# DAG SEMINAR, PROBLEM SET 1 (SEPT. 19-26).

**1.** Let $S$ be a simplicial set. Show that $S$ satisfies the *unique lifting property for inner horns* if and only if $S$ is the nerve of a usual category $\mathcal{C}$. Show that the assignment $\mathcal{C} \mapsto N(\mathcal{C})$ is a fully faithful embedding from the category of small categories (considered as a usual category, i.e., we discard natural transformations) to the category of simplicial sets.

**2.** Show that the assertion of Problem 1 remains valid when you replace the word "category" by "groupoid" and "inner horns" by "all horns".

**3.** Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories, and $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ be two functors. Show that the set of natural transformations $F \Rightarrow G$ can be identified with the set of maps of simplicial sets $\Delta^1 \times N(\mathcal{C}_1) \to N(\mathcal{C}_2)$ that restrict to $F$ and $G$ under $\Delta^0 \rightrightarrows \Delta^1$.

**4.** Recall the simplicial category $\mathfrak{C}([n])$.

(a) Convince yourself that for a simplicial category $\mathcal{C}$, maps of simplicial categories $\mathfrak{C}([2]) \to \mathcal{C}$ are the same as a triple of objects $c_1, c_2, c_3 \in \mathcal{C}$, maps $f_{i,j} \in \mathrm{Hom}(c_i, c_j)_0, i < j$ and a homotopy $g : f_{2,3} \circ f_{1,2} \Rightarrow f_{1,3} \in \mathrm{Hom}(c_1, c_3)_1$.

(b) Convince yourself that defining a map $\mathfrak{C}([3]) \to \mathcal{C}$ is the same as specifying objects $c_i \in \mathcal{C}$, $i = 1, 2, 3, 4$, maps $f_{i,j} \in \mathrm{Hom}(c_i, c_j)_0, i < j$, and a data of "coherent homotopy" between them (or, rather, take it as a definition of coherent homotopy and unravel what it it means).

**5.** Recall the simplicial nerve functor $N. : \mathrm{Cat}_\Delta \to \mathcal{S}et_\Delta$,

$$N_n(\mathcal{C}) := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}([n]), \mathcal{C}).$$

(a) What's the precise connection between the simplicial and usual nerve constructions?

(b) Show that the functor $N$ above admits a left adjoint, denoted $\mathfrak{C}$. What's the value of $\mathfrak{C}$ on $\Delta^n \in \mathcal{S}et_\Delta$?

(c) Put (b) in the following framework. Let $\mathcal{C}_1$ be a small category, and $\mathcal{C}_2$ category closed under colimits. Then colimit preserving functors $\mathrm{Funct}(\mathcal{C}_1^{op}, \mathcal{S}et) \to \mathcal{C}_2$ are in bijection with functors $\mathcal{C}_1 \to \mathcal{C}_2$.

(d) Let $A$ be a poset, considered as an ordinary category. Consider the simplicial set $N(A)$. Describe $\mathfrak{C}(N(A))$.

**6.** Let $\mathcal{C}$ be a simplicial category such that for any $c_1, c_2 \in \mathcal{C}$, the simplicial set $\mathrm{Hom}.(c_1, c_2)$ is Kan. Show that $N.(\mathcal{C}) \in \mathcal{S}et_\Delta$ is a quasi-category.

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*Date*: October 11, 2010.