alpha_p).$$

In particular, $\mathrm{cone}(\mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \alpha_p) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(Y, \alpha_p))$ does not lie in $D^{\geq d}(\mathbf{Z})$. □

[4]We do not know if there are such examples of higher dimension.

Recall that if G is a locally constant constructible sheaf of $\mathbf{F}_\ell$-modules on $Y_{\text{ét}}$ with $\ell \neq \operatorname{char} k$, then the Lefschetz hyperplane theorem holds for G if Y is smooth. One may wonder if there is an analogous result for flat coefficients. We do not know whether there is any result like this, and in general we do not know the correct coefficient theory for flat cohomology. In any event, Theorem 2.4.5 is false if one does not assume that G comes from a base field, as the following example shows.

Example 2.4.7. Let k be any field of characteristic $p > 0$. Then, for any $N > 1$, there is a commutative finite flat rank p group scheme G on $\mathbf{P}^N_k$ such that

(1) Zariski-locally on $\mathbf{P}^N_k$, G is defined over k,
(2) for any hyperplane $H \subset \mathbf{P}^N_k$, the cone
$$\operatorname{cone}\left(\mathrm{R}\Gamma_{\mathrm{fppf}}\left(\mathbf{P}^N_k, G\right) \to \mathrm{R}\Gamma_{\mathrm{fppf}}\left(H, G\right)\right)$$
does not lie in $D^{\geq N-1}(\mathbf{Z})$.

Proof. Let $\mathbf{G}_a(n)$ be the $\mathbf{P}^N$-group scheme associated with the line bundle $\mathcal{O}(n)$. Then we define $G = \ker(Fr\colon \mathbf{G}_a(1) \to \mathbf{G}_a(p))$. It is clear that, Zariski locally on $\mathbf{P}^N$, G is isomorphic to α_p. In particular, it is defined over the ground field k.

Now using that $\mathrm{H}^i_{\mathrm{fppf}}(\mathbf{P}^N_k, \mathbf{G}_a(n)) = \mathrm{H}^i(\mathbf{P}^N_k, \mathcal{O}(n))$, Serre's calculation of cohomology groups of $\mathcal{O}(n)$, and the short exact sequence
$$0 \to G \to \mathbf{G}_a(1) \to \mathbf{G}_a(p) \to 0,$$
we conclude that
$$\dim_k \mathrm{H}^1_{\mathrm{fppf}}(\mathbf{P}^N_k, G) = \binom{N+p}{N} - N - 1$$
$$\dim_k \mathrm{H}^1_{\mathrm{fppf}}(H, G) = \mathrm{H}^1_{\mathrm{fppf}}(\mathbf{P}^{N-1}_k, G) = \binom{N+p-1}{N-1} - N.$$
In particular, the map $\mathrm{H}^1_{\mathrm{fppf}}(\mathbf{P}^N_k, G) \to \mathrm{H}^1_{\mathrm{fppf}}(H, G)$ can not be injective by dimension reasons. Therefore, the cone C cannot lie in $D^{\geq N-1}$ for any $N > 1$. □

Question 2.4.8. Let $X \subset \mathbf{P}^N_k$ be a complete intersection of dimension at least 2, and G a finite (not necessarily commutative) k-group scheme. Is the natural morphism $\mathrm{H}^1(\mathbf{P}^N_k, G) \to \mathrm{H}^1(X, G)$ a bijection? The same question may be asked for $X \subset Y$ a sufficiently ample Cartier divisor (possibly with a different definition of "sufficiently ample").

Remark 2.4.9. If both X, Y are smooth and the ground field k is algebraically closed, Question 2.4.8 has a positive answer. This follows from the Lefschetz type result for Nori's fundamental group (see [BH07, Theorem 1.1])

2.5. **The torsion part of the Picard scheme.** In this section, we use the results of Section 2.2 and Section 2.3 to get a Lefschetz hyperplane theorem for the torsion part of Picard group. We show that, for a complete intersection $X \subset \mathbf{P}^N_k$ scheme of dimension at least 2, the torsion part of the Picard group $\operatorname{Pic}(X)_{\mathrm{tors}}$ and the torsion component $\mathbf{Pic}^\tau_{X/k}$ vanish. We also give a version of this result for a general sufficiently ample divisor.

If X is of dimension at least 3, Grothendieck proved the stronger result that $\operatorname{Pic}(X) \simeq \mathbf{Z}$ in [SGA2, Exp. XII, Corollaire 3.2] which can be also used to deduce that $\operatorname{Pic}_{X/k} \simeq \underline{\mathbf{Z}}$ as k-group schemes. These results are sharp in the sense that the whole Picard group $\operatorname{Pic}(X)$ may not be isomorphic to $\mathbf{Z}$ if $\dim X = 2$. For instance, the Segre embedding realizes $\mathbf{P}^1_k \times \mathbf{P}^1_k$ as a hypersurface in $\mathbf{P}^3_k$, but

$\mathrm{Pic}(\mathbf{P}^1_k \times \mathbf{P}^1_k) \simeq \mathbf{Z} \oplus \mathbf{Z}$. However, it turns out that one can still control the torsion part of Picard group in dimension 2. If X is a smooth surface and k is algebraically closed, then these results were established in [SGA7$_{\mathrm{II}}$, Exp. XI, Th. 1.8]. The general case was proven in [CS21, Corollary 7.2.3] by different methods.

For the rest of the section, we fix a field k of arbitrary characteristic.

Theorem 2.5.1. Let $X \subset \mathbf{P}^N_k$ be a complete intersection of dimension at least 2. Then

1. $\mathrm{Pic}(X)_{\mathrm{tors}} = 0$;
2. the group scheme $\mathbf{Pic}^{\tau}_{X/k}$ is trivial;
3. the class of $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^N}(1)$ is a non-divisible element of $\mathrm{Pic}(X)$.

Proof. For the first point, it suffices to show that $\mathrm{Pic}(X)[p] = 0$ for *all* prime numbers p. Using the Kummer exact sequence (in the fppf topology)

$$0 \to \mu_p \to \mathbf{G}_m \to \mathbf{G}_m \to 0,$$

we see that it suffices to show that $\mathrm{H}^1_{\mathrm{fppf}}(X, \mu_p) = 0$. If p is equal to the characteristic of k, the claim follows from Corollary 2.4.3 and Theorem 2.1.4. If p is different from the characteristic of k, it follows from Theorem 2.3.2.

Now we show the second point. Recall that the Picard functor $\mathbf{Pic}_{X/k}$ is (representable by) a locally finite type k-group scheme by [SGA6, Exp. XII, 1.5] and the functorial criterion for local finite presentation. By [SGA6, Exp. XIII, 4.7], $\mathbf{Pic}^{\tau}_{X/k}$ is an open subfunctor of $\mathbf{Pic}_{X/k}$, so we have an isomorphism of tangent spaces

$$\mathrm{T}_e(\mathbf{Pic}^{\tau}_{X/k}) = \mathrm{T}_e(\mathbf{Pic}_{X/k}) \simeq \mathrm{H}^1(X, \mathcal{O}_X).$$

Since X is a complete intersection in $\mathbf{P}^N_k$ of dimension ≥ 2, we have $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ by Theorem 2.1.4, and hence $\mathbf{Pic}^{\tau}_{X/k}$ is etale. Thus we need only show that $\mathbf{Pic}^{\tau}_{X/k}(\overline{k}) = 0$.

Since the Picard functor commutes with base change, we can assume that k is algebraically closed. In particular, X has a rational point. Therefore, $\mathbf{Pic}_{X/k}(k) = \mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. Thus in fact we need to show that $\mathrm{Pic}(X)_{\mathrm{tors}} = 0$, which was already shown above.

