NOTES FOR THE BIG MONODROMY LEMMA FROM VENKATESH-LAWRENCE PAPER

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Abstract. We provide some details to the proof of the big monodromy lemma ([LV, Lemma 4.3]) in the celebrated paper "Diophantine Problems And p-adic Period Mappings" by B. Lawrence and A. Venkatesh.

1. Introduction

These are the notes written after my talk at the Stanford Number Theory Working Seminar in 2019. The topic of the seminar was the recent paper [LV] of Lawrence and Venkatesh where they develop new techniques to approach certain global number theoretic problems by considering certain period maps both p-adically and complex-analytically. In particular, their methods are robust enough to give a new proof of the Mordell Conjecture. I did not discuss the Mordell Conjecture in my talk, rather the talk was mainly concerned with the so-called S-unit equation. This is a toy example of an interesting number theoretic problem that can be solved using the p-adic methods introduced by Lawrence and Venkatesh. The proof illustrates the strategy they use to handle more difficult problems such as Mordell Conjecture.

My talk closely followed the exposition in [LV, §4] that I found pretty good written. However, there is Lemma 4.3 that confused me and the audience a lot during the talk (probably due to my misunderstanding of the proof). Since it played a crucial role in the argument, I decided to write some notes that provide slightly more details. The notes might be useful to people like me who did not have much experience with computing topological monodromy representations explicitly.

We start by recalling the statement we want to prove and the context for it. The story starts with the S-unit equation:

Theorem 1.1. Let K be a number field with the ring of integers $\mathcal{O}$. Suppose that S be a finite set of places of K containing all archimedean places with a ring of S-integers $\mathcal{O}_S$. Then the set

$$(\mathbf{A}^1_{\mathcal{O}_S} \setminus \{0,1\})(\mathcal{O}_S)$$

is finite. In other words, the set

$$\{t \in \mathcal{O}_S^* \mid 1 - t \in \mathcal{O}_S^*\}$$

is finite.

Here is the outline of the proof of this theorem in [LV]. They make a sequence of reductions on K, S and a place $v \notin S$ to reduce to finiteness of a slightly different set $U_{1,L} \subset \mathbf{A}^1_K \setminus \{0,1\}$ that we are not going to recall here, but rather refer to [LV, Lemma 4.2] for its definition. Then the crux of their argument is to show that $U_{1,L}$ considered as a subset of the open disc $\Omega_v \subset \mathbf{A}^1_K(K_v)$ lies in a proper closed *analytic* subset. Therefore it must be finite as a Zariski closed subset in a connected 1-dimensional rigid-space Ω_v!

Now the question boils down to show that a certain subset $U_{1,L}$ (that we have not defined here) lies in a closed analytic subset of Ω_v. Their main approach to this question is via the period map for the Legendre family over $\mathbf{A}^1_K \setminus \{0,1\}$ (or a slightly modified version of this family to be defined later). Then they analyze it both using the complex *and* p-adic geometry to show that $U_{1,L}$ must lie in a proper closed subset of Ω_v. The essential idea is to show that p-adically the (Zariski closure) of the image of $U_{1,L}$ has smaller dimension that the dimension of $\Omega_{\mathbf{C}}$ complex-analytically. This would imply that imply that $U_{1,L}$ lies in a proper closed *analytic* subset of Ω_v.

All in all, we see that the main ingredients in Lawrence-Venkatesh proof is to understand the period map for certain version of the Legendre family both p-adically and complex analytically. Here are two lemmas from their paper that do the job.

Lemma 1.2 (Lemma 4.4, Generic simplicity)**.** Let L be a number field and p a rational prime, larger than 2, and unramified in L. There are only finitely many $z \in L$ such that $z, 1-z$ are both p-units, but for which the Galois representation of G_L on the Tate module $T_p(E_z) = \mathrm{H}^1_{\text{ét}}(E_{z,\overline{L}}, \mathbf{Q}_p)$ of the elliptic curve

$$E_z \colon y^2 = x(x-1)(x-z),$$

fails to be simple.

Lemma 1.3 (Lemma 4.3, Big monodromy)**.** Consider the family of curves over $\mathbf{C} \setminus \{0,1\}$ whose fiber over $t \in \mathbf{C}$ is the union of the elliptic curves $E_z \colon y^2 = x(x-1)(x-z)$, over all m-th roots $z^m = t$. Then the action of monodromy

$$\pi_1(\mathbf{C} \setminus \{0,1\}, t_0) \to \mathrm{Aut}\left(\bigoplus_{z^m = t_0} \mathrm{H}^1_{\mathrm{B}}(E_z, \mathbf{Q})\right)$$

has Zariski closure containing $\prod_z SL(\mathrm{H}^1_B(E_z, \mathbf{Q}))$.

The rest of the paper is devoted to the proof of Lemma 2.2 following the ideas from [LV].

2. The setup and Outline of the proof

2.1. **The setup.** First of all, let us formulate the result we want to proof more precisely and also set up some notations for the rest of the notes.

We start with the usual Legendre family over $X'' \to S''$ over $S'' \coloneqq \mathbf{A}^1_{\mathbf{C}} \setminus \{0,1\}$ with a coordinate t. More precisely, this family is defined by the equation

$$X'' \coloneqq \mathrm{V}\left(Y^2 Z - X\left(X - Z\right)\left(X - tZ\right)\right) \subset \mathbf{P}^2_{S''}.$$

Now we need to slightly modify this family according to the strategy of Lawrence and Venkatesh.

We define $S' \coloneqq \mathbf{A}^1 \setminus \{0, \mu_m(\mathbf{C})\}$[1] and consider the restriction of the Legendre family $g \colon X \coloneqq X'' \times_{S''} S' \to S'$. Finally, we consider the $m-th$ power map

$$h \colon S' \to \mathbf{A}^1 \setminus \{0,1\} =\colon S,$$

defined by

$$z \mapsto z^n.$$

Now the *modified Legendre family* is simply the composition

$$X \xrightarrow{g} S' \xrightarrow{h} S, \qquad p = h \circ g.$$

Remark 2.1. It is rather easy to show that $X \to S'$ is the Stein factorization of $X \to S$ (*prove it!*).

[1]Here $\mu_m(\mathbf{C})$ is a set of m-roots of unity considered as a set of closed points of S''.

