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INTRODUCTION TO THE THEORY OF $\mathscr{D}$-MODULES

LECTURE NOTES (NANKAI 2011)

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Abstract. These lectures introduce basic concepts of the theory of $\mathscr{D}$-modules. The setting is that of complex analytic manifolds (but most of the results can be adapted in a straightforward way to $\mathbb{C}$-algebraic varieties). *In all the lectures, X will denote a complex analytic manifold of complex dimension n.*

(1) In Lecture 1, we introduce the sheaf of differential operators and its (left or right) modules. Our main concern is to develop the relationship between two *a priori* different notions:
- the classical notion of a $\mathscr{O}_X$-module with an *integrable* connection ($\nabla^2 = 0$),
- the notion of a left $\mathscr{D}_X$-module.

Both notions are easily seen to be equivalent. The relationship between left and right $\mathscr{D}_X$-modules, although simple, is also somewhat subtle, and we insist on the basic isomorphisms.

(2) In Lecture 2, we recall the notion coherence and use classical theorems of Cartan and Oka to prove the coherence of $\mathscr{D}_X$. The main new ingredient is that of a *good filtration* of a coherent $\mathscr{D}_X$-module. The characteristic variety is defined. The main results concern to its involutiveness and the notion of non-characteristic restriction.

(3) In Lecture 3, we motivate and introduce the notion of direct images of $\mathscr{D}$-modules, and compare it with the classical notion of Gauss-Manin connection. The main result is Kashiwara's estimate of the behaviour of the characteristic variety by direct image.

(4) Lecture 4 is devoted to holonomic $\mathscr{D}$-modules. After basic definitions and properties, we review recent results concerning vector bundles with meromorphic integrable connections, which are fundamental examples of holonomic $\mathscr{D}$-modules.

CONTENTS

LECTURE 1

BASIC CONSTRUCTIONS

In this first lecture, we introduce the sheaf of differential operators and its (left or right) modules. Our main concern is to develop the relationship between two *a priori* different notions:

(1) the classical notion of a $\mathscr{O}_X$-module with an *integrable* connection ($\nabla^2 = 0$),

(2) the notion of a left $\mathscr{D}_X$-module.

Both notions are easily seen to be equivalent.

The relationship between left and right $\mathscr{D}_X$-modules, although simple, is also somewhat subtle, and we insist on the basic isomorphisms.

The results in this lecture are mainly algebraic, and do not involve any analytic property. They can be translated easily to the algebraic situation. One can find many of these notions in the classical books [**Kas95, Bjö79, Bor87, Meb89, MS93a, MS93b, Bjö93, Cou95**]. Some of them are also directly inspired from the work of M. Saito [**Sai88, Sai90, Sai89**] about Hodge $\mathscr{D}$-modules.

1.1. The sheaf of holomorphic differential operators

We will denote by Θ_X the sheaf of holomorphic vector fields on X. This is the $\mathscr{O}_X$-locally free sheaf generated in local coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$. It is a sheaf of $\mathscr{O}_X$-Lie algebras which is locally free as a $\mathscr{O}_X$-module.

Dually, we denote by Ω^1_X the sheaf of holomorphic 1-forms on X. We will set $\Omega^k_X = \wedge^k \Omega^1_X$. We denote by $d : \Omega^k_X \to \Omega^{k+1}_X$ the differential.

The vector fields act (on the left) on functions by derivation, in a compatible way with the Lie algebra structure. On the other hand, let ω_X denote the sheaf $\Omega^{\dim X}_X$ of forms of maximal degree. Then there is a natural right action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X: the action is given by $\omega \cdot \xi = -\mathscr{L}_\xi \omega$, where $\mathscr{L}_\xi$ denotes the Lie derivative, equal to the composition of the interior product ι_ξ by ξ with the differential d, as it acts on forms of maximal degree.

***Definition 1.1.1* (The sheaf of holomorphic differential operators).** For any open set U of X, the ring $\mathscr{D}_X(U)$ of *holomorphic differential operators on* U is the subring of $\mathrm{Hom}_{\mathbb{C}}(\mathscr{O}_U, \mathscr{O}_U)$ generated by

- multiplication by holomorphic functions on U,
- derivation by holomorphic vector fields on U.

The sheaf $\mathscr{D}_X$ is defined by $\Gamma(U, \mathscr{D}_X) = \mathscr{D}_X(U)$ for any open set U of X.

By construction, the sheaf $\mathscr{D}_X$ acts on the left on $\mathscr{O}_X$, i.e., $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module.

***Definition 1.1.2* (The filtration of $\mathscr{D}_X$ by the order).** The increasing family of subsheaves $F_k\mathscr{D}_X \subset \mathscr{D}_X$ is defined inductively:

- $F_k\mathscr{D}_X = 0$ if $k \leqslant -1$,
- $F_0\mathscr{D}_X = \mathscr{O}_X$ (*via* the canonical injection $\mathscr{O}_X \hookrightarrow \mathscr{H}om_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)$),
- the local sections P of $F_{k+1}\mathscr{D}_X$ are characterized by the fact that $[P, \varphi]$ is a local section of $F_k\mathscr{D}_X$ for any holomorphic function φ.

Exercises E.1.1, E.1.2.

***Definition 1.1.3* (Integrable connection).** Let $\mathscr{M}$ be a $\mathscr{O}_X$-module. A holomorphic connection is a $\mathbb{C}$-linear morphism $\nabla : \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$ which satisfies the *Leibniz rule*: $\nabla(fm) : df \otimes m + f\nabla m$ for any local section m of $\mathscr{M}$ and any holomorphic function f, both defined on the same open set of X.

We say that ∇ is *integrable* if its curvature $R_\nabla := \nabla \circ \nabla$ vanishes identically.

Proposition 1.1.4. *Giving a left $\mathscr{D}_X$-module $\mathscr{M}$ is equivalent to giving a $\mathscr{O}_X$-module $\mathscr{M}$ together with an integrable connection ∇.*

Proof. Exercises E.1.5, E.1.6 and E.1.7. □

Remark 1.1.5. The sheaf $\mathscr{D}_X$ is not commutative. The lack of commutativity of $\mathscr{D}_X$ is analyzed in Exercise E.1.4. On the other hand, it has no non-trivial two-sided ideals (see Exercise E.1.3), hence it is simple. This leads us to consider left or right $\mathscr{D}_X$-modules (or ideals), and the theory of two-sided objects is empty.

1.2. Left and right

The categories of left (resp. right) $\mathscr{D}_X$-modules are denoted by ${}^{\ell}\mathsf{M}(\mathscr{D}_X)$ (resp. ${}^{\mathrm{r}}\mathsf{M}(\mathscr{D}_X)$). We analyze the relations between both categories in this section. Let us first recall the basic lemmas for generating left or right $\mathscr{D}$-modules. We refer for instance to [**Cas93**, § 1.1] for more details.

***Lemma 1.2.1* (Generating left $\mathscr{D}_X$-modules).** *Let $\mathscr{M}^1$ be a $\mathscr{O}_X$-module and let $\varphi^1 : \Theta_X \otimes_{\mathbb{C}_X} \mathscr{M}^1 \to \mathscr{M}^1$ be a $\mathbb{C}$-linear morphism such that, for any local sections f of $\mathscr{O}_X$, ξ, η of Θ_X and m of $\mathscr{M}^1$, one has*

(1) $\varphi^1(f\xi \otimes m) = f\varphi^1(\xi \otimes m)$,

(2) $\varphi^{\mathrm{l}}(\xi\otimes fm)=f\varphi^{\mathrm{l}}(\xi\otimes m)+\xi(f)m$,

(3) $\varphi^{\mathrm{l}}([\xi,\eta]\otimes m)=\varphi^{\mathrm{l}}(\xi\otimes\varphi^{\mathrm{l}}(\eta\otimes m))-\varphi^{\mathrm{l}}(\eta\otimes\varphi^{\mathrm{l}}(\xi\otimes m))$.

Then there exists a unique structure of left $\mathscr{D}_X$-module on $\mathscr{M}^{\mathrm{l}}$ such that $\xi m=\varphi^{\mathrm{l}}(\xi\otimes m)$ for any ξ, m.

***Lemma 1.2.2* (Generating right $\mathscr{D}_X$-modules).** *Let $\mathscr{M}^{\mathrm{r}}$ be a $\mathscr{O}_X$-module and let $\varphi^{\mathrm{r}}:\mathscr{M}^{\mathrm{r}}\otimes_{\mathbb{C}_X}\Theta_X\to\mathscr{M}^{\mathrm{r}}$ be a $\mathbb{C}$-linear morphism such that, for any local sections f of $\mathscr{O}_X$, ξ,η of Θ_X and m of $\mathscr{M}^{\mathrm{r}}$, one has*

(1) $\varphi^{\mathrm{r}}(mf\otimes\xi)=\varphi^{\mathrm{r}}(m\otimes f\xi)$ *(φ^{r} is in fact defined on $\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\Theta_X$)*,

(2) $\varphi^{\mathrm{r}}(m\otimes f\xi)=\varphi^{\mathrm{r}}(m\otimes\xi)f-m\xi(f)$,

(3) $\varphi^{\mathrm{r}}(m\otimes[\xi,\eta])=\varphi^{\mathrm{r}}(\varphi^{\mathrm{r}}(m\otimes\xi)\otimes\eta)-\varphi^{\mathrm{r}}(\varphi^{\mathrm{r}}(m\otimes\eta)\otimes\xi)$.

Then there exists a unique structure of right $\mathscr{D}_X$-module on $\mathscr{M}^{\mathrm{r}}$ such that $m\xi=\varphi^{\mathrm{r}}(m\otimes\xi)$ for any ξ, m.

***Example 1.2.3* (Most basic examples).**

(1) $\mathscr{D}_X$ is a left and a right $\mathscr{D}_X$-module.

(2) $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module (Exercise E.1.8).

(3) $\omega_X:=\Omega_X^{\dim X}$ is a right $\mathscr{D}_X$-module (Exercise E.1.9).

***Definition 1.2.4* (Right-left transformation).** Any left $\mathscr{D}_X$-module $\mathscr{M}^{\mathrm{l}}$ gives rise to a right one $\mathscr{M}^{\mathrm{r}}$ by setting (see [**Cas93**] for instance) $\mathscr{M}^{\mathrm{r}}=\omega_X\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{l}}$ and, for any vector field ξ and any function f,

$$(\omega\otimes m)\cdot f=f\omega\otimes m=\omega\otimes fm,\quad (\omega\otimes m)\cdot\xi=\omega\xi\otimes m-\omega\otimes\xi m.$$

Conversely, set $\mathscr{M}^{\mathrm{l}}=\mathscr{H}\!om_{\mathscr{O}_X}(\omega_X,\mathscr{M}^{\mathrm{r}})$, which also has in a natural way the structure of a left $\mathscr{D}_X$-module.

Exercises E.1.8 to E.1.17.

1.3. Examples of $\mathscr{D}$-modules

We list here some classical examples of $\mathscr{D}$-modules. One may get many other examples by applying various operations on $\mathscr{D}$-modules.

1.3.a. Let $\mathscr{I}$ be a sheaf of left ideals of $\mathscr{D}_X$. We will see in Lecture 2 that, locally on X, $\mathscr{I}$ is generated by a finite set $\{P_1,\dots,P_k\}$ of differential operators (this follows from the noetherianity and coherence properties of $\mathscr{D}_X$). Then the quotient $\mathscr{M}=\mathscr{D}_X/\mathscr{I}$ is a left $\mathscr{D}_X$-module. Locally, $\mathscr{M}$ is the $\mathscr{D}_X$-module associated with $P_1,\dots,P_k$.

Notice that different choices of generators of $\mathscr{I}$ give rise to the same $\mathscr{D}_X$-module $\mathscr{M}$. It may be sometimes difficult to guess that two sets of operators generate the same ideal. Therefore, it is useful to develop a systematic procedure to construct from a

system of generators a *division basis* of the ideal in order to have a decision algorithm (this uses Gröbner bases).
Exercise E.1.18.

1.3.b. Let $\mathscr{L}$ be a $\mathscr{O}_X$-module. There is a very simple way to get a right $\mathscr{D}_X$-module from $\mathscr{L}$: consider $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ equipped with the natural right action of $\mathscr{D}_X$. This is called an *induced* $\mathscr{D}_X$-module. Although this construction is very simple, it is also very useful to get cohomological properties of $\mathscr{D}_X$-modules. One can also consider the left $\mathscr{D}_X$-module $\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{L}$ (however, this is not the left $\mathscr{D}_X$-module attached to the right one $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ by the left-right transformation of Definition 1.2.4).

1.3.c. One of the main geometrical examples of $\mathscr{D}_X$-modules are the vector bundles on X equipped with an *integrable* connection. Recall that left $\mathscr{D}_X$-modules are $\mathscr{O}_X$-modules with an integrable connection (Proposition 1.1.4). Among them, the coherent $\mathscr{D}_X$-modules are particularly interesting. We will see (see Exercise E.2.9), that such modules are $\mathscr{O}_X$-locally free, i.e., correspond to holomorphic vector bundles of finite rank on X.

It may happen that, for some X, such a category does not give any interesting geometric object. Indeed, if for instance X has a trivial fundamental group (e.g. $X=\mathbb{P}^1(\mathbb{C})$), then any vector bundle of rank r with integrable connection is isomorphic to the trivial bundle $\mathscr{O}_X^{\mathrm{r}}$ with the connection d. However, on Zariski open sets of X, there may exist interesting vector bundles with connections. This leads to the notion of meromorphic vector bundle with connection.

Let D be a divisor in X and denote by $\mathscr{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left $\mathscr{D}_X$-modules, being a $\mathscr{O}_X$-module equipped with the natural connection $d:\mathscr{O}_X(*D)\to\Omega^1_X(*D)$.

By definition, a *meromorphic bundle* is a locally free $\mathscr{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left $\mathscr{D}_X$-module.

1.3.d. One may *twist* the previous examples. Assume that there exists a *closed* holomorphic form ω on X. Define $\nabla:\mathscr{O}_X\to\Omega^1_X$ by the formula $\nabla=d+\omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle $\mathscr{O}_X$.

Usually, there only exist meromorphic closed form on X, with poles on some divisor D. Then ∇ is an integrable connection on $\mathscr{O}_X(*D)$.

If ω is exact, $\omega=df$ for some meromorphic function f on X, then ∇ may be written as $e^{-f}\circ d\circ e^f$.

More generally, if $\mathscr{M}$ is any meromorphic bundle with an integrable connection ∇, then, for any such ω, $\nabla+\omega$ defines a new $\mathscr{D}_X$-module structure on $\mathscr{M}$.

1.3.e. Denote by $\mathfrak{Db}_X$ the sheaf of distributions on X: given any open set U of X, $\mathfrak{Db}_X(U)$ is the space of distributions on U, which is by definition the week dual of the space of C^∞ forms with compact support on U, of type $(\dim U,\dim U)$. By

Exercise E.1.9, there is a right action of $\mathscr{D}_X$ on such forms. The left action of $\mathscr{D}_X$ on distributions is defined by adjunction: denote by $\langle\varphi, u\rangle$ the natural pairing of a compactly supported C^∞-form φ with a distribution u on U; let P be a holomorphic differential operator on U; define then $P \cdot u$ such that, for any φ, on has

$$\langle\varphi, P \cdot u\rangle = \langle\varphi \cdot P, u\rangle.$$

Given any distribution u on X, the subsheaf $\mathscr{D}_X \cdot u \subset \mathfrak{Db}_X$ is the $\mathscr{D}_X$-module generated by this distribution. Saying that a distribution is a solution of a family $P_1, \dots, P_k$ of differential equation is equivalent to saying that the morphism $\mathscr{D}_X \to \mathscr{D}_X \cdot u$ sending 1 to u induces a surjective morphism $\mathscr{D}_X/(P_1, \dots, P_k) \to \mathscr{D}_X \cdot u$.

Similarly, the sheaf $\mathfrak{C}_X$ of currents of maximal degree on X, dual to $\mathscr{C}^\infty_{c,X}$, is a right $\mathscr{D}_X$-module.

In local coordinates $x_1, \dots, x_n$, a current of maximal degree is nothing but a distribution times the volume form $dx_1 \wedge \cdots \wedge dx_n \wedge d\overline{x}_1 \wedge \cdots \wedge d\overline{x}_n$.

As we are now working with C^∞ forms or with currents, it is natural not to forget the anti-holomorphic part of these objects. Denote by $\mathscr{O}_{\overline{X}}$ the sheaf of anti-holomorphic functions on X and by $\mathscr{D}_{\overline{X}}$ the sheaf of anti-holomorphic differential operators. Then $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) are similarly left (resp. right) $\mathscr{D}_{\overline{X}}$-modules. Of course, the $\mathscr{D}_X$ and $\mathscr{D}_{\overline{X}}$ actions do commute, and they coincide when considering multiplication by constants.

It is therefore natural to introduce the following sheaves of rings:

$$\mathscr{O}_{X,\overline{X}} := \mathscr{O}_X \otimes_{\mathbb{C}} \mathscr{O}_{\overline{X}},$$
$$\mathscr{D}_{X,\overline{X}} := \mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}},$$

and consider $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) as left (resp. right) $\mathscr{D}_{X,\overline{X}}$-modules.

1.3.f. One may construct new examples from old ones by using various operations.

- Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ has a natural structure of right $\mathscr{D}_X$-module. Using a resolution $\mathscr{N}^\bullet$ of $\mathscr{M}$ by *left* $\mathscr{D}_X$-modules which are acyclic for $\mathscr{H}om_{\mathscr{D}_X}(\bullet, \mathscr{D}_X)$, one gets a right $\mathscr{D}_X$-module structure on the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$.
- Given two left (resp. a left and a right) $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, a similar argument allows one to put on the various $\mathscr{T}or_{i,\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ a left (resp. a right) $\mathscr{D}_X$-module structure.
- We will see in Lecture 3 the geometric operation "push-forward" of a $\mathscr{D}_X$-module by a holomorphic map.

1.3.g. Solutions. Let $\mathscr{M}, \mathscr{N}$ be two left $\mathscr{D}_X$-modules.

Definition 1.3.1. The sheaf of *solutions* of $\mathscr{M}$ in $\mathscr{N}$ is the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$.

Remark 1.3.2.

(1) The sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ has no structure more than that of a sheaf of $\mathbb{C}$-vector spaces in general, because $\mathscr{D}_X$ is not commutative.

(2) According to Exercise E.1.11(1), $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$ is a left $\mathscr{D}_X$-module, that is, a $\mathscr{O}_X$-module with an integrable connection (Proposition 1.1.4). Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ is the subsheaf of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$ consisting of local morphisms $\mathscr{M}\to\mathscr{N}$ which commute with the connections on $\mathscr{M}$ and $\mathscr{N}$, in other words local sections which are annihilated by the connection on $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$.

Example 1.3.3. Let $U\subset X$ be a coordinate chart and let $P\in\mathscr{D}_X(U)$. Let $\mathscr{I}=\mathscr{D}_U\cdot P$ be the left ideal of $\mathscr{D}_U$ generated by P and let $\mathscr{M}=\mathscr{D}_U/\mathscr{I}$. We have a canonical isomorphism $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})\simeq\operatorname{Ker}[P\cdot:\mathscr{N}\to\mathscr{N}]$, and this explains the terminology "solutions of $\mathscr{M}$ in $\mathscr{N}$".

If $\mathscr{N}=\mathscr{O}_X$, we get the sheaf of holomorphic solutions of P. If $\mathscr{N}=\mathfrak{Db}_X$, we get the sheaf of distributions solutions of P.

