# NOTES ON ADIC GEOMETRY

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Contents



## 1. Introduction

The main goal of these notes is to provide a reference for some "standard" results about analytic adic spaces that seem not to be present in the existing literature. Even though probably none of the results of these notes are surprising, some of them are crucial for the arguments in our other paper [Zav22]. For this reason, we have decided to write a separate note collecting all necessary background.

We now summarize the content of each section of this paper:

(1) In Sections (2), (3), and (4), we study the notions of connected components, dimensions, and coherent sheaves in analytic geometry. The content of these sections is mostly expository;

(2) In Section (5), we develop the notion of lci Zariski-closed immersions in analytic geometry. In the case of rigid-analytic variety over a non-archimedean field, this theory has been worked out in [GL21]. However, the notion of lci (immersions) on more general analytic adic spaces seems to be missing in the literature;

(3) In Section (6), we define the Proj construction in the world of analytic adic spaces. In the case of rigid-analytic variety over a non-archimedean field, this theory has been worked out in [Con07]. However, the definition of the relative Proj on more general analytic adic spaces seems to be missing in the literature;

(4) In Section (7), we study line bundles on the relative projective bundles;

(5) In Section (8), we construct the 6-functor formalism of étale sheaves on (locally noetherian) analytic adic spaces. Modulo the results from [Man22, Appendix A.5], this 6-functor formalism was already constructed in [Hub96]. The main work is to get rid of some boundedness assumptions in [Hub96].

(6) In Sections (9) and (10), we study some basic properties of étale sheaves. In particular, we give a categorical description of lisse and constructible sheaves. This description is certainly well-known to the experts, but we do not know if it is explicitly spelled out anywhere in the literature.

1.1. **Terminology.** We say that an analytic adic space $X$ is *locally noetherian* if there is an open covering by affinoids $X = \bigcup_{i \in I} \mathrm{Spa}(A_i, A_i^+)$ with strongly noetherian Tate $A_i$. Sometimes, such spaces are called locally *strongly* noetherian.

We follow [Hub96, Def. 1.3.3] for the definition of a locally finite type, locally weakly finite type, and locally $+$-weakly finite type morphisms of locally noetherian adic spaces.

For a Grothendieck abelian category $\mathcal{A}$, we denote by $D(A)$ its *triangulated derived category* and by $\mathcal{D}(\mathcal{A})$ its $\infty$-enhancement.

## 2. Connected Components

In this section, we study connected components of locally noetherian analytic adic spaces.

**Lemma 2.1.** Let $X = \mathrm{Spa}(A, A^+)$ be a strongly noetherian Tate affinoid. Then $X$ is connected if and only if $\mathrm{Spec}\, A$ is connected.

*Proof.* Both connectivity of $\mathrm{Spa}(A, A^+)$ and of $\mathrm{Spec}\, A$ are equivalent to the fact that $A$ does not admits any non-trivial idempotents[1]. In particular, they are equivalent to each other. □

**Lemma 2.2.** Let $X$ be a locally noetherian analytic adic space. Then any point $x \in X$ admits a fundamental system of connected affinoid open neighborhoods. In particular, $X$ is locally connected.

*Proof.* It suffices to show that, for any strongly noetherian Tate affinoid $X = \mathrm{Spa}(A, A^+)$ and a point $x \in S$, the connected component of $x$ is clopen. For this, note that the ring $A$ is noetherian, and so admits only a finite number of mutually orthogonal non-trivial idempotent. Therefore, $S$ has only a finite number of connected components, thus they all must be open and closed. □

**Corollary 2.3.** Let $X$ be a locally noetherian analytic adic space. Then each connected component of $X$ is closed and open.

*Proof.* Connected components are always closed (see [Sta23, Tag 004T]), so it suffices to show that they are open. This follows from [Sta23, Tag 04ME] and Lemma 2.2. □

[1] Here, we crucially use that $\mathrm{Spa}(A, A^+)$ is sheafy.

## 3. Dimension

In this section, we study different possible definitions of dimension in the adic analytic geometry.

**Definition 3.1.** ([Hub96, Def. 1.8.1]) The *dimension* of a locally spectral $X$ is the supremum of the length $d$ of the chains of specializations $x_0 \succ x_1 \succ \cdots \succ x_d$ of points of $X$.

A locally spectral space $X$ is of *pure dimension* $d$ if every non-empty open subset $U \subset X$ has dimension $d$.

The *(relative) dimension* $\dim f$ of a morphism of analytic adic spaces $f \colon X \to Y$ is the supremum of the dimensions of the fibers of $f$,

$$\dim f \coloneqq \sup_{y \in Y} \dim f^{-1}(y) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}.$$

A morphism $f \colon X \to Y$ is of *relative pure dimension* $d$ if all non-empty fibers $f^{-1}(y)$ is of pure dimension $d$.

Firstly, it turns out that one can only consider fibers over rank-1 points.

**Lemma 3.2.** ([Hub96, Cor. 1.8.7]) Let $S$ be a locally noetherian analytic adic space, and $f \colon X \to S$ a locally finite type morphism. Then $f$ is of pure relative dimension $d$ if and only if, for each rank-1 point $s \in S$, the fiber

$$X_s \coloneqq X \times_S \operatorname{Spa}\left(K(s), \mathcal{O}_{K(s)}\right)$$

is either empty or of pure dimension $d$.

It turns out that, in the case of rigid-analytic varieties, Definition 3.1 recovers the usual notion of dimension:

**Lemma 3.3.** Let $K$ be a non-archimedean field, and $X$ is a rigid-analytic $K$-variety. Then $X$ is of pure dimension $d$ if and only if, for each classical point $x \in X$, $\dim \mathcal{O}_{X,x} = d$.

*Proof.* First we note that [Hub96, Lemma 1.8.6] implies that $X$ is of pure dimension $d$ if and only if, for every open affinoid subspace $\operatorname{Spa}(A, A^\circ) \subset X$, $\dim A = d$. Then [FK18, Prop. II.10.1.9 and Cor. 10.1.10] imply that this condition is equivalent to the condition that $\dim \mathcal{O}_{X,x} = d$ for any classical point $x \in X$. □

**Corollary 3.4.** Let $S$ be a locally noetherian analytic adic space, and $f \colon X \to S$ is a morphism that factors as the compotision

$$X \xrightarrow{g} \mathbf{D}^d_S \xrightarrow{p} S,$$

where $g$ is étale and $p$ is the natural projection. Then $f$ is of pure relative dimension $d$.

*Proof.* By Lemma 3.2, it suffices to assume that $S = \operatorname{Spa}(K, \mathcal{O}_K)$ for some non-archimedean field $K$. Then the result follows Lemma 3.3 and [Zav21a, Lemma D.3]. □

Now we wish to show that any weakly finite type morphism $f$ is of finite (relative) dimension. Surprisingly, this claim seems to be missing in [Hub96]. For this, we need a number of preliminary lemmas that will allow us to reduce the general case to the case when $f$ is of finite type. The motivation for considering non-finite type morphisms comes from the theory of universal compactifications that are (essentially) never finite type (and merely +-weakly finite type).

**Lemma 3.5.** Let $(A, A^+) \to (B, B^+)$ be a morphism of strongly noetherian Tate pairs such that $B$ is finite over $A$. Denote by $B'^+$ the integral closure of $A^+$ in $B$. Then

(1) $(B, B'^+)$ is a Huber pair;
(2) $(A, A^+) \to (B, B'^+)$ is a finite morphism.

*Proof.* The subring $B'^+$ of $B$ is clearly integrally closed. It is also contained in $B^\circ$ because $B'^+ \subset B^+ \subset B^\circ$. So, in order to show that $(B, B'^+)$ is a Huber pair, we only need to show that it is open.

Choose a ring of definition $A_0$, a pseudo-uniformizer $\varpi \in A_0$, and $(b_1, \dots, b_n)$ a finite set of $A$-module generators of $B$. Since $B$ is finite over $A$, for each generator $b_i \in B$, we can choose monic polynomials

$$b_i^{m_i} + a_{i,1} b_i^{m_i - 1} + \dots + a_{i,m_i} = 0 \tag{1}$$

with $a_{i,j} \in A$. By construction, there is an integer $N$ such that $\varpi^N a_{i,j} \in A_0$ for all $i, j$. Using Equation (1), it is easy to see that all elements $\varpi^N b_i$ are *integral over* $A_0$. Thus, we can replace each $b_i$ with $\varpi^N b_i$ to assume that the $A$-module generators $b_i$ are integral over $A_0$. In particular, we can assume that each $b_i$ is integral over $A^+$, so they all lie in $B'^+ \subset B^\circ$.

Now consider the unique $A$-linear morphism

$$\varphi \colon A\langle T_1, \dots, T_n\rangle \to B$$

that sends $T_i$ to $b_i$. It is clearly surjective, and therefore it is open by the Open Mapping theorem (see [Hub93, Lemma 2.4(i)]), so we define

$$B_0 \coloneqq \varphi(A_0\langle T_1, \dots, T_n\rangle).$$

This is then a ring of definition in $B$ with a pseudo-uniformizer given by $\varpi$. By construction, the morphism

$$A_0/\varpi A_0 \to B_0/\varpi B_0$$

is finite. Therefore, using that $A_0$ and $B_0$ are complete, we conclude that $B_0$ is finite over $A_0$. In particular, elements of $B_0$ are integral over $A^+$, so $B_0 \subset B'^+$. This ensures that $B'^+$ is open. This finishes the proof that $(B, B'^+)$ is a Huber pair.

The morphism $(A, A^+) \to (B, B'^+)$ is now clearly finite. Indeed, $A \to B$ is finite by the assumption, and $A^+ \to B'^+$ is integral by construction. □

**Corollary 3.6.** Let $f \colon (A, A^+) \to (B, B^\circ)$ be weakly finite type morphism of strongly noetherian Tate affinoids. Then there is a Huber pair $(B, B^+)$ such that $f$ factors through $(B, B^+) \to (B, B^\circ)$, and $(B, B^+)$ is topologically finite type over $(A, A^+)$.

*Proof.* Since $(B, B^\circ)$ is weakly finite type over $(A, A^+)$, there is a surjective morphism

$$g \colon A\langle T_1, \dots, T_n\rangle \to B.$$

Since any morphism of Tate rings is adic, and adic morphisms preserve bounded elements (see [Hub94, Lemma 1.8]), we conclude $g$ induces a morphism of Huber pairs

$$g\colon (A\langle T_1, \dots, T_n\rangle, A\langle T_1, \dots, T_n\rangle^+) \to (B, B^\circ)$$

Thus we can apply Lemma 3.5 to $g$ to get a Huber sub-pair $(B, B^+) \subset (B, B^\circ)$ such that $B^+$ is integral over the image $A\langle T_1, \dots, T_n\rangle^+$ in $B$. In particular, the morphism $(A, A^+) \to (B, B^+)$ is topologically of finite type. □

**Lemma 3.7.** Let $f\colon X \to Y$ be a weakly finite type morphism and $Y$ is quasi-compact. Then $f$ is of finite dimension.