Our ultimate goal is to compute the monodromy of the modified Legendre family $p\colon X \to S$. What does this really mean? Note that f is a smooth projective morphism as it is the composition of a smooth projective morphism and a finite etale one. Therefore, we know that $\mathrm{R}^1 p_* \underline{\mathbf{Q}}$ is a $\mathbf{Q}$-local system on $S(\mathbf{C})^{an}$ by Lemma 4.2. Then Lemma 4.1 and the topological Proper Base Change Theorem (which is essentially obvious) show that this local system corresponds[2] to a representation of the *topological* fundamental group of $S(\mathbf{C})^{an}$:

$$\rho_S : \pi_1(S(\mathbf{C})^{an}, t) \to \mathrm{GL}\left(\mathrm{H}^1\left(X_t\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right).$$

So what we really want to understand is the image of $\pi_1(S(\mathbf{C})^{an}, t)$ in $\mathrm{GL}(\mathrm{H}^1(X_t(\mathbf{C})^{an}, \mathbf{Q}))$. Namely, we want to prove that it is big enough, or rather that its Zariski closure is big enough[3]

Even to state the precise result that we need to show, we need to understand the fiber X over a point t. We do this in two steps: we firstly understand the fibers of the Legendre family $X \to S'$ and then the fibers of the m-th power map $S' \to S$.

It is basically evident from the definition that the fiber of $X \to S'$ over any point $z \in S'(\mathbf{C})$ is the elliptic curve E_z defined by the equation

$$E_z := \mathrm{V}\left(Y^2 Z - X\left(X - Z\right)\left(X - zZ\right)\right) \subset \mathbf{P}^2_{\mathbf{C}}.$$

As for the fibers of m-the power map $S' \to S$, we see that the fiber over a point $t \in S(\mathbf{C})$ is the disjoint union of points

$$\{z \in S'(\mathbf{C}) | z^n = t\}.$$

Therefore, we conclude that the fiber of the modified Legendre family $X \to S$ over a point $t \in S(\mathbf{C})$ is simply the disjoint union $\bigsqcup_{z\colon z^m = t} E_z$. Thus we have a decomposition

$$\mathrm{H}^1(X_t(\mathbf{C})^{an}, \mathbf{Q}) = \bigoplus_{z\colon z^m = t} \mathrm{H}^1(E_z(\mathbf{C})^{an}, \mathbf{Q}).$$

In particular, this implies that $\mathrm{GL}\left(\mathrm{H}^1\left(X_t\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$ contains a copy of

$$\prod_{z\colon z^m = t} \mathrm{SL}\left(\mathrm{H}^1\left(E_z\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right) = \prod_{z\colon z^m = t} \mathrm{SL}_2.$$

The Big Monodromy Lemma states that Zariski closure of the image of ρ contains the product of SL_2 from above.

Theorem 2.2 (The Big Monodromy Lemma)**.** In the notation as above, the action of monodromy

$$\rho_S : \pi_1(S(\mathbf{C})^{an}, t) \to \mathrm{GL}\left(\mathrm{H}^1\left(E_z\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$$

has Zariski closure containing $\prod_z \mathrm{SL}\left(\mathrm{H}^1\left(E_z\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$.

2.2. **Outline of the Proof.** The proof of Theorem 2.2 contains two steps. Firstly, one needs to explicitly compute the action of monodromy in the case of the usual Legendre family. And then one has to express the action of monodromy for the modified Legendre family as the induced representation

$$\mathrm{Ind}_{\pi_1(S'(\mathbf{C})^{an}, t)}^{\pi_1(S(\mathbf{C})^{an}, z)} \mathrm{H}^1(E_z, \mathbf{Q}),$$

[2] At least after fixing a point $t \in S(\mathbf{C})$.

[3] Note that $\mathrm{GL}(\mathrm{H}^1(X_t(\mathbf{C})^{an}, \mathbf{Q}))$ has a natural structure of an algebraic group over $\mathbf{Q}$.

where $z \in S'(\mathbf{C})$ is any point lying over $t \in S(\mathbf{C})$ and the map[4] $\pi_1(S'(\mathbf{C})^{an}, z) \to \pi_1(S(\mathbf{C})^{an}, t)$ is just the natural homomorphism induced by the map

$$(S'(\mathbf{C}), z) \to (S(\mathbf{C}), t).$$

Unfortunately, the computation of the monodromy action for the usual Legendre family is quite lengthly if it is done completely rigorous. On the other hand, this computation is usually considered to be "well-known", so we only sketch it in these notes.

The essential idea is to identify $\mathbf{A}^1_{\mathbf{C}} \setminus \{0, 1\}$ with the quotient of the upper half place $\mathbb{H}/\Gamma(2)'$ where

$$\Gamma(2)' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid a, d \equiv 1 \bmod 4; \text{and } b, c \equiv 0 \bmod 2 \right\}$$

with the standard action on $\mathbb{H}$. One way to do this is to provide both $\mathbb{H}$ and $\mathbf{A}^1_{\mathbf{C}} \setminus \{0, 1\}$ with the moduli interpretation in the category of *complex-analytic spaces*. It is essential to work with the analytic category here as $\mathbb{H}$ is not an algebraic variety. This can be done using the methods from [Cona, §1, §2].

Once one can understand the monodromy representation $\mathrm{H}^1(E_z, \mathbf{Q})$ quite well, one can use representation-theoretic methods together with some (easy) topological analysis of the map $\pi_1(S'(\mathbf{C})^{an}, z) \to \pi_1(S(\mathbf{C})^{an}, t)$ to study the induced representation $\mathrm{Ind}_{\pi_1(S'(\mathbf{C})^{an},t)}^{\pi_1(S(\mathbf{C})^{an},z)} \mathrm{H}^1(E_z, \mathbf{Q})$. This will eventually show that the Zariski closure $\overline{\mathrm{Im}(\rho)}$ contains $\prod_z \mathrm{SL}\left(\mathrm{H}^1\left(E_z\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$.

3. Proof of The Big Monodromy Lemma

3.1. **The Monodromy Representation of The Standard Legendre Family.** The main goal of this section is to sketch the computation of the monodromy reprepresentation of the standard Legendre family. Our strategy is to use the moduli-theoretic interpretation of this family to identify this family with the quotient of the "universal family of elliptic curves" over the upper half plane by the free action of $\Gamma(2)'$.