If $\mathscr{N}=\mathscr{D}_X$ (with its standard left structure), then $P\cdot:\mathscr{D}_X\to\mathscr{D}_X$ is injective (Exercise E.1.2), so $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)=0$. It maybe therefore interesting to consider higher $\mathscr{H}om$, namely, $\mathscr{E}xt$ sheaves. We consider the free resolution of $\mathscr{M}$ defined as

$$0\longrightarrow\mathscr{D}_U\xrightarrow{\ \cdot P\ }\mathscr{D}_U\longrightarrow\mathscr{M}\longrightarrow 0.$$

The map $\cdot P$ is injective (same argument as for $P\cdot$), so this is indeed a resolution. By definition, $\mathscr{E}xt^1(\mathscr{M},\mathscr{N})$ is the cokernel of

$$\begin{aligned}\mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U,\mathscr{N})&\longrightarrow\mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U,\mathscr{N})\\ \varphi(\star)&\longmapsto\varphi(\star\cdot P).\end{aligned}$$

If one identifies $\mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U,\mathscr{N})$ with $\mathscr{N}$ by $\varphi\mapsto\varphi(1)$, the previous morphism reads

$$\mathscr{N}\xrightarrow{\ P\cdot\ }\mathscr{N},$$

so we recover that $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})=\operatorname{Ker}[P\cdot\ :\mathscr{N}\to\mathscr{N}]$, and we find that $\mathscr{E}xt^1(\mathscr{M},\mathscr{N})=\operatorname{Coker}[P\cdot\ :\mathscr{N}\to\mathscr{N}]$. In other words, $\mathscr{E}xt^1(\mathscr{M},\mathscr{N})$ measures the obstruction to the solvability of the differential equation $Pm=n$ for $n\in\mathscr{N}$.

Notice that, in this example, since the free resolution of $\mathscr{M}$ has length two, we have $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})=0$ for $k\geqslant 2$, for any $\mathscr{N}$.

When $\mathscr{N}$ has a supplementary structure which commutes with its left $\mathscr{D}_X$-structure, then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ inherit this supplementary structure.

Example 1.3.4.

(1) Assume $\mathscr{N} = \mathscr{D}_X$. Then the definition of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ uses the left $\mathscr{D}_X$-module structure of $\mathscr{D}_X$, which commutes with the right one, so these solution sheaves are right $\mathscr{D}_X$-modules.
(2) Assume that $\mathscr{N} = \mathfrak{Db}_X$. Then the definition of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ uses the left $\mathscr{D}_X$-module structure of $\mathfrak{Db}_X$, which commutes with the left $\mathscr{D}_{\overline{X}}$-structure, so these solution sheaves are left $\mathscr{D}_{\overline{X}}$-modules.

1.4. De Rham and Spencer

Let $\mathscr{M}^{\mathrm{l}}$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^{\mathrm{r}}$ be a right $\mathscr{D}_X$-module.

***Definition 1.4.1* (De Rham).** The *de Rham complex* $\Omega_X^{n+\bullet}(\mathscr{M}^{\mathrm{l}})$ of $\mathscr{M}^{\mathrm{l}}$ is the complex having as terms the $\mathscr{O}_X$-modules $\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{l}}$ and as differential the $\mathbb{C}$-linear morphism $(-1)^n\nabla$ defined in Exercise E.1.7.

Notice that the de Rham complex is shifted $n = \dim X$ with respect to the usual convention. The shift produces, by definition, a sign change in the differential, which is then equal to $(-1)^n\nabla$.

***Definition 1.4.2* (Spencer).** The *Spencer complex* $(\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{r}}), \delta)$ is the complex having as terms the $\mathscr{O}_X$-modules $\mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_X$ (with $\bullet \leqslant 0$) and as differential the $\mathbb{C}$-linear map δ given by

$$
\begin{aligned}
m \otimes \xi_1 \wedge \cdots \wedge \xi_k \stackrel{\delta}{\longmapsto} & \sum_{i=1}^{k} (-1)^{i-1} m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \\
& + \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k.
\end{aligned}
$$

Exercise E.1.19.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\mathscr{D}_X$, considered as a left or right $\mathscr{D}_X$-module. Notice that, in $\Omega_X^{n+\bullet}(\mathscr{D}_X)$, the differentials are *right* $\mathscr{D}_X$-linear, and in $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ they are *left* $\mathscr{D}_X$-linear.

Proposition 1.4.3. *The Spencer complex $\mathrm{Sp}(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module by locally free left $\mathscr{D}_X$-modules. Similarly, the complex $\Omega_X^{n+\bullet}(\mathscr{D}_X)$ is a resolution of ω_X as a right $\mathscr{D}_X$-module by locally free right $\mathscr{D}_X$-modules.*

Proof. Exercises E.1.20 and E.1.21. □

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^{\mathrm{r}}$ the associated right module. We will now compare $\Omega_X^{n+\bullet}(\mathscr{M})$ and $\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{r}})$.

Proposition 1.4.4. *There is a functorial isomorphism $\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{r}}) \xrightarrow{\sim} \Omega_X^{n+\bullet}(\mathscr{M})$ for any left $\mathscr{D}_X$-module $\mathscr{M}$, which is termwise $\mathscr{O}_X$-linear.*

Proof. Exercises E.1.22–E.1.25. □

Remark 1.4.5. We will denote by ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{r}})$ the Spencer complex $\mathrm{Sp}^{\bullet}_X(\mathscr{M}^{\mathrm{r}})$ and by ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{l}})$ the de Rham complex $\Omega_X^{n+\bullet}(\mathscr{M}^{\mathrm{l}})$. The previous exercise gives an isomorphism ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{r}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{l}})$ and justifies this convention. We will use this notation below. Exercise E.1.22 clearly shows that ${}^{\mathrm{p}}\mathrm{DR}_X$ is a functor from the category of right (resp. left) $\mathscr{D}_X$-modules to the category of complexes of sheaves of $\mathbb{C}$-vector spaces. It can be extended to a functor between the corresponding derived categories.

1.5. Filtered objects

***Definition 1.5.1* (of a filtered $\mathscr{D}_X$-module).** A *filtration* $F_\bullet\mathscr{M}$ of a $\mathscr{D}_X$-module $\mathscr{M}$ will mean an increasing filtration satisfying (for left modules for instance)

$$F_k\mathscr{D}_X \cdot F_\ell\mathscr{M} \subset F_{k+\ell}\mathscr{M} \quad \forall\, k,\ell \in \mathbb{Z}.$$

We usually assume that $F_\ell\mathscr{M} = 0$ for $\ell \ll 0$ and that the filtration is exhaustive, i.e., $\bigcup_\ell F_\ell\mathscr{M} = \mathscr{M}$.

***Definition 1.5.2* (of the de Rham complex of a filtered $\mathscr{D}_X$-module).** Let $F_\bullet\mathscr{M}$ be a filtered $\mathscr{D}_X$-module. The de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is filtered by sub-complexes $F_p{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ defined by

$$F_p{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M} = \begin{cases} \cdots \xrightarrow{\ \delta\ } F_{p-k}\mathscr{M}^{\mathrm{r}} \otimes \wedge^k\Theta_X \xrightarrow{\ \delta\ } F_{p-k+1}\mathscr{M}^{\mathrm{r}} \otimes \wedge^{k-1}\Theta_X \xrightarrow{\ \delta\ } \cdots \\ \cdots \xrightarrow{\ \nabla\ } \Omega_X^{n+k} \otimes F_{p+k}\mathscr{M}^{\mathrm{l}} \xrightarrow{\ \nabla\ } \Omega_X^{n+k+1} \otimes F_{p+k+1}\mathscr{M}^{\mathrm{l}} \xrightarrow{\ \nabla\ } \cdots \end{cases}$$

and the filtered de Rham complex is denoted by ${}^{\mathrm{p}}\mathrm{DR}\,F_\bullet\mathscr{M}$.

Exercises and complements

***Exercise E.1.1* (The sheaf $\mathscr{H}\!om$).** Let X be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves of $\mathscr{A}$-modules on X, $\mathscr{A}$ being a sheaf of rings on X. We denote by $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ the $\Gamma(X,\mathscr{A})$-module of morphisms of sheaves of $\mathscr{A}$-modules from $\mathscr{F}$ to $\mathscr{G}$. An element ϕ of $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ is a collection of morphisms $\phi(U) \in \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$, on open subsets U of X, compatible with the restrictions.

Show that the presheaf $\mathscr{H}\!om_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ defined by

$$\Gamma(U,\mathscr{H}\!om_{\mathscr{A}}(\mathscr{F},\mathscr{G})) = \mathrm{Hom}_{\mathscr{A}_{|U}}(\mathscr{F}_{|U},\mathscr{G}_{|U})$$

is a *sheaf* (notice that $U \mapsto \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$ is not a presheaf, because there are no canonical morphisms of restriction).

***Exercise E.1.2* (Local computations).** Let U be an open set of $\mathbb{C}^n$ with coordinates $x_1,\dots,x_n$. Denote by $\partial_{x_1},\dots,\partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathscr{D}(U)$:

$$\begin{aligned}
[\partial_{x_i}, \varphi] &= \frac{\partial \varphi}{\partial x_i}, \quad \forall\, \varphi \in \mathscr{O}(U),\ \forall\, i \in \{1, \dots, n\},\\
[\partial_{x_i}, \partial_{x_j}] &= 0 \quad \forall\, i, j \in \{1, \dots, n\},\\
\partial_x^\alpha \cdot \varphi &= \sum_{0 \leqslant \beta \leqslant \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} \partial_x^{\alpha-\beta}(\varphi) \partial_x^\beta,\\
\varphi \cdot \partial_x^\alpha &= \sum_{0 \leqslant \beta \leqslant \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} (-1)^{|\alpha-\beta|} \partial_x^\beta \partial_x^{\alpha-\beta}(\varphi),
\end{aligned}$$

with standard notation concerning multi-indices α, β.

(2) Show that any element $P \in \mathscr{D}(U)$ can be written in a unique way as $\sum_\alpha a_\alpha \partial_x^\alpha$ or $\sum_\alpha \partial_x^\alpha b_\alpha$with $a_\alpha, b_\alpha \in \mathscr{O}(U)$. Conclude that $\mathscr{D}_X$ is a locally free left and right module over $\mathscr{O}_X$.

(3) Show that $\max\{|\alpha|\ ;\ a_\alpha \neq 0\} = \max\{|\alpha|\ ;\ b_\alpha \neq 0\}$. It is denoted by $\mathrm{ord}_x P$.

(4) Show that $\mathrm{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Show that $PQ = 0$ in $\mathscr{D}(U) \Rightarrow P = 0$ or $Q = 0$.

(6) Identify $F_k\mathscr{D}_X$ with the subsheaf of local sections of $\mathscr{D}_X$ having order $\leqslant k$ (in some or any local coordinate system). Show that it is a locally free $\mathscr{O}_X$-module of finite rank.

(7) Show that the filtration $F_\bullet\mathscr{D}_X$ is exhaustive (i.e., $\mathscr{D}_X = \bigcup_k F_k\mathscr{D}_X$) and that it satisfies

$$F_k\mathscr{D}_X \cdot F_\ell\mathscr{D}_X = F_{k+\ell}\mathscr{D}_X.$$

(8) Show that the bracket $[P, Q] := PQ - QP$ induces for each k, ℓ a $\mathbb{C}$-bilinear morphism $F_k\mathscr{D}_X \otimes_{\mathbb{C}} F_\ell\mathscr{D}_X \to F_{k+\ell-1}\mathscr{D}_X$.

(9) Conclude that the graded ring $\mathrm{gr}^F\mathscr{D}_X$ is commutative.

Exercise E.1.3 **(The sheaf of rings $\mathscr{D}_X$ has no non-trivial two-sided ideals).** Let $\mathscr{I}$ be a non-zero two-sided ideal of $\mathscr{D}_X$.

(1) Let $x \in X$ and $0 \neq P \in \mathscr{I}_x$. Show that there exists $f \in \mathscr{O}_{X,x}$ such that $[P, f] \neq 0$. [Hint: use local coordinates to express P].

(2) Conclude by induction on the order that $\mathscr{I}_x$ contains a non-zero $g \in \mathscr{O}_{X,x}$.

(3) Show that $\mathscr{I}_x$ contains all iterated differentials of g, and conclude that $\mathscr{I}_x$ contains $h \in \mathscr{O}_{X,x}$ such that $h(x) \neq 0$.

(4) Conclude that $\mathscr{I}_x \ni 1$, hence $\mathscr{I}_x = \mathscr{D}_{X,x}$.

Exercise E.1.4 **(The graded sheaf $\mathrm{gr}^F\mathscr{D}_X$).** The goal of this exercise is to show that the sheaf of graded rings $\mathrm{gr}^F\mathscr{D}_X$ may be canonically identified with the sheaf of graded rings $\mathrm{Sym}\,\Theta_X$. If one identifies Θ_X with the sheaf of functions on the cotangent space

T^*X which are linear in the fibres, then $\operatorname{Sym}\Theta_X$ is the sheaf of functions on T^*X which are polynomial in the fibres.

(1) Identify the $\mathscr{O}_X$-module $\operatorname{Sym}^k\Theta_X$ with the sheaf of symmetric $\mathbb{C}$-linear forms $\boldsymbol{\xi} : \mathscr{O}_X \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathscr{O}_X$ on the k-fold tensor product, which behave like a derivation with respect to each factor.

(2) Show that $\operatorname{Sym}\Theta_X := \oplus_k \operatorname{Sym}^k\Theta_X$ is a sheaf of graded $\mathscr{O}_X$-algebras on X and identify it with the sheaf of functions on T^*X which are polynomial in the fibres.

(3) Show that the map $F_k\mathscr{D}_X \to \mathscr{H}\!om_{\mathbb{C}}\big(\overset{k}{\otimes}_{\mathbb{C}}\mathscr{O}_X, \mathscr{O}_X\big)$ which sends any section P of $F_k\mathscr{D}_X$ to
$$\varphi_1 \otimes \cdots \otimes \varphi_k \longmapsto [\cdots[[P,\varphi_1]\varphi_2]\cdots\varphi_k]$$
induces an isomorphism of $\mathscr{O}_X$-modules $\operatorname{gr}^F_k\mathscr{D}_X \to \operatorname{Sym}^k\Theta_X$.

(4) Show that the induced morphism
$$\operatorname{gr}^F\mathscr{D}_X := \oplus_k \operatorname{gr}^F_k\mathscr{D}_X \longrightarrow \operatorname{Sym}\Theta_X$$
is an isomorphism of sheaves of $\mathscr{O}_X$-algebras.

Exercise E.1.5. Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of rank d and let $\mathscr{E}^\vee$ be its dual. Show that, given any local basis $\boldsymbol{e} = (e_1,\dots,e_d)$ of $\mathscr{E}$ with dual basis $\boldsymbol{e}^\vee$, the section $\sum_{i=1}^d e_i\otimes e_i^\vee$ of $\mathscr{E}\otimes\mathscr{E}^\vee$ does not depend on the choice of the local basis $\boldsymbol{e}$ and extends as a global section of $\mathscr{E}\otimes\mathscr{E}^\vee$. Show that it defines, up to a constant, a $\mathscr{O}_X$-linear section $\mathscr{O}_X \to \mathscr{E}\otimes\mathscr{E}^\vee$ of the natural duality pairing $\mathscr{E}\otimes\mathscr{E}^\vee \to \mathscr{O}_X$. Conclude that we have a natural global section of $\Omega^1_X\otimes\Theta_X$ given, in local coordinates, by $\sum_i dx_i\otimes\partial_{x_i}$.

***Exercise E.1.6* (The universal connection).**

(1) Show that the natural left multiplication of Θ_X on $\mathscr{D}_X$ can be written as a *connection*
$$\nabla : \mathscr{D}_X \longrightarrow \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{D}_X,$$
i.e., as a $\mathbb{C}$-linear morphism satisfying the *Leibniz rule* $\nabla(fP) = df\otimes P + f\nabla P$, where f is any local section of $\mathscr{O}_X$ and P any local section of $\mathscr{D}_X$. (Hint: $\nabla(1)$ is the global section of $\Omega^1_X\otimes_{\mathscr{O}_X}\Theta_X$ considered in Exercise E.1.5.)

(2) Extend this connection for any $k\geqslant 1$ as a $\mathbb{C}$-linear morphism
$${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$$
satisfying the Leibniz rule written as
$${}^{(k)}\nabla(\omega\otimes P) = d\omega\otimes P + (-1)^k\omega\wedge\nabla P.$$

(3) Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$ for any $k\geqslant 0$ (i.e., ∇ is *flat*).

(4) Show that the morphisms ${}^{(k)}\nabla$ are *right* $\mathscr{D}_X$-linear (but not left $\mathscr{O}_X$-linear).

Exercise E.1.7. More generally, show that a left $\mathscr{D}_X$-module $\mathscr{M}$ is nothing but a $\mathscr{O}_X$-module with an *integrable* connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$. (Hint: to get the connection, tensor the left $\mathscr{D}_X$-action $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{M}$ by Ω^1_X on the left and compose with the universal connection to get $\mathscr{D}_X \otimes \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$; compose it on the left with $\mathscr{M} \to \mathscr{D}_X \otimes \mathscr{M}$ given by $m \mapsto 1 \otimes m$.) Define similarly the iterated connections ${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{M}$. Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$.

Exercise E.1.8 **($\mathscr{O}_X$ is a *simple* left $\mathscr{D}_X$-module).**

(1) Let f be a nonzero holomorphic function on $\mathbb{C}^n$. Show that there exists a multi-index $\alpha \in \mathbb{N}^n$ such that $(\partial_\alpha f)(0) \neq 0$.

(2) Conclude that $\mathscr{O}_X$ is a simple left $\mathscr{D}_X$-module, i.e., does not contain any proper non trivial $\mathscr{D}_X$-submodule. Is it simple as a left $\mathscr{O}_X$-module?

Exercise E.1.9 **(ω_X is a *simple* right $\mathscr{D}_X$-module).**

(1) Use the right action of Θ_X on ω_X to define on ω_X the structure of a right $\mathscr{D}_X$-module.

(2) Show that it is simple as a right $\mathscr{D}_X$-module.

Exercise E.1.10 **(Tensor products over $\mathscr{O}_X$).**

(1) Let $\mathscr{M}$ and $\mathscr{N}$ be two left $\mathscr{D}_X$-modules.

   (a) Show that the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has the structure of a left $\mathscr{D}_X$-module by setting, by analogy with the Leibniz rule,
$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

   (b) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\mathscr{D}_X$-action on the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$.

   (c) Let $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{N} \to \mathscr{N}'$ be $\mathscr{D}_X$-linear morphisms. Show that $\varphi \otimes \psi$ is $\mathscr{D}_X$-linear.

(2) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and $\mathscr{N}$ be a right $\mathscr{D}_X$-module. Show that $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$ has the structure of a right $\mathscr{D}_X$-module by setting
$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m.$$

Remark: one can define a right $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ by using the natural involution $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \xrightarrow{\sim} \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$, so this brings no new structure.

(3) Assume that $\mathscr{M}$ and $\mathscr{N}$ are right $\mathscr{D}_X$-modules. Does there exist a (left or right) $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ defined with analogous formulas?

Exercise E.1.11 **($\mathscr{H}om$ over $\mathscr{O}_X$).**

(1) Let $\mathscr{M}, \mathscr{N}$ be left $\mathscr{D}_X$-modules. Show that $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of Θ_X, m of $\mathscr{M}$ and φ of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$.

(2) Similarly, if $\mathscr{M}, \mathscr{N}$ are right $\mathscr{D}_X$-modules, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise E.1.12 **(Tensor product of a left $\mathscr{D}_X$-module with $\mathscr{D}_X$).** Let $\mathscr{M}^1$ be a left $\mathscr{D}_X$-module. Notice that $\mathscr{M}^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two commuting structures of $\mathscr{O}_X$-module. Similarly $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^1$ has two such structures. The goal of this exercise is to extend them as $\mathscr{D}_X$-structures and examine their relations.

(1) Show that $\mathscr{M}^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has the structure of a left and of a right $\mathscr{D}_X$-module *which commute*, given by the formulas:

$$\text{(left)} \qquad \begin{cases} f \cdot (m \otimes P) = (fm) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes \xi P, \end{cases}$$

$$\text{(right)} \qquad \begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

for any local vector field ξ and any local holomorphic function f. Show that a left $\mathscr{D}_X$-linear morphism $\varphi : \mathscr{M}_1^1 \to \mathscr{M}_2^1$ extends as a bi-$\mathscr{D}_X$-linear morphism $\varphi \otimes 1 : \mathscr{M}_1^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M}_2^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

(2) Similarly, show that $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^1$ also has such structures *which commute* and are functorial, given by formulas:

$$\text{(left)} \qquad \begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

$$\text{(right)} \qquad \begin{cases} (P \otimes m) \cdot f = P \otimes (fm) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P\xi \otimes m - P \otimes \xi m. \end{cases}$$

(3) Show that both morphisms

$$\begin{aligned} \mathscr{M}^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X &\longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^1 \\ m \otimes P &\longmapsto (1 \otimes m) \cdot P \end{aligned} \qquad \begin{aligned} \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^1 &\longrightarrow \mathscr{M}^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ P \otimes m &\longmapsto P \cdot (m \otimes 1) \end{aligned}$$

are left and right $\mathscr{D}_X$-linear, induce the identity $\mathscr{M}^1 \otimes 1 = 1 \otimes \mathscr{M}^1$, and their composition is the identity of $\mathscr{M}^1 \otimes_{\mathscr{O}_X} \mathscr{D}_X$ or $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^1$, hence both are reciprocal isomorphisms. Show that this correspondence is functorial (i.e., compatible with morphisms).