*Proof.* An easy argument with quasi-compactness reduces the general case to the case of a weakly finite type morphism of affinoid spaces $f\colon X = \mathrm{Spa}(B, B^+) \to Y = \mathrm{Spa}(A, A^+)$, i.e., $B$ is topologically of finite type over $A$. Now note that the natural inclusion

$$X' = \mathrm{Spa}(B, B^\circ) \to X = \mathrm{Spa}(B, B^+)$$

is a bijection on rank-1 points. Therefore, $\dim.\mathrm{tr}(X/Y) = \dim.\mathrm{tr}(X'/Y)$ (see [Hub96, Def. 1.8.4]). In particular, we can replace $B^+$ with $B^\circ$.

In this case, we apply Corollary 3.6 and a similar argument once again to reduce to the case of a finite type morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$. In this case, there is closed immersion

$$X \to \mathbf{D}^n_Y,$$

so it suffices to show the claim for the relative closed unit disk $\mathbf{D}^n_Y \to Y$. This case follows from Corollary 3.4. □

## 4. Coherent sheaves

In this section, we review the basic theory of coherent sheaves on locally noetherian analytic adic spaces.

We first recall the construction of the $\mathcal{O}_X$-module $\widetilde{M}$ on a strongly noetherian affinoid $X = \mathrm{Spa}(A, A^+)$ associated to a finite $A$-module $M$. For each rational subset $U \subset X$, we have

$$\widetilde{M}(U) = \mathcal{O}_X(U) \otimes_A M;$$

[Ked19, Thm. 1.4.16] and [Ked19, Thm. 1.2.11] (see also [Zav21d, Cor. 1.3]) guarantee that this assignment is indeed a sheaf.

**Definition 4.1.** An $\mathcal{O}_X$-module $\mathcal{F}$ on a locally strongly noetherian analytic adic space $X$ is *coherent* if there is an open covering $X = \cup_{i\in I} U_i$ by strongly noetherian affinoids such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for a finite $\mathcal{O}_X(U_i)$-module $M_i$.

This construction can be clearly promoted to a functor $\widetilde{(-)}\colon \mathbf{Mod}_A^{\mathrm{coh}} \to \mathbf{Coh}_X$. Similarly to the algebraic situation, this functor turns out to be an equivalence.

**Theorem 4.2.** Let $X = \mathrm{Spa}(A, A^+)$ be a strongly noetherian affinoid, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then

(1) the functor $\widetilde{(-)}\colon \mathbf{Mod}_A^{\mathrm{fg}} \to \mathbf{Mod}_X$ is exact;

(2) the functor $\widetilde{(-)}\colon \mathbf{Mod}_A^{\mathrm{fg}} \to \mathbf{Coh}_X$ is an equivalence with quasi-inverse taking $\mathcal{F}$ to $\Gamma(X, \mathcal{F})$;

(3) for any $\mathcal{F} \in \mathbf{Coh}_X$, $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i \geq 1$.

(4) the inclusion $\mathbf{Coh}_X$ is a Weak Serre subcategory of $\mathbf{Mod}_X$. In other words, coherent sheaves are closed under kernels, cokernels, and extensions;

*Proof.* First note that $A$ is sheafy by [Ked19, Thm. 1.2.11] or [Zav21d, Cor. 1.3]. So (1) can be easily deduced from [Ked19, Thm. 1.4.14]. (2) and (3) follow from [Ked19, Thm. 1.4.18]. Finally, (4) can be deduced from all (1-3) by a standard argument. □

**Lemma 4.3.** Let $f\colon X \to Y$ be a morphism of locally noetherian analytic adic spaces and $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module. Then

(1) the pullback $f^*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module;

(2) if $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$ are affinoid and $\mathcal{F} = \widetilde{M}$ for a finite $A$-module $M$, then $f^*\mathcal{F} \simeq \widetilde{M \otimes_A B}$.

*Proof.* Clearly, (1) follows from (2). To prove (2), we use noetherianness of $A$ to find a partial resolution

$$A^n \to A^m \to M \to 0.$$

Then a standard argument using Theorem 4.2(1, 2) and right exactness of $f^*$ shows that $f^*\mathcal{F} \simeq (\widetilde{M \otimes_A B})$. □

## 5. Regular closed immersions

We first review the theory of closed subspaces of locally noetherian analytic adic spaces. Then we discuss the theory of lci subspaces and, in particular, effective Cartier divisors on such spaces. In the case of rigid-analytic varieties over a non-archimedean field, (a more general) theory of lci morphisms is developed in [GL21].

**Definition 5.1.** A morphism $i\colon X \to Y$ of locally noetherian analytic adic spaces is a *Zariski-closed immersion* if $i$ is a homeomorphism of $X$ onto a closed subset of $Y$, the map $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective, and the kernel $\mathcal{I} \coloneqq \ker(\mathcal{O}_Y \to i_*\mathcal{O}_X)$ is coherent.

We refer to [Zav21c, Appendix B.6] for a detailed discussion of this notion (studied under the name of closed immersions). In particular, we point out [Zav21c, Cor. B.6.9] that guarantees that a Zariski-closed subspace of a strongly noetherian Tate affinoid $X = \mathrm{Spa}(A, A^+)$ is a (strongly noetherian Tate) affinoid. Furthermore, Zariski-closed subspaces of $X$ are parametrized by the ideals $I \subset A$.

Now we show that this definition of Zariski-closed immersions (specific to the locally noetherian case) is compatible with the definition of Zariski-closed subsets from [Sch17, Def. 5.7]:

**Lemma 5.2.** Let $X = \mathrm{Spa}(A, A^+)$ be a strongly noetherian Tate affinoid over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, $Z \subset X$ a Zariski-closed immersion (in the sense of Definition 5.1), $Y = \mathrm{Spa}(R, R^+)$ an affinoid perfectoid space with a morphism $Y \to X$. Then the fiber product $Z' \coloneqq Z \times_X Y$ is a Zariski-closed perfectoid subspace of $Y$ (in the sense of [Sch17, Def. 5.7]).

We note that a priori it is not even clear whether $Z'$ is a perfectoid space.

*Proof.* By [Zav21c, Cor. B.6.9], there is an (necessarily finitely generated) ideal $I \subset A$ such that $Z = \mathrm{Spa}(A/I, (A^+/I \cap A^+)^c)$. Choose some generators $I = (f_1, \dots, f_m)$ and a pseudo-uniformizer $\varpi \in A$. Then we have

$$Z \sim \lim_n X\left(|f_1| \le |\varpi|^n, \dots, |f_m| \le |\varpi|^n\right),$$

where $\sim$ stands for the $\sim$-limit in the sense of [Hub96, Def. 2.4.2] or [SW13, Def. 2.4.1]. Then [SW13, Prop. 2.4.3] ensures that

$$Z' \sim \lim_m Y\left(|f_1| \le |\varpi|^n, \dots, |f_m| \le |\varpi|^n\right).$$

Now we note that each $Y(|f_1| \le |\varpi|^m, \dots, |f_1| \le |\varpi|^m)$ is an affinoid perfectoid space, so $Z'$ is also an affinoid perfectoid space. Moreover, one easily sees from the above description that $Z'$ is a Zariski-closed subspace of $Y$ corresponding to the ideal $IR \subset R$. □

In this section, we concentrate on a particular class of Zariski-closed immersions.

**Definition 5.3.** A Zariski-closed immersion $i \colon X \to Y$ of strongly noetherian Tate affinoids is a *regular immersion of pure codimension* $c$ if the ideal of immersion $\mathcal{I}(Y) \subset \mathcal{O}_Y(Y)$ is generated by a regular sequence $(g_{i,1}, \dots, g_{i,c}) \subset \mathcal{O}_Y(Y)$.

A Zariski-closed immersion $i \colon X \to Y$ is an *lci immersion (of pure codimension* $c$*)* if there is an open affinoid covering $Y = \sqcup_{i \in I} U_i$ by strongly noetherian Tate affinoids such that the base change $X_{U_i} \to U_i$ is a regular immersion (of pure codimension $c$) for every $i \in I$.

A Zariski-closed immersion $i \colon X \to Y$ is an *effective Cartier divisor* if it an lci immersion of pure codimension 1.

**Lemma 5.4.** Let $Y = \mathrm{Spa}(A, A^+)$ be a strongly noetherian Tate affinoid, $i \colon X \to Y$ a regular immersion of pure codimension $c$, and $U = \mathrm{Spa}(A_U, A_U^+) \subset Y$ an open affinoid. Then the base change $i_T \colon X_U \to U$ is also a regular immersion of pure codimension $c$.

*Proof.* We first note that [Zav21c, Cor. B.6.9] implies that $\mathcal{O}_X(X) = A/I$ for an ideal $I \subset A$. Then [Zav21c, Lemma B.6.7] guarantees that the ideal of $i_T$ is equal to the ideal $IA_U \subset A_U$. Finally, the fact that $IA_U \subset A_U$ is generated by a regular sequence of length $c$ follows from flatness of $A \to A_U$ (see [Zav21c, Lemma B.4.3]) and [Sta23, Tag 00LM]. □

**Lemma 5.5.** Let $i \colon X \to Y$ be an lci immersion (of pure codimension $c$), and $f \colon Y' \to Y$ a flat morphism of locally noetherian analytic adic spaces. Then the base change

$$i' \colon X' \coloneqq Y' \times_Y X \to Y'$$

is also an lci immersion (of pure codimension $c$).

*Proof.* Lemma 5.4 ensures that the question is local on $X$, $Y$, and $T$. So we can assume that $X = \mathrm{Spa}(B, B^+)$, $Y = \mathrm{Spa}(A, A^+)$, and $Y' = \mathrm{Spa}(C, C^+)$ are strongly noetherian Tate affinoids. We can also assume that $i$ is a regular immersion of pure codimension $c$, so the ideal of immersion $I = \ker(A \to B)$ is generated by a regular sequence $(f_1, \dots, f_c) \subset A$. Then [Zav21c, Cor. B.6.9] implies that $B = A/I$, while [Zav21c, Lemma B.6.7] ensures that $i'$ is a Zariski-closed immersion and

$$\mathcal{O}_{X'}(X') \simeq C \otimes_A A/I \simeq C/IC. \tag{2}$$

We denote by $\mathcal{I}_S \coloneqq \ker(\mathcal{O}_T \to i'_*\mathcal{O}_S)$ the ideal of the Zariski-closed immersion $i'$. Then (2) and Theorem 4.2(3) ensure that $\mathcal{I}_S(T) \simeq IC$. Thus the question boils down to showing that $IC$ is generated by a regular sequence of length $c$. This follows from flatness of $A \to C$ (see [Zav21c, Lemma B.4.3]) and [Sta23, Tag 00LM]. □

**Remark 5.6.** Any smooth morphism of locally noetherian analytic adic spaces is flat by [Zav21c, Remark B.4.7]. In particular, Lemma 5.5 holds for any smooth morphism $f\colon Y' \to Y$.

**Lemma 5.7.** Let $i\colon X \to Y$ be an lci immersion of pure codimension $c$, and $f\colon Y' \to Y$ a morphism of locally noetherian analytic adic spaces. Suppose that the base change

$$i'\colon X' \coloneqq Y' \times_Y X \to Y'$$

is an lci immersion of pure codimension $c$. Then the natural morphism

$$f^*\mathcal{I}_X \to \mathcal{I}_{X'}$$

is an isomorphism, where $\mathcal{I}_X$ and $\mathcal{I}_{X'}$ are the ideal sheaves of the Zariski-closed immersions $i$ and $i'$ respectively.