More precisely, we start with the universal property of the upper half plane $\mathbb{H}$. It comes with "the universal" family of elliptic curves $f^{an} \colon \mathcal{E} \to \mathbb{H}$ and "the universal" i-permissible trivialization $\alpha^{an} \colon \mathrm{R}^1 f^{an}_* \underline{\mathbf{Z}} \to \underline{\mathbf{Z}}^2$. Indeed, the family is defined as the quotient $\mathcal{E} \coloneqq (\mathbf{C} \times \mathbb{H})/\Lambda$ where Λ is the relative lattice

$$\Lambda \coloneqq \mathbf{Z}^2 \times \mathbb{H} \hookrightarrow \mathbf{C}^2 \times \mathbb{H},$$

$$(m, n; z) \mapsto (mz + n; z) \; .$$

Now we note that this family of elliptic curves comes with the canonical trivialization of the integral homology fiberwise. Namely, the oriented pair $\{z, 1\}$ provides an isomorphism $\mathbf{Z}^2 \to \mathrm{H}_1(E_z, \mathbf{Z})$. Using the Poincare Duality, we get an isomorphism $\alpha^{an,\vee}_z \colon \mathbf{Z}^2 \to \mathrm{H}^1(E_z, \mathbf{Z})^\vee$ that can be glued in the family to define the map $\alpha^{an,\vee} \colon \underline{\mathbf{Z}}^2 \to (\mathrm{R}^1 f^{an}_* \underline{\mathbf{Z}})^\vee$. Taking dual we get a trivialization

$$\alpha^{an} \colon \mathrm{R}^1 f^{an}_* \underline{\mathbf{Z}} \to \underline{\mathbf{Z}}^2$$

that is i-permissible in the sense of [Cona, Section 1.2.4, page 44].

[4]Implicit in the construction of Induced Representation

Theorem 3.1. [Cona, Theorem 1.4.3.1] The family $f^{an}\colon \mathcal{E} \to \mathbb{H}$ with the i-permissible trivialization $\alpha^{an}\colon \mathrm{R}^1 f^{an}_* \underline{\mathbf{Z}} \to \underline{\mathbf{Z}}^2$ represents the functor

$$S \mapsto \left\{f\colon E \to S,\ \alpha\colon \mathrm{R}^1 f_* \mathbf{Z} \to \mathbf{Z}^2\right\} / \sim$$

where $f\colon E \to S$ is a relative elliptic curve and α is an i-permissible trivialization of $\mathrm{R}^1 f_* \mathbf{Z}$.

We now discuss the universal property of the Legendre family. Then first thing we note is that the Legendre family $X'' \to S''$ comes with the trivialization of the 2-torsion S''-group space $X''[2]$. Namely, we recall that X'' is given by

$$X'' \coloneqq \mathrm{V}\left(Y^2 Z - X\left(X - Z\right)\left(X - tZ\right)\right) \subset \mathbf{P}^2_{S''}\ ,$$

so we can trivialize $X''[2]$ by using the sections $P = [0:0:1]$ and $Q = [1:0:1]$. It is tempting to say that X'' represents the functor of elliptic families with the trivialization of the relative 2-torsion. However, it can not be literally correct as this functor is not rigid, any such family has a non-trivial automorphism given by $[-1]$. The way we fix this problem is by introducing the so-called "Legendre structures":

Definition 3.2. Let E be an elliptic curve over a complex-analytic space T. A *Legendre structure* on E is a pair $(\omega, \varphi = (P, Q))$ consisting of a trivializing section ω of $\omega_{E/T}$ and a full level-2 structure (P, Q) of E such that the unique adapted coordinates $\{x, y\}$ satisfying $y|_{E[2]} = 0$ and $x(P) = 0$ also satisfy $x(Q) = 1$.

We refer to [Conb, §2, Definition 2.3, and the discussion after it] for more details. The main result is that the Legendre family represents the functor of elliptic curves with Legendre structure on it. And, what is probably more important, the data of Legendre structure is equivalent to $\Gamma(2)'$-structure in the following sense.

Definition 3.3. An elliptic curve E with a *level* $\Gamma(2)'$*-structure* over a complex-analytic space T is an elliptic curve $f\colon E \to T$ with a choice of a section $s \in \Gamma(2)' \backslash \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$, where $\underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$ is the sheaf of isomorphisms $\alpha^\vee\colon \underline{\mathbf{Z}}^2 \to \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee$ such that its dual is i-permissible.

Remark 3.4. We recall that the group $\Gamma(2)'$ is defined as

$$\Gamma(2)' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid a, d \equiv 1 \bmod 4; \text{and } b, c \equiv 0 \bmod 2 \right\}.$$

It is easy to see that this is an index 2 subgroup of $\Gamma(2)$. In particular, it is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

Theorem 3.5.
- The Legendre family $X'' \to S''$ together with the full level 2-structure $\{P, Q\}$ and the trivialization of $\omega_{X''/S''}$ given by $-\mathrm{d}x/2y$ represents the functor of relative elliptic curves together with the choice of Legendre structure on it.
- A choice of a Legendre structure on an elliptic curve $E \to T$ is the same as a choice of a level $\Gamma(2)'$-structure on it.
- The standard action of $\Gamma(2)'$ on $\mathbb{H}$ is free.
- There are canonical compatible isomorphisms $\Gamma(2)' \backslash \mathbb{H} \xrightarrow{\sim} S''$ and $\Gamma(2)' \backslash \mathcal{E} \xrightarrow{\sim} X''$.

Proof. The *first claim* is proven in [Conb, Proposition 3.1] in the algebraic set-up. The same proof can be adapted to the analytic situation using the cohomological methods developed in [Cona, §1.1].