Now we show that $[\mathcal{O}_X(1)] \in \mathrm{Pic}(X)$ is non-divisible. It suffices to show that this class has non-zero image in $\mathrm{Pic}(X)/p$. Consider the first Chern class

$$c_1^X \colon \mathrm{Pic}(X)/p \to \mathrm{H}^2_{\mathrm{fppf}}(X, \mu_p)$$

which comes from the Kummer exact sequence

$$0 \to \mu_p \to \mathbf{G}_m \to \mathbf{G}_m \to 0.$$

By definition, c_1^X is injective, so it is enough to show that $c_1^X([\mathcal{O}_X(1)]) \neq 0$ in $\mathrm{H}^2_{\mathrm{fppf}}(X, \mu_p)$. The commutative square

$$\begin{array}{ccc} \mathrm{Pic}(\mathbf{P}^N_k)/p & \xrightarrow{c_1^{\mathbf{P}^N}} & \mathrm{H}^2_{\mathrm{fppf}}(\mathbf{P}^N_k, \mu_p) \\ \downarrow{\scriptstyle \mathrm{res}^{\mathrm{Pic}}} & & \downarrow{\scriptstyle \mathrm{res}^{\mathrm{fppf}}} \\ \mathrm{Pic}(X)/p & \xrightarrow{c_1^X} & \mathrm{H}^2_{\mathrm{fppf}}(X, \mu_p). \end{array}$$

shows that we have

$$c_1^X([\mathcal{O}_X(1)]) = c_1^X(\mathrm{res}^{\mathrm{Pic}}([\mathcal{O}_{\mathbf{P}^N}(1)])) = \mathrm{res}^{\mathrm{fppf}}(c_1^{\mathbf{P}^n}([\mathcal{O}_{\mathbf{P}^N}(1)])). \tag{2.1}$$

We know that $c_1^{\mathbf{P}^N}([\mathcal{O}_{\mathbf{P}^N}(1)])$ is non-zero because $\mathcal{O}_{\mathbf{P}^N}(1)$ is a generator of $\mathrm{Pic}(\mathbf{P}^N_k) \simeq \mathbf{Z}$, and $\mathrm{res}^{\mathrm{fppf}}$ is injective by Corollary 2.2.2 or Theorem 2.3.2 depending on whether p is equal to the characteristic of k. □

Our next aim is to prove an analogue of Theorem 2.5.1 for sufficiently ample divisors. Specifically, we have the following statement.

Theorem 2.5.2. Let Y be a projective syntomic k-scheme of dimension $d \geq 3$, and let $X \subset Y$ be a sufficiently ample Cartier divisor. The natural map $\mathbf{Pic}^{\tau}_{Y/k} \to \mathbf{Pic}^{\tau}_{X/k}$ is an isomorphism.

The derivation of this is slightly more involved than in the previous case, and we must begin with a series of general results about algebraic groups. We begin with the following folkloric lemma, which is implicit in some of the arguments of this section.

Lemma 2.5.3. Let S be a scheme, and let $f : G \to H$ be a homomorphism of finitely presented S-group schemes. The following are equivalent.

(1) f is faithfully flat,

(2) f is an epimorphism of fppf sheaves and $\ker f$ is flat.

Proof. Note that in any case we have $G \times_H G \cong G \times_S \ker f$ as G-schemes via the map $(g, g') \mapsto (g, g^{-1}g')$, which has inverse $(g, k) \mapsto (g, gk)$. In particular, f becomes a trivial $\ker f$-torsor after base change along f. Now if f is fppf, we see that it admits a section fppf-locally. To check that f is an epimorphism of fppf sheaves, it is enough to check after fppf base change, and this property is clear when f admits a section. Moreover, $\ker f$ is clearly flat by base change.

Conversely, suppose that f is an epimorphism of fppf sheaves and $\ker f$ is flat. The epimorphism property implies that there is some fppf cover $X \to H$ such that $G \times_H X \to X$ admits a section. Since f becomes a trivial $\ker f$-torsor after base change along f, it is also a trivial torsor after base change along $G \times_H X \to H$, and hence also along $X \to H$. Thus f is an fppf $\ker f$-torsor, and since $\ker f$ is flat it follows that f is faithfully flat. □

Lemma 2.5.4. Let G be a commutative group scheme locally of finite type over a field k, and let H be a finite type closed k-subgroup scheme of G. Let n be a positive integer. For each $M \geq 1$, there exists a closed subgroup scheme G_M of G killed by some power of n such that the natural map

$$G_M \to G/H$$

factors through $(G/H)[n^M]$ and such that the factored map is faithfully flat. (Note that G/H exists as a group scheme by [SGA3, $\mathrm{VI_A}$, 3.2].)

Proof. Fixing the positive integer M, we may replace G by the schematic preimage of $(G/H)[n^M]$ in G to assume that G/H is n^M-torsion. Consider for any N the commutative diagram of commutative locally finite type k-group schemes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n^N} & & \downarrow{\scriptstyle n^N} & & \downarrow{\scriptstyle n^N} & & \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0 \end{array}$$

with exact rows. By the snake lemma, this gives an exact sequence of group schemes

$$G[n^N] \to (G/H)[n^N] \to H/n^N H \to G/n^N G.$$

So it suffices to show that there is some $N \geq M$ such that $H/n^N H \to G/n^N G$ is a monomorphism. Equivalently, one must show that there is some $N \geq M$ such that

$$H \cap n^N G = n^N H.$$

To prove the existence of some such N, note that the sequence $\{n^N H\}$ of closed k-subgroup schemes of H is decreasing, so the fact that H is noetherian (being finite type over k) implies that there exists some N_0 such that $n^{N_0} H = n^{N_0+1} H$. Since G/H is n^M-torsion, we have

$$H \cap n^{N_0+M} G \subset n^{N_0} H = n^{N_0+M} H$$

by choice of N_0. Thus taking $N = N_0 + M$ completes the proof. □

Lemma 2.5.5. Let G be a commutative group scheme locally of finite type over an algebraically closed field k of characteristic $p \geq 0$.

(1) The natural map $G(k)_{\mathrm{tors}} \to (G(k)/G^0(k))_{\mathrm{tors}}$ is surjective.
(2) There exists some integer $N \geq 1$ such that the natural map $G[p^N] \to G/G_{\mathrm{red}}$ is faithfully flat.
(3) If G is smooth, $p > 0$, and G/G^0 is torsion (e.g., G is of finite type over k), then the set $G(k)_{\mathrm{tors}}$ is schematically dense in G.

Proof. For the first point, we may evidently replace G by the preimage of $(G/G^0)_{\mathrm{tors}}$ in G to assume that G/G^0 is torsion. Note that it suffices to show that for every integer $n \geq 1$, the natural map $G(k)[n^\infty] \to (G(k)/G^0(k))[n^\infty]$ is surjective, and this follows from Lemma 2.5.4. (Recall that G^0 is of finite type over k since it is a connected group scheme locally of finite type.)

For the second point, there is nothing to prove if $p = 0$. If instead $p > 0$, then G/G_{red} is a finite k-group scheme whose order is a power of p. By a theorem of Deligne [TO70, Sec. 1], G/G_{red} is killed by its order, so the result follows again from Lemma 2.5.4.

Finally we consider the third point. By the first point of this lemma, we may and do reduce to the case that G is connected, in which case we will show that if $\ell \neq p$ is any prime number then $G(k)[(\ell p)^\infty]$ is schematically dense in G. By a theorem of Chevalley [Con02], since k is perfect there is a short exact sequence

$$0 \to H \to G \to A \to 0,$$

where H is a linear algebraic group over k and A is an abelian variety over k. Moreover, since H is a commutative linear algebraic group over a perfect field, we have $H = T \times U$ for some k-torus T and a smooth commutative unipotent k-group scheme U. It is standard that $T(k)[\ell^\infty]$ and $A(k)[\ell^\infty]$ are schematically dense in T and A, respectively, and $U = U[p^M]$ for some $M \geq 1$. (This is where we use that $p > 0$; in characteristic 0, $\mathbf{G}_a$ has no torsion.)