*Proof.* Arguing as in the proof of Lemma 5.5, we reduce the question to proving the following claim:

*Claim: Let $A$ be a noetherian ring, $I \subset A$ an ideal generated by a regular sequence of length $c$, and $A \to B$ is a ring homomorphism such that $IB$ is still generated by a regular sequence of length $c$. Then $I \otimes_A B \to IB$ is an isomorphism.*

By induction, one can assume that $c = 1$. In this case, $I = (g) \subset A$ is a free $A$-module of rank-1. The assumption on $B$ tells us that $gB$ is a free $B$-module of rank-1. Therefore, $I \otimes_A B \to IB$ is a surjection of free $B$-modules of rank-1. Hence it is an isomorphism. □

Before we give some non-trivial examples of lci immersions, we need to discuss the notion of dimension in the adic geometry. In general, there are different ways to formalize the notion of dimension, so we explicitly spell our definitions.

**Lemma 5.8.** Let $S$ be a locally noetherian analytic adic space, $f_X\colon X \to S$ a smooth morphism of pure dimension $d_X$, $f_Y\colon Y \to S$ a smooth morphism of pure dimension $d_Y$, and $i\colon X \to Y$ a Zariski-closed immersion of adic $S$-spaces. Then, for each point $x \in X$, there is an open affinoid $x \in U_x \subset Y$ and an étale morphism $h\colon U_x \to \mathbf{D}^{d_Y}_S$ such that there is a cartesian diagram

$$\begin{array}{ccc} U_x \cap X & \xrightarrow{i|_{U_x\cap X}} & U_x \\ \downarrow{\scriptstyle h|_{U_x\cap X}} & & \downarrow{\scriptstyle h} \\ \mathbf{D}^{d_X}_S & \xrightarrow{j} & \mathbf{D}^{d_Y}_S, \end{array}$$

where $j\colon \mathbf{D}^{d_X}_S \to \mathbf{D}^{d_Y}_S$ is the inclusion of $\mathbf{D}^{d_X}_S$ into $\mathbf{D}^{d_Y}_S$ as the vanishing locus of the first $d_Y - d_X$ coordinates.

*Proof.* Let us denote by $\mathcal{I}$ the ideal sheaf of the Zariski-closed immersion $i$. The claim is local on $S$, so we clearly can assume that $S$ is a Tate affinoid with a pseudo-uniformizer $\varpi$.

Now [Hub96, Prop. 1.6.9(ii)] and a standard approximation argument imply that there is an open affinoid $x \in U_x = \mathrm{Spa}(B, B^+) \subset Y$ and generators of

$$g_1, \dots, g_{d'} \in \mathcal{I}(U_x) \subset B$$

that can be extended to a basis $\{g_1, \dots, g_{d'}, \dots, g_d\}$ of $\Omega^1_{U_x/S}$. In particular, $X \cap U_x$ is the vanishing locus of the functions $g_1, \dots, g_{d'}$.

We can simultaneously multiply $g_1, \dots, g_d$ by some power of $\varpi$ to assume that $g_i \in B^+$ and consider the unique $\mathcal{O}_S(S)$-linear morphism

$$h^\sharp \colon \left(\mathcal{O}_S(S)\langle T_1, \dots, T_d\rangle, \mathcal{O}_S(S)^+\langle T_1, \dots, T_d\rangle\right) \to (B, B^+)$$

sending $T_i$ to $g_i$. It defines a morphism of $S$-adic spaces

$$h \colon U_x \to \mathbf{D}^d_S.$$

that is étale by [Hub96, Prop. 1.6.9(iii)]. By construction (and [Zav21c, Lemma B.6.7]), $h$ fits into the Cartesian diagram

$$\begin{array}{ccc} U_x \cap X & \xrightarrow{i|_{U_x \cap X}} & U_x \\ \downarrow{\scriptstyle h|_{U_x \cap X}} & & \downarrow{\scriptstyle h} \\ \mathbf{D}^{d'}_S & \xrightarrow{j} & \mathbf{D}^d_S, \end{array} \tag{3}$$

where $j$ is the inclusion of $\mathbf{D}^{d'}_S$ into $\mathbf{D}^d_S$ as the vanishing locus of the first $d - d'$ coordinates. We are only left to show that $d = d_Y$ and $d' = d_X$. This follows from Corollary 3.4. ☐

**Remark 5.9.** In general, a similar argument shows that, for any smooth morphism $f \colon X \to S$ and a point $x \in X$, there is an open $x \in U$ and an integer $d$ such that $f|_U$ factors as the composition

$$U \to \mathbf{D}^d_S \to S.$$

In particular, analytically locally on the source, any smooth morphism is relatively pure of some dimension $d$.

**Corollary 5.10.** In the notation of Lemma 5.8, $i$ is an lci immersion of pure codimension $d_Y - d_X$. In particular, a section $s \colon S \to X$ of a separated smooth morphism $f \colon X \to S$ (of pure relative dimension $d$) is an lci immersion of (pure codimension $d$).

*Proof.* The first claim directly from Lemma 5.8, Lemma 5.5, and Remark 5.6. The "in particular" part follows from the previous claim if we can show that a section of a separated morphism is a Zariski-closed immersion. This, in turn, follows from the pullback diagram

$$\begin{array}{ccc} S & \xrightarrow{s} & X \times_S S \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{id}_X \times s} \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X, \end{array}$$

the fact that $\Delta_{X/S}$ is a Zariski-closed immersion (see [Zav21c, Cor. B.7.4]), and the fact that Zariski-closed immersions are closed under pullbacks (see [Zav21c, Cor. B.6.10]). ☐

## 6. Analytic Proj construction

This section is devoted to the discussion of the relative Proj construction in the world of adic spaces. In the case of rigid-analytic varieties over a non-archimedean field, this notion has been studied in [Con07].

We start with the relative analytification construction. Let $S = \mathrm{Spa}(A, A^+)$ an strongly noetherian affinoid space. Then the universal property of affine schemes (see [Sta23, Tag 01I1]) says that

$$\mathrm{Map}_{\mathbf{LRS}}(S, \mathrm{Spec}\, A) = \mathrm{Map}_{\mathbf{Rings}}(\mathcal{O}_S(S), A) = \mathrm{Map}_{\mathbf{Rings}}(A, A),$$

where **LRS** is the category of locally ringed spaces. Thus, the identity morphism $\mathrm{Id}_A$ defines a morphism of locally ringed spaces

$$\varphi \colon S \to \mathrm{Spec}\, A.$$

**Definition 6.1.** A *relative analytification* of a locally finite type $A$-scheme $X$ is an adic $S$-space $X^{\mathrm{ad}/S} \to S$ with a morphism of locally ringed $\mathrm{Spec}\, A$-spaces $\phi_X \colon X^{\mathrm{an}/S} \to X$ such that, for every adic $S$-space $U$, $\phi_X$ induces a bijection

$$\mathrm{Map}_{\mathbf{Adic}_{/S}}(U, X^{\mathrm{an}/S}) \simeq \mathrm{Map}_{\mathbf{LRS}_{/\mathrm{Spec}\, A}}(U, X).$$

**Remark 6.2.** Clearly, a relative analatytification is unique if it exists. It always exists (for locally finite type $A$-schemes) by [Hub94, Prop. 3.8].

**Remark 6.3.** ([Hub96, Lemma 5.7.3]) If $X$ is a proper $\mathcal{O}_S(S)$-scheme, then $X^{\mathrm{an}/S}$ is a proper adic $S$-space.

For the next definition, we fix a locally noetherian analytic adic space $S$.

**Definition 6.4.** A *locally coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$* is a graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet = \bigoplus_{d \geq 0} \mathcal{A}_d$ such that each $\mathcal{A}_d$ is a coherent $\mathcal{O}_S$-module, and $\mathcal{A}_\bullet$ is locally finitely generated as an $\mathcal{O}_S$-algebra.

Let $S$ be an affinoid. A *coherent graded $\mathcal{O}_S(S)$-algebra $A_\bullet$* is a graded $\mathcal{O}_S(S)$-algebra $A_\bullet = \bigoplus_{d \geq 0} A_d$ such that each $A_d$ is a coherent $\mathcal{O}_S(S)$-module, and $A_\bullet$ is locally finitely generated as an $\mathcal{O}_S(S)$-algebra.

Now we wish to show that there is an equivalence between locally coherent graded $\mathcal{O}_S$-algebras and coherent graded $\mathcal{O}_S(S)$-algebras for a strongly noetherian affinoid space $S$. For this, we will need the following lemma:

**Lemma 6.5.** Let $f \colon S' = \mathrm{Spa}(B, B^+) \to S = \mathrm{Spa}(A, A^+)$ be a flat (resp. surjective flat) morphism of strongly noetherian affinoid spaces. Then $f^\sharp \colon A \to B$ is flat (resp. faithfully flat).

*Proof.* Flatness of $A \to B$ follows from [Zav21c, Lemma B.4.3][2]. Now we assume that $f$ is also surjective, and show that $f^\sharp$ is faithfully flat. It suffices to show that $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective onto the closed points of $\mathrm{Spec}\, A$. This follows from the fact that, for any maximal

[2][Zav21c, Lemma B.4.2 and B.4.3] are formulated for Tate affinoids. However, the same proofs work for analytic affinoids. One only needs to use [Ked19, Thm. 1.4.14] in place of [Hub94, (II.1), (iv) on page 530] in the proof of [Zav21c, Lemma B.4.2].

ideal of $\mathfrak{m} \subset A$, there is a point $v \in \operatorname{Spa}(A, A^+)$ such that $\operatorname{supp}(v) = \mathfrak{m}$ (see [Hub94, Lemma 1.4]) and surjectivity of $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$. □

**Lemma 6.6.** Let $S$ be a strongly noetherian affinoid. Then $\Gamma(S, -)$ defines an equivalence

$$\Gamma(S,-)\colon \left\{ \begin{array}{c} \text{locally coherent graded} \\ \mathcal{O}_S\text{-algebras} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{coherent graded} \\ \mathcal{O}_S(S)\text{-algebras} \end{array} \right\}.$$

*Proof.* The proof essentially follows from Lemma 4.2. One easily sees that $\widetilde{(-)}$ provides a quasi-inverse to $\Gamma(S, -)$ provided that, for a locally coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$, the $\mathcal{O}_S$-algebra

$$\Gamma(S, \mathcal{A}_\bullet)$$

is naturally graded and coherent as a graded $\mathcal{O}_S(S)$-algebra. For the purposes of proving the first claim, it suffices to show $\Gamma(S, -)$ commutes with infinite direct sums. This follows from spectrality of $S$ and [Sta23, Tag 009F].

Now we need to show that $\Gamma(S, \mathcal{A}_\bullet)$ is a coherent graded $\mathcal{O}_S(S)$-algebra for any locally coherent graded $\mathcal{O}_S$-algebra. The locally coherent assumption together with Lemma 4.3(2) and Lemma 6.5 imply that there is a faithfully flat ring homomorphism $\mathcal{O}_S(S) \to \mathcal{O}_{S'}(S')$ such that

$$\Gamma(S, \mathcal{A}_\bullet) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S'}(S')$$

is a finitely generated $\mathcal{O}_{S'}(S')$-algebra. Therefore, [Sta23, Tag 00QP] ensures that $\Gamma(S, \mathcal{A}_\bullet)$ is a finite type $\mathcal{O}_S(S)$-algebra. □

For the next definition, we fix a stronly noetherian affinoid $S$, a locally coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$, and a corresponding coherent graded $\mathcal{O}_S(S)$-algebra $A_\bullet$.