Now we go to the *second claim*. We fix a relative elliptic curve $f\colon E \to T$. Then weto recall that an element $\alpha^\vee \in \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$ provides us with an trivialization of the Hodge bundle $\omega_{E/T}$. Indeed, using the isomorphism

$$E[4] \xrightarrow{\sim} (\mathrm{R}^1 f_* \underline{\mathbf{Z}/4\mathbf{Z}})^\vee \cong \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}/4\mathbf{Z}}$$

we get isomorphisms

$$\alpha_4^\vee \colon \underline{\mathbf{Z}/4\mathbf{Z}}^2 \xrightarrow{\sim} E[4] \ ,$$
$$\alpha_2^\vee \colon \underline{\mathbf{Z}/2\mathbf{Z}}^2 \xrightarrow{\sim} E[2] \ .$$

Then one can use [Conb, Proposition 3.2] to find the desired canonical(!) trivialization ω of the Hodge bundle $\omega_{E/T}$. Moreover, $(\omega, \alpha_2^\vee(1,0), \alpha_2^\vee(0,1))$ is a Legendre structure on E that extends the full level 2-structure $(\alpha_2^\vee(1,0), \alpha_2^\vee(0,1))$.

Suppose now that we have a section $s \in \Gamma(2)' \backslash \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$. This means that we have a covering of T by opens U_i and on each open we have a section $\widetilde{s_i} \in \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)|_{U_i}$ defined up to the action of $\Gamma(2)'$. Now note that since $\Gamma(2)'$ is a subgroup of $\Gamma(2)$, we see that each s_i is well-defined modulo 2. So they glue to a well-defined global isomorphism

$$s \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} \colon (\mathbf{Z}/2\mathbf{Z})^2 \xrightarrow{\sim} E[2]$$

and the fact that each s_i is i-permissible transfers exactly to the condition that $s \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z}$ defines the full level-2 structure (compatible with the Weil pairing). This argument implies that $\Gamma(2)$-structure is exactly the same as the full level-2 structure. But we want to understand what it means to have $\Gamma(2)'$-structure, this is a slightly more subtle question.

To answer that question, we observe that each $\widetilde{s_i}$ provides us (locally) with the 2-torsion point $P_i \coloneqq (\widetilde{s_i} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z})(1,0)$ and the 4-torsion point $Q_i \coloneqq (\widetilde{s_i} \otimes_{\mathbf{Z}} \mathbf{Z}/4\mathbf{Z})(0,1)$. These two points satisfy the condition that $(P_i, [2]Q_i)$ define the full level-2 structure on $E_{U_i} \to U_i$. Now [Conb, Proposition 3.2] guarantees that this data constructs a canonical section $\omega \in \mathrm{H}^0(U_i, \omega_{E/T})$ that trivializes the Hodge bundle $\omega_{E/T}|_{U_i}$. We note that this trivialization does depend on a choice of P_i and Q_i, and the choice of Q_i does depend on a choice of $\widetilde{s_i}$. However, if one goes back to the construction of ω in [Conb, Proposition 3.2], one can see that this construction is preserved by changing $\widetilde{s_i}$ by $g\widetilde{s_i}$ for $g \in \Gamma(2)'$. This implies that the global one-form ω defined locally by $\widetilde{s_i}$ is well-defined, even though the intermediate steps are not, and it provides us with a trivialization of the Hodge bundle $\omega_{E/T}$ such that $(\omega, (s \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z})(1,0), (s \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z})(0,1))$ is a Legendre structure. The latter claim follows from the construction and the fact that we can check that something is a Legendre structure analytically locally on T.

The discussion above constructs a Legendre structure by a $\Gamma(2)'$-level structure on $E \to T$. This construction can be reversed by using the fact that $\omega_{\widetilde{s_i}}$ by $\omega_{g\widetilde{s_i}}$ for $g \in \Gamma(2)$ if and only if $g \in \Gamma(2)'$. This can be checked by noting that each element of $g \in \Gamma(2)$ can be written as $g = (\pm 1)g'$ with $g' \in \Gamma(2)'$. Now we know that the action of $\Gamma(2)'$ does not change the construction of ω, but it can be checked by inspection that the action of $g = -\mathrm{E}_2$ replaces ω by $-\omega$. The the data of a Legendre structure allows us to locally define $\widetilde{s_i} \in \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$ that are uniquely well-defined up to an action of $\Gamma(2)'$. Therefore, they uniquely glue to a section $s \in \Gamma(2)' \backslash \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_* \underline{\mathbf{Z}}^\vee)$. This finishes the proof that the notions of $\Gamma(2)'$-level structure and Legendre structure coincide[5].

The *third claim* is a standard computation that has nothing to do with "geometry". Namely, we use [Miy89, Theorem 4.2.10] to say that $\Gamma(2)$ has no "elliptic points", i.e. any point $z \in \mathbb{H}$ has either

[5]Strictly speaking, one needs to check that the two maps constructed above are inverse to each other. We leave it to the interested reader.

the trivial stabilizer or equal to the group $\{\pm \mathrm{E}_2\}$. Since the action of $\Gamma(2)'$ and $\Gamma(2)$ on $\mathbb{H}$ coincide[6], and $-\mathrm{E}_2 \notin \Gamma(2)'$, we conclude that the action of $\Gamma(2)'$ on $\mathbb{H}$ is free.

The *fourth claim* is the standard consequence of all previous claims. Namely, claims 1 and 2 guarantee that the Legendre family represents the functor of relative elliptic curves with a choice of $\Gamma(2)'$-level structure. The third claim shows that $\Gamma(2)'$ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ acting freely on $\mathbb{H}$. Thus we can use [Cona, Lemma 2.1.1.1] to see that the pair $\Gamma(2)' \backslash \mathcal{E} \to \Gamma(2)' \backslash \mathbb{H}$ with the evident choice of $\Gamma(2)'$-level structure also represents the functor of relative elliptic curves with $\Gamma(2)'$-level structure. Therefore, there must be compatible isomorphisms $\Gamma(2)' \backslash \mathbb{H} \xrightarrow{\sim} S''$, $\Gamma(2)' \backslash \mathcal{E} \xrightarrow{\sim} X''$. ☐

Corollary 3.6. There is a natural isomorphism $\pi_1(\mathbf{C} \setminus \{0,1\}, z) \xrightarrow{r} \Gamma(2)'$, such that

$$r(l_0) = \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } r(l_1) = \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix},$$

where l_i is an element of the fundamental group corresponding to a homotopy class of a loop around i oriented counterclockwise[7]. Moreover, under this identification the monodromy action for the standard Legendre family becomes a natural injection of $\Gamma(2)'$ into $\mathrm{GL}_2(\mathbf{Q})$. Namely, the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(\mathbf{C} \setminus \{0,1\}, z) & & \\ \downarrow r & \overset{\rho_{S''}}{\searrow} & \\ \Gamma(2)' & \xrightarrow{\quad i \quad} & \mathrm{GL}\left(\mathrm{H}^1\left(E_z, \mathbf{Q}\right)\right). \end{array}$$

In particular, $\rho_{S''}(l_1)$ is an unipotent matrix and Zariski closure $\overline{\rho_{S''}(\pi_1(\mathbf{C} \setminus \{0,1\}, z))} = \mathrm{SL}_2$.