Let G_0 denote the schematic closure of $G(k)[(\ell p)^\infty]$ in G, so that G_0 is a smooth closed k-subgroup scheme of G. We aim to show that $G_0 = G$. Every connected commutative finite type k-group scheme is ℓ-divisible, so by Lemma 2.5.4 we see that the natural map $G(k)[\ell^\infty] \to A(k)[\ell^\infty]$ is surjective. By schematic density of $A(k)[\ell^\infty]$ in A, it follows that the induced map $G_0 \to A$ is dominant, hence surjective by [SGA3, $\mathrm{VI_B}$, 1.2]. It suffices therefore to show that H is contained in G_0. But $H(k)[(\ell p)^\infty]$ is schematically dense in H, so indeed $H \subset G_0$ and so $G_0 = G$, establishing the result. □

Lemma 2.5.6. Let $f : G \to H$ be a homomorphism of commutative group schemes locally of finite type over an algebraically closed field k of characteristic $p \geq 0$. Suppose that G/G^0 and H/H^0 are torsion, and suppose

- $f[\ell^n](k) : G[\ell^n](k) \to H[\ell^n](k)$ is an isomorphism of groups for every prime number ℓ and every positive integer n,
- $\mathrm{Lie}\, f : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$ is an isomorphism,
- if $p > 0$, then $f[p^n] : G[p^n] \to H[p^n]$ is faithfully flat for every positive integer n.

Then f is an isomorphism.

Proof. The second bullet shows that $\ker f$ is finite etale, and so the first point implies that $\ker f = 0$. Thus f is a closed embedding by [SGA3, $\mathrm{VI_B}$, 1.4.2]. Moreover, the first point of Lemma 2.5.5 shows that the image of f intersects each connected component of H nontrivially. If $p = 0$ then G and H are smooth, so f is an isomorphism. Thus we may and do assume from now on that $p > 0$.

Let $\overline{G} = G/G_{\mathrm{red}}$ (which exists as a scheme by [SGA3, $\mathrm{VI_A}$, 3.2]) and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_{\mathrm{red}} & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{\mathrm{red}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & H_{\mathrm{red}} & \longrightarrow & H & \longrightarrow & \overline{H} & \longrightarrow & 0 \end{array}$$

with exact rows. Since $p > 0$ and f is surjective on torsion, the third point of Lemma 2.5.5 shows that f is dominant, and thus it is surjective by [SGA3, $\mathrm{VI_B}$, 1.2]. Now G_{red} and H_{red} are both smooth over k, so because f is a surjective closed embedding it follows that f_{red} is an isomorphism. Thus to show that f is an isomorphism, it suffices to show that $\overline{f}$ is an isomorphism.

Now by the second point of Lemma 2.5.5, there is some integer $N \geq 1$ such that the natural maps $G[p^N] \to \overline{G}$ and $H[p^N] \to \overline{H}$ are faithfully flat. Thus we find a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_{\mathrm{red}}[p^N] & \longrightarrow & G[p^N] & \longrightarrow & \overline{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{\mathrm{red}}[p^N]} & & \downarrow{\scriptstyle f[p^N]} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & H_{\mathrm{red}}[p^N] & \longrightarrow & H[p^N] & \longrightarrow & \overline{H} & \longrightarrow & 0 \end{array}$$

with exact rows. By assumption, $f[p^N]$ is faithfully flat, so it is an isomorphism since $\ker f = 0$. The previous paragraph shows that $f_{\mathrm{red}}[p^N]$ is an isomorphism, so also $\overline{f}$ is an isomorphism. Since both f_{red} and $\overline{f}$ are isomorphisms, we see that f is an isomorphism, as desired. ☐

Proof of Theorem 2.5.2. We may and do assume that k is algebraically closed of characteristic $p \geq 0$. We need only verify the hypotheses of Lemma 2.5.6 applied to $G = \mathbf{Pic}^{\tau}_{Y/k}$ and $H = \mathbf{Pic}^{\tau}_{X/k}$. The second bullet follows from the fact that the natural map $\mathrm{H}^1(Y, \mathcal{O}) \to \mathrm{H}^1(X, \mathcal{O})$ is an isomorphism by definition of sufficient ampleness. For the first bullet, we note that if ℓ is a prime number then $\mathbf{Pic}^{\tau}_{Y/k}[\ell^n](k) \to \mathbf{Pic}^{\tau}_{X/k}[\ell^n](k)$ is an isomorphism. Indeed, the map

$$\mathrm{H}^1(Y, \mu_{\ell^n}) \to \mathrm{H}^1(X, \mu_{\ell^n})$$

is an isomorphism by Theorem 2.3.2. Moreover, the ℓ^n-power map $\mathrm{H}^0(Y, \mathbf{G}_m) \to \mathrm{H}^0(Y, \mathbf{G}_m)$ is surjective because ℓ is invertible in the algebraically closed field k and $\mathrm{H}^0(Y, \mathbf{G}_m)$ is the group of units in the finite k-algebra $\mathrm{H}^0(Y, \mathcal{O})$. Thus from the exact sequence

$$\mathrm{H}^0(Y, \mathbf{G}_m) \to \mathrm{H}^0(Y, \mathbf{G}_m) \to \mathrm{H}^1(Y, \mu_{\ell^n}) \to \mathrm{Pic}(Y)[\ell^n] \to 0$$

we see that $\mathrm{H}^1(Y, \mu_{\ell^n}) \cong \mathrm{Pic}(Y)[\ell^n]$. The same reasoning applies to X in place of Y, so we see that the map $\mathrm{Pic}(Y)[\ell^n] \to \mathrm{Pic}(X)[\ell^n]$ is an isomorphism.

Finally, we must check that if $p > 0$ then $f : P_{Y,n} \coloneqq \mathbf{Pic}^{\tau}_{Y/k}[p^n] \to P_{X,n} \coloneqq \mathbf{Pic}^{\tau}_{X/k}[p^n]$ is faithfully flat for all n. For any syntomic k-scheme S, consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^0(Y_S, \mathbf{G}_m) & \longrightarrow & \mathrm{H}^1(Y_S, \mu_{p^n}) & \longrightarrow & \mathrm{Pic}(Y_S)[p^n] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{H}^0(X_S, \mathbf{G}_m) & \longrightarrow & \mathrm{H}^1(X_S, \mu_{p^n}) & \longrightarrow & \mathrm{Pic}(X_S)[p^n] & \longrightarrow & 0 \end{array}$$

with exact rows. The leftmost vertical arrow is an isomorphism by definition of sufficient ampleness and the fact that $\mathrm{H}^0(Y_S, \mathbf{G}_m) = \mathrm{H}^0(Y_S, \mathcal{O})^*$ (and similarly for X_S). The second vertical arrow is an isomorphism by Corollary 2.4.4, so the map $\mathrm{Pic}(Y_S)[p^n] \to \mathrm{Pic}(X_S)[p^n]$ is an isomorphism.

Now set $S = P_{X,n}$, so that S is a syntomic k-scheme by Lemma A.2. Since X has a rational point (k being algebraically closed) and $\mathrm{Pic}(P_{X,n}) = 0$ ($P_{X,n}$ being an extension of a finite k-group scheme by a unipotent group scheme), we have $\mathbf{Pic}_{X/k}(P_{X,n}) = \mathrm{Pic}(X_{P_{X,n}})$. Thus also $P_{X,n}(P_{X,n}) = \mathbf{Pic}^{\tau}_{X/k}[p^n](P_{X,n})$. Completely similar reasoning applies to $P_{Y,n}(P_{X,n})$.