**Definition 6.7.** The *analytic relative Proj space*

$$\underline{\operatorname{Proj}}_S^{\mathrm{an}} \mathcal{A}_\bullet := \left(\underline{\operatorname{Proj}}_{\operatorname{Spec} \mathcal{O}_S(S)} A_\bullet\right)^{\mathrm{an}/S}$$

is the relative analytification of the algebraic (relative) Proj scheme[3].

**Lemma 6.8.** Let $f\colon S' \to S$ be a morphism of strongly noetherian affinoids, and $\mathcal{A}_\bullet$ is locally coherent graded $\mathcal{O}_S$-algebra. Then there is a natural isomorphism

$$\psi_{S,S'}\colon \underline{\operatorname{Proj}}_{S'}^{\mathrm{an}}\left(f^*\mathcal{A}_\bullet\right) \xrightarrow{\sim} \underline{\operatorname{Proj}}_S^{\mathrm{an}}\left(\mathcal{A}_\bullet\right) \times_S S'.$$

Furthermore, if $g\colon S'' \to S'$ is another morphism of strongly noetherian affinoids, then the diagram

$$\begin{array}{ccccc} \underline{\operatorname{Proj}}_{S''}^{\mathrm{an}}\left(g^*\left(f^*\mathcal{A}_\bullet\right)\right) & \xrightarrow{\psi_{S',S''}} & \underline{\operatorname{Proj}}_{S'}^{\mathrm{an}}\left(f^*\mathcal{A}_\bullet\right) \times_{S'} S'' & \xrightarrow{\psi_{S,S'} \times \mathrm{id}} & \underline{\operatorname{Proj}}_S^{\mathrm{an}}\left(\mathcal{A}_\bullet\right) \times_S S' \times_{S'} S'' \\ \downarrow \sim & & & & \downarrow \wr \\ \underline{\operatorname{Proj}}_{S''}^{\mathrm{an}}\left(\left(f \circ g\right)^* \mathcal{A}_\bullet\right) & & \xleftarrow{\quad\psi_{S'',S}\quad} & & \underline{\operatorname{Proj}}_S^{\mathrm{an}}\left(\mathcal{A}_\bullet\right) \times_S S'' \end{array}$$

[3]See [EGA II, §2] for a detailed discussed of the algebraic Proj construction. In particular, use [EGA II, Prop. (2.7.1)] to ensure that $\underline{\operatorname{Proj}}_{\operatorname{Spec} \mathcal{O}_S(S)} A_\bullet$ is a finite type $\mathcal{O}_S(S)$-scheme, so its relative analytification is well-defined.

commutes.

*Proof.* Let $A_\bullet$ be a coherent graded $\mathcal{O}_S(S)$-algebra corresponding to $\mathcal{A}_\bullet$. Then, after unravelling the definitions, it suffices to show that there is a natural isomorphism of $\mathcal{O}_{S'}(S')$-schemes

$$\underline{\mathrm{Proj}}_{S'}\left(A_\bullet \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S'}(S')\right) \simeq \underline{\mathrm{Proj}}_S\left(A_\bullet\right) \times_S S'$$

that satisfies the "cocycle" formula. This follows from [Sta23, Tag 01N2] and [Sta23, Tag 01MZ]. □

**Lemma 6.9.** Let $S$ be a locally noetherian analytic adic space, and $\mathcal{A}_\bullet$ a locally coherent graded $\mathcal{O}_S$-algebra. Then there is an (essentially unique) analytic adic $S$-space

$$\pi\colon \underline{\mathrm{Proj}}^{\mathrm{an}}_S \mathcal{A}_\bullet \to S$$

with the following properties:

(1) for every affinoid $U \subset S$ there exists an isomorphism $i_U\colon \pi^{-1}(U) \xrightarrow{\sim} \underline{\mathrm{Proj}}^{\mathrm{an}}_U \mathcal{A}_\bullet|_U$, and

(2) for affinoid opens $V \subset U \subset S$ the composition

$$\underline{\mathrm{Proj}}^{\mathrm{an}}_V \mathcal{A}_\bullet|_V \xrightarrow{i_V^{-1}} \pi^{-1}(V) \xrightarrow{\sim} \pi^{-1}(U) \times_U V \xrightarrow{i_U \times_U V} \underline{\mathrm{Proj}}^{\mathrm{an}}_U \mathcal{A}_\bullet|_U \times_U V$$

is equal to $\psi_{U,V}$ from Lemma 6.8.

*Proof.* This follows formally from Lemma 6.8 and standard gluing arguments. □

For the next definition, we fix a locally noetherian analytic adic space $S$ and a locally coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$.

**Definition 6.10.** The *analytic relative Proj* of $\mathcal{A}_\bullet$ is the morphism

$$\underline{\mathrm{Proj}}^{\mathrm{an}}_S \mathcal{A}_\bullet \to S$$

is the adic $S$-space constructed in Lemma 6.9.

**Remark 6.11.** Lemma 6.8 easily implies that the formation of analytic Proj commutes with arbitrary base change. More precisely, for any morphism $f\colon S' \to S$ of locally noetherian analytic adic spaces and a locally coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$, there is a natural isomorphism

$$\mathrm{Proj}^{\mathrm{an}}_{S'}\left(f^*\mathcal{A}_\bullet\right) \simeq \left(\mathrm{Proj}^{\mathrm{an}}_S \mathcal{A}_\bullet\right) \times_S S'.$$

**Remark 6.12.** We note that $\underline{\mathrm{Proj}}^{\mathrm{an}}_S \mathcal{A}_\bullet$ is proper over $S$ by Remark 6.3 and a combination of [EGA II, Prop. (3.1.9)(i), Prop. (3.1.10), and Thm. (5.5.3)(i)].

**Remark 6.13.** Similarly to the algebraic situation, the analytic Proj-construction

$$P \coloneqq \underline{\mathrm{Proj}}^{\mathrm{an}}_S(\mathcal{A}_\bullet) \to S$$

comes equipped with the coherent sheaf $\mathcal{O}_{P/S}(1)$. If $S$ is affinoid, one just defines $\mathcal{O}_{P/S}(1)$ to be the relative analytification of the algebraic $\mathcal{O}(1)$. In general, one glues these line bundles locally on $S$. The formation of $\mathcal{O}_{P/S}(1)$ commutes with an arbitrary base change $S' \to S$. If $\mathcal{A}_\bullet$ is generated by $\mathcal{A}_1$, then $\mathcal{O}_{P/S}(1)$ is a line bundle.

Now we give two particularly interesting examples of the analytic Proj construction:

**Definition 6.14.** Let $\mathcal{E}$ be a vector bundle on $S$. The *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}_S(\mathcal{E}) \coloneqq \underline{\mathrm{Proj}}_S^{\mathrm{an}}\left(\mathrm{Sym}_S^\bullet \mathcal{E}\right) \to S.$$

Let $\mathcal{I}$ be a coherent ideal sheaf on $S$ and $Z \subset S$ be the associated closed adic subspace. The *blow-up of* $S$ *along* $Z$, or *the blow-up of* $S$ *in the ideal sheaf* $\mathcal{I}$, is the morphism

$$\mathrm{Bl}_Z(S) \coloneqq \underline{\mathrm{Proj}}_S^{\mathrm{an}} \bigoplus_{d\geq 0} \mathcal{I}^d \to S.$$

## 7. Line bundles on the relative projective bundle

In this section, we study line bundles on the relative projective bundle $\mathbf{P}_S(\mathcal{E}) \to S$ for any locally noetherian analytic adic space $S$ and a vector bundle $\mathcal{E}$ on $S$. The main goal is to prove the following theorem:

**Theorem 7.1.** Let $S$ be a connected locally noetherian analytic adic space, $\mathcal{E}$ a vector bundle on $S$, and $f\colon P \coloneqq \mathbf{P}_S(\mathcal{E}) \to S$ the corresponding projective bundle. Then the natural morphism

$$\mathrm{Pic}\,(S) \bigoplus \mathbf{Z} \to \mathrm{Pic}\,(P)$$

defined by the rule

$$(\mathcal{L}, n) \mapsto f^*\mathcal{L} \otimes \mathcal{O}_{P/S}(n)$$

is an isomorphism.

Let us begin with the case of a strongly noetherian Tate affinoid $S = \mathrm{Spa}(A, A^+)$. In this case, the relative projective space $\mathbf{P}^d_S \to S$ is the relative analytification of the relative algebraic projective space $\mathbf{P}^{d,\mathrm{alg}}_A \to \mathrm{Spec}\, A$. In particular, there is the analytification morphism

$$i\colon \mathbf{P}^d_S \to \mathbf{P}^{d,\mathrm{alg}}_A.$$

**Lemma 7.2.** In the notation as above, the natural morphism

$$\mathrm{Pic}\left(\mathbf{P}^{d,\mathrm{alg}}_A\right) \to \mathrm{Pic}\left(\mathbf{P}^d_S\right)$$

is an isomorphism.

*Proof.* The GAGA Theorem [FK18, Theorem 9.5.1] implies that the natural morphism

$$i^*\colon \mathbf{Coh}\left(\mathbf{P}^{d,\mathrm{alg}}_K\right) \to \mathbf{Coh}\left(\mathbf{P}^d_S\right)$$

is an equivalence of categories (respecting the symmetric monoidal structures on both sides). By identifying line bundles with invertible objects in **Coh**, we get that the pullback morphism

$$\mathrm{Pic}\left(\mathbf{P}^{d,\mathrm{alg}}_A\right) \to \mathrm{Pic}\left(\mathbf{P}^d_S\right).$$

is an isomorphism. ∎

Lemma 7.2 essentially proves Theorem 7.1 in the case of an affinoid base. In order to globalize the result, we will need to do some extra work.

**Corollary 7.3.** Let $K$ be a non-archimedean field with an open bounded valuation subring $K^+ \subset K$, $S = \mathrm{Spa}(K, K^+)$, and $f\colon \mathbf{P}^d_S \to S$ is the relative projective space. Then the natural morphism

$$\mathbf{Z} \to \mathrm{Pic}\left(\mathbf{P}^d_S\right),$$

defined by the rule

$$n \mapsto \mathcal{O}_{\mathbf{P}^d_S/S}(n),$$

is an isomorphism.

*Proof.* This follows directly from Lemma 7.2 and the standard algebraic computation

$$\mathrm{Pic}\left(\mathbf{P}^d_K\right) \simeq \mathbf{Z}[\mathcal{O}(1)].$$

□

**Notation 7.4.** Suppose that $f\colon \mathbf{P}_S(\mathcal{E}) \to S$ is a relative projective bundle over $S$ and $x \in S$ is a point. We will denote by $\mathbf{P}_x(\mathcal{E})$ the fiber product

$$\mathbf{P}_S(\mathcal{E}) \times_S \mathrm{Spa}\left(K(x), K(x)^+\right)$$

and call it the *fiber over* $x$.

**Warning 7.5.** Unless $x$ is a rank-1 point, the underlying topological space

$$|\,\mathrm{Spa}\left(K(x), K(x)^+\right)|$$

is not just one point $\{x\}$. Instead, it is the set of all generalizations of $x$. In particular, the adic space $\mathbf{P}_x(\mathcal{E})$ is not literally the fiber over $x$ unless $x$ is of rank-1.