(4) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{L}$ be a $\mathscr{O}_X$-module. Justify the following isomorphisms of left $\mathscr{D}_X$-modules and right $\mathscr{O}_X$-modules:

$$\begin{aligned}\mathscr{M}\otimes_{\mathscr{O}_X}(\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{L})&\simeq(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X)\otimes_{\mathscr{O}_X}\mathscr{L}\\ &\simeq(\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{M})\otimes_{\mathscr{O}_X}\mathscr{L}\simeq\mathscr{D}_X\otimes_{\mathscr{O}_X}(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{L}).\end{aligned}$$

Assume moreover that $\mathscr{M}$ and $\mathscr{L}$ are $\mathscr{O}_X$-locally free. Show that $\mathscr{M}\otimes_{\mathscr{O}_X}(\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{L})$ is $\mathscr{D}_X$-locally free.

Exercise E.1.13 **(Tensor product of a right $\mathscr{D}_X$-module with $\mathscr{D}_X$).** Let $\mathscr{M}^{\mathrm{r}}$ be a right $\mathscr{D}_X$-module.

(1) Show that $\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ has two structures of right $\mathscr{D}_X$-module denoted $\cdot_{\mathrm{r}}$ (trivial) and $\cdot_{\mathrm{t}}$ (the latter defined by using the left structure on $\mathscr{D}_X$ and Exercise E.1.10(2)), given by:

$$\text{(right)}_{\mathrm{r}}\qquad\begin{cases}(m\otimes P)\cdot_{\mathrm{r}}f=m\otimes(Pf),\\ (m\otimes P)\cdot_{\mathrm{r}}\xi=m\otimes(P\xi),\end{cases}$$

$$\text{(right)}_{\mathrm{t}}\qquad\begin{cases}(m\otimes P)\cdot_{\mathrm{t}}f=mf\otimes P=m\otimes fP,\\ (m\otimes P)\cdot_{\mathrm{t}}\xi=m\xi\otimes P-m\otimes(\xi P).\end{cases}$$

(2) Show that there is a unique involution $\iota:\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\mathscr{D}_X\xrightarrow{\sim}\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ which exchanges both structures and is the identity on $\mathscr{M}^{\mathrm{r}}\otimes 1$, given by $(m\otimes P)_{\mathrm{r}}\mapsto(m\otimes 1)\cdot_{\mathrm{t}}P$ (Hint: show first the properties of ι by using local coordinates, and glue the local constructions by uniqueness of ι).

(3) For each $p\geqslant 0$, consider the pth term $F_p\mathscr{D}_X$ of the filtration of $\mathscr{D}_X$ by the order (see Exercise 1.1.2) with both structures of $\mathscr{O}_X$-module (one on the left, one on the right) and equip similarly $\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X$ with two structures t and r of $\mathscr{O}_X$-modules. Show that, for each p, ι induces an isomorphism of $\mathscr{O}_X$-modules $(\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X)_{\mathrm{t}}\xrightarrow{\sim}(\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X)_{\mathrm{r}}$.

Exercise E.1.14 **(Compatibility of right-left transformations).** Show that the natural morphisms

$$\mathscr{M}^{\mathrm{l}}\longrightarrow\mathscr{H}om_{\mathscr{O}_X}(\omega_X,\omega_X\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{l}}),\qquad\omega_X\otimes_{\mathscr{O}_X}\mathscr{H}om_{\mathscr{O}_X}(\omega_X,\mathscr{M}^{\mathrm{r}})\longrightarrow\mathscr{M}^{\mathrm{r}}$$

are isomorphisms of $\mathscr{D}_X$-modules.

Exercise E.1.15 **(Compatibility of left-right transformation with tensor product).** Let $\mathscr{M}^{\mathrm{l}}$ and $\mathscr{N}^{\mathrm{l}}$ be two left $\mathscr{D}_X$-modules and denote by $\mathscr{M}^{\mathrm{r}},\mathscr{N}^{\mathrm{r}}$ the corresponding right $\mathscr{D}_X$-modules (see Definition 1.2.4). Show that there is a natural isomorphism of right $\mathscr{D}_X$-modules (by using the right structure given in Exercise E.1.10(2)):

$$\begin{gathered}\mathscr{N}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{l}}\xrightarrow{\sim}\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X}\mathscr{N}^{\mathrm{l}}\\ (\omega\otimes n)\otimes m\longmapsto(\omega\otimes m)\otimes n\end{gathered}$$

and that this isomorphism is functorial in $\mathscr{M}^{\mathrm{l}}$ and $\mathscr{N}^{\mathrm{l}}$.

***Exercise E.1.16* (Local expression of the left-right transformation).** Let U be an open set of $\mathbb{C}^n$.

(1) Show that there exists a unique $\mathbb{C}$-linear involution $P \mapsto {}^tP$ from $\mathscr{D}(U)$ to itself such that
- $\forall\, \varphi \in \mathscr{O}(U)$, ${}^t\varphi = \varphi$,
- $\forall\, i \in \{1, \dots, n\}$, ${}^t\partial_{x_i} = -\partial_{x_i}$,
- $\forall\, P, Q \in \mathscr{D}(U)$, ${}^t(PQ) = {}^tQ \cdot {}^tP$.

(2) Let $\mathscr{M}$ be a left (resp. right) $\mathscr{D}_X$-module and let ${}^t\mathscr{M}$ be $\mathscr{M}$ equipped with the right (resp. left) $\mathscr{D}_X$-module structure
$$P \cdot m := {}^tPm.$$
Show that ${}^t\mathscr{M} \xrightarrow{\sim} \mathscr{M}^{\mathrm{r}}$ (resp. ${}^t\mathscr{M} \xrightarrow{\sim} \mathscr{M}^{\mathrm{l}}$).

***Exercise E.1.17* (The left-right transformation is an isomorphism of categories).** To any left $\mathscr{D}_X$-linear morphism $\varphi^{\mathrm{l}} : \mathscr{M}^{\mathrm{l}} \to \mathscr{N}^{\mathrm{l}}$ is associated the $\mathscr{O}_X$-linear morphism $\varphi^{\mathrm{r}} = \mathrm{Id}_{\omega_X} \otimes \varphi^{\mathrm{l}} : \mathscr{M}^{\mathrm{r}} \to \mathscr{N}^{\mathrm{r}}$.

(1) Show that φ^{r} is right $\mathscr{D}_X$-linear.

(2) Define the reverse correspondence $\varphi^{\mathrm{r}} \mapsto \varphi^{\mathrm{l}}$.

(3) Conclude that the left-right correspondence ${}^{\ell}\mathsf{M}(\mathscr{D}_X) \mapsto \mathsf{M}(\mathscr{D}_X)$ is a functor, which is an isomorphism of categories, having the right-left correspondence $\mathsf{M}(\mathscr{D}_X) \mapsto {}^{\ell}\mathsf{M}(\mathscr{D}_X)$ as inverse functor.

Exercise E.1.18. Show that the two sets of differential operators $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ and $\{\partial_{x_1}, x_1\partial_{x_2} + \cdots + x_{n-1}\partial_{x_n}\}$ generate the same ideal of $\mathscr{D}_{\mathbb{C}^n}$.

Exercise E.1.19. Check that $(\mathrm{Sp}^{\bullet}_X(\mathscr{M}^{\mathrm{r}}), \delta)$ is indeed a complex, i.e., that $\delta \circ \delta = 0$.

***Exercise E.1.20* (The Spencer complex is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module).** Let $F_{\bullet}\mathscr{D}_X$ be the filtration of $\mathscr{D}_X$ by the order of differential operators. Filter the Spencer complex $\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ by the subcomplexes $F_k(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X))$ defined as
$$\cdots \xrightarrow{\delta} F_{k-\ell}\mathscr{D}_X \otimes \wedge^{\ell}\Theta_X \xrightarrow{\delta} F_{k-\ell+1}\mathscr{D}_X \otimes \wedge^{\ell-1}\Theta_X \xrightarrow{\delta} \cdots$$

(1) Show that, locally on X, using coordinates $x_1, \dots, x_n$, the graded complex $\mathrm{gr}^F \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X) := \oplus_k \mathrm{gr}^F_k \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is equal to the Koszul complex of the ring $\mathscr{O}_X[\xi_1, \dots, \xi_n]$ with respect to the regular sequence $\xi_1, \dots, \xi_n$.

(2) Conclude that $\mathrm{gr}^F \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$.

(3) Check that $F_k \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X) = 0$ for $k < 0$, $F_0 \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X) = \mathrm{gr}^F_0 \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is isomorphic to $\mathscr{O}_X$ and deduce that the complex
$$\mathrm{gr}^F_k \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X) := \{\cdots \xrightarrow{[\delta]} \mathrm{gr}^F_{k-\ell}\mathscr{D}_X \otimes \wedge^{\ell}\Theta_X \xrightarrow{[\delta]} \mathrm{gr}^F_{k-\ell+1}\mathscr{D}_X \otimes \wedge^{\ell-1}\Theta_X \xrightarrow{[\delta]} \cdots\}$$
is acyclic (i.e., quasi-isomorphic to 0) for $k > 0$.

(4) Show that the inclusion $F_0 \operatorname{Sp}^\bullet_X(\mathscr{D}_X) \hookrightarrow F_k \operatorname{Sp}^\bullet_X(\mathscr{D}_X)$ is a quasi-isomorphism for each $k \geqslant 0$ and deduce, by passing to the inductive limit, that the Spencer complex $\operatorname{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module by locally free left $\mathscr{D}_X$-modules.

Exercise E.1.21. Similarly, show that the complex $\Omega^{n+\bullet}_X(\mathscr{D}_X)$ is a resolution of ω_X as a right $\mathscr{D}_X$-module by locally free right $\mathscr{D}_X$-modules.

Exercise E.1.22. Let $\mathscr{M}^{\mathrm{r}}$ be a right $\mathscr{D}_X$-module

(1) Show that the natural morphism

$$\mathscr{M}^{\mathrm{r}} \otimes_{\mathscr{D}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k \Theta_X) \longrightarrow \mathscr{M}^{\mathrm{r}} \otimes_{\mathscr{O}_X} \wedge^k \Theta_X$$

defined by $m \otimes P \otimes \xi \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\mathscr{M}^{\mathrm{r}} \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_X(\mathscr{D}_X) \xrightarrow{\ \sim\ } \operatorname{Sp}^\bullet(\mathscr{M}^{\mathrm{r}}).$$

(2) Similar question for $\Omega^{n+\bullet}_X(\mathscr{D}_X) \otimes_{\mathscr{D}_X} \mathscr{M}^{\mathrm{l}} \to \Omega^{n+\bullet}_X(\mathscr{M}^{\mathrm{l}})$.

Exercise E.1.23. Consider the function

$$\mathbb{Z} \xrightarrow{\ \varepsilon\ } \{\pm 1\}, \quad a \longmapsto \varepsilon(a) = (-1)^{a(a-1)/2},$$

which satisfies in particular

$$\varepsilon(a+1) = \varepsilon(-a) = (-1)^a \varepsilon(a), \quad \varepsilon(a+b) = (-1)^{ab} \varepsilon(a)\varepsilon(b).$$

Given any $k \geqslant 0$, the *contraction* is the morphism

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \wedge^k \Theta_X &\longrightarrow \Omega^{n-k}_X \\ \omega \otimes \xi &\longmapsto \varepsilon(n-k)\omega(\xi \wedge \bullet). \end{aligned}$$

Show that the isomorphism of right $\mathscr{D}_X$-modules

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k \Theta_X\big) &\xrightarrow[\sim]{\ \iota\ } \Omega^{n-k}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ \big[\omega \otimes (1 \otimes \xi)\big] \cdot P &\longmapsto \big(\varepsilon(n-k)\omega(\xi \wedge \bullet)\big) \otimes P \end{aligned}$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the left structure after going from left to right) induces an isomorphism of complexes of right $\mathscr{D}_X$-modules

$$\iota : \omega_X \otimes_{\mathscr{O}_X} \big(\operatorname{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) \xrightarrow{\ \sim\ } \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X, \nabla\big).$$

Exercise E.1.24. Similarly, if $\mathscr{M}$ is any left $\mathscr{D}_X$-module and $\mathscr{M}^{\mathrm{r}} = \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is the associated right $\mathscr{D}_X$-module, show that there is an isomorphism

$$\begin{aligned} \mathscr{M}^{\mathrm{r}} \otimes_{\mathscr{D}_X} \big(\operatorname{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) &\simeq \big(\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet} \Theta_X, \delta\big) \\ &\xrightarrow{\ \sim\ } \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla\big) \simeq \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X, \nabla\big) \otimes_{\mathscr{D}_X} \mathscr{M} \end{aligned}$$

given on $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^k \Theta_X$ by

$$\omega \otimes m \otimes \xi \longmapsto \varepsilon(n-k)\omega(\xi \wedge \bullet) \otimes m.$$

Exercise E.1.25. Using Exercise E.1.24, show that there is a functorial isomorphism $\mathrm{Sp}^{\bullet}_X(\mathscr{M}^{\mathrm{r}}) \xrightarrow{\sim} \Omega^{n+\bullet}_X(\mathscr{M})$ for any left $\mathscr{D}_X$-module $\mathscr{M}$, which is termwise $\mathscr{O}_X$-linear.

Exercise E.1.26. Let $\mathscr{L}$ be a $\mathscr{O}_X$-module.

(1) Show that, for any k, we have a (termwise) exact sequence of complexes (see Exercise E.1.20)

$$0 \to \mathscr{L}\otimes_{\mathscr{O}_X} F_{k-1}(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)) \to \mathscr{L}\otimes_{\mathscr{O}_X} F_k(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)) \to \mathscr{L}\otimes_{\mathscr{O}_X} \mathrm{gr}^F_k(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)) \to 0.$$

(Hint: use that the terms of the complexes $F_j(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X))$ and $\mathrm{gr}^F_k(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X))$ are $\mathscr{O}_X$-locally free.)

(2) Show that $\mathscr{L}\otimes_{\mathscr{O}_X} \mathrm{gr}^F \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{L}$ as a $\mathscr{O}_X$-module.

(3) Show that $\mathscr{L}\otimes_{\mathscr{O}_X} \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{L}$ as a $\mathscr{O}_X$-module.

Exercise E.1.27. Show that the isomorphisms in Exercises E.1.12 and E.1.13 are isomorphisms of filtered objects $\mathscr{M}^{\mathrm{l}}\otimes_{\mathscr{O}_X} F_{\bullet}\mathscr{D}_X$, $F_{\bullet}\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{l}}$ and $\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{O}_X} F_{\bullet}\mathscr{D}_X$.

LECTURE 2

COHERENCE AND CHARACTERISTIC VARIETIES

Let us begin by recalling the definition of coherence. Let $\mathscr{A}$ be a sheaf of rings on a space X. A sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is said to be *$\mathscr{A}$-coherent* if it is locally of finite type and if, for any open set U of X and any $\mathscr{A}$-linear morphism $\varphi : \mathscr{A}^r_{|U} \to \mathscr{F}_{|U}$, the kernel of φ is locally of finite type. The sheaf $\mathscr{A}$ is a coherent sheaf of rings if it is coherent as a (left and right) module over itself. If $\mathscr{A}$ is coherent, a sheaf $\mathscr{F}$ is $\mathscr{A}$-coherent if and only if it has locally a finite presentation.

Classical theorems of Cartan and Oka claim the coherence of $\mathscr{O}_X$. Although it would be natural to develop the theory of coherent $\mathscr{D}_X$-modules in a similar way, some points of the theory are not known to extend to $\mathscr{D}_X$-modules (the lemma on holomorphic matrices). The approach which is therefore classically used consists in using the $\mathscr{O}_X$-theory, and the main tools for that purpose are the good filtrations.

This lecture is much inspired from [**GM93**].

2.1. Coherence of $\mathscr{D}_X$

Let K be a compact subset of X. We say K is a *compact polycylinder* if there exist a neighbourhood Ω of K, an analytic chart $\phi : \Omega \to W$ of X, and $(\rho_1, \dots, \rho_n) \in (\mathbb{R}^+)^n$ such that

$$\phi(K) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \forall\, i \in \{1, \dots, n\},\ |x_i| \leqslant \rho_i\}.$$

In particular a point $x \in X$ is a compact polycylinder. Let $\mathscr{F}$ be a sheaf on X and K a polycylinder. We know by [**God64**], that

$$\varinjlim_{\substack{U \supset K \\ U \text{open}}} \mathscr{F}(U) \simeq \mathscr{F}_{|K}(K)$$

denoted by $\mathscr{F}(K)$. We have $\mathscr{D}_{X|\Omega} \simeq \mathscr{D}_{\mathbb{C}^n|W}$ and this isomorphism is compatible with the filtrations. Thus, to study local properties of $\mathrm{gr}^F\mathscr{D}_X$ or of $\mathscr{D}_X$ in the neighbourhood of a polycylinder K we can assume that $K \subset \mathbb{C}^n$ is a usual polycylinder.

We have $\mathscr{D}_X(K) \subset \mathscr{H}\!om_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)(K)$ and any element of $\mathscr{D}_X(K)$ can be written in a unique way as $\sum_{\alpha\in I} c_\alpha \partial^\alpha$, with $c_\alpha \in \mathscr{O}_X(K)$ and $I \subset \mathbb{N}^n$ finite. The relations in Exercise E.1.2 remain true when we replace U by K. We also have

$$\varinjlim_{\substack{U\supset K\\ U\text{open}}} F_k\mathscr{D}_X(U) = \big\{P \in \mathscr{D}_X(K) \mid P = \textstyle\sum_{|\alpha|\leqslant k} c_\alpha\partial^\alpha \quad \text{with } c_\alpha \in \mathscr{O}_X(K)\big\}.$$

Let $F_k\mathscr{D}_X(K)$ be this $\mathscr{O}(K)$-submodule of $\mathscr{D}(K)$. We get a filtration of $\mathscr{D}_X(K)$ having the same properties as that of $\mathscr{D}_X(U)$. Finally, we deduce from Exercise E.1.4 the existence of a canonical ring isomorphism

$$\mathrm{gr}^F(\mathscr{D}_X(K)) \xrightarrow{\ \sim\ } (\mathrm{gr}^F\mathscr{D}_X)(K).$$

We thus have an isomorphism

$$\mathrm{gr}^F(\mathscr{D}_X(K)) \simeq \mathscr{O}_{\mathbb{C}^n}(K)[\xi_1, \dots, \xi_n]$$

by an inductive limit on $U \supset K$. By a theorem of Frisch [**Fri67**], $\mathscr{O}_{\mathbb{C}^n}(K)$ is a Noetherian ring and, for any $x \in K$, the ring $\mathscr{O}_{\mathbb{C}^n,x}$ is flat over $\mathscr{O}_{\mathbb{C}^n}(K)$. We therefore get:

Proposition 2.1.1. *If K is a compact polycylinder, $\mathrm{gr}^F\mathscr{D}_X(K)$ is a Noetherian ring.* □

Proposition 2.1.2. *The ring $\mathscr{D}_X(K)$ is Noetherian.*

Proof. Let $I \subset \mathscr{D}_X(K)$ be a left ideal. We have to prove that it is finitely generated. Set $F_kI = I \cap F_k\mathscr{D}_X(K)$. Then $\mathrm{gr}^F I = \oplus_{k\in\mathbb{N}} F_kI/F_{k-1}I$ is an ideal in $\mathrm{gr}^F\mathscr{D}_X(K)$, thus is of finite type. Let $e_1, \dots, e_\ell$ be homogeneous generators of $\mathrm{gr}^F I$, of degrees $d_1, \dots, d_\ell$ and $P_1, \dots, P_\ell$ elements of I with $\sigma(P_j) = e_j$. It is easy to prove, by induction on the order of $P \in I$, that $I = \sum_{i=1}^\ell \mathscr{D}_X(K)\cdot P_i$ (left to the reader). □

Theorem 2.1.3. *The sheaf of rings $\mathscr{D}_X$ is coherent.*

Proof. If $U \subset X$ is open and

$$\phi : (\mathscr{D}_{X|U})^q \longrightarrow (\mathscr{D}_{X|U})^p$$

is a morphism of left $\mathscr{D}_{X|U}$-modules, we have to prove that $\operatorname{Ker}\phi$ is locally of finite type. We may assume that U is an open chart, thus in fact an open subset of $\mathbb{C}^n$. Let $\varepsilon_1, \dots, \varepsilon_q$ be the canonical base of $\mathscr{D}_X(U)^q$ and $k \in \mathbb{N}$ be such that, for all $i \in \{1, \dots, q\}$, $\phi(\varepsilon_i) \in F_k\mathscr{D}_X(U)^p$. We then have $\phi(F_\ell\mathscr{D}_U^q) \subset F_{k+\ell}\mathscr{D}_U^p$, and $\operatorname{Ker}\phi \cap F_\ell\mathscr{D}_U^q$ is the kernel of a morphism between locally free $\mathscr{O}_U$-modules of finite type

$$F_\ell\mathscr{D}_U^q \longrightarrow F_{k+\ell}\mathscr{D}_U^p.$$

Thus $\operatorname{Ker}\phi \cap F_\ell\mathscr{D}_U^q$ is $\mathscr{O}_U$ coherent, and $\operatorname{Ker}\phi$ is the union of these $\mathscr{O}_U$-modules.

