DAG SEMINAR, PROBLEM SET 2 (OCT. 3-10).

1. Let S be a simplicial set. Consider the simplicial sets $\mathrm{Maps}(\Delta^1, S)$, $\mathrm{Maps}(\Lambda^2_1, S)$, and $\mathrm{Hom}(\Delta^2, S)$. Consider the maps

$$\mathrm{Maps}(\Delta^2, S) \to \mathrm{Maps}(\Lambda^2_1, S) \text{ and } \mathrm{Maps}(\Delta^1, S) \to S$$

obtained by restriction (in the latter case we restrict with respect to either $\{0\}$ or $\{1\}$ mapping into Δ^1).

(a) Consider the following conditions: (i) S is weakly Kan, and (ii) $\mathrm{Maps}(\Delta^2, S) \to \mathrm{Maps}(\Lambda^2_1, S)$ is a trivial Kan fibration. [1] Show that these conditions are equivalent when evaluated on simplices of dimensions 0, 1 and 2.

(b) Consider the following conditions: (i) S is Kan, and (ii) $\mathrm{Maps}(\Delta^1, S) \to S$ is a trivial Kan fibration. Show that these conditions are equivalent when evaluated on simplices of dimensions 0, 1 and 2.

2. Consider the functor $\mathcal{S}et_\Delta \to \mathrm{Cat}$ that assigns to a simplicial set S the homotopy category of the simplicial category hS defined by $\mathrm{Hom}_{hS}(x, y) := \mathrm{Hom}_{\mathfrak{C}[S]}(x, y)$, regarded as a mere category (i.e., we turn a simplicial category into an ordinary category by considering π_0 of the mapping space of simplicial sets). Show that this functor is the left adjoint to the usual nerve functor: $\mathrm{Cat} \to \mathcal{S}et_\Delta$.

3. Let hS be as above (again, considered as a mere category).

(a) Show that it can be described by generators are relations as follows: its objects are vertices of S. It's morphisms are *generated* by edges of S. The compositions are *generated* by 2-simplices in S. [2]

(b) Assume that S is a quasi-category. Show that in this case hS is much more explicit: its morphisms are edges in S *up to homotopy* (formulate what the latter means). Show that the weak Kan assumption allows you to define the composition of arrows, up to homotopy.

4. Let S be a quasi-category. Show that if S is Kan, then hS is a groupoid. Conversely, assume that hS is a groupoid, and show that S satisfies the condition of being a Kan simplicial set on 0, 1 and 2 simplicies.

5. Let $f : S_1 \to S_2$ be a map between Kan simplicial sets. Assuming the fact that $\mathrm{Hom}^R(x, y) \simeq \mathrm{Hom}_{\mathfrak{C}[S]}(x, y)$, prove that if f is a *categorical equivalence*, then it's a weak homotopy equivalence. [3]

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[1] Recall that for two simplicial sets S_1 and S_2 the mapping space $\mathrm{Maps}(S_1, S_2)$ is the simplicial set defined by $\mathrm{Hom}_{\mathcal{S}et_\Delta}(\Delta^n, \mathrm{Maps}(S_1, S_2)) = \mathrm{Hom}_{\mathcal{S}et_\Delta}(\Delta^n \times S_1, S_2)$.

[2] It's part of the exercise to formulate what it means for a category to be generated by a set of morphisms and compositions. Hint: this category satisfies a universal property.

[3] Recall that a map $S_1 \to S_2$ between quasi-categories is called a categorical equivalence if it induces a (weak) equivalence $\mathfrak{C}[S_1] \to \mathfrak{C}[S_2]$ as simplicial categories.

6. Let S be a quasi-category and K an arbitrary simplicial set. We have the following theorem: $\mathrm{Maps}(K, S)$ is a quasi-category. Check this for low-dimensional simplicial sets K and low-dimensional inner horns. Can you conceive a strategy to prove it in general?

7.(a) Let K be one of the simplicial sets Δ^0, Δ^1, Δ^2, and Λ^2_0. Describe the simplicial set $K * \Delta^0$ in the above cases.

(b) Describe $\Delta^n * \Delta^1$ for any n.

(c) Describe $\Delta^n * \Delta^m$ for any n and m.

(d) Let S_1 and S_2 be quasi-categories. Show that $S_1 \star S_2$ is also a quasi-category.

8. Let $p : \mathcal{K} \to \mathcal{C}$ be a map between ordinary categories.

(a) Define the category $\mathcal{C}_{/p}$ of "objects of $\mathcal{C}$ over p". Show that for $\mathcal{K} = \mathrm{pt}$ and p corresponding to an object $x \in \mathcal{C}$, this reduces to the familiar notion of "objects over x".

(b) Show that the quasi-category $N(\mathcal{C}_{/p})$ is canonically *isomorphic* to $N(\mathcal{C})_{/N(p)}$.

9. Let S be a simplicial set and $x \in S$ a vertex. Show that the following conditions are equivalent:

(i) The map $S_{/x} \to S$ is a trivial Kan fibration.

(ii) Any map $\partial\Delta^n \to S$ that maps $n \mapsto x$ can be extended to a map $\Delta^n \to S$.

(iii) Assuming that S is a quasi-category, for any $y \in S$, the (Kan) simplicial set $\mathrm{Hom}^R_S(y, x)$ is contractible.

(iv) Assuming that S is a quasi-category, x is a final object in hS as a category enriched over the homotopy category of spaces. [4]

NB: objects x satisfying the equivalent conditions above are called "strongly final", or just "final".

10. Uniqueness of final objects. Let S be a quasi-category. Let $S' \subset S$ be the *full* quasi-sub-category spanned by vertices that are strongly final. Show that if S' is non-empty, then it's a contractible Kan complex.

[4] As part of the exercise, formulate what it means to be a final object in an enriched setting.