Proof. The first proof follows from the identification of the Legendre family as the quotient of the universal family over $\mathbb{H}$ by the action of $\Gamma(2)'$. More precisely, we know that $\mathbb{H}$ is simply-connected and the monodromy of the universal family $f^{an} \colon \mathcal{E} \to \mathbb{H}$ is canonically trivialized. Therefore, when we take a quotient of this picture by the free action of $\Gamma(2)'$, we see that $\pi_1(\mathbf{C} \setminus \{0,1\}, z)$ is canonically identified with $\Gamma(2)'$ and this is exactly the monodromy action corresponding to the local system[8]

$$\mathrm{R}^1 p_* \mathbf{Q} = \left(\pi_* \left(\mathrm{R}^1 f^{an}_* \mathbf{Q}\right)\right)^{\Gamma(2)'}$$

where $\pi \colon \mathbb{H} \to \mathbf{C} \setminus \{0,1\}$ is the projection map from Theorem 3.5.

In order to see that $\overline{\rho_{S''}(\pi_1(\mathbf{C} \setminus \{0,1\}, z))} = \mathrm{SL}_2$, we need to note that this Zariski closure must contain the subgroups of upper and lower triangular matrices as the matrices

$$r(l_0) = \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } r(l_1) = \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}$$

are in the image. Now the claim follows that upper and lower triangular matrices generate $\mathrm{SL}_2(\overline{\mathbf{Q}})$. ☐

[6] The matrix $-\mathrm{E}_2$ acts trivially on $\mathbb{H}$.

[7] There is no real sign ambiguity. The problem is that one needs to do harder work to compute that sign. It crucially depends on a choice of orientation and other conventions. We will never need to know what this sign is, so we just ignore that issue.

[8] Look at [Cona, Lemma 1.5.6.1, 1.5.6.2] for more details.

3.2. **The Monodromy Representation of The Modified Legendre Family.** This section is devoted to the computation of the monodromy representation of the modified Legendre family $p\colon X \to S$. We will be able to understand it well-enough to eventually show that its image is sufficiently large giving a proof of Theorem 2.2.

We start the section with some review. Recall that we constructed the modified Legendre family $p\colon X \to S$ as the composition of the restriction of the Legendre family $g\colon X \to S'$ with $S' = \mathbf{A}^1_{\mathbf{C}} \setminus \{0, \mu_m(\mathbf{C})\}$ and the m-th power map $h\colon S' \to S$. In what follows, we denote by Γ' the fundamental group of $\pi_1(S'(\mathbf{C})^{an}, z)$ for some $z \in S'(\mathbf{C})^{an}$. This group is abstractly isomorphic to F_{m+1} – the free group on $m+1$ generators and these generators correspond to homotopy class of loops around punctured points oriented counterclockwise. We also fix a primitive m-th root of unity ζ_m with smallest angle with the x-coordinate on the complex plane[9] and denote by $\alpha_i \in \Gamma'$ a generator corresponding to a homotopy class of a loop around ζ_m^i oriented counterclockwise. We also define $\beta \in \Gamma'$ as a generator corresponding to a homotopy class of a loop around 0. The monodromy action for this family we denote by

$$\rho_{S'} : \pi_1\left(S'\left(\mathbf{C}\right)^{an}, z\right) \to \mathrm{GL}\left(\mathrm{H}^1\left(E_z, \mathbf{Q}\right)\right).$$

Now we consider the modified Legendre family $p\colon X \to S$. This is the family we are actually interested in. In the same manner as above, we denote by Γ the fundamental group $\pi_1(S(\mathbf{C})^{an}, t)$ for some $t \in S(\mathbf{C})^{an}$. This group is abstractly isomorphic to F_2 – a free group on 2 generators. Let us denote by γ_0 a homotopy class of a loop around 0 and by γ_1 a homotopy class of a loop around 1. A group Γ is freely generated by γ_0 and γ_1. Finally, we denote the monodromy action for this family as

$$\rho_S : \pi_1\left(S\left(\mathbf{C}\right)^{an}, t\right) \to \mathrm{GL}\left(\mathrm{H}^1\left(X_t, \mathbf{Q}\right)\right).$$

In what follows, a point z will always mean some point $S(\mathbf{C})$, and $t \in h^{-1}(t)$ is fixed point in a fiber of g over a point t.

The first thing we need to understand is the map $\Gamma' \to \Gamma$ induced by the map $S' \to S$. The explicit "formula" for this map will be the crux of all our discussion later.

Lemma 3.7. The natural map $\varphi\colon \Gamma' \to \Gamma$ induced by a map of pointed topological spaces $(S'(\mathbf{C})^{an}, z) \to (S(\mathbf{C})^{an}, t)$ is injective and coincides with the map defined by

$$\varphi(\beta) = \gamma_0^m, \varphi(\alpha_i) = \gamma_0^{-i}\gamma_1\gamma_0^i.$$

In particular, the image $\varphi(\Gamma')$ is normal in Γ and the quotient space has representatives $\{e, \gamma_0, \dots, \gamma_0^{m-1}\}$.

Proof. The map $S' \to S$ is finite étale, thus the associated map $S'(\mathbf{C})^a n \to S(\mathbf{C})^a n$ is a finite covering map. Thus the induced map $\Gamma' \to \Gamma$ is injective by the formalism of fundamental groups.

Now we need to establish the explicit formula for φ. I don't know to make this into an absolutely rigorous argument, but the idea is quite easy and intuitive.