Let $K \subset U$ be a compact polycylinder. By Theorem A of Cartan, for any $x \in K$, the sheaf $[\operatorname{Ker}\phi \cap F_\ell \mathscr{D}^q_{X|U}]_x$ is generated by $\Gamma(K, \operatorname{Ker}\phi \cap F_\ell \mathscr{D}^q_{X|U})$, which is included in $\Gamma(K, \operatorname{Ker}\phi)$. Thus for any $x \in K$, $(\operatorname{Ker}\phi)_x$ is generated by $\Gamma(K, \operatorname{Ker}\phi)$, i.e., any germ of section of $\operatorname{Ker}\phi$ at x is a linear combination with coefficients in $\mathscr{O}_{X,x}$ of sections of $\operatorname{Ker}\phi$ over K. By left exactness of $\Gamma(K, \bullet)$ we have an exact sequence of left $\mathscr{D}_X(K)$-modules:

$$0 \longrightarrow \Gamma(K, \operatorname{Ker}\phi) \longrightarrow \Gamma(K, \mathscr{D}^q_U) = \mathscr{D}_X(K)^p \xrightarrow{\ \Gamma(K,\phi)\ } \mathscr{D}_X(K)^p.$$

Because $\mathscr{D}_X(K)$ is Noetherian, $\Gamma(K, \operatorname{Ker}\phi)$ is then of finite type as a left $\mathscr{D}_X(K)$-module. It is then easy to build a surjective morphism of left $\mathscr{D}_{X|K}$-modules

$$(\mathscr{D}_{X|K})^r \longrightarrow (\operatorname{Ker}\phi)_{|K} \longrightarrow 0$$

using the two properties above. This proves that $\operatorname{Ker}\phi$ is locally of finite type. $\square$

2.2. Coherent $\mathscr{D}_X$-modules and good filtrations

Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. From Theorem 2.1.3 and the preliminary reminder on coherence, we know that $\mathscr{M}$ is *$\mathscr{D}_X$-coherent* if it is locally finitely presented, i.e., if for any $x \in X$ there exists an open neighbourhood U of x an an exact sequence $\mathscr{D}^q_{X|U} \to \mathscr{D}^p_{X|U} \to \mathscr{M}_{|U}$.

***Definition 2.2.1* (Good filtrations).** Let $F_\bullet\mathscr{M}$ be a filtration of $\mathscr{M}$ (see § 1.5). We say that the filtration is *good* if the graded module $\operatorname{gr}^F\mathscr{M}$ is coherent over the coherent sheaf $\operatorname{gr}^F\mathscr{D}_X$ (i.e., locally finitely presented).

Good filtrations are the main tool to get results on coherent $\mathscr{D}_X$-modules from theorems on coherent $\mathscr{O}_X$-modules. There are equivalent definitions: Exercise E.2.3.

***Proposition 2.2.2* (Local existence of good filtrations).** *If $\mathscr{M}$ is $\mathscr{D}_X$-coherent, then it admits locally on X a good filtration.*

Proof. Exercise E.2.4. $\square$

Remark 2.2.3. It is not known whether any coherent $\mathscr{D}_X$-module has *globally* a good filtration. Nevertheless, it is known that any *holonomic* $\mathscr{D}_X$-module (see Definition 4.2.1) has a good filtration (see [**Mal94a, Mal94b, Mal96**]); in fact, if such is the case, there even exists a coherent $\mathscr{O}_X$-submodule $\mathscr{F}$ of $\mathscr{M}$ which generates $\mathscr{M}$, i.e., such that the natural morphism $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{M}$ is onto (this is a little stronger than the existence of a good filtration, if the manifold X is not compact).

This leads to the following definition, although it is not known whether any coherent $\mathscr{D}_X$-module is good.

***Definition 2.2.4* (Good $\mathscr{D}$-modules, see [SS94]).** A $\mathscr{D}_X$-module is *good* if, for any compact set $K \subset X$, there exists, on some neighbourhood U of K, a finite filtration of $\mathscr{M}_{|U}$ by $\mathscr{D}_U$-submodules such that each successive quotient has a good filtration.

The main results concerning coherent $\mathscr{D}_X$-modules are obtained from the theorems of Cartan and Oka for $\mathscr{O}_X$-modules.

***Theorem 2.2.5* (Theorems of Cartan-Oka for $\mathscr{D}_X$-modules).** *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let K be a compact polycylinder contained in an open subset U of X, such that $\mathscr{M}$ has a good filtration on U. Then,*

(1) *$\Gamma(K,\mathscr{M})$ generates $\mathscr{M}_{|K}$ as a $\mathscr{O}_K$-module,*
(2) *For any $i \geqslant 1$, $H^i(K,\mathscr{M}) = 0$.*

Proof. This is easily obtained from the theorems A and B for $\mathscr{O}_X$-modules, by using inductive limits (for A it is obvious and, for B, see [**God64**, Th. 4.12.1]). □

***Theorem 2.2.6* (Characterization of coherence for $\mathscr{D}_X$-modules).**

(1) *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then, for any small enough compact polycylinder K, we have the following properties:*
    (a) *$\mathscr{M}(K)$ is a finite type $\mathscr{D}(K)$-module,*
    (b) *For any $x \in K$, $\mathscr{O}_x \otimes_{\mathscr{O}(K)} \mathscr{M}(K) \to \mathscr{M}_x$ is an isomorphism.*
(2) *Conversely, if there exists a covering $\{K_\alpha\}$ by polycylinders K_α such that $X = \cup \overset{\circ}{K}_\alpha$ and that on any K_α the properties* (1a) *and* (1b) *are fulfilled, then $\mathscr{M}$ is $\mathscr{D}_X$-coherent.*

Proof. Let $U \subset X$ be an open subset small enough for $\mathscr{M}$ to have a presentation

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{D}_U^p \longrightarrow \mathscr{M}_{|U} \longrightarrow 0.$$

The $\mathscr{D}_U$-module $\mathscr{N}$ is coherent, therefore we have $H^1(K,\mathscr{N}) = 0$ for any small enough compact polycylinder $K \subset U$, and $\mathscr{D}^p(K) \to \mathscr{M}(K)$ is surjective. This proves (1a).

The $\mathscr{O}_X$-module $F_k\mathscr{M}_{|U} :=$ image $F_k\mathscr{D}_U^p$ being coherent we also have for any k an isomorphism $\mathscr{O}_x \otimes F_k\mathscr{M}(K) \to F_k\mathscr{M}_x$, by Theorem A of Cartan-Oka. From this we get (1b) by using an inductive limit

$$\mathscr{O}_x \otimes_{\mathscr{O}(K)} \mathscr{M}(K) \simeq \varinjlim_k \mathscr{O}_x \otimes_{\mathscr{O}(K)} F_k\mathscr{M}(K).$$

Conversely, if Condition (1a) is fulfilled we have, since $\mathscr{D}(K)$ is Noetherian, a finite presentation

$$\mathscr{D}^q(K) \xrightarrow{\ \phi\ } \mathscr{D}^p(K) \xrightarrow{\ \pi\ } \mathscr{M}(K) \longrightarrow 0,$$

which gives sheaf morphisms which we denote again by ϕ, π:

$$\mathscr{D}_{X|K}^q \xrightarrow{\ \phi\ } \mathscr{D}_{X|K}^p \xrightarrow{\ \pi\ } \mathscr{M}_{|K} \longrightarrow 0,$$

and, by the exactness of the functor $\mathscr{D}_x \otimes_{\mathscr{D}(K)}$, an exact sequence

$$\mathscr{D}_x^q \xrightarrow{\ \phi_x\ } \mathscr{D}_x^p \xrightarrow{\ p_x\ } \mathscr{D}_x \otimes_{\mathscr{D}(K)} \mathscr{M}(K) \longrightarrow 0.$$

By Condition (1b) and the lemma below, the morphism

$$c_x : \mathscr{D}_x \otimes_{\mathscr{D}(K)} \mathscr{M}(K) \longrightarrow \mathscr{M}_x$$

is an isomorphism. We deduce from this, and from the equality $\pi_x = c_x \circ p_x$ that $\mathscr{M}_{|K} = \mathrm{Coker}(\phi)$ is finitely presented on K. □

Lemma 2.2.7. *For any left $\mathscr{D}(K)$-module N, the canonical homomorphism*

$$\mathscr{O}_x \otimes_{\mathscr{O}(K)} N \longrightarrow \mathscr{D}_x \otimes_{\mathscr{D}(K)} N$$

is an isomorphism.

Proof. This is clear when $N = \mathscr{D}(K)$, hence for any free module and finally, in the general case, by the right exactness of the functors $\mathscr{O}_x \otimes_{\mathscr{O}(K)} \bullet$ and $\mathscr{D}_x \otimes_{\mathscr{D}(K)} \bullet$. □

A first application of Theorem 2.2.6 is a variant of the classical Artin-Rees Lemma:

Corollary 2.2.8. *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be a short exact sequence of coherent $\mathscr{D}_X$-modules. Let $F_\bullet\mathscr{M}$ be a good filtration. Then the filtrations $F_\bullet\mathscr{M}' := \mathscr{M}' \cap F_\bullet\mathscr{M}$ and $F_\bullet\mathscr{M}'' := \mathrm{Im}\, F_\bullet\mathscr{M} \subset \mathscr{M}''$ are good.* □

2.3. Support

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Being a sheaf on X, $\mathscr{M}$ has a support $\mathrm{Supp}\,\mathscr{M}$, which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which $\mathscr{M}$ is zero. Recall that the support of a coherent $\mathscr{O}_X$-module is a closed analytic subset of X. Such a property extends to coherent $\mathscr{D}_X$-modules:

Proposition 2.3.1. *The support $\mathrm{Supp}\,\mathscr{M}$ of a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is a closed analytic subset of X.*

Proof. The property of being an analytic subset being local, we may assume that $\mathscr{M}$ is generated over $\mathscr{D}_X$ by a coherent $\mathscr{O}_X$-submodule $\mathscr{F}$ (see Exercise E.2.4(3)). Then the support of $\mathscr{M}$ is equal to the support of $\mathscr{F}$. □

Let $Y \subset X$ be a complex submanifold. The following is known as “Kashiwara’s equivalence”.

Proposition 2.3.2. *There is a natural equivalence between coherent $\mathscr{D}_X$-modules supported on Y and coherent $\mathscr{D}_Y$-modules.*

Proof. We will prove this in the special case where X is an open set in $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$ and Y is defined by $x_n = 0$. Given a coherent $\mathscr{D}_X$-module $\mathscr{M}$ supported on Y, we set $\mathscr{N} := \mathrm{Ker}[x_n \cdot : \mathscr{M} \to \mathscr{M}]$: this is a $\mathscr{D}_Y$-module. We also set $\mathscr{N}[\partial_{x_n}] := \mathscr{N} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{x_n}]$: this is a $\mathscr{D}_X$-module by the following rule; let $f(x_1, \ldots, x_n) = \sum_k f_k(x_1, \ldots, x_{n-1}) x_n^k$ be a holomorphic function and n be a section of $\mathscr{N}$; then we set

$$f \cdot (n \otimes \partial_j) = \sum_{k \leqslant j} \frac{(-1)^k j!}{(j-k)!} (f_k \cdot n) \otimes \partial_{x_n}^{j-k}.$$

We first claim that $\mathscr{M} = \mathscr{N}[\partial_{x_n}]$. Indeed, let m be a local section of $\mathscr{M}$. We will prove that it decomposes uniquely as $\sum_j \otimes \partial_{x_n}^j n_j$ for some local sections n_j of $\mathscr{N}$. The section m generates a $\mathscr{O}_X$-submodule of $\mathscr{M}$, which is finitely generated, hence coherent since $\mathscr{M}$ is coherent, and is supported on Y. Therefore, locally, there exists ℓ such that $x_n^{\ell+1} m = 0$. Then

$$0 = \partial_{x_n}(x_n^{\ell+1} m) = (\ell+1) x_n^\ell m + x_n^{\ell+1} \partial_{x_n} m = x_n^\ell \big((\ell+1) + x_n \partial_{x_n}\big) m = x_n^\ell (\ell + \partial_{x_n} x_n) m,$$

so $m = m_1 + \partial_{x_n} m_2$, $m_1 = (\ell + \partial_{x_n} x_n) m/\ell$, $m_2 = -x_n m/\ell$, with $x_n^\ell m_1 = x_n^\ell m_2 = 0$. This gives the existence by decreasing induction on ℓ. Uniqueness is obtained similarly. It remains to be proved that the natural $\mathscr{D}_Y$-linear morphism $\mathscr{N}[\partial_{x_n}] \to \mathscr{M}$ defined by $\sum_j n_j \otimes \partial_{x_n}^j \mapsto \sum_j \partial_{x_n}^j n_j$ is a $\mathscr{D}_X$-linear isomorphism, which is straightforward.

Lastly, the proof that $\mathscr{N}$ is $\mathscr{D}_Y$-coherent is obtained by using the coherence criterion given by Theorem 2.2.6. $\square$

2.4. Characteristic variety

The support is usually not the right geometric object attached to a $\mathscr{D}_X$-module $\mathscr{M}$, as it does not provide enough information on $\mathscr{M}$. A finer object is the *characteristic variety* that we introduce below. The following lemma will justify its definition.

Lemma 2.4.1. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then there exists a coherent sheaf $\mathscr{I}(\mathscr{M})$ of ideals of $\mathrm{gr}^F \mathscr{D}_X$ such that, for any open set U of X and any good filtration $F_\bullet \mathscr{M}_{|U}$, we have $\mathscr{I}(\mathscr{M})_{|U} = \mathrm{Rad}(\mathrm{ann}_{\mathrm{gr}^F \mathscr{D}_U} \mathrm{gr}^F \mathscr{M}_{|U})$.*

We denote by $\mathrm{Rad}(I)$ the radical of the ideal I and by ann the annihilator ideal of the corresponding module. Hence, for any $x \in U$, we have

$$\mathrm{Rad}(\mathrm{ann}_{\mathrm{gr}^F \mathscr{D}_{X,x}} \mathrm{gr}^F \mathscr{M}_x) = \{\varphi \in \mathrm{gr}^F \mathscr{D}_{X,x} \mid \exists\, \ell,\ \varphi^\ell \mathrm{gr}^F \mathscr{M}_x = 0\}.$$

Proof. It is a matter of showing that, if $F_\bullet \mathscr{M}_{|U}$ and $G_\bullet \mathscr{M}_{|U}$ are two good filtrations, then the corresponding ideals coincide. Notice first that these ideals are homogeneous, i.e., if φ belongs to the ideal, then so does any homogeneous component of φ. Let φ be a homogeneous element of degree j in the ideal corresponding to $F_\bullet \mathscr{M}$ and let $\widetilde{\varphi}$

be a lifting of φ in $F_j\mathscr{D}_X$. Then, locally, there exists ℓ such that, for any k, we have $\widetilde{\varphi}^\ell F_k\mathscr{M} \subset F_{k+j\ell-1}\mathscr{M}$ and thus, for any $p \geqslant 0$,

$$\widetilde{\varphi}^{(p+1)\ell}F_k\mathscr{M} \subset F_{k+j(p+1)\ell-p-1}\mathscr{M}.$$

Taking k_0 as in Exercise E.2.3, associated to $F_\bullet\mathscr{M}, G_\bullet\mathscr{M}$, we have

$$\begin{aligned}\widetilde{\varphi}^{(2k_0+1)\ell}G_k\mathscr{M} \subset \widetilde{\varphi}^{(2k_0+1)\ell}F_{k+k_0}\mathscr{M} &\subset F_{k+k_0+j(2k_0+1)\ell-2k_0-1}\mathscr{M}\\ &\subset G_{k+2k_0+j(2k_0+1)\ell-2k_0-1}\mathscr{M}\\ &= G_{k+j(2k_0+1)\ell-1}\mathscr{M}.\end{aligned}$$

This shows that, by setting $\ell' = (2k_0+1)\ell$, $\widetilde{\varphi}^{\ell'}G_k\mathscr{M} \subset G_{k+j\ell'-1}\mathscr{M}$, and thus φ is in the ideal corresponding to $G_\bullet\mathscr{M}$. By a symmetric argument, we find that both ideals are identical. □

Notice that we consider the radicals of the annihilator ideals, and not these annihilator ideals themselves, because of the shift k_0. In fact, the annihilator ideals may not be equal, as shown by the example given in Exercise E.2.6.

***Definition 2.4.2* (Characteristic variety).** The *characteristic variety* $\operatorname{Char}\mathscr{M}$ is the subset of the cotangent space T^*X defined by the ideal $\mathscr{I}(\mathscr{M})$.

Locally, given any good filtration of $\mathscr{M}$, the characteristic variety is defined as the set of common zeros of the elements of $\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_X}\operatorname{gr}^F\mathscr{M}$.

Assume that $\mathscr{M}$ is the quotient of $\mathscr{D}_X$ by the left ideal $\mathscr{J}$. Then one may choose for $F_\bullet\mathscr{M}$ the filtration induced by $F_\bullet\mathscr{D}_X$, so that $\operatorname{Char}\mathscr{M}$ is the locus of common zeros of the elements of $\operatorname{gr}^F\mathscr{J}$. In general, finding generators of $\operatorname{gr}^F\mathscr{J}$ from generators of $\mathscr{J}$ needs the use of Gröbner bases.

In local coordinates $x_1,\dots,x_n$, denote by $\xi_1,\dots,\xi_n$ the complementary symplectic coordinates in the cotangent space. Then $\operatorname{gr}^F\mathscr{J}$ is generated by a finite set of homogeneous elements $a_\alpha(x)\xi^\alpha$, where α belongs to a finite set of multi-indices. Hence the homogeneity of the ideal $\mathscr{I}(\mathscr{M})$ implies that

$$\operatorname{Supp}\mathscr{M} = \pi(\operatorname{Char}\mathscr{M}) = \operatorname{Char}\mathscr{M}\cap T^*_XX, \tag{2.4.3}$$

where $\pi : T^*X \to X$ denotes the bundle projection and T^*_XX denotes the zero section of the cotangent bundle.

2.5. Involutiveness of the characteristic variety

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, $\operatorname{Char}\mathscr{M} \subset T^*X$ its characteristic variety and $\operatorname{Supp}\mathscr{M}$ its support. For $(x,0) \in T^*X$, we denote by $\dim_{(x,0)}\operatorname{Char}\mathscr{M}$ the dimension at $(x,0)$ of the analytic space $\operatorname{Char}\mathscr{M}$.