**Lemma 7.6.** Let $S = \mathrm{Spa}(A, A^+)$ be a connected strongly noetherian Tate affinoid, $f\colon \mathbf{P}^d_S \to S$ the relative projective space, and $\mathcal{N}$ a line bundle on $\mathbf{P}^d_S$. Suppose that there is a point $x \in S$ such that

$$\mathcal{N}|_{\mathbf{P}^d_x} \simeq \mathcal{O}.$$

Then

1. $f_*\mathcal{N}$ is a line bundle on $S$;
2. the natural morphism $f^*f_*\mathcal{N} \to \mathcal{N}$ is an isomorphism.

In particular, the restriction of $\mathcal{N}$ onto any fiber is trivial.

*Proof.* Using Lemma 7.2 and the GAGA Theorem (see [FK18, Thm. 9.4.1]), we easily reduce the claim to an analogous claim for the algebraic relative projective space

$$g\colon \mathbf{P}^{d,\mathrm{alg}}_A \to \mathrm{Spec}\, A.$$

Then the results are well-known (and left as an exercise to the reader) as long as we know that $\mathrm{Spec}\, A$ is connected. However, connectedness of $\mathrm{Spec}\, A$ follows from Lemma 2.1. □

**Corollary 7.7.** Let $S$ be a locally noetherian analytic adic space, $f\colon \mathbf{P}_S(\mathcal{E}) \to S$ a projective bundle, and $\mathcal{N}$ a line bundle on $\mathbf{P}_S(\mathcal{E})$. For each integer $n$, let $E_n(\mathcal{N})$ be the set

$$E_n(\mathcal{N}) \coloneqq \{x \in S \mid \mathcal{N}|_{\mathbf{P}_x(\mathcal{E})} \simeq \mathcal{O}(n)\} \subset S.$$

Then $E_n(\mathcal{N})$ is a clopen subset of $S$ for each integer $n$.

*Proof.* Since the subsets $E_n(\mathcal{N})$ are disjoint, it suffices to show that each of them is open. This follows directly from Lemma 7.6 and Lemma 2.2. $\square$

**Lemma 7.8.** Let $S$ be a locally noetherian analytic adic space, and $f\colon \mathbf{P}_S(\mathcal{E}) \to S$ a relative projective bundle. Then, for any line bundle $\mathcal{L} \in \mathrm{Pic}(S)$, the natural morphism

$$\mathcal{L} \to f_* f^* \mathcal{L}$$

is an isomorphism.

*Proof.* The proof is clearly local on $S$, so we can assume that $S$ is affine and both $\mathcal{L}$ and $\mathcal{E}$ are trivial. In this case, it suffices to show that the natural morphism

$$\mathcal{O}_S \to f_* \mathcal{O}_{\mathbf{P}^d_S}$$

is an isomorphism. This is standard and follows, for example, from the analogous algebraic results and the (relative) GAGA Theorem (see [FK18, Thm. 9.4.1]). $\square$

Now we are ready to give a proof of Theorem 7.1.

**Theorem 7.9.** Let $S$ be a connected locally noetherian analytic adic space, and $f\colon P \coloneqq \mathbf{P}_S(\mathcal{E}) \to S$ a projective bundle. Then the natural morphism

$$\alpha\colon \mathrm{Pic}\,(S) \bigoplus \mathbf{Z} \to \mathrm{Pic}\,(P)$$

defined by the rule

$$(\mathcal{L}, n) \mapsto f^*\mathcal{L} \otimes \mathcal{O}_{P/S}(n)$$

is an isomorphism.

*Proof. Step* 1. *Injectivity of* $\alpha\colon \mathrm{Pic}\,(S) \bigoplus \mathbf{Z} \to \mathrm{Pic}\,(P)$. Suppose that the map is not injetive, so there is a line bundle $\mathcal{N} = f^*\mathcal{L} \otimes \mathcal{O}_{P/S}(n)$ that is isomorphic to $\mathcal{O}_P$. Then Corollary 7.3 implies that $n = 0$ by restricting $\mathcal{N}$ onto the fiber over some rank-1 point $x \in S$. Thus

$$\mathcal{O}_P \simeq \mathcal{N} \simeq f^*\mathcal{L}.$$

In this case, Lemma 7.8 implies that

$$\mathcal{L} \simeq f_* f^* \mathcal{L} \simeq f_* \mathcal{O}_P \simeq \mathcal{O}_S$$

finishing the proof.

*Step* 2. *Surjectivity of* $\alpha\colon \mathrm{Pic}\,(S) \bigoplus \mathbf{Z} \to \mathrm{Pic}\,(P)$. Pick any object $\mathcal{N} \in \mathrm{Pic}(P)$ and a point $x \in S$. By Corollary 7.3, we know that $\mathcal{N}_x \simeq \mathcal{O}_{P_x}(n)$ for some integer $n$. Then Corollary 7.7 implies that, for any point $y \in S$,

$$\mathcal{N}_y \simeq \mathcal{O}_{P_y}(n).$$

Therefore, by replacing $\mathcal{N}$ with $\mathcal{N} \otimes \mathcal{O}_{P/S}(-n)$, we can assume that the restriction of $\mathcal{N}$ on any fiber is trivial. In this case, it suffices to show that

$$f_*\mathcal{N}$$

is a line bundle on $S$, and that the natural morphism

$$f^* f_* \mathcal{N} \to \mathcal{N}$$

is an isomorphism. This question is local on $S$, so the result follows from Lemma 7.6 and Lemma 2.2. $\square$

**Corollary 7.10.** Let $S$ be a locally noetherian analytic adic space, $f\colon \mathbf{P}_S(\mathcal{E}) \to S$ a projective bundle, and $\mathcal{N}$ a line bundle on $\mathbf{P}_S(\mathcal{E})$. Then there is a disjoint decomposition of $S$ into clopen subsets $S = \sqcup_{i\in I} S_i$ with the induced morphisms

$$f_i\colon \mathbf{P}_{S_i}(\mathcal{E}|_{S_i}) \to S_i$$

such that

$$\mathcal{N}|_{\mathbf{P}_{S_i}(\mathcal{E}|_{S_i})} \simeq f_i^*\mathcal{L}_i \otimes \mathcal{O}(n_i)$$

for some $\mathcal{L}_i \in \mathrm{Pic}(S_i)$ and integers $n_i$.

*Proof.* This follows directly from Theorem 7.9 and Corollary 7.10. □

## 8. Étale 6-functor formalism

In this section, we construct an étale 6-functor formalism on the category of locally noetherian adic spaces. We refer to [Man22, Appendix A.5] for the extensive discussion of 6-functor formalisms and to [Zav22, Def. 2.3.10 and Rem. 2.3.11] for the precise definition of a 6-functor formalism that we are going to use in these notes. Here, we only say that a data of a 6-functor formalism is a formal way of encoding the 6-functors

$$\left(f^*,\ \mathrm{R}f_*,\ \otimes^L,\ \mathrm{R}\underline{\mathrm{Hom}},\ \mathrm{R}f_!,\ \mathrm{R}f^!\right)$$

with all (including "higher") coherences between these functors. In particular, this encodes the projection formula and proper base-change.

This formalism was essentially constructed by R. Huber in [Hub96]. However, at some places he had to work with bounded derived categories and a restricted class of morphisms. We eliminate all these extra assumptions in this section, and also make everything $\infty$-categorical. The main new input is the formalism developed in [Man22, Appendix A.5].

In the rest of the section, we fix an integer $n$. For each locally noetherian analytic adic space $X$, we denote by $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ the $\infty$-derived category of étale sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules on $X$. This is a stable, presentable $\infty$-category with the standard $t$-structure (see [HA, Prop. 1.3.5.9 and Prop. 1.3.5.21]).

We wish to define 6-functors on $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. We note that 4-functors come for free. The category $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ admits the natural symmetric monoidal structure by deriving the usual tensor product on $\mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ (see [LZ17, Lemma 2.2.2 and Notation 2.2.3] for details). We denote this functor by

$$- \otimes^L - \colon \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \times \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}).$$

By deriving the inner-Hom functor, we also get the functor

$$\mathrm{R}\underline{\mathrm{Hom}}_X(-,-)\colon \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})^{\mathrm{op}} \times \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$$

that is right adjoint to the tensor product functor. Similarly, for any morphism $f\colon X \to Y$ of locally noetherian analytic adic spaces, we get a pair of adjoint functors

$$f^*\colon \mathcal{D}(Y_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}),$$
$$\mathrm{R}f_*\colon \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(Y_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}).$$

Thus, the question of constructing 6-functor essentially reduces to the question of constructing $\mathrm{R}f_!$ and $f^!$-functors and showing certain compatibilities.

**Lemma 8.1.** Let $f\colon X \to Y$ be a weakly finite type morphism of locally noetherian analytic adic spaces, and $n$ invertible in $\mathcal{O}_Y^+$. Then

(1) if $Y$ is quasi-compact, $\mathrm{R}f_*$ is of finite cohomological dimension;
(2) $\mathrm{R}f_*$ commutes with all (homotopy) colimits. So it admits a right adjoint $\mathrm{R}f^!$;
(3) for any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the natural morphism
$$g^*\mathrm{R}f_* \to \mathrm{R}f'_* \circ (g')^*$$
is an isomorphism of functors $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(Y'_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$.

*Proof. Step* 1. *We show (3) for bounded below complexes.* We wish to show that the natural morphism
$$\psi_{\mathcal{F}}\colon g^*\mathrm{R}f_*\mathcal{F} \to \mathrm{R}f'_* \circ (g')^*\mathcal{F}$$
is an isomorphism for any $\mathcal{F} \in \mathcal{D}^+(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. An easy argument with spectral sequences reduces the question to the case of a sheaf $\mathcal{F} \in \mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. It suffices to show that $\psi_{\mathcal{F}}$ is an isomorphism on stalks at geometric points of $Y$. Then [Hub96, Lem. 2.5.12 and Prop. 2.6.1] reduce the question to the case of a surjective morphism
$$Y' = \mathrm{Spa}(C', C'^+) \to Y = \mathrm{Spa}(C, C^+)$$
for some algebraically closed non-archimedean fields $C$ and $C'$ and open, bounded valuation subrings $C^+ \subset C$ and $C'^+ \subset C'$. In this case, the result follows from [Hub96, Cor. 4.3.2].

*Step* 2. *We show (1).* We wish to show that there is an integer $N$ such that, for every $\mathcal{F} \in \mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$, the complex $\mathrm{R}f_*\mathcal{F}$ lies in $\mathcal{D}^{\leq N}(Y; \mathbf{Z}/n\mathbf{Z})$. We claim that $N = \mathrm{tr.deg}(f)$ does the job ($N$ is finite by Lemma 3.7). Indeed, Step 1 allows us to reduce to the case when $Y = \mathrm{Spa}(C, C^+)$ for an algebraically closed field $C$ and an open, bounded valuation subring $C^+ \subset C$. Then the result follows from [Hub96, Cor. 2.8.3].