By the above, the natural map $\mathrm{Pic}(Y_{P_{X,n}})[p^n] \to \mathrm{Pic}(X_{P_{X,n}})[p^n]$ is an isomorphism. Thus by the previous paragraph, $P_{Y,n}(P_{X,n}) \to P_{X,n}(P_{X,n})$ is an isomorphism, and thus there is a morphism $g : P_{X,n} \to P_{Y,n}$ such that $f \circ g = \mathrm{id}_{P_{X,n}}$. Therefore f is an epimorphism of fppf sheaves. The hypotheses of Lemma 2.5.6 are now seen to hold, so $\mathbf{Pic}^{\tau}_{Y/k} \to \mathbf{Pic}^{\tau}_{X/k}$ is an isomorphism. ☐

Corollary 2.5.7. Let Y be a projective syntomic k-scheme of dimension $d \geq 4$, and let $X \subset Y$ be a sufficiently ample Cartier divisor. The natural map $\mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$ is an isomorphism.

Proof. In view of Theorem 2.5.2, we see that $\mathbf{Pic}^{\tau}_{Y/k} \to \mathbf{Pic}^{\tau}_{X/k}$ is an isomorphism. Considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Pic}^{\tau}_{Y/k} & \longrightarrow & \mathbf{Pic}_{Y/k} & \longrightarrow & \mathrm{NS}(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Pic}^{\tau}_{X/k} & \longrightarrow & \mathbf{Pic}_{X/k} & \longrightarrow & \mathrm{NS}(X) & \longrightarrow & 0 \end{array}$$

we see that it suffices to show that the natural map $\mathrm{NS}(Y) \to \mathrm{NS}(X)$ is an isomorphism, and for this it suffices to show that $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is an isomorphism, a consequence of [SGA2, Exp. XII, 3.6] (whose hypotheses are satisfied because Y is syntomic and X is sufficently ample in Y). ☐

Remark 2.5.8. It is not difficult to see that an example of a pair $X \subset Y$ as in Example 2.4.6 with $d \geq 3$ (resp. $d \geq 4$) would give an example of an ample divisor such that the map $\mathbf{Pic}^{\tau}_{Y/k} \to \mathbf{Pic}^{\tau}_{X/k}$ (resp. $\mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$) is not an isomorphism. If one additionally assumes that the natural map $\mathrm{H}^0(Y, \Omega^1_Y) \to \mathrm{H}^0(X, \Omega^1_X)$ is an isomorphism, then $\mathrm{Pic}(Y)[p] \to \mathrm{Pic}(X)[p]$ is not an isomorphism. However, we are not aware of any such examples in the literature.

3. Examples

3.1. **Weird deformations of commutative group schemes.** The main goal of this section is to construct an example of a finite flat commutative group scheme G over any discrete valuation ring of equicharacteristic $p > 0$ with connected p-torsion special fiber G_s and etale non-p-torsion generic fiber G_η. We will use such G to construct interesting smooth projective varieties X as in the following theorem, usually called *Godeaux-Serre varieties*.

Theorem 3.1.1. ([Ray79, Theorem 4.2.3] or [CZ] for a detailed exposition) Let S be a local scheme and let G be a finite locally free S-group scheme. For any $d \geq 1$, there is an S-flat relative complete intersection Y of dimension d (inside of some relative projective space $\mathbf{P}^N_S$) with a free S-action of G such that the quotient $X = Y/G$ is smooth and projective over S with geometrically connected fibers. If $d \geq 2$ then for such X we have $\mathbf{Pic}^\tau_{X/S} \cong G^\vee$, where $G^\vee$ is the Cartier dual of G.

Proof. We only give some remarks on the hypotheses. In [CZ] it is assumed that G is commutative, but this does not play a role in the proof of the above theorem. Regarding the final point, in [CZ] it is shown that if $\pi : Y \to X$ is a G-torsor over S, then the kernel of the pullback map $\pi^* : \mathbf{Pic}_{X/S} \to \mathbf{Pic}_{Y/S}$ is $G^\vee$. If Y is a complete intersection of dimension $d \geq 2$, then $\mathbf{Pic}^\tau_{Y/S} = 0$: it suffices to check that the identity section $e : S \to \mathbf{Pic}^\tau_{Y/S}$ is an isomorphism, and this may be checked on fibers by the fibral isomorphism criterion. Thus this claim follows from Theorem 2.5.1, and we see that $\mathbf{Pic}^\tau_{X/S} \cong G^\vee$, as desired. ☐

For the rest of the section, we fix a discrete valuation ring R characteristic $p > 0$. In what follows, we denote $\operatorname{Spec} R$ by S.

Before we start the construction of the desired group scheme G, we need to recall some basic facts about supersingular elliptic curves.

Lemma 3.1.2. There is always a smooth family of elliptic curves $f\colon \mathcal{E} \to S$ with supersingular special fiber and ordinary generic fiber.

Proof. We note that there is always a supersingular elliptic curve E_s over the residue field $k(s)$. If $k(s)$ contains $\mathbf{F}_{p^2}$, this is clear as any supersingular curve over $\overline{k(s)}$ is defined over $\mathbf{F}_{p^2}$. In general, this follows from [Brö09, Theorem 1.1] or [Wat69, Theorem 4.1]. We will show that this always admits an ordinary deformation. At this point it is not difficult to conclude by considering Weierstrass equations, but we offer the following different proof.

Now fix a supersingular elliptic curve E_s and an integer $N \geq 3$ not divisible by p, and let $\Gamma = E_s[N]$ denote the N-torsion subgroup of E_s. We consider the moduli stack $\mathcal{Y}(\Gamma)$ whose fiber over a $k(s)$-scheme T consists of those pairs $(\mathcal{E}, \gamma)$ with $\mathcal{E}$ an elliptic curve over S and $\gamma : \Gamma_T \to \mathcal{E}[N]$ an isomorphism of T-group schemes. If $k'/k(s)$ is a finite separable extension splitting Γ, then we note that $\mathcal{Y}(\Gamma) \otimes_{k(s)} k' \cong Y(N)$ is a smooth affine k'-scheme of dimension 1 with smooth regular compactification $X(N)$ [DR73, IV, Corollaire 2.9, Théorème 3.4]. In particular, Galois descent shows that $\mathcal{Y}(\Gamma)$ is representable by a smooth affine $k(s)$-scheme $Y(\Gamma)$. We recall that there are well-known formulas for the genus of $X(N)$ [DR73, VI, Section 4.2], and in particular $X(N)$ is of genus 0 if $N \in \{3, 4\}$.

Let $X(\Gamma)$ be the regular compactification of $Y(\Gamma)$, so that $X(\Gamma) \otimes_{k(s)} k' \cong X(N)$. It is clear that $X(\Gamma)$ is a smooth projective curve over $k(s)$, and it has genus 0 if $N \in \{3, 4\}$. It follows that in fact the $k(s)$-point corresponding to E_s lies in a connected component of $X(\Gamma)$ isomorphic to $\mathbf{P}^1$. Thus E_s lies in a connected component $Y(\Gamma)^0$ of $Y(\Gamma)$ which is an open subscheme of $\mathbf{A}^1$, and it follows immediately that there is an R-point of $Y(\Gamma)^0$ mapping the special point of R to E_s and mapping the generic point of R to the generic point of $Y(\Gamma)^0$. Since there are only finitely many isomorphism classes of supersingular elliptic curves over $\overline{k(s)}$, the generic point of $Y(\Gamma)$ corresponds to an ordinary elliptic curve. ☐

Lemma 3.1.3. Let E be a supersingular elliptic curve over a field k, and let $H \subset E$ be a finite subgroup scheme of order p^2. Then $H = E[p]$.

Proof. It suffices to check equality after base change to the algebraic closure of k, so we may and do assume that k is algebraically closed.