In order to get a loop around ζ_m^i, we firstly need to go to the origin, then go by a small circle around it by $2\pi i/n$ angle counterclockwise, then go to ζ_m^i and then go back to the origin, go by a small circle around it by $2\pi i/n$ angle clockwise and come back to the starting point z by the same path. Each rotation by $2\pi i/n$ angles in this circle contributes to a factor of γ_0. This part of the path has a total contribution γ_0^i, a path to ζ_m^i itself contributes a factor of γ_1. And then we go back after approaching ζ_m^i, this contributes γ_0^{-i} (because we are going in the opposite direction now). Totally we get that $\varphi(\alpha_i) = \gamma_0^{-i}\gamma_1\gamma_0^i$ and one can show that $\varphi(\beta) = \gamma_0^n$ by a similar argument.

[9] it makes sense only after choosing $i \in \mathbf{C}$ but this choice will not be of any importance.

[I should add a picture here of what happens for $m = 3$, it clarifies what is going on. But I don't know how to add pictures to LaTeX at the moment]

$\square$

Lemma 3.8. In the notation as above, all higher pushforwards $\mathrm{R}^i h_* \mathcal{L}$ vanish for any sheaf of abelian group $\mathcal{L}$ on $S'(\mathbf{C})^{an}$ and any $i > 0$. In particular, $\mathrm{R}^i p_* \underline{\mathbf{Q}} = h_*(\mathrm{R}^i g_*(\underline{\mathbf{Q}}))$. Moreover, $\mathrm{R}^1 p_* \underline{\mathbf{Q}}$ is a $\mathbf{Q}$-local system of rank $2m$, and $R^1 g_* \underline{\mathbf{Q}}$ is a $\mathbf{Q}$-local system of rank 2.

Proof. We start by noting that $h : S' \to S$ is a finite etale map. This implies that its analytification is a finite covering space $h : S'(\mathbf{C})^{an} \to S(\mathbf{C})^{an}$. In particular, it can be trivialized locally (in the analytic topology) on $S(\mathbf{C})^{an}$. This easily implies that h_* is exact. As a result, $R^i h_*$ vanish for all $i > 0$.

To prove the second claim we just write down a Leray-Serre spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{R}^i h_*(\mathrm{R}^j g_* \underline{\mathbf{Q}}) \Rightarrow \mathrm{R}^{i+j} p_* \underline{\mathbf{Q}}$$

and use the fact that $R^i h_*$ is zero for any $i > 0$.

As for the last part of Lemma, we observe that both f and g are smooth and proper morphisms of finite type $\mathbf{C}$-schemes. This means that their analytifications are proper submersions. Then Ehresmann's theorem and the topological proper base change theorem imply that $\mathrm{R}^i f_*$ and $\mathrm{R}^i g_*$ send local systems to local systems and commute with base change. Thus in order to compute the ranks of $\mathrm{R}^1 f_* \underline{\mathbf{Q}}$ and $\mathrm{R}^1 g_* \underline{\mathbf{Q}}$ we can compute ranks of corresponding cohomology group on fibres. Since the fiber of f is equal to the disjoint union of m copies of (connected) elliptic curves, and the fiber of g is equal to a (connected) elliptic curve, we conclude that ranks of $\mathrm{R}^1 f_*^{an} \underline{\mathbf{Q}}$ and $\mathrm{R}^1 g_*^{an} \underline{\mathbf{Q}}$ are equal to $2m$ and 2, correspondingly. $\square$

Now we want to understand the action of monodromy for the modified Legendre family. We will try to do this in a clear way. Since it is usually rather difficult (at least to me) to make precise arguments in terms of loops and topological spaces, we will try to forget as much as possible about topology and work on the level of representations instead of local systems.

Lemma 3.9. Let $g\colon X \to S'$ be the map as above. Let us denote by $(V, \rho_{S'})$ the representation of Γ' corresponding to the $\mathbf{Q}$-local system $R^1 g_* \underline{\mathbf{Q}}$. Then $\rho_{S'}(\alpha_i) = e$ for any $i > 0$ and

$$\rho_{S'}(\alpha_0) = \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } \rho_{S'}(\beta) = \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}.$$

Proof. Observe that we can extend the smooth proper morphism $X \to S'$ to the standard Legendre family $X'' \to S'' = \mathbf{A}^1_{\mathbf{C}} \setminus \{0, 1\}$ that is smooth and proper. Lemma 4.3 implies that we have the commutative diagram

$$\begin{array}{ccc} \pi_1(S'(\mathbf{C})^{an}, z) & \xrightarrow{\pi_1(j)} & \pi_1(S''(\mathbf{C})^{an}, z) \\ & \searrow^{\rho_{S'}} & \downarrow \rho_{S''} \\ & & \mathrm{GL}(\mathrm{H}^1(E_z, \mathbf{Q})), \end{array}$$

where $j : (S'(\mathbf{C})^{an}, z) \to (S''(\mathbf{C})^{an}, z)$ is the natural open immersion. It is easy to see that $\pi_1(j)(\alpha_i) = e$ for any $i > 0$, $\pi_1(j)(\alpha_0) = l_1$ and $\pi_1(j)(\beta) = l_0$. Therefore, Corollary 3.6 implies that

$$\rho_{S'}(\alpha_0) = \rho_L(\pi_1(j)(\alpha_0)) = \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix},$$

$$\rho_{S'}(\beta) = \rho_L(\pi_1(j)(\beta)) = \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}.$$

□

We want to understand a representation $(\mathrm{H}^1(X_t, \mathbf{Q}), \rho_S)$ of the fundamental group $\pi_1(S(\mathbf{C})^{an}, t)$ is terms of the representation $(V, \rho_{S'})$ of the group $\pi_1(S'(\mathbf{C})^{an}, z)'$.

Lemma 3.10. The representation $(\mathrm{H}^1(X_t, \mathbf{Q}), \rho_S)$ is isomorphic as a Γ-representation to the induced representation $\mathrm{Ind}_{\Gamma'}^{\Gamma}(V, \rho_{S'})$[10].