*Step* 3. *We show (2).* Since $\mathrm{R}f_*$ is a right adjoint, it commutes with all finite limits. Therefore, it commutes with all finite colimits by [HA, Prop. 1.1.4.1], so it suffices to show that $\mathrm{R}f_*$ commutes with infinite direct sums. Therefore, it suffices to show that, for any collection of objects $\mathcal{F}_i \in \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$, the natural morphism
$$\bigoplus_{i\in I} \mathrm{R}f_*(\mathcal{F}_i) \to \mathrm{R}f_*\left(\bigoplus_{i\in I} \mathcal{F}_i\right)$$
is an isomorphism. If all $\mathcal{F}_i \in \mathcal{D}^{\geq 0}(X; \mathbf{Z}/n\mathbf{Z})$, this follows from [Hub96, Lemma 2.3.13(ii)]. In general, the claim is local on $Y$, so we can assume that $Y$ is quasi-compact. Then $\mathrm{R}f_*$ is of finite cohomological dimension by Step 2. Therefore, the unbounded version follows from the bounded one by a standard argument with truncations.

Existence of a right adjoint follows directly from the fact that $\mathrm{R}f_*$ commutes with colimits and [HTT, Cor. 5.5.2.9].

*Step* 4. *We show (3).* The question is clearly analytically local on $Y$ and $Y'$, so we can assume that both spaces are quasi-compact. Therefore, Step 2 ensures that both $\mathrm{R}f_*$ and $\mathrm{R}f'_*$ have finite cohomological dimension. Furthermore, Step 1 guarantees that the base change morphism is an isomorphism for all bounded below complexes. Therefore, a standard argument with truncations allows to formally deduce the unbounded version. □

Now we discuss the fifth functor $f_!$. The idea is to define it separately for an étale morphism and a proper morphism, and then show that these two functors "glue" together.

**Lemma 8.2.** Let $j\colon U \to X$ be an étale morphism of locally noetherian analytic adic spaces. Then the functor $j^*\colon \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(U_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ admits a left adjoint

$$j_!\colon \mathcal{D}(U_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$$

such that

(1) for any Cartesian diagram

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\ X' & \xrightarrow{g} & X \end{array}$$

of locally noetherian analytic adic spaces, the natural morphism

$$j'_! \circ (g')^* \to g^* \circ j_!$$

is an isomorphism of functors $\mathcal{D}(U) \to \mathcal{D}(X')$;

(2) the natural morphism

$$j_!(- \otimes^L j^*(-)) \to j_!(-) \otimes^L -$$

is an isomorphism of functors $\mathcal{D}(U_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \times \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$.

*Proof. We first show existence of $j_!$.* For this, we note the étale topos $U_{\text{ét}}$ is the slice topos $(X_{\text{ét}})_{/h_U}$. Therefore, the pullback functor $j^*$ commutes with both limits and colimits. Since both $\mathcal{D}(U; \mathbf{Z}/n\mathbf{Z})$ and $\mathcal{D}(X; \mathbf{Z}/n\mathbf{Z})$ are presentable, the adjoint functor Theorem (see [HTT, Cor. 5.5.2.9]) implies that $j^*$ admits a left adjoint $j_!$.

*Base-change.* By adjunction, it suffices to show that the natural morphism

$$j^*\mathrm{R}g_* \to \mathrm{R}g'_* j'^*$$

is an isomorphism of functors. This is essentially obvious because $U_{\text{ét}}$ is the slice topos of $X_{\text{ét}}$.

*Projection Formula.* This follows from Yoneda's Lemma and the following sequence of isomorphisms

$$\begin{aligned}\mathrm{Hom}_X\left(j_!\left(A\otimes^L j^*B\right),C\right) &\simeq \mathrm{Hom}_U\left(A\otimes^L j^*B, j^*C\right)\\ &\simeq \mathrm{Hom}_U\left(A, \mathrm{R}\underline{\mathrm{Hom}}_U\left(j^*B, j^*C\right)\right)\\ &\simeq \mathrm{Hom}_U\left(A, j^*\mathrm{R}\underline{\mathrm{Hom}}_X\left(B, C\right)\right)\\ &\simeq \mathrm{Hom}_X\left(j_!A, \mathrm{R}\underline{\mathrm{Hom}}_X\left(B, C\right)\right)\\ &\simeq \mathrm{Hom}_X\left(j_!A\otimes^L B, C\right).\end{aligned}$$

□

Now we discuss the hardest part of the construction: we show that $j_!$ and $\mathrm{R}f_*$ are compatible in some precise sense:

**Proposition 8.3.** Let $Y$ be a locally noetherian analytic adic space,

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{j} & Y \end{array}$$

a Cartesian diagram such that $f$ is proper and $j$ is étale, and $n$ an integer invertible in $\mathcal{O}_Y^+$. Then

(1) there is a natural isomorphism of functors
$$j_!\circ \mathrm{R}f'_*\to \mathrm{R}f_*\circ j'_!\colon \mathcal{D}(X'_{\text{ét}};\mathbf{Z}/n\mathbf{Z})\to\mathcal{D}(Y_{\text{ét}};\mathbf{Z}/n\mathbf{Z}).$$

(2) (Projection Formula) The natural morphism of functors
$$\mathrm{R}f_*((-)\otimes^L f^*(-))\to \mathrm{R}f_*(-)\otimes^L(-)\colon \mathcal{D}(X)\times\mathcal{D}(Y)\to\mathcal{D}(Y)$$
is an isomorphism.

*Proof. Part (1).* Firstly, we define the morphism

$$\alpha\colon j_!\circ\mathrm{R}f'_*\to\mathrm{R}f_*\circ j'_!$$

to be adjoint to the natural morphism

$$f^*\circ j_!\circ\mathrm{R}f'_*\simeq j'_!\circ f'^*\circ\mathrm{R}f'_*\xrightarrow{j'_!(\mathrm{adj})} j'_!,$$

where the first map comes from the base-change established in Lemma 8.2. The question whether $\alpha$ is an isomorphism is étale local on $Y$ and $Y'$, so we may assume that both spaces are affinoids. Then [Hub96, Lemma 2.2.8] ensures that, after possibly passing to an open covering of $Y$, there is a decomposition of $j$ into a composition $j=g\circ i$ such that $i$ is an open immersion and $g$ is a finite étale morphism.

It suffices to treat these two cases separately. Suppose first that $j$ is finite étale. We can check that $\alpha$ is an isomorphism étale locally on $Y$, so we can reduce to the case when $Y'$ is a disjoint union of copies of $Y$. Then the result is evident.

Now we deal with the case when $j$ is an open immersion. Since $\mathrm{R}f_*$ and $j_!$ both have finite cohomological dimesion, a standard argument reduces the question to showing that the natural morphism

$$\alpha^i_{\mathcal{F}}\colon j_! \circ \mathrm{R}^i f'_*\mathcal{F} \to \mathrm{R}^i f_* \circ j'_!\mathcal{F}$$

is an isomorphism for any $\mathcal{F} \in \mathrm{Shv}(X'_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ and $i \geq 0$. It suffices to verify this claim on stalks. Since both $\mathrm{R}f_*$ and $j_!$ commute with arbitrary base change, we can reduce the question to showing that the natural morphism

$$\mathrm{R}\Gamma(X, j'_!\mathcal{F}) \to \mathrm{R}\Gamma(Y, j_!\mathrm{R}'_*\mathcal{F})$$

is an isomorphism, where $Y = \mathrm{Spa}(C, C^+)$ for an algebraically closed non-archimedean field $C$ and an open and bounded valuation subring $C^+ \subset C$. If $Y' = Y$, then the claim is evident. Otherwise, we see that

$$\mathrm{R}\Gamma(Y, j_!\mathrm{R}'_*\mathcal{F}) = 0$$

since the stalk of $j_!\mathrm{R}'_*\mathcal{F}$ at the unique closed point of $\mathrm{Spa}(C, C^+)$ is zero. Therefore, it suffices to show that

$$\mathrm{R}\Gamma(X, j'_!\mathcal{F}) = 0$$

in this case. This is follows from [Hub96, Prop. 4.4.3] since the restriction of $j'_!\mathcal{F}$ onto the fiber[4] of $f$ over the closed point of $Y$ is equal to 0.

*Part (2).* We wish to prove that the natural morphism

$$\mathrm{R}f_*(\mathcal{F} \otimes^L f^*\mathcal{G}) \to \mathrm{R}f_*(\mathcal{F}) \otimes^L \mathcal{G}$$

is an isomorphism for any $\mathcal{F} \in \mathcal{D}(X; \mathbf{Z}/n\mathbf{Z})$ and $\mathcal{G} \in \mathcal{D}(Y; \mathbf{Z}/n\mathbf{Z})$. Now we choose any complex $\mathcal{G}^\bullet$ representing $\mathcal{G}$. Then we note that the natural morphism

$$\mathrm{hocolim}_N \sigma^{\geq -N}\mathcal{G}^\bullet \to \mathcal{G}$$

is an isomorphism. Since all functors commute with (homotopy) colimits, it suffices to prove the result for $\sigma^{\geq -N}\mathcal{G}^\bullet$, i.e., we can assume that $\mathcal{G} \in \mathcal{D}^+(Y; \mathbf{Z}/n\mathbf{Z})$. Then we may similarly use that

$$\mathrm{hocolim}_N \tau^{\leq N}\mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

to reduce to the case of a bounded complex $\mathcal{G} \in \mathcal{D}^b(Y; \mathbf{Z}/n\mathbf{Z})$. This, in turn, can be reduced to the case when $\mathcal{G} \in \mathrm{Shv}(Y_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ by an easy induction on the number of non-zero cohomology sheaves. Then [Sta23, Tag 0GLW] implies that $\mathcal{G}$ is a colimit of sheaves of the form $j_!\mathbf{Z}/n\mathbf{Z}$ for some étale morphism $j\colon U \to X$. Again, since all functors in the question commute with all (homotopy) colimits, it suffices to prove the claim for $\mathcal{G} = j_!\underline{\mathbf{Z}/n\mathbf{Z}}$. In this case, this follows from the following sequence of isomorphisms

$$\begin{aligned}
\mathrm{R}f_*(\mathcal{F} \otimes f^* j_!\underline{\mathbf{Z}/n\mathbf{Z}}) &\simeq \mathrm{R}f_*(\mathcal{F} \otimes^L j'_!\underline{\mathbf{Z}/n\mathbf{Z}}) \\
&\simeq \mathrm{R}f_* \circ j'_! \circ (j')^*\mathcal{F} \\
&\simeq j_!\mathrm{R}f'_*(j')^*\mathcal{F} \\
&\simeq j_! j^* \mathrm{R}f_*\mathcal{F} \\
&\simeq (\mathrm{R}f_*\mathcal{F}) \otimes^L j_!(\underline{\mathbf{Z}/n\mathbf{Z}}).
\end{aligned}$$

[4]This fiber is merely a pseudo-adic space, and not an adic space.

□

Now we fix a locally noetherian adic space $S$, $\mathcal{C}'$ the category of locally $+$-weakly finite type adic $S$-spaces, and an integer $n$ invertible in $\mathcal{O}_S^+$. The next theorem shows that there is a 6-functor formalism $\mathcal{D}(-;\mathbf{Z}/n\mathbf{Z})$ on the category $\mathcal{C}'$. We also denote by $\mathcal{C}\mathrm{at}_\infty^\otimes$ the $\infty$-category of symmetric monoidal $\infty$-categories (see [HA, Variant 2.1.4.12]).