Since H is a commutative group scheme of order p^2, we conclude that $H = H[p^2] \subset E[p^2]$ is connected. Therefore, the relative Frobenius morphism $F\colon H \to H^{(p)}$ is nilpotent. We use that H is of order p^2 again to conclude that $F^2 = 0$. Therefore,

$$H \subset E[F^2] = \ker(F^2\colon E \to E^{(p^2)}).$$

Both H and $E[F^2]$ are finite group schemes of order p^2. Therefore, the inclusion $H \subset E[F^2]$ must be an isomorphism. It is a classical result that $E[F^2] = E[p]$ for any supersingular E. □

Now let $f\colon \mathcal{E} \to S$ be a family of elliptic curves provided by Lemma 3.1.2. In particular, its special fiber is supersingular, and its generic fiber is ordinary. We consider the subgroup

$$H_\eta \coloneqq \left(\mathcal{E}_\eta[p^2]\right)^0 \subset E_\eta.$$

Since $\mathcal{E}_\eta$ is ordinary, we conclude that $H_{\overline{\eta}} \simeq \mu_{p^2}$. In particular, H_η is a commutative group scheme of order p^2 such that $H_\eta \neq G_\eta[p]$.

We define H to be the schematic closure of H_η inside $\mathcal{E}$, which is clearly a finite flat commutative group scheme over S. We define G to be the Cartier dual of H.

Lemma 3.1.4. Let G be the S-group scheme defined above. Then G is a finite flat commutative S-group scheme of order p^2 with connected p-torsion special fiber G_s and etale non-p-torsion generic fiber G_η.

Proof. With notation as above, H_s is a finite group scheme of order p^2 inside the supersingular elliptic curve E_s. Therefore Lemma 3.1.3 guarantees that $H_s = E_s[p]$. Since $E_s[p]$ is self-dual, we see that $G_s \cong E[p]$ as well, so in particular G_s is connected and p-torsion. Moreover, since $H_{\overline{\eta}} \cong \mu_{p^2}$ we see that $G_{\overline{\eta}} \cong \mathbf{Z}/p^2\mathbf{Z}$, so G_η is etale and not p-torsion. □

3.2. **First example. Jump of de Rham numbers.** In this section we aim to give a relatively elementary (if somewhat ad hoc) example of de Rham cohomology jumping, inspired by [Ray79, Section 4.2].

For the rest of the section, we fix an R-group scheme G from Lemma 3.1.4. By construction $G_{\overline{\eta}} \simeq \mathbf{Z}/p^2\mathbf{Z}$, so there is a generically finite extension of discrete valuation rings $R \subset R'$ such that $G_{\mathrm{Frac}(R')} \simeq \mathbf{Z}/p^2\mathbf{Z}$. For the rest of the section, we replace R with R' and assume that $G_\eta \simeq \mathbf{Z}/p^2\mathbf{Z}$.

Let U be the subgroup of $\mathrm{SL}_3(\mathbf{Z}/p^2\mathbf{Z})$ consisting of strictly upper-triangular matrices, so that U is a non-split extension of $\mathbf{Z}/p^2\mathbf{Z}$ by $(\mathbf{Z}/p^2\mathbf{Z})^2$. A section of $G(R) = G_\eta(k(\eta))$ of order p^2 defines a morphism $\mathbf{Z}/p^2\mathbf{Z} \to G$ which is an isomorphism on the generic fiber and the zero map on the special fiber. Form the pushout H of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/p^2\mathbf{Z} & \longrightarrow & U & \longrightarrow & (\mathbf{Z}/p^2\mathbf{Z})^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & (\mathbf{Z}/p^2\mathbf{Z})^2 & \longrightarrow & 0 \end{array}$$

where the rightmost vertical map is the identity. The generic fiber of H satisfies $H_\eta \cong U_\eta$, whereas the special fiber satisfies $H_s \cong G_s \times (\mathbf{Z}/p^2\mathbf{Z})^2$.

By Theorem 3.1.1, there exists a smooth projective R-scheme X with fibers of pure dimension 2 such that $\mathbf{Pic}^\tau_{X/k}$ is isomorphic to the Cartier dual $H^\vee$. In fact, we may find X as the quotient of a complete intersection Y by a free action of H. Beware that in the non-commutative case (as above), $H^\vee$ is not generally flat over R. In fact, in our setting we note that $(H^\vee)_s = (H_s)^\vee = G_s \times \mu_{p^2}^2$ (since G is self-dual), whereas $(H^\vee)_\eta = \mu_{p^2}^2$, so indeed $H^\vee$ is not flat. We have the following result.

Theorem 3.2.1. Let k be a field of characteristic $p > 0$, and let $X \to S$ be as above. We have $\dim_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(X_s/k(s)) > \dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(X_\eta/k(\eta))$.

Proof. First, we note that both geometric fibers of H lift to the length-2 Witt vectors. Indeed, the special fiber of H is $E_s[p] \times (\mathbf{Z}/p^2\mathbf{Z})^2$ (which lifts since elliptic curves have no obstructions to lifting), and the generic fiber of H is equal to U, a finite constant group scheme. Consequently both geometric fibers of X lift to the length-2 Witt vectors: see the construction in [CZ, Section 3]. By [DI87, Corollaire 2.5], the Hodge-de Rham spectral sequences degenerate on both fibers in degree 1, so that

$$\mathrm{H}^1_{\mathrm{dR}}(X_t/k(t)) = \bigoplus_{i+j=1} \mathrm{H}^i(X_t, \Omega^j_{X_t/k(t)})$$

where $t \in \{s, \eta\}$ and $k(t)$ is the residue field of t.

We now compute

$$\mathrm{H}^1(X_t, \mathcal{O}_{X_t}) = \mathrm{T}_e\mathbf{Pic}_{X_t/k(t)},$$

so that $h^{0,1}(X_s) = 3$ and $h^{0,1}(X_\eta) = 2$ via the above description of $\mathbf{Pic}^\tau_{X_t/k(t)}$. Moreover, since H_η is etale, we see from the smoothness of X that Y_η is also smooth. This shows that the map $\mathrm{H}^0(X_\eta, \Omega_{X_\eta/k(\eta)}) \to \mathrm{H}^0(Y_\eta, \Omega_{Y_\eta/k(\eta)})$ is injective, and the latter is 0 because Y_η is a complete intersection. Thus we find

$$\dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(X_\eta/k(\eta)) = 2,$$

whereas $\mathrm{H}^1_{\mathrm{dR}}(X_s/k(s))$ contains $\mathrm{H}^1(X_s, \mathcal{O}_{X_s})$, which is 3-dimensional over $k(s)$. Thus the dimension of $\mathrm{H}^1_{\mathrm{dR}}$ jumps from the generic fiber to the special fiber. □

3.3. **Second example I. Jump of de Rham numbers. Stacks.** The main goal of this section is to construct a smooth and proper morphism $f \colon \mathcal{X} \to S$ of algebraic stacks such that the rank of the first de Rham cohomology group of the fiber is not (locally) constant on the base.

The construction is quite easy. We take the base to be $S = \operatorname{Spec} R$ for some discrete valuation ring R of equicharacteristic $p > 0$, and define our family to be $f \colon BG \to S$ for G as in Lemma 3.1.4.

Our main tool to get control over $\mathrm{H}^1_{\mathrm{dR}}(BG_t/k(t))$ for $t \in \{\eta, s\}$ is [Mon21, Theorem 1.2, Proposition 3.14] that say that $\mathrm{H}^1_{\mathrm{crys}}(BG/W(k)) = 0$ and $\mathrm{H}^2_{\mathrm{crys}}(BG/W(k)) \simeq M(G)$ where $M(G)$ is the Dieudonné module of G. Together with an isomorphism (see Section 1.2)

$$\mathrm{R\Gamma}_{\mathrm{crys}}(BG/W(k)) \otimes^L_{W(k)} k \simeq \mathrm{R\Gamma}_{\mathrm{dR}}(BG/k),$$

we get that $\mathrm{H}^1_{\mathrm{dR}}(BG/k) \simeq M(G)[p]$, so the question boils down to computing Dieudonné modules of geometric fibers.