Proof. We recall that $(V, \rho_{S'})$ corresponds to the $\mathbf{Q}$-local system $\mathcal{L} := R^1 f_* \underline{\mathbf{Q}}$ by its very construction. Moreover, the representation $(\mathrm{H}^1(X_t, \mathbf{Q}), \rho_S)$ corresponds to the $\mathbf{Q}$-local system $R^1 g_* \mathbf{Q} = h_*(R^1 f_* \underline{\mathbf{Q}}) = h_* \mathcal{L}$ by Lemma 3.8. We know that on the representation-theoretic side the pushforward of a $\mathbf{Q}$-local system corresponds to the Induced representation of the corresponding representation (Lemma 4.4). This means that

$$\left(\mathrm{H}^1(X_t, \mathbf{Q}), \rho_S\right) = \mathrm{Ind}_{\Gamma'}^{\Gamma}\left(V, \rho_{S'}\right).$$

□

We reduced the problem of proving Theorem 2.2 to a purely algebraic question now. Our strategy now is to understand the induced representation by algebraic means using Lemma 3.7 to understand explicitly the map $\varphi\colon \Gamma' \to \Gamma$. Then we use some group theoric arguments to show that Zariski closure of $\overline{\rho_S(\Gamma)}$ is big enough. We start with the computation of this induced representation.

Lemma 3.11. In the notation as above, we have an isomorphism

$$\mathrm{Ind}_{\Gamma'}^{\Gamma}(V, \rho) \cong \bigoplus_{i=0}^{m-1} \gamma_0^i . V,$$

where the action of Γ is defined in the following way: for any $v = \sum_{i=0}^{m-1} \gamma_0^i . v_i$ we define

$$\rho_S(\gamma) v = \sum_{i=0}^{m-1} \gamma_0^{j(i)} . \left(\rho_{S'}\left(h_i\right) v_i\right) \text{ for any } \gamma \in \Gamma,$$

where $j(i)$ is a unique number in $\{0, \dots, m-1\}$ such that there exists $h_i \in \Gamma'$ with a property $\gamma \gamma_0^i = \gamma_0^{j(i)} \varphi(h_i)$.

Proof. This is basically the definition of the induced representation (or at least one of its explicit constrictions). The main point is the we have already computed the map $\Gamma' \xrightarrow{\varphi} \Gamma$ in Lemma 3.7. In particular, we know that the quotient $\Gamma/\varphi(\Gamma')$ has representatives $\{e, \gamma_0, \dots, \gamma_0^{m-1}\}$. This implies that we have an identification of $\mathbf{Q}$-vector spaces $\mathrm{Ind}_{\Gamma'}^{\Gamma}(V, \rho) \cong \bigoplus_{i=0}^{m-1} \gamma_0^i . V$. The computation of the action of Γ is an easy exercise that is left to the reader. □

Here is the fist application of Lemma 3.11. Using the identification

$$\left(\mathrm{H}^1\left(X_t, \mathbf{Q}\right), \rho_S\right) \cong \mathrm{Ind}_{\Gamma'}^{\Gamma}\left(V, \rho\right) \cong \bigoplus_{i=0}^{m-1} \gamma_0^i . V$$

[10]We recall that $\Gamma = \pi_1(S(\mathbf{C})^{an}, t)$ and $\Gamma' = \pi_1(S'(\mathbf{C})^{an}, z)$

we get the containment

$$\mathrm{GL}\left(\mathrm{H}^1\left(X_t,\mathbf{Q}\right)\right)=\mathrm{GL}\left(\bigoplus_{i=0}^{m-1}\gamma_0^iV\right)\supset\prod_{i=0}^{m-1}\mathrm{SL}_2\,.$$

We note that this is the same containment as the one that was explained just before Theorem 2.2. We will use this fact but we will not prove it (at least in this version of the notes). It boils down to the compatibility of two different topological constructions and, hopefully, the proof can be recovered by the interested reader. Let us now go the proof of Theorem 2.2.

Lemma 3.12. In the notation as above, we have a formula $\rho_S(\gamma_1)=(u,1,\dots,1)$, where $u\in\mathrm{SL}_2(\mathbf{C})$ a unipotent matrix.

Proof. Let us use Lemma 3.11 to compute the action of γ_1. Observe that Lemma 3.7 guarantees that

$$\gamma_1\gamma_0^i=\gamma_0^i(\gamma_0^{-i}\gamma_1\gamma_0^i)=\gamma_0^i\varphi(\alpha_i).$$

Thus Lemma 3.11 shows that

$$\rho_S(\gamma_1)(\gamma_0^i.v)=\gamma_0^i.(\rho_{S'}(\alpha_i))v\ .$$

Then Lemma 3.9 implies that

$$\rho_S(\gamma_1)(\gamma_0^i.v)=\gamma_0^i.v,\ \text{if } i>0,\ \text{and}$$
$$\rho_S(\gamma_1)(e.v)=e.(\rho_{S''}(l_1)v)\ \text{for every } v\in V.$$

This means that an action of ρ_S preserves direct decomposition $\bigoplus_{i=1}^{m-1}\gamma_0^iV$ and it acts trivially on all but one factor. Moreover, Lemma 3.6 guarantees that it acts as an unipotent matrix on the first factor. □

Lemma 3.13. In the notation as above, the action of $\rho_S(\gamma_0)$ permutes factors of the direct decomposition $\bigoplus_{i=0}^{m-1}\gamma_0^i.V$. More precisely, for all $w\in\gamma_0^i.V$ the image $\rho_S(\gamma_0)w\in\gamma_0^{i+1}.V$, where we understand $\gamma_0^m.V$ as $e.V$.

Proof. Again, we just use Lemma 3.11 to compute this action. By the very definition $\gamma_0\gamma_0^i=\gamma_0^{i+1}$. This means that

$$\rho_S(\gamma_0)(\gamma_0^i.v)=\gamma_0^{i+1}.v,\ \text{if } i<m-1,\ \text{and}$$
$$\rho_S(\gamma_0)(\gamma_0^{m-1}.v)=e.(\rho_{S'}(\beta)v)\ \text{for every } v\in V.$$

The last equality uses that $\varphi(\beta)=\gamma_0^m$ that follows from Lemma 3.7. Thus $\rho_S(\gamma_0)$ does permute factors (but also provides us with a non-trivial isomorphism of $\gamma_0^{m-1}.V$ and $e.V$). □

Lemma 3.14. Fix any element $A\in\mathrm{SL}_2(\mathbf{C})$, then Zariski closure $\overline{\rho_S(\Gamma)}\subset\mathrm{GL}\left(\bigoplus_{i=1}^{m-1}\gamma_0^iV\right)$ contains an element that preserves this decomposition and whose action on the 1-st factor coincides with A.