Before reading the proof of the next theorem, we strongly advise the reader to take a look at [Man22, Appendix A.5] and [Zav22, §2.1, 2.3].

**Theorem 8.4.** Let $S$, $\mathcal{C}'$, and $n$ be as above. Then there is a 6-functor formalism (in the sense[5] of [Zav22, Def. 2.3.10 and Rmk. 2.3.11])

$$\mathcal{D}_{\text{ét}}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}') \to \mathcal{C}\mathrm{at}_\infty$$

such that

(1) there is a canonical isomorphism of symmetric monoidal $\infty$-categories $\mathcal{D}_{\text{ét}}(X;\mathbf{Z}/n\mathbf{Z}) \simeq \mathcal{D}(X_{\text{ét}};\mathbf{Z}/n\mathbf{Z})$ for any $X \in \mathcal{C}'$;

(2) the functor $\mathcal{D}_{\text{ét}}(-;\mathbf{Z}/n\mathbf{Z})$ sends a 1-edge $[X \xleftarrow{f} Y \xrightarrow{\text{id}} Y]$ to the pullback functor $f^*\colon \mathcal{D}(Y;\mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}(X;\mathbf{Z}/n\mathbf{Z})$.

*Proof.* We use [LZ17, Lemma 2.2.2 and Notation 2.2.3] to get a functor

$$\mathcal{D}^*(-;\mathbf{Z}/n\mathbf{Z})\colon \mathcal{C}'^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty^\otimes$$

that sends a locally $+$-weakly finite type adic $S$-space $X$ to $\mathcal{D}(X_{\text{ét}};\mathbf{Z}/n\mathbf{Z})$. We extend it to the desired functor

$$\mathcal{D}_{\text{ét}}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{\mathrm{all},\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty$$

in four steps:

*Step 1. We define $\mathcal{D}_{ét}$ on "compatifiable" morphisms.* More precisely, we define $E \subset \mathrm{Hom}(\mathcal{C}')$ to be the class of $+$-weakly finite type, separated, taut morphisms (in the sense of [Hub96, Def. 5.1.2]). We also define the subclasses

$$I, P \subset E$$

to be quasi-compact open immersions and proper morphisms, respectively. Now [Hub96, Cor. 5.1.6] implies that any morphism $f \in E$ admits a decomposition $f = p \circ i$ such that $i \in I$ and $p \in P$. One easily checks that $I, P \subset E$ defines a *suitable decomposition* of $E$ in the sense of [Man22, Def. A.5.9]. Now Lemma 8.1, Lemma 8.2, and Proposition 8.3 ensure that all the conditions of [Man22, Prop. A.5.10] are satisfied, and so it defines a weak 6-functor formalism (see [Zav22, Def. 2.1.2])

$$\mathcal{D}_{\text{ét}}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{E,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

Now recall that a 6-functor formalism $\mathcal{D}_{\text{ét}}$ defines a lower-shriek functor $f_!$ for any morphism $f \in E$ (see [Man22, Def. A.5.6]). In this case, the construction tells us that the lower shriek functor $f_!$ is equal to $\mathrm{R}g_* \circ j_!$, where $f = g \circ j$ is the decomposition of $f$ into a composition of an open immersion $j$ and a proper morphism $g$. In particular, for a proper morphism $f$,

[5] Note that this definition slightly differs from [Man22, Def. A.5.7]

we get an equality $f_! = \mathrm{R}f_*$. In particular, any proper morphism is cohomologically proper in the sense of [Zav21b, Def. 2.3.4].

*Step* 2. *We extend* $\mathcal{D}_{ét}$ *to separated, locally* $+$*-weakly finite type morphisms.* We define $E_1$ to be the class of morphisms of the form $\sqcup_{i\in I} X_i \to Y$ such that each $X_i \to Y$ lies in $E$. Then [Man22, Prop. A.5.12] ensures that $\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})$ uniquely extends to a weak 6-functor formalism

$$\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{E_1,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

Now we define a new class of morphisms $E_1'$ to be the class of locally $+$-weakly finite type, separated morphism. We also define a subclass $S_1 \subset E_1$ to consist of morphisms $\sqcup_{i\in I} U_i \to X$ for covers $X = \cup_{i\in I} U_i$ by quasi-compact open immersions. Then [Man22, Prop. A.5.14] implies that $\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})$ uniquely extends to a weak 6-functor formalism

$$\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{E_1',\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

*Step* 3. *We extend* $\mathcal{D}_{ét}$ *to all locally* $+$*-weakly finite type morphisms.* This reduction is pretty similar to Step 2. We define $E''$ to be the collection of all locally $+$-weakly finite type morphisms, and $S \subset E'$ to be the collection of morphisms $\sqcup_{i\in I} U_i \to X$ for covers $X = \cup_{i\in I} U_i$ by open immersions. Then [Man22, Prop. A.5.14] implies that $\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})$ uniquely extends to a weak 6-functor formalism

$$\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{\mathrm{all},\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

*Step* 4. *We show that* $\mathcal{D}_{ét}$ *is a* 6*-functor formalism in the sense of* [Zav22, Def. 2.3.10 and Rmk. 2.3.11]. We already have a weak 6-functor formalism

$$\mathcal{D}_{ét}(-;\mathbf{Z}/n\mathbf{Z})\colon \mathrm{Corr}(\mathcal{C}')_{\mathrm{all},\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

By construction, the categories $\mathcal{D}_{ét}(X;\mathbf{Z}/n\mathbf{Z}) \simeq \mathcal{D}(X_{ét};\mathbf{Z}/n\mathbf{Z})$, so they are stable and presentable. Clearly, $\mathcal{D}_{ét}$ satisfies analytic descent; it even satisfies étale descent. By Step 1, we know that any proper morphism $f\colon X \to Y$ is cohomologically proper (in the sense of [Zav22, Def. 2.3.4]). Therefore, we are only left to check that any étale morphism $j\colon X \to Y$ is cohomologically étale in the sense of [Zav22, Def. 2.3.4].

For this, we set up $E = \text{ét}$ to be the class of all étale morphisms, and restrict $\mathcal{D}_{ét}$ onto $\mathrm{Corr}(\mathcal{C}')_{ét,\mathrm{all}}$ to get a weak 6-functor formalism

$$\mathcal{D}'_{ét}\colon\ \mathrm{Corr}(\mathcal{C}')_{ét,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

Alternatively, we can can apply [Man22, Prop. A.5.8] to $E = I$ being the class of all étale morphisms and the class $P$ consisting only of the identity morphisms to get another weak 6-functor formalism

$$\mathcal{D}''_{ét}\colon\ \mathrm{Corr}_{ét,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty.$$

By construction, any étale morphism is cohomologically étale with respect to $\mathcal{D}''_{ét}$. Thus, the question boils down to showing that $\mathcal{D}'_{ét}$ and $\mathcal{D}''_{ét}$ coincide. Using the uniqueness statements from [Man22, Prop. A.5.12, A.5.14, A.5.16], we can repeat the same arguments as in Steps 2 and 3 to reduce the question to showing that the restrictions

$$\mathcal{D}'_{ét}|_{ét qcsep}\colon\ \mathrm{Corr}_{ét qcsep,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty,$$

$$\mathcal{D}''_{ét}|_{ét qcsep}\colon\ \mathrm{Corr}_{ét qcsep,\mathrm{all}} \to \mathcal{C}\mathrm{at}_\infty$$

coincide, where étqcsep stands for the class of étale quasi-compact, separated morphisms. Now we note that étale quasi-compact, separated morphisms are taut by [Hub96, Lemma 5.1.3(iv)]. Therefore, after unravelling the definitions, we see that both $\mathcal{D}'_{\text{ét}}|_{\text{étqcsep}}$ and $\mathcal{D}''_{\text{ét}}|_{\text{étqcsep}}$ are obtained by applying [Man22, Prop. A.5.8] to $I = $ étqcsep and $P = \text{id}$. Therefore they coincide. □

**Remark 8.5.** Let $S$ be a locally noetherian analytic adic space, $\mathcal{C}$ the category of locally finite type adic $S$-spaces, and $n$ is an integer invertible in $\mathcal{O}_S^+$. Then we can restrict the functor $\mathcal{D}_{\text{ét}}(-; \mathbf{Z}/n\mathbf{Z})\colon \text{Corr}(\mathcal{C}') \to \mathcal{C}\text{at}_\infty$ onto $\text{Corr}(\mathcal{C})$ to get the étale 6-functor formalism

$$\mathcal{D}_{\text{ét}}(-; \mathbf{Z}/n\mathbf{Z})\colon \text{Corr}(\mathcal{C}) \to \mathcal{C}\text{at}_\infty.$$

**Remark 8.6.** Let $S$ is a scheme, $\mathcal{C}$ the category of locally finitely presented $S$-schemes, and $n$ any integer. Then one can similarly construct the étale 6-functor formalism

$$\mathcal{D}_{\text{ét}}(-; \mathbf{Z}/n\mathbf{Z})\colon \text{Corr}(\mathcal{C}) \to \mathcal{C}\text{at}_\infty.$$

The proof of Theorem 8.4 applies essentially verbatim. The main non-trivial input needed is:

(1) ([Con07, Thm. 4.1]) Nagata's compatification;
(2) ([Fu11, Prop. 5.9.6]) the natural morphism

$$\bigoplus_I \mathrm{R}f_*\mathcal{F}_i \to \mathrm{R}f_*\left(\bigoplus_I \mathcal{F}_i\right)$$

for a proper morphism $f$ and a collection of sheaves $\{\mathcal{F}_i \in \text{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})\}_{i\in I}$;
(3) ([Fu11, Thm 7.3.1]) proper base-change for bounded below complexes;
(4) projection formula for proper $f$ and bounded below complexes (in this case, it follows automatically from (2) and (3) by arguing on stalks, see [Fu11, 7.4.7]);
(5) finite cohomological dimension of $f_*$ for a proper $f$ (one can either adapt the proof of [Fu11, Thm. 7.4.5][6] or [Fu11, Corollary 7.5.6]).

## 9. Overconvergent sheaves

In this section, we prove two basic facts about overconvergent sheaves. Both facts can be deduced from the results in [Hub96]. However, the proofs in [Hub96] seem to be unnecessary difficult, so we prefer to include alternative proofs of these facts in these notes.

**Definition 9.1.** ([Hub96, Def. 8.2.1]) An étale sheaf $\mathcal{F} \in \text{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ on a locally noetherian analytic adic space is *overconvergent* if for every specialization of geometric points $u\colon \overline{\eta} \to \overline{s}$, the specialization morphism

$$\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}}$$

is an isomorphism.

[6] For this, one notices that (1) and (2) already imply that $\mathrm{R}f_!$ is a well-defined functor

**Lemma 9.2.** Let $Y$ be a locally noetherian analytic adic space, $j\colon X \to Y$ a quasi-compact dense pro-open immersion, $n$ an integer, and $\mathcal{F} \in \mathrm{Shv}(Y_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ an overconvergent étale sheaf on $Y$. Then the natural morphism $\mathcal{F} \to \mathrm{R}j_*j^*\mathcal{F}$ is an isomorphism. In particular, the natural morphism

$$\mathrm{R}\Gamma(Y, \mathcal{F}) \to \mathrm{R}\Gamma(X, j^*\mathcal{F})$$

is an isomorphism.