Theorem 3.3.1. Let $f \colon BG \to S$ be classifying stack over $S = \operatorname{Spec} R$ for G coming from Lemma 3.1.4. Then $\dim_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(BG_s/k(s)) = 2$ and $\dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(BG_\eta/k(\eta)) = 1$. In particular $\mathrm{H}^1_{\mathrm{dR}}$ is not (locally) constant on S.

Proof. Flat base change guarantees that

$$\dim_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(BG_s/k(s)) = \dim_{k(\overline{s})} \mathrm{H}^1_{\mathrm{dR}}(BG_{\overline{s}}/k(\overline{s})) \text{ and}$$

$$\dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(BG_\eta/k(\eta)) = \dim_{k(\overline{\eta})} \mathrm{H}^1_{\mathrm{dR}}(BG_{\overline{\eta}}/k(\overline{\eta})),$$

where $\overline{s}$ and $\overline{\eta}$ are geometric points over s and η, respectively. Therefore, the discussion before the theorem ensures that it suffices to show that $\dim_{k(\eta)} M(G_{\overline{\eta}})[p] > \dim_{k(s)} M(G_{\overline{s}})[p]$.

Now recall that $G_{\overline{t}}$ is of order p^2 for $t \in \{\eta, s\}$. Therefore, both $M(G_{\overline{t}})$ are $W(k(\overline{t}))$-modules of length 2. We want to compute these Dieudonne modules as $W(k)$-modules.

Now observe that $M(G_{\overline{s}})$ must be p-torsion since $G_{\overline{s}}$ is so. The only p-torsion $W(k(\overline{s}))$-module is $k(\overline{s}) \oplus k(\overline{s})$. Therefore, $M(G_{\overline{s}}) \cong k(\overline{s}) \oplus k(\overline{s})$. In particular,

$$\dim_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(BG_s/k(s)) = \dim_{k(\overline{s})} M(G_{\overline{s}})[p] = 2.$$

Now we recall that [Fon77, Chapitre III, Théorème 1] ensures that the Dieudonne functor $M(-)$ is an anti-equivalence between finite commutative group $k(\overline(\eta))$-schemes of order p^2 and finite Dieudonne modules of length p^2. In particular, it implies that $M(G_{\overline{\eta}}) \neq M(G_{\overline{\eta}})[p]$. Then the classification of finite $W(k(\overline{\eta}))$-modules implies that $M(G_{\overline{\eta}}) \cong W_2(k(\overline{\eta}))$. In particular,

$$\dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(BG_\eta/k(\eta)) = \dim_{k(\overline{\eta})} M(G_{\overline{\eta}})[p] = 1.$$

□

3.4. **Second example II. Jump of de Rham numbers. Schemes.** In this section, we use the approximation results from [ABM21] and [CZ] to approximate $BG \to S$ from Section 3.3 by a smooth, projective family $f: X \to S$ with the same first de Rham cohomology of fibers. In particular, it will give a smooth projective family over a connected S with nonconstant de Rham numbers of fibers.

For the rest of the section, we fix $S = \operatorname{Spec} R$ for some discrete valuation ring R of equicharacteristic $p > 0$, and G a finite flat S-group scheme from Lemma 3.1.4.

Theorem 3.4.1. Let $f: X \to S$ be a smooth, projective morphism coming from Theorem 3.1.1 for G from Lemma 3.1.4 and $d \geq 2$. Then $\mathrm{H}^1_{\mathrm{dR}}(X_s/k(s)) = 2$ and $\mathrm{H}^1_{\mathrm{dR}}(X_\eta/k(\eta)) = 1$.

Proof. Theorem 2.1.4 guarantees that the inclusion[5] $Y_t \to \mathbf{P}^N_{k(t)}$ is a Hodge d-equivalence for any $t \in \{\eta, s\}$. Clearly the morphism $\mathbf{P}^N_{k(t)} \to \operatorname{Spec} k(t)$ is a Hodge 2-equivalence. Therefore, the composition morphism $Y_t \to \operatorname{Spec} k(t)$ is a Hodge 2-equivalence as well. Finally, we use Theorem 2.1.10 to conclude that

$$X_t \simeq Y_t/G_t \simeq [Y_t/G_t]^{6} \to [\operatorname{Spec} k(t)/G_t] = BG_t$$

is a Hodge 2-equivalence. In particular, $\mathrm{H}^1_{\mathrm{dR}}(X_t/k(t)) = \mathrm{H}^1_{\mathrm{dR}}(BG_t/k(t))$[7] for any $t \in \{\eta, s\}$. □

Corollary 3.4.2. Let d be any integer bigger than 1, and R any discrete valuation ring of equicharacteristic $p > 0$. Then there is a smooth projective family $f: X \to \operatorname{Spec} R$ with geometrically connected fibers of pure dimension d such that

$$\dim_{k(s)} \mathrm{H}^1_{\mathrm{dR}}(X_s/k(s)) > \dim_{k(\eta)} \mathrm{H}^1_{\mathrm{dR}}(X_\eta/k(\eta)).$$

In particular, $\mathrm{H}^2_{\mathrm{dR}}(X/R)$ has non-trivial torsion.

Proof. The first part is simply Theorem 3.4.1. To see that $\mathrm{H}^2_{\mathrm{dR}}(X/R)$ has non-trivial torsion classes, we first note that

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/R) \otimes^L_R k(t) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_t/k(t))$$

for $t \in \{\eta, s\}$. Therefore, we see that

$$\dim_{k(s)} \mathrm{H}^i_{\mathrm{dR}}(X_s/k(s)) > \dim_{k(\eta)} \mathrm{H}^i_{\mathrm{dR}}(X_\eta/k(\eta))$$

---

[5] Use the notation from Theorem 3.1.1 for Y.

[6] Both isomorphisms follow from [SGA3, Exposé V, Théorème 4.1]. It is crucial that G acts freely on Y.

[7] Note that abstractly defined de Rham cohomology of k-syntomic stacks coincide with usual de Rham cohomology for k-smooth schemes (see Section 1.2).

if and only if $\mathrm{H}^i_{\mathrm{dR}}(X/R)$ or $\mathrm{H}^{i+1}_{\mathrm{dR}}(X/R)$ have non-trivial torsion classes.

Now note the Hodge-de Rham spectral sequence and [Sta21, Tag 0FW5] imply that

$$\mathrm{H}^0_{\mathrm{dR}}(X_t/k(t)) \simeq \mathrm{H}^0(X_t, \mathcal{O}_{X_t}) \tag{3.1}$$

for $t \in \{\eta, s\}$. Since the fibers of f are geometrically integral, we conclude that

$$\dim_{k(s)} \mathrm{H}^0_{\mathrm{dR}}(X_s/k(s)) = 1 = \dim_{k(\eta)} \mathrm{H}^0_{\mathrm{dR}}(X_\eta/k(\eta)).$$

using Equation (3.1) and [Sta21, Tag 0FD2]. Therefore, none of $\mathrm{H}^0_{\mathrm{dR}}(X/R)$ and $\mathrm{H}^1_{\mathrm{dR}}(X/R)$ has non-trivial torsion elements. However, there is a jump in $\mathrm{H}^1_{\mathrm{dR}}$, so there must be non-trivial torsion classes in $\mathrm{H}^2_{\mathrm{dR}}(X/R)$. ☐

Corollary 3.4.3. Let d be any integer bigger than 1, and k any field of characteristic $p > 0$. Then there exist a k-smooth scheme S and a smooth projective family $f\colon X \to S$ with connected geometric fibers of pure dimension d such that $\mathcal{H}^2_{\mathrm{dR}}(X/S)$ does not admit any stratification (see [BO78, Definition 2.10] for a definition). In particular, the Gauss-Manin connection on $\mathcal{H}^2_{\mathrm{dR}}(X/S)$ (see [KO68, Theorem 1]) cannot be promoted to a stratification.

