

DAG SEMINAR, PROBLEM SET 4 (OCT. 17-24).

References: Sect. 2.2, A.2.1-A.2.5, A.2.8, A.3.1 in the book

1. Let \mathcal{C} be a simplicial category.

(a) Assume that \mathcal{C} is endowed with a model structure, which is compatible with the simplicial structure, making \mathcal{C} into a simplicial model category. Show that if $X \in \mathcal{C}$ is a cofibrant object and $Y \in \mathcal{C}$ is fibrant, then $\text{Maps}_{\mathcal{C}}(X, Y)$ is Kan.¹

(b) Show that under the above assumptions on X and Y , the map $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y)$ factors as

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \pi_0(\text{Maps}_{\mathcal{C}}(X, Y)) \rightarrow \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y),$$

and that the latter arrow is an isomorphism.

(c) Show that \mathcal{C} admits at most one compatible model structure, once we (i) require that all objects be cofibrant, (ii) specify which morphisms are cofibrations, and (iii) specify which objects are fibrant.

2. Recall that to a model category \mathcal{C} we can assign an ordinary category $\text{Ho}(\mathcal{C})$ by inverting the weak equivalences. Recall that to a simplicial category \mathcal{C}_1 we can assign the ordinary category $\pi_0(H(\mathcal{C}_1))$, by taking π_0 of simplicial Hom sets between objects. Let now \mathcal{C} be a simplicial model category; let $\mathcal{C}^\circ \subset \mathcal{C}$ be the full subcategory spanned by fibrant-cofibrant objects.

(a) Construct a canonical equivalence of ordinary categories $\text{Ho}(\mathcal{C}) \simeq \pi_0(H(\mathcal{C}^\circ))$.

(b) The assignment $\mathcal{C} \mapsto \text{Ho}(\mathcal{C})$ obviously loses information (it doesn't remember the higher category structure). However, we'll show that in some sense it doesn't:

Let $\mathcal{C}_i, i = 1, 2$ be simplicial model categories, and let

$$F : \mathcal{C}_1 \rightleftarrows \mathcal{C}_2 : G$$

be a Quillen adjunction (as model categories). Assume that the functor G is endowed with a structure of simplicial functor.

Assume now that every object of \mathcal{C}_1 is cofibrant. Hence, G defines a simplicial functor $G^\circ : \mathcal{C}_2^\circ \rightarrow \mathcal{C}_1^\circ$. Show that (F, G) is a Quillen equivalence if and only if G° is a weak equivalence of simplicial categories.

3. Recall the functors St and Un (see PS 3, Problems 5, 6). Assume that $S = \Delta_0$. In this case we are dealing with an adjoint pair of functors

$$St_{pt} : \text{Set}_\Delta \rightleftarrows \text{Set}_\Delta : Un_{pt}.$$

(a) Show that, as any colimit preserving functor $\text{Set}_\Delta \rightarrow \text{Bla}$, the functor St_{pt} is uniquely determined by a functor $Q : \Delta \rightarrow \text{Bla}$ ². Describe the functor Q as

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¹Here $\text{Maps}_{\mathcal{C}}(X, Y)$ denotes the simplicial set Hom coming from the structure of simplicial category on \mathcal{C} . This should be distinguished from $\text{Hom}_{\mathcal{C}}(X, Y)$, which equals, by definition, $\text{Maps}_{\mathcal{C}}(X, Y)_0$.

²Here Δ is the category of ordered finite sets and non-decreasing maps

explicitly as you can. Check for simplices of small dimension that all $Q([n])$ are contractible.

(b) Construct a natural transformation $St_{pt} \rightarrow \text{Id}$. Show that (a) combined with the fact that St sends injections to injections formally implies that $St_{pt}(S) \rightarrow S$ is a weak homotopy equivalence for any $S \in \text{Set}_\Delta$, and that for any S' , which is Kan, the adjunction map $St_{pt}(Un_{pt}(S')) \rightarrow S'$ is also a weak homotopy equivalence.

(c) Let \mathcal{C} be a simplicial category, and $x, y \in \mathcal{C}$. Construct an isomorphism of simplicial sets

$$\text{Hom}_{N(\mathcal{C})}^R(x, y) \simeq Un_{pt}(\text{Maps}_{\mathcal{C}}(x, y)).$$

(d) Deduce that for a Kan simplicial category, there is a canonically defined but that establishes a weak homotopy equivalence of Kan simplicial sets

$$\text{Hom}_{N(\mathcal{C})}^R(x, y) \simeq \text{Maps}_{\mathcal{C}}(x, y).$$

4. Let S be a simplicial set, and $x, y \in S$.

(a) Construct a canonical map

$$St_{pt}(\text{Hom}_S^R(x, y)) \rightarrow \text{Maps}_{\mathfrak{C}[S]}(x, y).$$

(b) In the lecture we showed that if S is a quasi-category, then the map from (a) is a weak homotopy equivalence. Deduce from here that for any Kan simplicial category \mathcal{C} , the functor

$$\mathfrak{C}(N(\mathcal{C})) \rightarrow \mathcal{C}$$

is a weak equivalence of simplicial categories.

5. Let's return to the setting of Problem 1. Let $\mathcal{C} = \text{Set}_\Delta$, considered as a simplicial category in the usual way. We shall now try (and fail) to define a new model structure on it compatible with the simplicial structure. First, recall that it does have a genuine simplicial model structure where we (i) set all objects to be cofibrant, (ii) declare cofibrations to be monomorphisms, and (iii) declare Kan simplicial sets to be the fibrant objects. Let's leave (i) and (ii) intact, but replace (iii) by the weak Kan condition. I.e., we want our fibrant objects to be quasi-categories. Show, however, that this doesn't work: i.e., one of the conditions of simplicial model category will be violated. ³

³There exists in fact a model structure on Set_Δ with the above fibrant objects; it's just that it is not simplicial. We'll see how to deal with the situation in the next two lectures.