*Proof.* We can check that

$$\mathcal{F} \to \mathrm{R}j_*j^*\mathcal{F} \tag{4}$$

is an isomorphism at the geometric points of $Y$. Therefore, we can assume that $j$ is of the form $j\colon \mathrm{Spa}(C, C'^+) \to \mathrm{Spa}(C, C^+)$ for an algebraically closed non-archimedean field $C$ and open and bounded valuation subrings $C'^+ \subset C^+ \subset C$. In this case, it suffices to show that

$$\mathrm{H}^i(\mathrm{Spa}(C, C'^+), j^*\mathcal{F}) = 0$$

for $i \geq 1$, and

$$\mathrm{H}^0(\mathrm{Spa}(C, C^+), \mathcal{F}) \to \mathrm{H}^0(\mathrm{Spa}(C, C'^+), j^*\mathcal{F})$$

is an isomorphism. The first follows from the fact that any *surjective* étale morphism $S \to \mathrm{Spa}(C, C'^+)$ has a section[7].

Now we show the second claim. Let $\overline{s} = \mathrm{Id}\colon \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(C, C^+)$ be the "closed" geometric point of $\mathrm{Spa}(C, C^+)$ and $\overline{s}' = j\colon \mathrm{Spa}(C, C'^+) \to \mathrm{Spa}(C, C^+)$ the geometric point corresponding to the closed point of $\mathrm{Spa}(C, C'^+)$. Then the argument as above implies that $\mathrm{H}^0(\mathrm{Spa}(C, C^+), \mathcal{F}) = \mathcal{F}_s$ and $\mathrm{H}^0(\mathrm{Spa}(C, C'^+), j^*\mathcal{F}) \simeq \mathcal{F}_{s'}$. So the overconvergent assumption implies that the natural morphism

$$\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{s}'}$$

is an isomorphism finishing the proof. □

**Lemma 9.3.** Let $f\colon X \to S$ be a finite type, quasi-separated morphism of locally noetherian analytic adic spaces, $n$ an integer, and $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/n\mathbf{Z})$ an overconvergent sheaf. Then $\mathrm{R}^i f_*\mathcal{F}$ is overconvergent for any $i \geq 0$.

*Proof.* By [Hub96, Prop. 2.6.1], it suffices to show that, for any algebrically closed non-archimedean field $C$ with an open bounded valuation ring $C^+$ and a morphism $\mathrm{Spa}(C, C^+) \to S$, the natural morphism

$$\mathrm{H}^i(X_{\mathrm{Spa}(C,C^+)}, \mathcal{F}) \to \mathrm{H}^i(X_{\mathrm{Spa}(C,\mathcal{O}_C)}, \mathcal{F})$$

is an isomorphism. This follows from Lemma 9.2. □

[7] First reduce to an affinoid $S$, then use [Hub96, Lemma 2.2.8] and an equivalence $\mathrm{Spa}(C, C'^+)_{\text{fét}} \simeq (\mathrm{Spec}\, C)_{\text{fét}}$ to construct a section.

## 10. Categorical properties of lisse and constructible sheaves

In this section, we show that lisse and constructible étale sheaves on a locally noetherian analytic adic space (resp. a scheme) $X$ admit a nice categorical description. The results of this section are well-known to the experts, but it seems hard to find them explicitly stated in the existing literature.

For the rest of the section, we fix a locally noetherian adic space (resp. a scheme) $X$ and an integer $n$.

We recall that the derived category $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ admits a natural structure of a symmetric monoidal category (with the monoidal structure given by $- \otimes^L -$). In particular, it there is a well-defined notion of dualizable objects in $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$, see [Sta23, Tag 0FFP].

**Lemma 10.1.** Let $X$ be a locally noetherian analytic adic space or a scheme, and $n$ an integer. Then an object $\mathcal{F} \in \mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ is dualizable if and only if $\mathcal{F}$ lies in $\mathcal{D}^{(b)}_{\text{lisse}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ and, for each geometric point $\overline{s} \to X$, the stalk $\mathcal{F}_{\overline{s}}$ is a perfect complex in $\mathcal{D}(\mathbf{Z}/n\mathbf{Z})$.

*Proof.* First, we note that [Sta23, Tag 0FPV] ensures that $\mathcal{F}$ is dualizable if and only if $\mathcal{F}$ is perfect.

Now suppose that $\mathcal{F}$ is dualizable, so it is perfect by the observation above. Then clearly all stalks $\mathcal{F}_{\overline{s}}$ are perfect objects of $\mathcal{D}(\mathbf{Z}/n\mathbf{Z})$. Furthermore, the definition of perfect complexes (see [Sta23, Tag 08G5]), [Sta23, Tag 08G9], and the fact that lisse sheaves form a Weak Serre subcategory of $\mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ imply that the object $\mathcal{F}$ is locally bounded and has lisse cohomology sheaves, i.e. $\mathcal{F} \in \mathcal{D}^{(b)}_{\text{lisse}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$.

Now we suppose that $\mathcal{F}$ lies in $\mathcal{D}^{(b)}_{\text{lisse}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ and all its stalks are perfects. We wish to show that $\mathcal{F}$ is perfect (and so is dualizable). This is a local question, so we can assume that $X$ is qcqs, and thus $\mathcal{F}$ lies in $\mathcal{D}^{b}_{\text{lisse}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. Now [Sta23, Tag 094G] implies that there is a (finite) covering $\{U_i\}_{i \in I} \to X$ such that

$$\mathcal{F}|_{U_i} \simeq \underline{M_i^\bullet}$$

for some finite complexes of finite $\mathbf{Z}/n\mathbf{Z}$-modules $M_i^\bullet$. Using that all stalks of $\mathcal{F}$ are perfect as objects of $\mathcal{D}(\mathbf{Z}/n\mathbf{Z})$, we conclude that each $M_i^\bullet$ must be perfect. This formally implies that $\mathcal{F}$ is a perfect object of $\mathcal{D}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. ◻

Now we discuss the categorical description of constructible sheaves:[8]

**Lemma 10.2.** Let $X$ be a qcqs noetherian analytic adic space or a qcqs scheme, and $n, N$ some integers. Then an object $\mathcal{F} \in \mathcal{D}^{\geq -N}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ is compact if and only if $\mathcal{F}$ lies in $\mathcal{D}^{b, \geq -N}_{\text{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$, i.e., $\mathcal{F}$ is bounded and all its cohomology sheaves are constructible.

*Proof.* Without loss of generality, we can assume that $N = 0$.

*Step* 1. *The "if" direction.* An easy spectral sequence argument implies that we can assume that $\mathcal{F}$ is an (abelian) constructible sheaf. Then the question boils down to showing that $\underline{\mathrm{Ext}}^i(\mathcal{F}, -)$ and $\mathrm{H}^j(X_{\text{ét}}, -)$ commute with arbitrary direct sums in $\mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ for all $i$ and $j$.

[8]We refer to [Hub96, §2.7] and [Sta23, Tag 05BE] for the definition of constructible sheaves in the adic and schematic setups respectively.

We show that $\underline{\mathrm{Ext}}^i(\mathcal{F}, -)$ commutes with direct sums for a constructible sheaf $\mathcal{F}$. If $\mathcal{F} = f_!\left(\underline{\mathbf{Z}/n\mathbf{Z}}\right)$ for a qcqs étale morphism $f\colon U \to X$, then the claim follows from the isomorphism

$$\mathrm{R}\underline{\mathrm{Hom}}_X(f_!\underline{\mathbf{Z}/n\mathbf{Z}}, -) \simeq \mathrm{R}f_*\mathrm{R}\underline{\mathrm{Hom}}_U(\underline{\mathbf{Z}/n\mathbf{Z}}, f^*-) \simeq \mathrm{R}f_*f^*(-).$$

Now the claim follows from the fact that both $\mathrm{R}f_*$ and $f^*$ commute with infinite direct sums in $\mathrm{Shv}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. Indeed, $f^*$ always commutes with direct sums, and $\mathrm{R}f_*$ commutes with direct sums by [Hub96, Lemma 2.3.13(ii)].

Now, for a general constructible sheaf $\mathcal{F}$, we use a resolution of the form

$$\cdots \to f_{1,!}\underline{\mathbf{Z}/n\mathbf{Z}} \to f_{0,!}\underline{\mathbf{Z}/n\mathbf{Z}} \to \mathcal{F} \to 0,$$

with $f_i\colon X_i \to X$ being qcqs étale maps (this presentation follows from [Sta23, Tag 095N] in the scheme case and from the proof of [Hub96, Lemma 2.7.8] in the adic case). Then an easy spectral sequence argument implies that $\mathcal{F}$ is compact since each $f_{n,!}\left(\underline{\mathbf{Z}/n\mathbf{Z}}\right)$ is so.

To finish the proof, we note that $\mathrm{H}^j(X_{\text{ét}}, -)$ commutes with arbitrary direct sums by [Hub96, Lemma 2.7.8(i)].

*Step* 2. *We show that the natural morphism* $\mathrm{Ind}\left(\mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\acute{e}t}; \mathbf{Z}/n\mathbf{Z})\right) \to \mathcal{D}^{\geq 0}(X_{\acute{e}t}; \mathbf{Z}/n\mathbf{Z})$ *is an equivalence.* First we note, $\mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ admits all (small) filtered colimits, so the natural inclusion

$$\mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z}) \to \mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$$

extends to the functor

$$i\colon \mathrm{Ind}(\mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})) \to \mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$$

due to [HTT, Lemma 5.3.5.8]. Since each object of $\mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ is compact in $\mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$, we conclude that the functor $i$ is fully faithful, so it suffices to show that $i$ is essentially surjective.

Now note that any object $\mathcal{F} \in \mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ can be written as a (homotopy) colimit

$$\mathrm{colim}_n\, \tau^{\leq n}\mathcal{F} \to \mathcal{F}.$$

Since $i$ commutes with filtered colimits, it suffices to show that any $\mathcal{F} \in \mathcal{D}^{b,\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ lies in the essential image. For this, we note that the essential image of $i$ is closed under the shift $[-1]$ and "kernels" in $\mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. Therefore, an easy inductive argument reduces the question to showing that any abelian sheaf $\mathcal{F} \in \mathrm{Shv}\,(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ lies in the essential image of $\mathcal{F}$. This follows from [Hub96, Lemma 2.7.8] in the adic world and from [Sta23, Tag 09YU] in the scheme world.

*Step* 3. *Finish the proof.* Step 2 implies that any $\mathcal{F} \in \mathcal{D}^{\geq 0}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ can be written as a filtered (homotopy) colimit

$$\mathcal{F} = \mathrm{colim}_{i\in I}\, \mathcal{F}_i$$

with $\mathcal{F}_i \in \mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. If $\mathcal{F}$ is compact, we see that there is an equivalence

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathrm{colim}_{i\in I}\, \mathcal{F}_i) = \mathrm{colim}_{i\in I}\, \mathrm{Hom}(\mathcal{F}, \mathcal{F}_i).$$

In particular, we note that the identity morphism $\mathrm{id}\colon \mathcal{F} \to \mathcal{F}$ factors through some $\mathcal{F}_i \to \mathcal{F}$. Thus, $\mathcal{F}$ is a direct summand of $\mathcal{F}_i$, so it must lie in $\mathcal{D}^{b,\geq 0}_{\mathrm{cons}}(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$. □

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