Proof. Apply Corollary 3.4.2 to $R = k[T]_{(T)} = \mathcal{O}_{\mathbf{A}^1_k, 0}$, and standard spreading out techniques to find an open subscheme $S \subset \mathbf{A}^1_k$ and a smooth projective morphism $f\colon X \to S$ with geometrically connected fibers of pure dimension d such that $\mathcal{H}^2_{\mathrm{dR}}(X/S)$ is not locally free. However, [BO78, Note 2.17] guarantees that any coherent $\mathcal{O}_S$-module with a structure of a stratification must be locally free. Therefore, $\mathcal{H}^2_{\mathrm{dR}}(X/S)$ can not admit any stratification. ☐

Remark 3.4.4. There are easier examples of smooth, projective families $f\colon X \to S$ such that the Gauss-Manin connection $\mathcal{H}^1_{\mathrm{dR}}(X/S)$ does not prolong to a stratification. Namely, this phenomenon already occurs for the Legendre family of elliptic curves $f\colon \mathcal{E} \to S \coloneqq \mathbf{A}^1_k \setminus \{0,1\}$. Indeed, using [BO78, Proposition 2.11], it is not hard to see that if

$$\nabla_{\mathrm{GM}}\colon \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/S) \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$$

admits a structure of a stratification, its p-curvature vanishes. Now [Kat72, Theorem 3.2] implies that p-curvature of ∇_{GM} is non-zero if the Kodaira-Spencer class of f does not vanish. It is a classical computation that the Kodaira-Spencer class does not vanish for the Legendre family of elliptic curves. However, note that $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/S)$ is a trivial vector bundle, so it admits a "trivial" stratification.

3.5. **Example. Higher pushforwards of crystals in characteristic p.** In this section, we give an example of a smooth projective morphism $f\colon X \to S$ of $\mathbf{F}_p$-schemes such that $\mathrm{R}^1 f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)$ is not a crystal on S. This negatively answers the question raised in [BO78, Remark 7.10]. Before we discuss the construction, we recall some important definitions.

We fix an $\mathbf{F}_p$-scheme S, and an S-scheme X. Objects of a *(small) crystalline site* $(X/S)_{\mathrm{crys}}$ of an S-scheme X are given by triples (U, T, γ) where $U \subset T$ is a Zariski open, $U \to T$ is a nilpotent thickening, and γ is a PD-structure on the ideal of the thickening. Morphisms are defined as morphisms of triples, and coverings are defined to be Zariski coverings in T. See [Sta21, Tag 07GI] and [BO78, §6] for more details.

A crystalline site comes with the crystalline *structure sheaf* $\mathcal{O}_{X/S}$ defined by the rules

$$\mathcal{O}_{X/S}(U, T, \gamma) = \mathcal{O}_T(T).$$

For any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ and a PD-thickening $T \coloneqq (U, T, \gamma)$, we can define a Zariski $\mathcal{O}_T$-module $\mathcal{F}_{(U,T,\gamma)}$ (or just $\mathcal{F}_T$ if there cannot be any confusion) by restricting F on T with its Zariski topology.

Definition 3.5.1. An $\mathcal{O}_{X/S}$-module $\mathcal{F}$ is a *crystal in (quasi-)coherent modules* if

(1) for every PD-thickening (U, T, γ), an $\mathcal{O}_T$-module $\mathcal{F}_T$ is (quasi-)coherent;
(2) for every morphism $u\colon (U', T', \gamma') \to (U, T, \gamma)$, the natural morphism $u^*(\mathcal{F}_T) \to \mathcal{F}_{T'}$ is an isomorphism.

An object $\mathcal{F} \in D(\mathbf{Mod}_{\mathcal{O}_{X/S}})$ is a *derived crystal in (quasi-)coherent modules* if

(1) for every PD-thickening (U, T, γ), $\mathcal{F}_T \in D_{qc}(\mathbf{Mod}_{\mathcal{O}_T})$ (resp. $\mathcal{F}_T \in D_{coh}(\mathbf{Mod}_{\mathcal{O}_T})$);
(2) for every morphism $u\colon (U', T', \gamma') \to (U, T, \gamma)$, the natural morphism $\mathrm{L}u^*(\mathcal{F}_T) \to \mathcal{F}_{T'}$ is an isomorphism.

Theorem 3.5.2. [BO78, Theorem 7.16] Let S be a noetherian $\mathbf{F}_p$-scheme, $f\colon X \to S$ a smooth, proper morphism. Then $\mathrm{R}f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)$ is a derived crystal in coherent modules on S.

Remark 3.5.3. [BO78, Theorem 7.16] proves a stronger result. In particular, they allow $\mathcal{E}$ to be any "locally free, finite rank" crystal. We will not need this result.

If S is a $\mathbf{Q}$-scheme, then the correct analog of the crystalline site would be the infinitesimal site whose objects are pairs (U, T) of an open $U \subset X$ and a nilpotent thickening. Using [BO78, Note 2.17] and methods used in [BO78, Theorem 7.16], one can show that each individual $\mathcal{O}_{S/S}$-module $\mathrm{R}^i f_{*,\mathrm{inf}}\left(\mathcal{O}_{X/S}\right)$ is a crystal in coherent modules.

It is natural to ask whether the same result holds in characteristic p, or even in mixed characteristic. It is relatively easy to construct a counter-example in mixed characteristic. However, it is somewhat harder to do in the characteristic p situation and is essentially equivalent to the question raised in [BO78, Remark 7.10]. The theorem below gives a counter-example in characteristic p which can be adapted to mixed characteristic with little work.

Theorem 3.5.4. Let $d > 1$ be an integer, and R any discrete valuation ring of equicharacteristic $p > 0$. Then there is a smooth projective family $f\colon X \to S = \operatorname{Spec} R$ with geometrically connected fibers of pure dimension d such that at least one of $\mathrm{R}^1 f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)$ is not a crystal.

Proof. Take $f\colon X \to S$ as in Corollary 3.4.2. In what follows, we will denote by S the trivial PD-thickening of S. Then [BO78, Corollary 7.9] implies that

$$M \coloneqq \mathrm{R}f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)(S) = \mathrm{R}\Gamma(X, \Omega^\bullet_{X/S}).$$

Since $\mathrm{R}f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)(S, S, \mathrm{trivial})$ is a derived crystal by Theorem 3.5.2, we conclude that, for any morphism of PD-thickenings $(S, \operatorname{Spec} A, \gamma) \to (S, S, \mathrm{trivial})$, we have a natural isomorphism

$$\mathrm{R}f_{*,\mathrm{crys}}\left(\mathcal{O}_{X/S}\right)(S, \operatorname{Spec} A, \gamma) \simeq M \otimes^L_R S. \tag{3.2}$$

We wish to show that $\mathrm{R}^1 f_{*,\mathrm{crys}}(\mathcal{O}_{X/S})$ is not a crystal. Using equation (3.2), we see that it is equivalent to

$$\mathrm{H}^1(M) \otimes_R A \not\simeq \mathrm{H}^1(M \otimes^L_R A)$$

for some morphism of PD-thickenings $(S, \operatorname{Spec} A, \gamma) \to (S, S, \mathrm{triv})$.