Proposition 2.5.1. *Let $\mathscr{M}$ be a nonzero coherent $\mathscr{D}_X$-module. Then, for any $x \in X$, $\dim_{(x,0)}\operatorname{Char}\mathscr{M} \geqslant \dim X$.*

This inequality is called *Bernstein's inequality.*

Proof. We can assume that $X = \mathbb{C}^n$. The proposition is proved by induction on $\dim X$. If $\operatorname{Supp}\mathscr{M}$ is n-dimensional, the inequality is obvious. Then, it is enough to prove the proposition for every x in the smooth part of $\operatorname{Supp}\mathscr{M}$. Therefore, we have to consider the case where $\operatorname{Supp}\mathscr{M}$ is contained in the hypersurface $x_n = 0$. The proposition follows from Kashiwara's equivalence of categories between coherent $\mathscr{D}_{\mathbb{C}^n}$-modules supported on $x_n = 0$ and coherent $\mathscr{D}_{\mathbb{C}^{n-1}}$-modules (Proposition 2.3.2) (for the details, see [**GM93**, p. 129]). $\square$

But there exists a more precise result. In order to state it, consider on T^*X the fundamental 2-form ω. In local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, it is written $\omega = \sum_{1=1}^n d\xi_i \wedge dx_i$. For any $(x, \xi) \in T^*X$, ω defines on $T_{(x,\xi)}(T^*X)$ a nondegenerate bilinear form. We denote by $E^\perp$ the orthogonal space in the sense of ω of the vector subspace E of $T_{(x,\xi)}(T^*X)$. Recall that if V is a reduced analytic subspace of T^*X, with smooth part V_0,

- V is said to be *involutive* if, for any $a \in V_0$, we have $(T_aV)^\perp \subset T_aV$,
- V is said to be *isotropic* if, for any $a \in V_0$, we have $T_aV \subset (T_aV)^\perp$,
- V is said to be *Lagrangean* if, for any $a \in V_0$, we have $(T_aV)^\perp = T_aV$.

We observe that if V is involutive, the dimension of any irreducible components of V is bigger than $\dim X$.

Theorem 2.5.2. *Let $\mathscr{M}$ be a nonzero coherent $\mathscr{D}_X$-module. Then* $\operatorname{Char}\mathscr{M}$ *is an involutive set in T^*X.* $\square$

The first proof has been given by Sato, Kawai, Kashiwara [**SKK73**]. Next, Malgrange gave a very simple proof in a seminar Bourbaki talk ([**Mal78**], see also [**GM93**, p. 165]). And finally, Gabber gave the proof of a general algebraic version of this theorem (see [**Gab81**], see also [**Bjö93**, p. 473]).

A consequence is that any irreducible component of the characteristic variety of a coherent $\mathscr{D}_X$-module has a dimension $\geqslant \dim X$. On the other hand, we can get homological consequences of this result by using the homological theory of dimension.

Let $\mathscr{M}$ be a $\mathscr{D}_X$-module coherent and $x \in X$ and let $F_\bullet\mathscr{M}$ a local good filtration of $\mathscr{M}$. The dimension of the characteristic variety $\operatorname{Char}\mathscr{M}$ at $x \in X$ can be determined with $\operatorname{gr}^F\mathscr{M}$. Let $(x, \xi) \in \operatorname{Char}\mathscr{M}$ and let $\mathfrak{m}_{(x,\xi)}$ be the maximal ideal defining (x, ξ). For d sufficiently large, $\dim \operatorname{gr}^F\mathscr{M}/\mathfrak{m}^d_{(x,\xi)}$ is a polynomial in d. Let $d(x, \xi)$ be its degree. We have

$$\dim_x \operatorname{Char}\mathscr{M} = \sup\{d(x, \xi) \mid (x, \xi) \in T^*X\}.$$

Then, the following results can be proved using algebraic properties of $\operatorname{gr}^F\mathscr{D}_X$ (see e.g. [**Bjö79, GM93**]).

Proposition 2.5.3. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. We have*

$$\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0 \quad \textit{for } i \geqslant n+1. \qquad \square$$

Theorem 2.5.4. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and $x \in \operatorname{Supp} \mathscr{M}$. Then*

$$2n - \dim_x \operatorname{Char} \mathscr{M} = \inf\{i \in \mathbb{N} \mid \mathscr{E}xt^i_{\mathscr{D}_{X,x}}(\mathscr{M}_x, \mathscr{D}_{X,x}) = 0\}. \qquad \square$$

Another useful consequence of the homology theory of the dimension is the following proposition:

Proposition 2.5.5. *Let $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Then, the $\mathscr{D}_X$-submodule of $\mathscr{M}$ consisting of local sections m such that $\dim \mathscr{D}_X m \leqslant k$ is coherent.* $\square$

2.6. Non-characteristic restrictions

Let $i : Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal $\mathscr{I}_Y$ (in local coordinates $(x_1, \dots, x_n)$, $\mathscr{I}_Y$ is generated by $x_1, \dots, x_p$, where $p = \operatorname{codim} Y$). A local section ξ of $i^{-1}\Theta_X$ (vector field on X, considered at points of Y only; we denote by i^{-1} the sheaf-theoretic pull-back) is said to be tangent to Y if, for any local section f of $\mathscr{I}_Y$, $\xi(f) \in \mathscr{I}_Y$. This defines a subsheaf $\Theta_{X|Y}$ of $i^{-1}\Theta_X$. Then $\Theta_Y = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} \Theta_{X|Y} = i^*\Theta_{X|Y}$ is a subsheaf of $i^*\Theta_X$.

Given a left $\mathscr{D}_X$-module, the action of $i^{-1}\Theta_X$ on $i^{-1}\mathscr{M}$ restricts to an action of Θ_Y on $i^*\mathscr{M} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{M}$. The criterion of Lemma 1.2.1 is fulfilled since it is fulfilled for Θ_X and $\mathscr{M}$, defining therefore a left $\mathscr{D}_Y$-module structure on $i^*\mathscr{M}$. We denote this left $\mathscr{D}_Y$-module by $i^+\mathscr{M}$.

Without any other assumption, coherence is not preserved by i^+. For example, $i^+\mathscr{D}_X$ is not $\mathscr{D}_Y$-coherent if $\operatorname{codim} Y \geqslant 1$. A criterion for coherence of the pull-back is given below.

The cotangent map to the inclusion defines a natural bundle morphism $\varpi : T^*X_{|Y} \to T^*Y$, the kernel of which is by definition the conormal bundle $T^*_Y X$ of Y in X.

Lemma-Definition 2.6.1 **(Non-characteristic property).** We say that Y is *non-characteristic* with respect to the coherent $\mathscr{D}_X$-module $\mathscr{M}$ if one of the following equivalent conditions is satisfied:

- $T^*_Y X \cap \operatorname{Char} \mathscr{M} \subset T^*_X X$,
- $\varpi : \operatorname{Char} \mathscr{M}_{|Y} \to T^*Y$ is finite, i.e., proper with finite fibres.

Theorem 2.6.2 **(Coherence of non-characteristic restrictions).** *Assume that $\mathscr{M}$ is $\mathscr{D}_X$-coherent and that Y is non-characteristic with respect to $\mathscr{M}$. Then $i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent and* $\operatorname{Char} i^+\mathscr{M} \subset \varpi(\operatorname{Char} \mathscr{M}_{|Y})$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that $\mathscr{M}$ has a good filtration $F_\bullet\mathscr{M}$.

(1) Set $F_k i^+\mathscr{M} = \text{image}[i^* F_k\mathscr{M} \to i^*\mathscr{M}]$. Then, using Exercise E.2.5(2), one shows that $F_\bullet i^+\mathscr{M}$ is a good filtration with respect to $F_\bullet i^+\mathscr{D}_X$.

(2) The module $\mathrm{gr}^F i^+\mathscr{M}$ is a quotient of $i^*\mathrm{gr}^F\mathscr{M}$, hence its support is contained in $\mathrm{Char}\,\mathscr{M}_{|Y}$. By Remmert's Theorem, it is a coherent $\mathrm{gr}^F\mathscr{D}_Y$-module.

(3) The filtration $F_\bullet i^+\mathscr{M}$ is thus a good filtration of the $\mathscr{D}_Y$-module $i^+\mathscr{M}$. By Exercise E.2.4(1), $i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent. Using the good filtration above, it is clear that $\mathrm{Char}\, i^+\mathscr{M} \subset \varpi(\mathrm{Char}\,\mathscr{M}_{|Y})$. □

Exercises and complements

Exercise E.2.1.

(1) Let $D \subset X$ be a hypersurface and let $\mathscr{O}_X(*D)$ be the sheaf of meromorphic functions on X with poles on D at most (with arbitrary order). Prove similarly that $\mathscr{O}_X(*D)$ is a coherent sheaf of rings.

(2) Prove that $\mathscr{D}_X(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a coherent sheaf of rings.

(3) Let $i : Y \hookrightarrow X$ denote the inclusion of a smooth submanifold. Show that $i^*\mathscr{D}_X := \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a coherent sheaf of rings on Y.

Exercise E.2.2.

(1) Let $\mathscr{M} \subset \mathscr{N}$ be a $\mathscr{D}_X$-submodule of a coherent $\mathscr{D}_X$-module $\mathscr{N}$. Show that, if $\mathscr{M}$ is locally finitely generated, then it is coherent.

(2) Let $\phi : \mathscr{M} \to \mathscr{N}$ be a morphism between coherent $\mathscr{D}_X$-modules. Show that $\mathrm{Ker}\,\phi$ and $\mathrm{Coker}\,\phi$ are coherent.

***Exercise E.2.3* (Characterization of good filtrations).** Show that the following properties are equivalent:

(1) $F_\bullet\mathscr{M}$ is a good filtration;

(2) for any $k \in \mathbb{Z}$, $F_k\mathscr{M}$ is $\mathscr{O}_X$-coherent, and, for any $x \in X$, there exists a neighbourhood U of x and $k_0 \in \mathbb{Z}$ such that, for any $k \geqslant 0$, $F_k\mathscr{D}_{X|U} \cdot F_{k_0}\mathscr{M}_{|U} = F_{k+k_0}\mathscr{M}_{|U}$.

Conclude that, if $F_\bullet\mathscr{M}, G_\bullet\mathscr{M}$ are two good filtrations of $\mathscr{M}$, then, locally on X, there exists k_0 such that, for any k, we have

$$F_{k-k_0}\mathscr{M} \subset G_k\mathscr{M} \subset F_{k+k_0}\mathscr{M}.$$

***Exercise E.2.4* (Local existence of good filtrations).**

(1) Show that, if $\mathscr{M}$ has a good filtration, then it is $\mathscr{D}_X$-coherent and $\mathrm{gr}^F\mathscr{M}$ is $\mathrm{gr}^F\mathscr{D}_X$-coherent. In particular, a good $\mathscr{D}_X$-module is coherent.

(2) Conversely, show that any coherent $\mathscr{D}_X$-module admits locally a good filtration. [Hint: choose a local presentation $\mathscr{D}^q_{X|U} \xrightarrow{\varphi} \mathscr{D}^p_{X|U} \to \mathscr{M}_{|U} \to 0$, and show that the filtration induced on $\mathscr{M}_{|U}$ by $F_\bullet\mathscr{D}^p_{X|U}$ is good by using Exercise E.2.3: Set $\mathscr{K} = \operatorname{Im}\varphi$ and reduce the assertion to showing that $F_j\mathscr{D}_X \cap \mathscr{K}$ is $\mathscr{O}_X$-coherent; prove that, up to shrinking U, there exists $k_o \in \mathbb{N}$ such that $\varphi(F_k\mathscr{D}^q_{X|U}) \subset F_{k+k_o}\mathscr{D}^p_{X|U}$ for each k; deduce that $\varphi(F_k\mathscr{D}^q_{X|U})$, being locally of finite type and contained in a coherent $\mathscr{O}_X$-module, is $\mathscr{O}_X$-coherent for each k; conclude by using the fact that an increasing sequence of coherent $\mathscr{O}_X$-modules in a coherent $\mathscr{O}_X$-module is locally stationary.]

(3) Show that, locally, any coherent $\mathscr{D}_X$-module is generated over $\mathscr{D}_X$ by a coherent $\mathscr{O}_X$-submodule.

(4) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let $\mathscr{F}$ be a $\mathscr{O}_X$-submodule which is locally finitely generated. Show that $\mathscr{F}$ is $\mathscr{O}_X$-coherent. (Hint: choose a good filtration $F_\bullet\mathscr{M}$ and show that, locally, $\mathscr{F} \subset F_k\mathscr{M}$ for some k; apply then the analogue of Exercise E.2.2(1) for $\mathscr{O}_X$-modules.)

Exercise E.2.5.

(1) Show statements similar to those of Theorem 2.2.6 for $\operatorname{gr}^F\mathscr{D}_X$-modules, $\mathscr{O}_X(*D)$-modules, $\mathscr{D}_X(*D)$-modules and $i^*\mathscr{D}_X$-modules (see Exercise E.2.1).

(2) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Show that $\mathscr{D}_X(*D) \otimes_{\mathscr{D}_X} \mathscr{M}$ is $\mathscr{D}_X(*D)$-coherent and that $i^*\mathscr{M}$ is $i^*\mathscr{D}_X$-coherent.

Exercise E.2.6. Let t be a coordinate on $\mathbb{C}$ and set $\mathscr{M} = \mathscr{O}_{\mathbb{C}}(*0)/\mathscr{O}_{\mathbb{C}}$. Consider the two elements $m_1 = [1/t]$ and $m_2 = [1/t^2]$, where $[\bullet]$ denotes the class modulo $\mathscr{O}_{\mathbb{C}}$. Show that the good filtrations generated respectively by m_1 and m_2 do not give rise to the same annihilator ideals.

Exercise E.2.7. Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence of $\mathscr{D}_X$-modules. Show that $\operatorname{Char}\mathscr{M} = \operatorname{Char}\mathscr{M}' \cup \operatorname{Char}\mathscr{M}''$. (Hint: take a good filtration on $\mathscr{M}$ and induce it on $\mathscr{M}'$ and $\mathscr{M}''$.)

Exercise E.2.8. Show that both conditions in Definition 2.6.1 are indeed equivalent. (Hint: use the homogeneity property of $\operatorname{Char}\mathscr{M}$.)

Exercise E.2.9 **(Coherent $\mathscr{D}_X$-modules with characteristic variety T^*_XX).** Recall that a local section m of a left $\mathscr{D}_X$-module $\mathscr{M}$ is said to be horizontal if $\nabla m = 0$, i.e., in local coordinates, forall $i \in \mathbb{N}$, $(\partial/\partial x_i)m = 0$. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module such that $\operatorname{Char}\mathscr{M} = T^*_XX$. Show that

(1) for every $x \in X$, $\mathscr{M}_x$ is a $\mathscr{O}_{X,x}$-module of finite type;

(2) $\mathscr{M}_x$ is therefore free over $\mathscr{O}_{X,x}$;

(3) $\mathscr{M}_x$ has a $\mathscr{O}_{X,x}$-basis made of horizontal sections;

(4) $\mathscr{M}$ is locally isomorphic, *as a $\mathscr{D}_X$-module*, to $\mathscr{O}_X^d$ for some d.

***Exercise E.2.10* (Coherent $\mathscr{D}_X$-modules with characteristic variety contained in $T^*_Y X$).** Let $i : Y \hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathscr{O}_X = \varinjlim_k \mathscr{E}xt^p(\mathscr{O}_X/\mathscr{J}_Y^k, \mathscr{O}_X)$, where $\mathscr{J}_Y$ is the ideal defining Y. $R^p\Gamma_{[Y]}\mathscr{O}_X$ has a natural structure of $\mathscr{D}_X$-module. In local coordinates $(x_1, \dots, x_n)$ where Y is defined by $x_1 = \cdots = x_p = 0$, we have

$$R^p\Gamma_{[Y]}\mathscr{O}_X \simeq \frac{\mathscr{O}_{\mathbb{C}^n}[1/x_1 \cdots x_n]}{\sum_{i=1}^p \mathscr{O}_{\mathbb{C}^n}(x_i/x_1 \cdots x_n)}.$$

Denote this $\mathscr{D}_X$-module by $\mathscr{B}_{Y|X}$.

(1) Show that $\mathscr{B}_{Y|X}$ has support contained in Y and characteristic variety equal to $T^*_Y X$.

(2) Identify $\mathscr{B}_{Y|X}$ with $i_+\mathscr{O}_Y$.

(3) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety equal to $T^*_Y X$. Show that $\mathscr{M}$ is locally isomorphic to $(\mathscr{B}_{Y|X})^d$ for some d.

LECTURE 3

DIRECT IMAGES OF $\mathscr{D}_X$-MODULES

The notion of direct image of a $\mathscr{D}$-module answers the following problem: given a C^∞ differential form η of maximal degree on a complex manifold X, which satisfies a linear system of partial differential equations (recall that $\mathscr{D}_X$ acts on the right on the sheaf $\mathscr{E}_X^{n,n}$ of forms of maximal degree), what can be said of the form (or more generally the current) obtained by integrating η along the fibres of a holomorphic map $f : X \to Y$? Does it satisfy a finite (i.e., coherent) system of holomorphic differential equations on Y? How can one define intrinsically this system?

Such a question arises in many domains of algebraic geometry. The system of differential equations is often called the "Picard-Fuchs system", or the Gauss-Manin system. A way of "solving" a linear system of holomorphic or algebraic differential equations on a space Y consists in recognizing in this system the Gauss-Manin system attached to some holomorphic or algebraic function $f : X \to Y$. The geometric properties of f induce interesting properties of the system. Practically, this reduces to expressing solutions of the system as integrals over the fibers of f of some differential forms.

The definition of the direct image of a $\mathscr{D}$-module cannot be as simple as that of the direct image of a sheaf. One is faced to a problem which arises in differential geometry: the cotangent map of a holomorphic map $f : X \to Y$ is not a map from the cotangent space T^*Y of Y to that of X, but is a bundle map from the pull-back bundle f^*T^*Y to T^*X. In other words, a vector field on X does not act naturally as a derivation on functions on Y. The transfer module $\mathscr{D}_{X\to Y}$ will give a reasonable solution to this problem.

We have seen that the notion of a left $\mathscr{D}_X$-module is equivalent to that of a $\mathscr{O}_X$-module equipped with a flat connection. Correspondingly, there are two notions of direct images.

- The direct image of a $\mathscr{O}_X$-module with a flat connection is known as the *Gauss-Manin* connection attached to te original one. This notion is only cohomological. Although many examples were given some centuries ago (related to the differential equations satisfied by the periods of a family of elliptic curves), the systematic construction was only achieved in [**KO68**]. The construction with a filtration is due to Griffiths [**Gri70a, Gri70b**] (the main result is called *Griffiths' transversality theorem*). There is a strong constraint however: the map should be smooth (i.e., without critical points).

- The direct image of left $\mathscr{D}$-modules was constructed in [**SKK73**]. This construction has the advantage of being very functorial, and defined at the level of derived categories, not only at the cohomology level as is the first one. It is very flexible. The filtered analogue is straightforward. It appears as a basic tool in various questions in algebraic geometry.

3.1. Example of computation of a Gauss-Manin differential equation

Let $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial in n variables. Consider the following integral depending on a parameter t:

$$I(t) = \int_{\mathbb{T}^n} \frac{\omega}{f - t}, \qquad \omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

where $\mathbb{T}^n$ is the real torus $\{|x_i| = 1 \,\forall\, i\}$.

Proposition 3.1.1. *There exists a non-zero differential operator $q(t, \partial_t)$ with polynomial coefficients such that $q(t, \partial_t) I(t) = 0$.*

We will show how to compute algebraically this differential operator. Denote by $\Omega^k = \Omega^k((\mathbb{C}^*)^n)$ the space of differential forms of degree k with Laurent polynomials as coefficients and by $\Omega^k[\tau]$ the space of polynomials in the new variable τ with coefficients in Ω^k. The differential $d : \Omega^k \to \Omega^{k+1}$ gives rise to a twisted differential

$$d - \tau df \wedge : \Omega^k[\tau] \longrightarrow \Omega^{k+1}[\tau].$$

Lemma 3.1.2. *We have $(d - \tau df \wedge)^2 = 0$, hence $(\Omega^\bullet[\tau], d - \tau df \wedge)$ is a complex.*

Definition 3.1.3. The k-th Gauss-Manin system $\mathrm{GM}^k(f)$ is defined as the k-th cohomology $H^k\big(\Omega^\bullet[\tau], d - \tau df \wedge\big)$.