Proof. We argue in the same way as in the proofs of Lemma 3.12 and Lemma 3.13 to see that the action of $\rho_S(\gamma_0^m)$ and $\rho_S(\gamma_1)$ preserve the decomposition, and they act on eV in the following way:

$$\rho_S(\gamma_0^m)|_{e.V}=\rho_{S''}(l_0^m)=\begin{bmatrix}1 & \pm 2m\\ 0 & 1\end{bmatrix},$$
$$\rho_S(\gamma_1)|_{e.V}=\rho_{S''}(l_1)=\begin{bmatrix}1 & 0\\ \pm 2 & 1\end{bmatrix}.$$

Since matrices of the form $\begin{bmatrix} 1 & 2mn \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2k & 1 \end{bmatrix}$ are Zariski-dense in the positive and negative root spaces of SL_2 correspondingly, we conclude that $\overline{\rho_S(\Gamma)}$ contains block diagonal matrixes, whose action on $e.V$ is given by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Now recall the standard fact that the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ generate $\mathrm{SL}_2(\mathbf{C})$. This finishes the proof. □

These three lemmas allow us to prove Theorem 2.2.

Theorem 3.15. In the notations as above, the action of monodromy

$$\rho_S : \pi_1\left(S\left(\mathbf{C}\right)^{an}, t\right) \to \mathrm{GL}\left(\mathrm{H}^1\left(X_t\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$$

has Zariski closure containing $\prod_z \mathrm{SL}\left(\mathrm{H}^1\left(E_z\left(\mathbf{C}\right)^{an}, \mathbf{Q}\right)\right)$.

Proof. Let us denote the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ by w. Then Lemma 3.12 guarantees that $\rho_S(\Gamma)$ contains a block diagonal matrix of the form $(u, 1, 1, \ldots, 1)$, where $u = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Then Zariski closure $\overline{\rho_S(\Gamma)}$ (by the same density argument as in Lemma 3.14) contains an element $U_- := (u_-, 1, 1, \ldots, 1)$, where $u_- = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Lemma 3.14 says that $\overline{\rho_S(\Gamma)}$ contains a block diagonal matrix of the form $W := (w, a_1, a_2, \ldots, a_m)$, where $a_i \in \mathrm{GL}_2$. Then if we conjugate U_- by W we get the element $U_+ \in \overline{\rho_S(\Gamma)}$, which is the block diagonal matrix of the form $(u_+, 1, 1, \ldots, 1)$ (where $u_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). Now we again exploit the fact that u_+ and u_- generate the whole $\mathrm{SL}_2(\overline{Q})$ to conclude that for any matrix $a \in \mathrm{SL}_2(\overline{Q})$ the block diagonal matrix of the form $(a, 1, 1, \ldots, 1) \in \overline{\rho_S(\Gamma)}$. Finally, we use permutations from Lemma 3.13 to conclude that $\overline{\rho_S(\Gamma)} \supset \prod_{i=0}^{m-1} \mathrm{SL}_2$ (we inductively show that SL_2 in the i-th spot lie in the closure of the image. And since the closure of the image is a subgroup (*prove it!*), we get that the product of SL_2's is inside the image).

□

4. Appendix: Some Facts from Topology

Here we list a bunch of "well-known" facts from topology.

Lemma 4.1. Let X be a connected locally simply-connected (for example, a CW complex) topological space, then there is an equivalence of categories between $\mathbf{Q}$-local systems on X and finite dimensional $\mathbf{Q}$-representations of $\pi_1(X, x)$.

Proof. [Sza09, Theorem 2.5.15 and Corollary 2.6.2] □

Lemma 4.2. Let $f \colon X \to Y$ be a proper submersion of complex manifolds and let $\mathcal{L}$ be a $\mathbf{Q}$-local system on X. Then higher pushforwards $R^i f_* \mathcal{L}$ are $\mathbf{Q}$-local systems for all i.

Proof. An exercise on Ehresmann's Theorem. □

Lemma 4.3. Let $h \colon (X, x) \to (Y, y)$ be a continuous map of connected locally simply-connected based topological spaces and let $\mathcal{L}$ be a $\mathbf{Q}$-local system on Y. Then $h^*(\mathcal{L})$ is a $\mathbf{Q}$-local system on X. And if (V, ρ) is a representation of $\pi_1(Y, y)$ that corresponds to $\mathcal{L}$, then $(V, \rho|_{\pi_1(X,x)})$ is a representation that corresponds to $h^*(\mathcal{L})$.

Lemma 4.4. Let $h\colon (X,x) \to (Y,y)$ be a finite covering space of connected locally simply-connected based topological spaces and let $\mathcal{L}$ be a $\mathbf{Q}$-local system on X. Let (V,ρ) be a representation of $\pi_1(X,x)$ that corresponds to $\mathcal{L}$. Then $h_*(\mathcal{L})$ is a local system on Y and it corresponds to the representation $\mathrm{Ind}_{\pi_1(X,x)}^{\pi(Y,h(x))}(V,\rho)$, where this representation is induced with respect to the natural homomorphism $\pi_1(h)\colon \pi_1(X,x) \to \pi_1(Y,h(x))$.

Proof. It is easy to check that $h_*\mathcal{L}$ is a local system as h is discrete and locally trivial on Y. Now we know that (h^*, h_*) is pair of adjoin functors:

$$\mathbf{Q}\text{-local systems on } Y \underset{h^*}{\overset{h_*}{\rightleftarrows}} \mathbf{Q}\text{-local systems on } X$$

We identify $\mathbf{Q}$-local systems on Y (resp. X) with a finite dimensional $\mathbf{Q}$-representations of $\pi_1(Y,y)$ (resp. $\pi_1(X,x)$) to get the pair of adjoint functors

$$\pi_1(Y,y) - \mathbf{Q}\text{-fin.Reps} \underset{h^*}{\overset{h_*}{\rightleftarrows}} \pi_1(X,x) - \mathbf{Q}\text{-fin.Reps}$$

Lemma 4.3 guarantees that h^* coincides with the restriction functor. Therefore, the uniqueness of the adjoint functor implies that h_* must be the Induction functor.

□

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