In order to prove that claim, we construct an explicit PD-thickening of S. We fix a uniformizer $t \in R$ and consider a square zero nilpotent thickening

$$S \to \operatorname{Spec} R[e]/(e^2, et) = \operatorname{Spec} R'.$$

The ideal of the thickening $(e) \subset R'$ admits a PD-structure γ by [Mes71, Chapter V, Lemma (2.3.4)] (for example, by taking $\pi = 0$). We denote by S' the PD-thickening $(S, \operatorname{Spec} R', \gamma)$. Note that $R' \simeq R \oplus R/t$ as an R-module. Therefore,

$$\mathrm{H}^1(M \otimes^L_R R') \simeq \mathrm{H}^1(M \oplus [M/t]) = \mathrm{H}^1(M) \oplus \mathrm{H}^1([M/t])$$

and

$$\mathrm{H}^1(M) \otimes_R R' \simeq \mathrm{H}^1(M) \otimes_R \mathrm{H}^1(M)/t.$$

Therefore, $\mathrm{R}^1 f_{*,\mathrm{crys}}(\mathcal{O}_{X/S})$ is not a crystal as long as the natural morphism

$$\mathrm{H}^1(M)/t \to \mathrm{H}^1([M/t])$$

is not an isomorphism. This is equivalent to the existence of non-trivial torsion classes in $\mathrm{H}^2(M) = \mathrm{H}^2_{\mathrm{dR}}(X/R)$ which exist by construction. $\square$

Appendix A. Syntomic morphisms

We recall the definition of a syntomic morphism from [Sta21] and give some examples. We also show that this definition is equivalent to the definition of a syntomic morphism in [ABM21].

For the rest of the section, we fix a commutative ring k. Throughout the paper, we will be mostly interested in the case of a field k of characteristic $p > 0$.

Definition A.1. We say that a morphism of schemes $f\colon X \to Y$ is *syntomic* if f is flat, locally finitely presented, and all fibers are local complete intersections (in the sense of [Sta21, Tag 00S9]).

The following two lemmas are entirely standard, and we prove it only for want of a reference.

Lemma A.2. Let G be a flat, finitely presented k-group scheme. Then G is k-syntomic.

Proof. Since G is already flat and finitely presented over k, it suffices to show that its fibers are local complete intersections. So we can assume that k is a field. Moreover, [Sta21, Tag 00SJ] ensures that we can assume that k is algebraically closed. In this case, G_{red} is a normal smooth subgroup scheme of G, and the map

$$G \to G/G_{\mathrm{red}}$$

is a smooth morphism of k-group schemes. Therefore, it suffices to show that G/G_{red} is k-syntomic.

Now note that G/G_{red} is a connected finite group scheme. Then the classification of such group schemes in [Wat79, §14.4, Theorem] ensures that there is an isomorphism of k-schemes

$$G/G_{\mathrm{red}} \simeq \operatorname{Spec} k[T_1, \dots, T_n]/(T_1^{p^{e_1}}, \dots, T_n^{p^{e_n}})$$

for some $n, e_i \in \mathbf{Z}$. In particular, it is a local complete intersection. □

Lemma A.3. A morphism $f\colon A \to B$ is syntomic if and only if it is flat, finitely presented, and $L_{B/A} \in D(B)$ has Tor amplitude in $[-1, 0]$.

Proof. One direction is easy. If f is syntomic, it is locally a complete intersection morphism by [Sta21, Tag 069K]. Then $L_{B/A}$ has Tor amplitude in $[-1, 0]$ by [Sta21, Tag 08SL].

Now we assume that f is flat, finitely presented, and $L_{B/A}$ has Tor amplitude in $[-1, 0]$. We want to conclude that the fibers of f are complete intersections.

Firstly, we note that the cotangent complex is pseudo-coherent. Indeed, if A is noetherian, it is a classical result that $L_{B/A} \in D^-_{coh}(B)$. In general, formation of the cotangent complex of a flat morphism commutes with any base change [Sta21, Tag 08QQ], so a standard spreading out argument reduces to the case of noetherian A.

Then *loc. cit.* and [Sta21, Tag 068V] ensure that $L_{B/A}$ has Tor amplitude in $[-1, 0]$ if and only if $L_{B\otimes_A k(\mathfrak{p})/k(\mathfrak{p})}$ has Tor amplitude in $[-1, 0]$ for any prime ideal $\mathfrak{p} \subset A$. Now [Avr99, (1.2) Second Vanishing Theorem] ensures that $L_{B\otimes_A k(\mathfrak{p})/k(\mathfrak{p})}$ has Tor amplitude in $[-1, 0]$ if and only if $B \otimes_A k(\mathfrak{p})$ is a local complete intersection. □

Remark A.4. Lemma A.3 guarantees that Definition A.1 coincides with the definition of syntomic morphisms given in [ABM21, Notation 2.1]. In particular, all results of their paper are applicable with our definition of syntomic morphism.

We note that syntomic morphisms are local on source-and-target by [Sta21, Tag 06FC], so we can extend the definition of syntomic morphisms of schemes to algebraic stacks by general nonsense.

Definition A.5. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *syntomic* if it is syntomic in the sense of [Sta21, Tag 06FM][8].

An algebraic k-stack $\mathcal{X}$ is *k-syntomic* if the structure morphism $\mathcal{X} \to \operatorname{Spec} k$ is syntomic.

Remark A.6. We leave it to the reader to check that an algebraic k-stack is syntomic if and only if there exists a syntomic cover $U \to \mathcal{X}$ with U a syntomic k-scheme. This guarantees that our definition of k-syntomic stacks coincides with the definition in [ABM21, Notation 2.1].

Example A.7. Let G be a flat, finitely presented k-group scheme. Then BG is a syntomic k-stack[9] with a syntomic cover $f\colon \operatorname{Spec} k \to BG$.

Indeed, $\operatorname{Spec} k$ is clearly k-syntomic. So it suffices to show f is syntomic. Note that

$$\operatorname{Spec} k \times_{BG} \operatorname{Spec} k \simeq G$$

is syntomic over $\operatorname{Spec} k$ by Lemma A.2. Therefore, $\operatorname{Spec} k \to BG$ is syntomic since being syntomic is fppf local on the base by [Sta21, Tag 0428].

Example A.8. More generally, if X is k-syntomic scheme with a k-action of a flat, finitely presented k-scheme G. Then $[X/G]$ is a k-syntomic stack with a syntomic cover $X \to [X/G]$.

Appendix B. Descent For Flat cohomology

We show that show that flat cohomology of finitely presented group schemes satisfy descend with respect to algebraic extension of the base field.

For the rest of the section, we fix a base field k.

Lemma B.1. Let X be a finite type k-scheme, X_n is the base change $X_{\overline{k}^{\otimes_k n}}$, and G a flat finitely presented commutative group X-scheme. Then the natural morphism

$$\mathrm{R}\Gamma_{\mathrm{fppf}}(X, G) \to \mathrm{R}\lim_{n\in\Delta} \left(\mathrm{R}\Gamma_{\mathrm{fppf}}(X_n, G)\right)$$

is an isomorphism.

Proof. For each finite extension $k \subset k'$, let us denote by $X_{n,k'}$ the fiber product $X_{k'^{\otimes_k n}}$. Then the natural morphism

$$\mathrm{R}\Gamma_{\mathrm{fppf}}(X, G) \to \mathrm{R}\lim_{n\in\Delta} \left(\mathrm{R}\Gamma_{\mathrm{fppf}}(X_{n,k'}, G)\right)$$

is an equivalence for any finite $k \subset k'$ because fppf cohomology satisfy fppf descent. Now a standard approximation result (for example, argue as in [Fu11, Proposition 5.9.2])) implies that the natural morphism

$$\operatorname{hocolim}_{k\subset k'\subset \overline{k}} \mathrm{R}\Gamma_{\mathrm{fppf}}(X_{n,k'}, G) \to \mathrm{R}\Gamma_{\mathrm{fppf}}(X_n, G)$$

is an equivalence for any $n \geq 0$. Thus the claim follows from the fact that totalization of coconnective cosimplicial objects commute with filtered (homotopy) colimits. ☐

[8] This definition makes sense as syntomic morphisms of algebraic spaces are smooth local on source-and-target by [Sta21, Tag 06FC].

[9] We note that, if G is not k-smooth the cover $\operatorname{Spec} k \to BG$ is not smooth, so it is not an atlas. Therefore, one needs to use [Art74, Theorem (6.1)] to show that BG is an algebraic stack.

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