Lemma 3.1.4. *The following action:*

$$\begin{aligned} \partial_t \cdot \big(\textstyle\sum \eta_i \tau^i\big) &= \textstyle\sum \eta_i \tau^{i+1} \\ t \cdot \big(\textstyle\sum \eta_i \tau^i\big) &= \textstyle\sum (f\eta_i - (i+1)\eta_{i+1})\tau^i \end{aligned}$$

defines an action of the Weyl algebra $\mathbb{C}[t]\langle \partial_t \rangle := \mathscr{D}(\mathbb{A}^1_t)$ on $\mathrm{GM}^k(f)$ for each k. □

The Gauss-Manin systems $\mathrm{GM}^\bullet(f)$ are an algebraic version of the direct image $f_+\mathscr{O}_{(\mathbb{C}^*)^n}$ that we will consider later.

***Theorem 3.1.5* (Bernstein).** *Each non-zero element of $\mathrm{GM}^k(f)$ is annihilated by a non-zero element of $\mathbb{C}[t]\langle \partial_t \rangle$.* □

Proposition 3.1.6. *Let $p \in \mathbb{C}[t]\langle\partial_t\rangle$ be such that $p\cdot[\omega] = 0$ in $\mathrm{GM}^n(f)$. Then there exists $N \geqslant 0$ such that*

$$\partial_t^N \cdot p(t,\partial_t)\cdot I(t) = 0.$$

Proof. We write p as $p = \sum_{i=0}^d \partial_t^i a_i(t)$. The relation $p\cdot[\omega] = 0$ shows that there exists $k \geqslant 0$ and $\eta_0,\dots,\eta_{d+k} \in \Omega^{n-1}$ such that

$$\begin{aligned} 0 &= \qquad\quad - df\wedge\eta_{d+k}\\ 0 &= d\eta_{d+k} - df\wedge\eta_{d+k-1}\\ &\;\;\vdots\\ 0 &= d\eta_{d+1} - df\wedge\eta_d\\ a_d\circ f\cdot\omega &= d\eta_d - df\wedge\eta_{d-1}\\ &\;\;\vdots\\ a_0\circ f\cdot\omega &= d\eta_0. \end{aligned} \tag{3.1.7}$$

Claim. *We have for all j,ℓ:*

$$\int_{\mathbb{T}^n}\frac{d\eta_j}{(f-t)^\ell} = \ell\int_{\mathbb{T}^n}\frac{df\wedge\eta_j}{(f-t)^{\ell+1}}.$$

This follows from Stokes formula: $\displaystyle\int_{\mathbb{T}^n} d\big(\eta_j/(f-t)^\ell\big) = 0$. From (3.1.7) we get

$$\int_{\mathbb{T}^n}\frac{d\eta_d}{(f-t)} = \int_{\mathbb{T}^n}\frac{df\wedge\eta_d}{(f-t)^2} = \int_{\mathbb{T}^n}\frac{d\eta_{d+1}}{(f-t)^2} = \cdots = 0.$$

On the other hand, let us work modulo $\mathbb{C}[t]$, and use the sign $\equiv$ instead of $=$. For any polynomial $a(t)$ we thus have

$$a(t)\int_{\mathbb{T}^n}\frac{\omega}{(f-t)} \equiv \int_{\mathbb{T}^n}\frac{a\circ f\cdot\omega}{(f-t)} \mod \mathbb{C}[t],$$

since $[a(t)-a\circ f]/(f-t) \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}][t]$. Now,

$$\frac{d}{dt}a_d(t)\int_{\mathbb{T}^n}\frac{\omega}{f-t} \equiv \frac{d}{dt}\int_{\mathbb{T}^n}\frac{a_d\circ f\cdot\omega}{f-t} \equiv \frac{d}{dt}\int_{\mathbb{T}^n} -\frac{df\wedge\eta_d}{f-t} = \int_{\mathbb{T}^n} -\frac{df\wedge\eta_d}{(f-t)^2},$$

and by using Stokes formula,

$$\frac{d}{dt}a_d(t)\int_{\mathbb{T}^n}\frac{\omega}{f-t} \equiv \int_{\mathbb{T}^n} -\frac{d\eta_{d-1}}{f-t} \equiv -a_{d-1}(t)\int_{\mathbb{T}^n}\frac{\omega}{f-t} - \int_{\mathbb{T}^n}\frac{df\wedge\eta_{d-2}}{f-t}.$$

Iterating this reasoning (by applying d/dt once more, etc.) gives:

$$p(t,\partial_t)\int_{\mathbb{T}^n}\frac{\omega}{f-t} \equiv 0, \quad \text{i.e.,} \ \in \mathbb{C}[t].$$

Applying now a sufficiently high power of d/dt to kill the polynomial, we get

$$\partial_t^N p(t,\partial_t)\int_{\mathbb{T}^n}\frac{\omega}{f-t} = 0. \qquad \square$$

3.2. Transfer modules and basic direct image

Let us begin with some relative complements to §1.2. Let $f : X \to Y$ be a holomorphic map between analytic manifolds. For any section ξ of the sheaf Θ_X of vector fields on X, $Tf(\xi)$ is a local section of $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$. We hence have a $\mathscr{O}_X$-linear map (tangent map to f)

$$Tf : \Theta_X \longrightarrow \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y.$$

If $\mathscr{M}_X$ is a left $\mathscr{D}_X$-module and $\mathscr{M}_Y$ is a left $\mathscr{D}_Y$-module, then $\mathscr{M}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}_Y$ may be equipped with a left $\mathscr{D}_X$-module structure: if ξ is a local vector field on X, we set

$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + m \otimes Tf(\xi)n,$$

and the assertion follows from Lemma 1.2.1.

Definition 3.2.1 **(Transfer modules, see, e.g. [Cas93] for details).** The sheaf $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ is a left-right $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule when using the natural right $f^{-1}\mathscr{D}_Y$-module structure and the left $\mathscr{D}_X$-module introduced above.

Example 3.2.2. Let $f : X \to \mathbb{C}$ be a holomorphic map and let t be the coordinate on $\mathbb{C}$. Then $\mathscr{D}_{X\to\mathbb{C}} = \mathscr{O}_X[\partial_t]$ with a suitable $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule structure.

More generally, notice that, $\mathscr{D}_Y$ being a locally free $\mathscr{O}_Y$-module, $\mathscr{D}_{X\to Y}$ is a locally free $\mathscr{O}_X$-module. Choose local coordinates $x_1, \dots, x_n$ on X and $y_1, \dots, y_m$ on Y. Then $\mathscr{D}_{X\to Y} = \mathscr{O}_X[\partial_{y_1}, \dots, \partial_{y_m}]$. The left $\mathscr{D}_X$-module structure is given by

$$\partial_{x_i} \cdot \sum_\alpha a_\alpha(x) \otimes \partial_y^\alpha = \sum_\alpha \frac{\partial a_\alpha}{\partial x_i} \otimes \partial_y^\alpha + \sum_\alpha \sum_{j=1}^m a_\alpha(x) \frac{\partial f_j}{\partial x_i} \otimes \partial_y^{\alpha+1_j}.$$

Let $\mathscr{M}$ be a (left or right) $\mathscr{D}_X$-module. In general, $f_*\mathscr{M}$ is not naturally equipped with a (left or right) $\mathscr{D}_Y$-module structure. If $\mathscr{N}$ is a right $\mathscr{D}_X$-module, its "basic" direct image by f is defined as

$$f_*(\mathscr{N} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}).$$

This is a right $\mathscr{D}_Y$-module. This definition has to be extended cohomologically in two ways:

(1) In general, $\mathscr{D}_{X\to Y}$ is not locally free as a left $\mathscr{D}_X$-module (it is only locally free as a $\mathscr{O}_X$-module), and some Tor's may be interesting.

(2) The higher direct images $R^k f_*$ should be considered.

3.3. Direct images of $\mathscr{D}$-modules

Recall that the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$, which was defined in 1.4.2, is a complex of left $\mathscr{D}_X$-modules. Denote by $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ the complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$ (the left $\mathscr{O}_X$-structure on each factor is used for the tensor product). It is a complex of $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodules: the right $f^{-1}\mathscr{D}_Y$ structure is the natural one; the left $\mathscr{D}_X$-structure is that defined by Exercise E.1.10(1).

Lemma 3.3.1. *The complex $\mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule by locally free left $\mathscr{D}_X$-modules.*

Proof. Exercise E.3.4. □

Examples 3.3.2.

(1) For $f = \mathrm{Id} : X \to X$, the complex $\mathrm{Sp}^{\bullet}_{X\to X}(\mathscr{D}_X) = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to X} = \mathscr{D}_X$ as a left and right $\mathscr{D}_X$-module (notice that the left structure of $\mathscr{D}_X$ is used for the tensor product).

(2) For $f : X \to \mathrm{pt}$, the complex $\mathrm{Sp}^{\bullet}_{X\to \mathrm{pt}}(\mathscr{D}_X) = \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to\mathrm{pt}} = \mathscr{O}_X$.

Definition 3.3.3 **(Direct images of right $\mathscr{D}$-modules).**

(1) The *direct image* $f_+\mathscr{N}$ is is defined as

$$f_+\mathscr{N} = \boldsymbol{R}f_*\Big(\mathscr{N} \otimes_{\mathscr{D}_X} \mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)\Big).$$

It is a complex of right $\mathscr{D}_Y$-modules.

(2) The *direct image with proper support* $f_\dagger$ is defined as

$$f_\dagger\mathscr{N} = \boldsymbol{R}f_!\Big(\mathscr{N} \otimes_{\mathscr{D}_X} \mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)\Big).$$

It is a complex of right $\mathscr{D}_Y$-modules.

Remarks 3.3.4.

(1) If $\mathscr{M}$ is a left $\mathscr{D}_X$-module, one *defines* $f_\dagger\mathscr{M}$ as $\mathscr{H}om_{\mathscr{O}_X}(\omega_X, f_\dagger(\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}))$.

(2) If f is proper, or proper on the support of $\mathscr{M}$, we have an isomorphism in the derived category $\mathsf{D}^+(\mathscr{D}_Y)$:

$$f_\dagger\mathscr{M} \xrightarrow{\sim} f_+\mathscr{M}.$$

(3) One may replace $\mathscr{N}$ with a complex of right $\mathscr{D}_X$-modules which is bounded from below. Then $\mathscr{N} \otimes_{\mathscr{D}_X} \mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)$ is first replaced with the associated single complex. Up to this modification, one defines similarly $f_+, f_\dagger$. These can be extended as functors from the derived category $\mathsf{D}^+(\mathscr{D}_X)$ to the derived category $\mathsf{D}^+(\mathscr{D}_Y)$.

(4) If $\mathscr{F}$ is any sheaf on X, we have $R^j f_*\mathscr{F} = 0$ and $R^j f_!\mathscr{F} = 0$ for $j \notin [0, 2\dim X]$. Therefore, taking into account the length $\dim X$ of the relative Spencer complex, we find that $\mathscr{H}^j f_+\mathscr{N}$ and $\mathscr{H}^j f_\dagger\mathscr{N}$ are zero for $j \notin [-\dim X, 2\dim X]$: we say that $f_+\mathscr{N}, f_\dagger\mathscr{N}$ have *bounded amplitude*. Similarly, if $\mathscr{N}^\bullet$ has bounded amplitude, then so has $f_\dagger\mathscr{N}^\bullet$.

Proposition 3.3.5 **(Direct image of left $\mathscr{D}_X$-modules).** *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. As $\mathscr{D}_{X\to Y}$ is a left $\mathscr{D}_X$-module, $\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y}$ has a natural structure of left $\mathscr{D}_X$-module (see Exercise* E.1.10(2)*) and of course a compatible structure of right $f^{-1}\mathscr{D}_Y$-module.*

(1) *The de Rham complex $\Omega_X^{n+\bullet}(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})$ is isomorphic, as a complex of right $f^{-1}\mathscr{D}_Y$-modules, to $\mathscr{M}^{\mathrm{r}}\otimes_{\mathscr{D}_X}\mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)$, $\mathscr{M}^{\mathrm{r}}:=\omega_X\otimes_{\mathscr{O}_X}\mathscr{M}$.*

(2) *$f_+\mathscr{M}$ is the complex of left $\mathscr{D}_Y$-modules associated to the double complex $\boldsymbol{R}f_*\Omega_X^{n+\bullet}(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})$ of right $\mathscr{D}_Y$-modules.*

Proof. Exercise. □

Theorem 3.3.6.

(1) *Let $f:X\to Y$ and $g:Y\to Z$ be two maps. There is a functorial canonical isomorphism of functors $(g\circ f)_\dagger=g_\dagger f_\dagger$. If f is proper, we also have $(g\circ f)_+=g_+f_+$.*

(2) *If f is a closed embedding, then $f_\dagger\mathscr{N}=f_+\mathscr{N}=f_*(\mathscr{N}\otimes_{\mathscr{D}_X}\mathscr{D}_{X\to Y})$.*

(3) *If $f:X=Y\times Z\to Y$ is the projection, we have*

$$f_+\mathscr{N}=\boldsymbol{R}f_*\big(\mathscr{N}\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}\big)\quad\textit{and}\quad f_\dagger\mathscr{N}=\boldsymbol{R}f_!\big(\mathscr{N}\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}\big).$$

This theorem reduces the computation of the direct image by any morphism $f:X\to Y$ by decomposing it as $f=p\circ i_f$, where $i_f:X\hookrightarrow X\times Y$ denotes the graph inclusion $x\mapsto(x,f(x))$. As i_f is an embedding, it is proper, so we have $f_+=p_+i_{f+}$.

3.4. Coherence of direct images

Let $f:X\to Y$ be a holomorphic map and $\mathscr{M}$ be a $\mathscr{D}_X$-module. We say that $\mathscr{M}$ is *f-good* if there exists a covering of Y by open sets V_j such that $\mathscr{M}$ is good on each $f^{-1}(V_j)$. As we indicated in Remark 2.2.3, any holonomic $\mathscr{D}_X$-module is good with respect to any holomorphic map.

Theorem 3.4.1. *Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module. Assume that f is proper on the support of $\mathscr{M}$. Then $f_\dagger\mathscr{M}=f_+\mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.*

This theorem is an application of Grauert's coherence theorem for $\mathscr{O}_X$-modules, and this is why we restrict to f-good $\mathscr{D}_X$-modules. In general, it is not known whether the theorem holds for any coherent $\mathscr{D}_X$-module or not. Notice, however, that one may relax the geometric condition on $f_{|\operatorname{Supp}\mathscr{M}}$ (properness) by using more specific properties of $\mathscr{D}$-modules: as we have seen, the characteristic variety is a finer geometrical object attached to the $\mathscr{D}$-module, and one should expect that the right condition on f has to be related with the characteristic variety. The most general statement in this direction is the coherence theorem for *elliptic pairs*, due to P. Schapira and J.-P. Schneiders [**SS94**]. For instance, if X is an open set of X' and f is the restriction of $f':X'\to Y$, and if the boundary of X is f-non-characteristic with respect to $\mathscr{M}$ (a relative variant of Definition 2.6.1) then the direct image of $\mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.

Proof of Theorem 3.4.1. As the coherence property is a local property on Y, the statement one proves is, more precisely, that the direct image of a good $\mathscr{D}_X$-module $\mathscr{M}$ is a good $\mathscr{D}_Y$-module when f is proper on $\operatorname{Supp}\mathscr{M}$. By an extension argument, it is even enough to assume that $\mathscr{M}$ has a good filtration and show that, locally on Y, the cohomology modules of $f_\dagger\mathscr{M}$ have a good filtration.

First step: induced $\mathscr{D}$-modules. Assume that $\mathscr{M} = \mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ and $\mathscr{L}$ is $\mathscr{O}_X$-coherent. By Exercise E.3.5, it is enough to prove that the cohomology of $\boldsymbol{R}f_!\mathscr{L}$ is $\mathscr{O}_Y$-coherent when f is proper on $\operatorname{Supp}\mathscr{L}$: this is Grauert's Theorem.

Second step: finite complexes of induced $\mathscr{D}$-modules. Let $\mathscr{L}_\bullet\otimes_{\mathscr{O}_X}\mathscr{D}_X$ be a *finite* complex of induced $\mathscr{D}_X$-modules. Recall that its direct image complex was defined in Remark 3.3.4(3). Assume that f restricted to the support of each term is proper. Using Exercise E.2.2(2) and Artin-Rees (Corollary 2.2.8), one shows by induction on the length of the complex that the cohomology modules of $f_\dagger(\mathscr{L}_\bullet\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ have a good filtration (hence are $\mathscr{D}_Y$-coherent).

Third step: general case. Fix a compact set K of Y. We will show that the cohomology modules of $f_\dagger\mathscr{M}$ have a good filtration in a neighbourhood of K. Fix a good filtration $F_\bullet\mathscr{M}$ of $\mathscr{M}$. As $f^{-1}(K)\cap\operatorname{Supp}\mathscr{M}$ is compact, there exists k such that $\mathscr{L}^0 := F_k\mathscr{M}$ generates $\mathscr{M}$ as a $\mathscr{D}_X$-module in some neighbourhood of $f^{-1}(K)$. Hence $\mathscr{L}^0$ is a coherent $\mathscr{O}_X$-module with support contained in $\operatorname{Supp}\mathscr{M}$ and we have a surjective morphism $\mathscr{L}^0\otimes_{\mathscr{O}_X}\mathscr{D}_X\to\mathscr{M}$ in some neighbourhood of $f^{-1}(K)$ that we still call X. The kernel of this morphism is therefore $\mathscr{D}_X$-coherent, has support contained in $\operatorname{Supp}\mathscr{M}$ and, by Artin-Rees (Corollary 2.2.8), has a good filtration.

The process may therefore be continued and leads to the existence, in some neighbourhood of K, of a (maybe infinite) resolution $\mathscr{L}^{-\bullet}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ by coherent induced $\mathscr{D}_X$-modules with support contained in $\operatorname{Supp}\mathscr{M}$.

Fix some ℓ and stop the resolution at the ℓ-th step. Denote by $\mathscr{N}^{-\bullet}$ this bounded complex and by $\mathscr{M}'$ the kernel of $\mathscr{N}^{-\ell}\to\mathscr{N}^{-\ell+1}$. We have an exact sequence of complexes

$$0\longrightarrow\mathscr{M}'[\ell]\longrightarrow\mathscr{N}^{-\bullet}\longrightarrow\mathscr{M}\longrightarrow 0,$$

where $\mathscr{M}$ is considered as a complex with only one term in degree 0, and $\mathscr{M}'[\ell]$ a complex with only one term in degree $-\ell$. This sequence induces a long exact sequence

$$\cdots\longrightarrow\mathscr{H}^{j+\ell}(f_\dagger\mathscr{M}')\longrightarrow\mathscr{H}^j(f_\dagger\mathscr{N}^{-\bullet})\longrightarrow\mathscr{H}^j(f_\dagger\mathscr{M})\longrightarrow\mathscr{H}^{j+\ell+1}(f_\dagger\mathscr{M}')\longrightarrow\cdots$$

Recall (see Remark 3.3.4(4)) that $\mathscr{H}^j(f_\dagger\mathscr{M})=0$ for $j\notin[-\dim X, 2\dim X]$. Choose then ℓ big enough so that, for any $j\in[-\dim X, 2\dim X]$, both numbers $j+\ell$ and $j+\ell+1$ do not belong to $[-\dim X, 2\dim X]$. With such a choice, we have $\mathscr{H}^j(f_\dagger\mathscr{M})\simeq$

$\mathscr{H}^j(f_\dagger \mathscr{N}^{-\bullet})$ for $j \in [-\dim X, 2\dim X]$ and $\mathscr{H}^j(f_\dagger \mathscr{M}) = 0$ otherwise. By the second step, $\mathscr{H}^j(f_\dagger \mathscr{M})$ has a good filtration in some neighbourhood of K. □

3.5. Kashiwara's estimate for the behaviour of the characteristic variety

Let $f: X \to Y$ be a holomorphic map and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety $\operatorname{Char} \mathscr{M}$. Assume that the cohomology modules $\mathscr{H}^j(f_\dagger \mathscr{M})$ are $\mathscr{D}_Y$-coherent (for instance, assume that all conditions in Theorem 3.4.1 are fulfilled). Is it possible to give an upper bound of the characteristic variety of each $\mathscr{H}^j(f_\dagger \mathscr{M})$ in terms of that of $\mathscr{M}$? There is such an estimate which is known as *Kashiwara's estimate*.

The most natural approach to this question is to introduce the sheaf of microdifferential operators and to show that the characteristic variety is nothing but the support of the microlocalized module associated with $\mathscr{M}$ (see [**Kas76**], see also [**Bjö79, Mal93, Bjö93, Kas03**]). The behaviour of the support of a microdifferential module with respect to direct images can then be obtained (see [**Kas76**] for such a proof, see [**SS94**] for a very general result and [**Lau85**] for an algebraic approach).

Nevertheless, we will not introduce here microdifferential operators (see however [**Sch85**] for a good introduction to the subject). Therefore, we will give a direct proof of Kashiwara's estimate, following [**Mal85**].

This estimate may be understood as a weak version of a general Riemann-Roch theorem for $\mathscr{D}_X$-modules (see, e.g. [**Sab97**] and the references given therein).

Let $f: X \to Y$ be a holomorphic map. We will consider the following associated cotangent diagram:

$$T^*X \xleftarrow{T^*f} f^*T^*Y \xrightarrow{\widetilde{f}} T^*Y.$$

***Theorem 3.5.1* (Kashiwara's estimate for the characteristic variety).** *Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module such that f is proper on* $\operatorname{Supp} \mathscr{M}$. *Then, for any $j \in \mathbb{Z}$, we have*

$$\operatorname{Char} \mathscr{H}^j(f_\dagger \mathscr{M}) \subset \widetilde{f}\big((T^*f)^{-1}(\operatorname{Char} \mathscr{M})\big).$$

Sketch of proof. As in the proof of Theorem 3.4.1, we first reduce to the case where $\mathscr{M}$ has a good filtration $F_\bullet \mathscr{M}$.

Notice first that it is possible to define a functor $f_\dagger$ for $\operatorname{gr}^F \mathscr{D}_X$-modules, by the formula $f_\dagger(\bullet) = \boldsymbol{R} f_!(\boldsymbol{L}(T^*f)^*(\bullet))$. Moreover, the inverse image $(T^*f)^*$ is nothing but the tensor product $\otimes_{f^{-1}\mathscr{O}_Y} \operatorname{gr}^F \mathscr{D}_Y$. We therefore clearly have the inclusion

$$\operatorname{Supp} \mathscr{H}^j f_\dagger \operatorname{gr}^F \mathscr{M} \subset \widetilde{f}\big((T^*f)^{-1}(\operatorname{Supp} \operatorname{gr}^F \mathscr{M})\big) = \widetilde{f}\big((T^*f)^{-1}(\operatorname{Char} \mathscr{M})\big).$$

The problem consists now in understanding the difference between $f_\dagger \operatorname{gr}^F$ and $\operatorname{gr}^F f_\dagger$. In order to analyse this difference, we will put $\mathscr{M}$ and $\operatorname{gr}^F \mathscr{M}$ in a one parameter family, i.e., we will consider the associated Rees module. Let z be a new variable. The Rees sheaf of rings $R_F\mathscr{D}_X$ is defined as the subsheaf $\bigoplus_p F_p\mathscr{D}_X z^p$ of $\mathscr{D}_X \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. Similarly, a filtered $\mathscr{D}_X$-module $(\mathscr{M}, F_\bullet\mathscr{M})$ gives rise to a module $R_F\mathscr{M} := \bigoplus_p F_p\mathscr{M} z^p$ over $R_F\mathscr{D}_X$. The following to relations are important:

$$R_F\mathscr{M}/(z-1)R_F\mathscr{M} = \mathscr{M}, \quad R_F\mathscr{M}/zR_F\mathscr{M} = \operatorname{gr}^F\mathscr{M}.$$

On the other hand, any graded $R_F\mathscr{D}_X$-module *without* $\mathbb{C}[z]$*-torsion* gives rise to a filtered $\mathscr{D}_X$-module.

The sheaf $R_F\mathscr{D}_X$ shares coherence properties with $\mathscr{D}_X$. Moreover, the equivalent definitions of a good filtration in Exercise E.2.3 are also equivalent to: *$R_F\mathscr{M}$ is $R_F\mathscr{D}_X$-coherent.*

Lastly, one can define direct images of $R_F\mathscr{D}_X$-modules, still denoted by $f_\dagger, f_+$, and show the R_F analogue of Theorem 3.4.1. Therefore, $f_\dagger R_F\mathscr{M}$ has $R_F\mathscr{D}_Y$-coherent cohomology. One has to be careful that the cohomology of $f_\dagger R_F\mathscr{M}$ may have $\mathbb{C}[z]$-torsion, hence does not take the form R_F of something. Nevertheless, as $\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ is $R_F\mathscr{D}_Y$-coherent,

- the kernel sequence $\operatorname{Ker}\big[z^\ell : \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \to \mathscr{H}^j(f_\dagger R_F\mathscr{M})\big]$ is locally stationary,
- the quotient of $\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ by its z-torsion (i.e., locally by $\operatorname{Ker} z^\ell$ for ℓ big enough) is $R_F\mathscr{D}_Y$-coherent, hence is the Rees module associated with some good filtration $F_\bullet$ on $\mathscr{H}^j(f_\dagger\mathscr{M})$.

Consider the exact sequence

$$\cdots \longrightarrow \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \xrightarrow{z^\ell} \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \longrightarrow \mathscr{H}^j\big(f_\dagger(R_F\mathscr{M}/z^\ell R_F\mathscr{M})\big) \longrightarrow \cdots$$

Then,

- $\mathscr{H}^j(f_\dagger R_F\mathscr{M})/z^\ell\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ is a submodule of $\mathscr{H}^j\big(f_\dagger(R_F\mathscr{M}/z^\ell R_F\mathscr{M})\big)$
- and, on the other hand, if ℓ is big enough, $R_F\mathscr{H}^j(f_\dagger\mathscr{M})/z^\ell R_F\mathscr{H}^j(f_\dagger\mathscr{M})$ is a quotient of $\mathscr{H}^j(f_\dagger R_F\mathscr{M})/z^\ell\mathscr{H}^j(f_\dagger R_F\mathscr{M})$.

For $\ell \geqslant 1$, let us denote by $\operatorname{gr}^F_{[\ell]}$ the grading with step ℓ, namely $\bigoplus_k F_k/F_{k-\ell}$, and let us define f_+ for $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-modules in a way similar to what is done for $\mathscr{D}_X$-modules. The conclusion is that $\operatorname{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$ is a $\operatorname{gr}^F_{[\ell]}\mathscr{D}_Y$-submodule of $\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$.

The sheaf of rings $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$ is filtered by the finite filtration $G_j\operatorname{gr}^F_{[\ell]}\mathscr{D}_X = \bigoplus_k F_{k+j-\ell}\mathscr{D}_X/F_{k-\ell}\mathscr{D}_X$, and there is the notion of a G-filtration of a $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-module (these filtrations should be finite). Moreover, $\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{D}_X \simeq \operatorname{gr}^F\mathscr{D}_X[u]/u^\ell$ by suitably defining the grading on the left-hand term. Given a coherent $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-module, the graded module with respect to any G-filtration is $\operatorname{gr}^F\mathscr{D}_X[u]/u^\ell$-coherent, hence $\operatorname{gr}^F\mathscr{D}_X$-coherent, and its support as such does not depend on the choice of such a filtration (same proof as that for the characteristic variety, in a simpler way).

Since the filtration $G_\bullet$ is finite, there is a finite spectral sequence having E_2 term equal to $\mathscr{H}^j(f_\dagger\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{M}) = \mathscr{H}^j(f_\dagger\operatorname{gr}^F\mathscr{M}[u]/u^\ell) \simeq \mathscr{H}^j(f_\dagger\operatorname{gr}^F\mathscr{M})^\ell$ abutting to $\mathrm{gr}^G\mathscr{H}^j(f_\dagger \mathrm{gr}^F_{[\ell]}\mathscr{M})$ for a suitable G-filtration on $\mathscr{H}^j(f_\dagger \mathrm{gr}^F_{[\ell]}\mathscr{M})$. It follows that the support of $\mathrm{gr}^G\mathscr{H}^j(f_\dagger \mathrm{gr}^F_{[\ell]}\mathscr{M})$ is contained in $\widetilde{f}\big((T^*f)^{-1}(\mathrm{Char}\,\mathscr{M})\big)$.

The filtration $G_\bullet\mathscr{H}^j(f_\dagger \mathrm{gr}^F_{[\ell]}\mathscr{M})$ induces in a natural way a G-filtration on any submodule and any quotient of it, and therefore on $\mathrm{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$. The support of $\mathrm{gr}^G\mathrm{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$ as a $\mathrm{gr}^F\mathscr{D}_X$-module is therefore included in that of $\mathrm{gr}^G\mathscr{H}^j(f_\dagger \mathrm{gr}^F_{[\ell]}\mathscr{M})$, hence in $\widetilde{f}\big((T^*f)^{-1}(\mathrm{Char}\,\mathscr{M})\big)$. Now, as already remarked, as $\mathrm{gr}^F\mathscr{D}_X$-modules we have $\mathrm{gr}^G\mathrm{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})\simeq\big(\mathrm{gr}^F\mathscr{H}^j(f_\dagger\mathscr{M})\big)^\ell$, which has the same support as $\mathrm{gr}^F\mathscr{H}^j(f_\dagger\mathscr{M})$, that is, $\mathrm{Char}\,\mathscr{H}^j(f_\dagger\mathscr{M})$. □

Exercises and complements

Exercise E.3.1 **($\mathscr{D}_{X\to Y}$ for a closed embedding).** Assume that X is a complex submanifold of Y of codimension d, defined by $g_1=\cdots=g_d=0$, where the g_i are holomorphic functions on Y. Show that

$$\mathscr{D}_{X\to Y}=\mathscr{D}_Y\big/\textstyle\sum_{i=1}^d g_i\mathscr{D}_Y$$

with its natural right $\mathscr{D}_Y$ structure. In local coordinates $(x_1,\dots,x_n,y_1,\dots,y_d)$ such that $g_i=y_i$, show that $\mathscr{D}_{X\to Y}=\mathscr{D}_X[\partial_{y_1},\dots,\partial_{y_d}]$.

Conclude that, if f is an embedding, the sheaves $\mathscr{D}_{X\to Y}$ and $\mathscr{D}_{Y\leftarrow X}$ are locally free over $\mathscr{D}_X$.

Exercise E.3.2 **(Filtration of $\mathscr{D}_{X\to Y}$).** Put $F_k\mathscr{D}_{X\to Y}=\mathscr{O}_X\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}F_k\mathscr{D}_Y$. Show that this defines a filtration (see Definition 1.5.1) of $\mathscr{D}_{X\to Y}$ as a left $\mathscr{D}_X$-module and as a right $f^{-1}\mathscr{D}_Y$-module, and that $\mathrm{gr}^F\mathscr{D}_{X\to Y}=\mathscr{O}_X\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathrm{gr}^F\mathscr{D}_Y$.

Exercise E.3.3 **(The chain rule).** Consider holomorphic maps $f:X\to Y$ and $g:Y\to Z$.

(1) Give an canonical isomorphism $\mathscr{D}_{X\to Y}\otimes_{f^{-1}\mathscr{D}_Y}f^{-1}\mathscr{D}_{Y\to Z}\xrightarrow{\sim}\mathscr{D}_{X\to Z}$ as right $(g\circ f)^{-1}\mathscr{D}_Z$-modules.

(2) Use the chain rule to show that this isomorphism is left $\mathscr{D}_X$-linear.

(3) Same question with filtrations $F_\bullet$.

Exercise E.3.4 **(The (filtered) relative Spencer complex).**

(1) Show that $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule.

(2) Show that the terms of the complex $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ are locally free left $\mathscr{D}_X$-modules. (Hint: use Exercise E.1.12(4).)

(3) Define the filtration of $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ by the formula

$$F_\ell\,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)=\sum_{j+k=\ell}F_j\,\mathrm{Sp}^\bullet_X(\mathscr{D}_X)\otimes_{f^{-1}\mathscr{O}_Y}F_k\mathscr{D}_{X\to Y},$$

where the filtration on the Spencer complex is defined in Exercise E.1.20. Show that, for any ℓ, $F_\ell\,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $F_\ell\mathscr{D}_{X\to Y}$.

Exercise E.3.5 **(Direct image of induced $\mathscr{D}$-modules, see [Sai89,** Lemma 3.2]**).** Let $\mathscr{L}$ be a $\mathscr{O}_X$-module and let $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ be the associated induced right $\mathscr{D}_X$-module. Let $f: X\to Y$ be a holomorphic map. Show that $f_\dagger(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ is quasi-isomorphic to $\boldsymbol{R}f_!\mathscr{L}\otimes_{\mathscr{O}_Y}\mathscr{D}_Y$. [Hint: use that $\mathscr{L}\otimes_{\mathscr{O}_X}\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\to\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y}$ is a quasi-isomorphism as $\mathscr{D}_X$ is $\mathscr{O}_X$-locally free, and use the projection formula.]

Exercise E.3.6. Explain more precisely the estimate of Theorem 3.5.1 when f is the inclusion of a closed submanifold.

Problem: Direct image of currents

Let φ be a C^∞ form of maximal degree on X. If $f: X\to Y$ is a proper holomorphic map which is *smooth*, then the integral of φ in the fibres of f is a C^∞ form of maximal degree on Y, that one denotes by $\int_f\varphi$.

If f is not smooth, then $\int_f\varphi$ is only defined as a current of maximal degree on Y, and the definition extends to the case where φ is itself a current of maximal degree on X (see § 1.3.e for the notion of current).

Question Q.3.1. Extend the notion and properties of direct image of a right (resp. left) $\mathscr{D}_{X,\overline{X}}$-module, by introducing the transfer module $\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}=\mathscr{D}_{X\to Y}\otimes_{\mathbb{C}}\mathscr{D}_{\overline{X}\to\overline{Y}}$. One denotes these direct images by f_{++} or $f_{\dagger\dagger}$. Using the canonical section 1 on $\mathscr{O}_X$, $\mathscr{O}_{\overline{X}}$, $\mathscr{D}_Y$, $\mathscr{D}_{\overline{Y}}$, show that $\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}$ has a canonical section $\mathbf{1}$.

Definition **(Integration of currents of maximal degree).** Let $f: X\to Y$ be a proper holomorphic map and let T be a current of maximal degree on X. The current $\int_f T$ of maximal degree on Y is defined by

$$\Big\langle\int_f T,\varphi\Big\rangle=\langle T,\varphi\circ f\rangle.$$

We continue to assume that f is proper. We will now show how the integration of currents is used to defined a natural $\mathscr{D}_{Y,\overline{Y}}$ morphism $\mathscr{H}^0 f_{++}\,\mathfrak{C}_X\to\mathfrak{C}_Y$. Let us first treat as an exercise the case of a closed embedding.

Question Q.3.2. Assume that X is a closed submanifold of Y and denote by $f: X\hookrightarrow Y$ the embedding (which is a proper map). Denote by $\mathbf{1}$ the canonical section of $\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}$. Show that the natural map

$$\mathscr{H}^0 f_{++}\,\mathfrak{C}_X=f_*\big(\mathfrak{C}_X\otimes_{\mathscr{D}_{X,\overline{X}}}\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}\big)\longrightarrow\mathfrak{C}_Y,\quad T\otimes\mathbf{1}\longmapsto\int_f T$$

induces an isomorphism of the right $\mathscr{D}_{Y,\overline{Y}}$-module $\mathscr{H}^0 f_{++}\,\mathfrak{C}_X$ with the submodule of $\mathfrak{C}_Y$ consisting of currents supported on X. (Hint: use a local computation).

By going from right to left, identify $\mathscr{H}^0 f_{++}\,\mathfrak{Db}_X$ with the sheaf of distributions on Y supported on X.

Denote by $\mathfrak{Db}_X^{n-p,n-q}$ or $\mathfrak{Db}_{X,p,q}$ the sheaf of currents of degree p,q, which are linear forms on C^∞ differential forms of degree p,q. The integration of currents is a morphism

$$\int_f : \mathfrak{Db}_X^{n-p,n-q} \longrightarrow \mathfrak{Db}_Y^{m-p,m-q},$$

if $m = \dim Y$ and $n = \dim X$, which is compatible with the d' or d'' differential of currents on X and Y.

Question Q.3.3.

(1) Show that the complex $f_{++}\,\mathfrak{C}_X$ is quasi-isomorphic to the single complex associated to the double complex $f_*(\mathfrak{Db}_X^{n-\bullet,n-\bullet} \otimes_{\mathscr{O}_{X,\overline{X}}} \mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}})$. (Hint: use Exercise 3.3.5(1)).

(2) Show that the integration of currents $\int_f$ induces a $\mathscr{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$\int_f : f_{++}\,\mathfrak{C}_X \longrightarrow \mathfrak{C}_Y \otimes_{\mathscr{D}_{Y,\overline{Y}}} \mathrm{Sp}^{\bullet,\bullet}_{Y\to Y,\overline{Y}\to\overline{Y}} \simeq \mathfrak{C}_Y\,.$$

LECTURE 4

HOLONOMIC $\mathscr{D}_X$-MODULES

4.1. Motivation: division of distributions

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-zero polynomial. In general the function $1/f$ is not locally integrable, hence does not define a distribution on $\mathbb{C}^n$.

Question. *Does there exist a distribution T on $\mathbb{C}^n$ (or, better, a temperate distribution) such that $f \cdot T = 1$? (More generally, given any distribution (resp. temperate distribution) S on $\mathbb{C}^n$, does there exists a distribution (resp. temperate distribution) T such that $fT = S$.)*

The solution given by J. Bernstein [**Ber72**] proceeds along the following steps.

(1) For $s \in \mathbb{C}$ such that $\operatorname{Re} s \geqslant 0$, the function $|f|^{2s}$ is continuous, hence defines a distribution T_s on $\mathbb{C}^n$: for each test (n,n)-form $\varphi \in C_c^\infty(\mathbb{C}^n) d\boldsymbol{x} \wedge d\overline{\boldsymbol{x}}$, set

$$T_s(\varphi) = \int |f|^{2s} \varphi.$$

One reduces the question to prove that, for each φ, the holomorphic function $s \mapsto T_s(\varphi)$ on the half plane $\operatorname{Re} s > 0$ extends as a meromorphic function on $\mathbb{C}$. One also shows that, denoting by $S(\varphi)$ the constant term in the Laurent expansion of $T_s(\varphi)$ at $s = -1$, the correspondence $\varphi \mapsto S(\varphi)$ defines a distribution (i.e., a continuous linear form on test (n,n)-forms). Lastly, $|f|^2 S(\varphi) = S(|f|^2 \varphi)$ is seen to be equal to the constant term of the Laurent expansion of $T_s(\varphi)$ at $s = 0$. This is nothing but $\int \varphi$. In other words, $|f|^2 S = 1$, hence $T := \overline{f} S$ is a solution to $fT = 1$.

(2) In order to obtain the meromorphic extension of $s \mapsto T_s(\varphi)$, one looks for a pair of differential operators $P \in \mathbb{C}[s][\boldsymbol{x}]\langle \partial_{\boldsymbol{x}} \rangle$ and $Q \in \mathbb{C}[s][\overline{\boldsymbol{x}}]\langle \partial_{\overline{\boldsymbol{x}}} \rangle$ and polynomials $b'(s), b''(s) \in \mathbb{C}[s]$ such that

$$b'(s)|f|^{2s} = P \cdot |f|^{2s} f,$$
$$b''(s)|f|^{2s} = Q \cdot |f|^{2s} \overline{f}.$$

Assume P, Q, b', b'' are found. Then, for $\operatorname{Re} s > 0$,

$$b'(s) \int |f|^{2s}\varphi = \int (P|f|^{2s}f)\varphi = \int |f|^{2s} f P^*\varphi,$$

where P^* denotes the adjoint differential operator (the previous formula corresponds to an iteration of integrations by parts). Therefore,

$$b''(s)b'(s) \int |f|^{2s}\varphi = b''(s) \int |f|^{2s} f P^*\varphi = \int (Q|f|^{2s}\overline{f}) f P^*\varphi = \int |f|^{2(s+1)}\psi,$$

where $\psi = Q^*P^*\varphi$. The right-hand term is holomorphic on $\operatorname{Re} s > -1$, and thus the expression

$$\frac{1}{b'(s)b''(s)} \int |f|^{2(s+1)}\psi$$

is a meromorphic function on $\operatorname{Re} s > -1$ which coincides with $\int |f|^{2s}\varphi$ on $\operatorname{Re} s > 0$. Iterating this process gives the desired meromorphic extension.

(3) It remains to find P, Q, b', b''. Let us try to find P and b'. Then Q and b'' are obtained similarly, by working with $\overline{\boldsymbol{x}}$ and $\partial_{\overline{\boldsymbol{x}}}$. Consider the ring of differential operators $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$. We wish to find $\widetilde{P} \in \mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$ such that $\widetilde{P}|f|^{2s}f = |f|^{2s}$ (we then get P, b' by eliminating denominators in $\widetilde{P}$). Note that $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ is naturally a left $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module.

The main observation of Bernstein is that this $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module *has finite length*. This means that any decreasing sequence of submodules is stationary.

Consider the decreasing sequence consisting of $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-submodules M_j of $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ generated by $f^j|f|^{2s}$ ($j \geqslant 0$). There exists therefore $k \geqslant 1$ such that $f^k|f|^{2s} \in M_{k+1}$, hence there exists $\widetilde{P}_k \in \mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$ such that $f^k|f|^{2s} = \widetilde{P}_k f^{k+1}|f|^{2s}$. Multiplying by $\overline{f}^k$ and using that $\widetilde{P}_k$ is holomorphic, we get $|f|^{2(s+k)} = \widetilde{P}_k f|f|^{2(s+k)}$. We can change the variable s to $s-k$ to get the desired relation.

The property that $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ has finite length as a $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module is the main property used, which follows from a finer property called *holonomy*, concerning dimension. In the next sections, we make explicit this notion in the analytic framework. We come back to the algebraic framework at the end of §4.4.

4.2. First properties of holonomic $\mathscr{D}_X$-modules

We consider a complex analytic manifold X of pure dimension n and we introduce in this general setting the notion of holonomic $\mathscr{D}_X$-module.

Definition 4.2.1. A coherent $\mathscr{D}_X$-module is said to be *holonomic* if its characteristic variety $\operatorname{Char}\mathscr{M}$ has dimension $\dim X$.

It follows from the involutiveness theorem 2.5.2 that, if $\mathscr{M}$ is holonomic, $\operatorname{Char}\mathscr{M}$ is a Lagrangean conical subspace of T^*X.

Let us recall a result on conical Lagrangean subspaces of T^*X. Denote by $\pi : T^*X \to X$ the canonical projection. Let Y be a closed analytic subset of X and Y_o the smooth part of Y. The conormal bundle $T^*_{Y_o}X \subset T^*X$ is the following vector subbundle of T^*X:

$$T^*_{Y_o}X = \{v \in T^*X \mid p = \pi(v) \in Y_o \text{ and } v \text{ annihilates } T_pY\}.$$

The *conormal space of* Y, denoted by T^*_YX, is by definition the closure in T^*X of $T^*_{Y_o}X$. We say that a subspace V of T^*X is *conical* if $(x,\xi) \in V \Rightarrow (x,\lambda\xi) \in V$, for any $\lambda \in \mathbb{C}$.

Lemma 4.2.2. *If $V \subset T^*X$ is an analytic conical and Lagrangean subset of T^*X, then there exists a locally finite set $(Y_\alpha)_{\alpha\in A}$ of closed irreducible analytic subsets of Y such that $V = \bigcup_\alpha T^*_{Y_\alpha}X$. Moreover, the $Y_\alpha \subset X$ are the projections of the irreducible components of V.*

As a consequence, if $\mathscr{M}$ is holonomic, there exists a locally finite family (Y_α) of irreducible closed analytic subset of X such that $\operatorname{Char}\mathscr{M} = \bigcup_\alpha T^*_{Y_\alpha}X$, and if we know $\operatorname{Char}\mathscr{M}$, we can recover the sets Y_α as the projections of the irreducible components of $\operatorname{Char}\mathscr{M}$.

Examples 4.2.3

(1) $\mathscr{O}_X$ is a holonomic $\mathscr{D}_X$-module and $\operatorname{Char}\mathscr{O}_X = T^*_XX$.

(2) For any smooth hypersurface H of X, $\mathscr{O}_X(*H)$ is holonomic, $\operatorname{Char}\mathscr{O}_X(*H) = T^*_XX \cup T^*_YX$.

(3) If $n = 1$, a $\mathscr{D}_X$-module is holonomic if and only if each local section of m is annihilated by a non-zero differential operator.

(4) For $n \geqslant 2$, if P is a section of $\mathscr{D}_X$, the quotient $\mathscr{D}_X/\mathscr{D}_XP$ is never holonomic. Its characteristic variety is a hypersurface of T^*X.

From Exercise E.2.7 we get:

Corollary 4.2.4. *In an exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of coherent $\mathscr{D}_X$-modules, $\mathscr{M}$ is holonomic if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are so.* $\square$

Remark 4.2.5. It is possible to make this result more precise. One can attach to each irreducible component of $\operatorname{Char}\mathscr{M}$ a multiplicity, which is a strictly positive number. This produces a cycle in T^*X, that is, a linear combination of irreducible analytic subsets of T^*X with multiplicities. Then one can prove that the characteristic cycle behaves in an *additive* way in exact sequences.

Corollary 4.2.6. *A decreasing sequence $\mathscr{M}_1 \supset \mathscr{M}_2 \supset \cdots$ of holonomic $\mathscr{D}_X$-modules is locally stationary.*

Proof. By considering the exact sequences $0 \to \mathscr{M}_{j+1} \to \mathscr{M}_j \to \mathscr{M}_j/\mathscr{M}_{j+1} \to 0$, one checks that the family of characteristic cycles is decreasing. In the neighbourhood of a given compact set, we have a decreasing family of cycles with a finite number of components and integral coefficients. It is therefore stationary. Now for $j \gg 0$, the characteristic cycle of $\mathscr{M}_j/\mathscr{M}_{j+1}$ is zero, hence this module is zero, so the sequence $\mathscr{M}_j$ is stationary. □

Corollary 4.2.7. *Each holonomic $\mathscr{D}_X$-module has a Jordan-Hölder sequence which is locally finite.* □

***Corollary 4.2.8* (of Th. 2.5.4).** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then $\mathscr{M}$ is holonomic if and only if $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ for $i \neq n$. In this case the natural right $\mathscr{D}_X$-module $\mathscr{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ is a holonomic $\mathscr{D}_X$-module. The associated left $\mathscr{D}_X$-module is called the dual of $\mathscr{M}$.*

4.3. Vector bundles with integrable connections

Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank r, equipped with an *integrable* connection $\nabla : \mathscr{E} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{E}$.

***Lemma 4.3.1* (Cauchy-Kowalevski's theorem).** *In the neighbourhood of each point of X there exist a local frame of $\mathscr{E}$ consisting of horizontal sections, i.e., annihilated by ∇.*

This classical theorem is equivalent to the property that, as a vector bundle with connection, $(\mathscr{E}, \nabla)$ is locally isomorphic to $(\mathscr{O}_X, d)^r$. As an immediate consequence, the corresponding $\mathscr{D}_X$-module (see Proposition 1.1.4) has characteristic variety equal to $T^*_X X$ (see Example 4.2.3(1)), and is therefore holonomic. In fact the converse is true (see Exercise E.2.9).

What happens now if the connection has a pole along a hypersurface $D \subset X$? In such a case, $\mathscr{E}(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathscr{E}$ has an integrable connection, hence is a left $\mathscr{D}_X$-module. Is it coherent, or holonomic, as such?

***Theorem 4.3.2* (Kashiwara [Kas78]).** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Assume that $\mathscr{M}_{|X \smallsetminus D}$ is holonomic. Then $\mathscr{M}(*D)$ is a holonomic (hence coherent) $\mathscr{D}_X$-module.*

Note that the coherence property of $\mathscr{M}(*D)$ is already not obvious. This theorem extends the algebraic result of Bernstein used in §4.1 to the analytic setting.

Corollary 4.3.3. *Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module equipped with an integrable meromorphic connection, having poles along a hypersurface $D \subset X$. Then $(\mathscr{E}(*D), \nabla)$ defines a holonomic (hence coherent) $\mathscr{D}_X$-module.*

Proof. We have $\mathscr{E} \subset \mathscr{E}(*D)$. Let us consider the $\mathscr{D}_X$-submodule $\mathscr{M} = \mathscr{D}_X \cdot \mathscr{E} \subset \mathscr{E}(*D)$ generated by $\mathscr{E}$. Consider the filtration $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot \mathscr{E}$. The criterion of Exercise E.2.3(2) shows that it is a good filtration, hence $\mathscr{M}$ is $\mathscr{D}_X$-coherent.

Moreover, $\mathscr{M}_{|X\smallsetminus D}$ is the left $\mathscr{D}_X$-module associated with the holomorphic bundle $\mathscr{E}_{|X\smallsetminus D}$ with holomorphic connection $\nabla_{|X\smallsetminus D}$. From Kashiwara's theorem 4.3.2 we conclude that $\mathscr{M}(*D)$ is $\mathscr{D}_X$-holonomic. But from the inclusions $\mathscr{E} \subset \mathscr{M} \subset \mathscr{E}(*D)$ we deduce that $\mathscr{M}(*D) = \mathscr{E}(*D)$. □

The following converse holds.

Theorem 4.3.4 **(Malgrange [Mal94a, Mal94b, Mal96]).** *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module.*

(1) *If $\mathscr{M}_{|X\smallsetminus D}$ is a vector bundle with integrable connection, there exists a coherent $\mathscr{O}_X$-module $\mathscr{E}$ equipped with an integrable meromorphic connection ∇ such that $\mathscr{M}(*D) = (\mathscr{E}(*D), \nabla)$.*

(2) *In general, $\mathscr{M}$ has a globally defined good filtration.*

4.4. Direct images of holonomic $\mathscr{D}_X$-modules

Theorem 4.4.1 **(Kashiwara [Kas76]).** *Let $f : X \to Y$ be a holomorphic map between complex manifolds and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Assume that $f_{|\operatorname{Supp}\mathscr{M}}$ is proper. Then the cohomology sheaves $\mathscr{H}^j(f_+\mathscr{M})$ are holonomic.*

Proof. Since $\mathscr{M}$ has a globally defined good filtration (Theorem 4.3.4), it is f-good, and $\mathscr{H}^j(f_+\mathscr{M})$ are coherent $\mathscr{D}_Y$-module whose characteristic variety is controlled by the estimate of Kashiwara 3.5.1. The result is then a consequence of the following geometric lemma.

Lemma 4.4.2. *Let $\Lambda = T^*_Z X$ be a Lagrangean closed analytic subvariety in T^*X (i.e., Z is a closed analytic set in X). Then each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\Lambda\big)$ is isotropic in T^*Y.*

Assume that the lemma is proved. It follows that each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\operatorname{Char}\mathscr{M}\big)$ is isotropic. As a consequence, according to Kashiwara's estimate, each irreducible component of $\operatorname{Char}\mathscr{H}^j(f_+\mathscr{M})$ is isotropic. Since such a component is also involutive (Theorem 2.5.2), it is therefore Lagrangean, so $\mathscr{H}^j(f_+\mathscr{M})$ is holonomic. □

Proof of Lemma 4.4.2. It is convenient to decompose f in the following way:

$$f : \underset{x}{X} \xhookrightarrow{\ i_f\ } \underset{(x,f(x))}{X \times Y} \xrightarrow{\ p_2\ } \underset{f(x)}{Y}$$

and we are reduced to show ing the lemma when f is an inclusion (like i_f) and when f is a projection (like p_2). The first case is easy, and we will only consider the second one. We will use the following property, which follows from a theorem due to H. Whitney: let $\Lambda' \subset \Lambda$ be to closed analytic subsets of T^*X; if Λ is isotropic (i.e., ω_X vanishes when restricted to the smooth part Λ^o), then Λ' is also isotropic.

Recall the basic diagram:

$$\begin{array}{ccccc} T^*X & \xleftarrow{T^*f =: \rho} & f^*T^*Y & \xrightarrow{\widetilde{f}} & T^*Y \\ \omega_X & & \rho^*\omega_X = \widetilde{f}^*\omega_Y & & \omega_Y \end{array}$$

Then ω_X vanishes on any $\Lambda'^o \subset (\Lambda \cap f^*T^*Y)$, hence so does $\rho^*\omega_X$, that is, $\widetilde{f}^*\omega_Y$. Argue now by contradiction: assume there is an irreducible component Λ_Y of $\widetilde{f}((T^*f)^{-1}\Lambda)$ such that $\omega_Y \not\equiv 0$ on Λ_Y^o; let Λ' be an irreducible component of $\Lambda \cap f^*TY$ whose image by $\widetilde{f}$ is Λ_Y; then $\widetilde{f} : \Lambda' \to \Lambda_Y$ is generically a submersion; therefore, $\widetilde{f}^*\omega_Y$ cannot vanish identically on Λ'^o, a contradiction. □

Application to the proof of Theorem 3.1.5. The algebraic analogues of the previous results are due to J. Bernstein (note however that the algebraic analogue of Malgrange's theorem is much easier than the analytic one). We explain now how they can be combined to obtain a proof of Theorem 3.1.5.

Firstly, $\mathrm{GM}^k(f)$ is identified with the algebraic direct image $\mathscr{H}^{k-n}(f_+\mathscr{O}_{(\mathbb{C}^*)^n})$. Since f is not proper, one cannot apply the coherence and holonomy theorem. However, choose a smooth quasi-projective variety X and a projective morphism $g : X \to \mathbb{C}$ such that $(\mathbb{C}^*)^n$ is a dense Zariski open set of X whose complement is a hypersurface D, and $g_{|(\mathbb{C}^*)^n} = f$. Let $i : (\mathbb{C}^*)^n \hookrightarrow X$ denote the inclusion. Since $g \circ i = f$, we have

$$\mathscr{H}^{k-n}(f_+\mathscr{O}_{(\mathbb{C}^*)^n}) = \mathscr{H}^{k-n}\big(g_+(\mathscr{O}_X(*D))\big).$$

Since $\mathscr{O}_X(*D)$ is $\mathscr{D}_X$-holonomic, these $\mathscr{D}_{\mathbb{C}}$-modules are holonomic, that is, $\mathrm{GM}^k(f)$ are holonomic. In particular (Exercise 4.2.3(3)), any section of $\mathrm{GM}^k(f)$ is annihilated by a non-zero differential operator. □

4.5. Recent advances

4.5.a. Local normal form of a meromorphic connection near a pole. Classical asymptotic analysis in one complex variable produces a normal form for a meromorphic connection in one variable near one of its pole. It is now standard to present this result in three steps:

(1) existence of a local formal normal form for the matrix of the connection,

(2) asymptotic – or multisummable – liftings of this normal form in sectors around the pole,

(3) comparison between the various liftings, which gives rise to the Stokes phenomenon.

In higher dimension, such results have only been obtained recently. More precisely, work of Majima [**Maj84**] and C.S. [**Sab93, Sab00**] answer the second point, if the first one is assumed to be solved.

A precise conjecture for a statement analogous to the first point is given in [**Sab00**]. It has been solved recently in two different ways:

- T. Mochizuki has solved the conjecture as stated in [**Sab00**], that is, in dimension two, by using a reduction to characteristic p and the notion of p-curvature [**Moc09**]. He then solved the analogue of the conjecture in arbitrary dimension by using techniques of differential geometry (Higgs bundles and harmonic metrics) [**Moc11b**].
- K. Kedlaya used techniques inspired from p-adic differential equations (in particular, a systematic use of Berkovich spaces) to solve the conjecture in any dimension [**Ked10, Ked11**].

The third point has been generalized by T. Mochizuki [**Moc11b, Moc11a**] and C.S. [**Sab10**] by developing the notion of Stokes filtration, following a previous approach in dimension one by P. Deligne [**Del07**] and B. Malgrange [**Mal91**]. This leads to a Riemann-Hilbert correspondence in arbitrary dimension, and in the setting of possibly irregular singularities.

4.5.b. A conjecture of Kashiwara. One of the consequences of the previous results is the following theorem, proved by M. Kashiwara [**Kas87**] under the assumption of regular singularity, and in general (a conjecture of Kashiwara) by T. Mochizuki T. Mochizuki [**Moc11a**] and C.S. [**Sab00, Sab10**].

Theorem 4.5.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathfrak{Db}_X)$, equipped with its left $\mathscr{D}_{\overline{X}}$-module structure coming from that on $\mathfrak{Db}_X$, see §*1.3.e*, is a holonomic (hence coherent) $\mathscr{D}_{\overline{X}}$-module. Moreover, for each $i>0$, $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M},\mathfrak{Db}_X)=0$.*

This statement has many consequences, already noted by M. Kashiwara [**Kas87**] (see also [**Bjö93**]). One of them says that any holonomic $\mathscr{D}_X$-module is locally a $\mathscr{D}_X$-submodule of $\mathfrak{Db}_X$.

4.5.c. Wild Hodge theory. The theory of Hodge $\mathscr{D}$-modules developed by M. Saito [**Sai88, Sai90**] allows one to consider Hodge theory for singular spaces. The basic objects are holonomic $\mathscr{D}_X$-modules equipped with a specific good filtration (the Hodge filtration). Following Deligne, Beilinson and Bernstein [**BBD82**], this makes an analogy with the theory of pure perverse ℓ-adic sheaves with tame ramification on an algebraic variety in characteristic p.

In order to extend such an analogy in the case of wild ramification, a generalization of Hodge theory (and thus of Hodge $\mathscr{D}$-modules) is needed. This has been developed by T. Mochizuki and C.S. in the tame case first [**Moc07, Sab05**] and then in the wild case [**Moc11b**], see also [**Sab09**].

Exercises and complements

Exercise E.4.1. Let $f(x_1, \dots, x_n) = x_1^{m_1} \cdots x_n^{m_n}$, $m_j \in \mathbb{N}$. Show that

$$\Big[\prod_{i=1}^{n}\prod_{k=1}^{m_i}(m_i s + k)\Big] \cdot |f|^{2s} = \Big[\partial_{x_1}^{m_1} \cdots \partial_{x_n}^{m_n}\Big](f|f|^{2s}).$$

Exercise E.4.2. Consider the quadratic form

$$f(x_1, \dots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2$$

with $a_i \neq 0$ for each i. By using

$$\partial_{x_i}^2(f|f|^{2s}) = 2a_i(s+1)\big(|f|^{2s} + 2s(a_i x_i^2)|f|^{2(s-1)}\overline{f}\big)$$

show that

$$4(s+1)(s+n/2)|f|^{2s} = \Big(\sum_i \frac{\partial_{x_i}^2}{a_i}\Big)(f|f|^{2s}).$$

Exercise E.4.3. Consider the semi-cubic parabola

$$f(x_1, x_2) = x_1^2 + x_2^3.$$

Show the following relations

$$\begin{aligned}\partial_{x_1}^2(f|f|^{2s}) &= 2(s+1)\big(|f|^{2s} + 2s x_1^2 |f|^{2(s-1)}\overline{f}\big)\\ \partial_{x_2}^2(f|f|^{2s}) &= 3(s+1)x_2\big(2|f|^{2s} + 3s x_2^3 |f|^{2(s-1)}\overline{f}\big).\end{aligned}$$

Deduce that

$$(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)(f|f|^{2s}) = 6(s+1)(6s+7)x_2|f|^{2s}$$

and then

$$\big(\partial_{x_2}(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)\big)(f|f|^{2s}) = 6(s+1)(6s+7)\big(|f|^{2s} + 3s x_2^3|f|^{2(s-1)}\overline{f}\big).$$

As in the previous exercise, show that

$$(6s+5)(6s+6)(6s+7)|f|^{2s} = \big(9(6s+7)\partial_{x_1}^2 + 2\partial_{x_2}(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)\big)(f|f|^{2s}).$$

Notice that the operator $P(s, \boldsymbol{x}, \partial_{\boldsymbol{x}})$ depends on s.

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