ALMOST COHERENT MODULES AND ALMOST COHERENT SHEAVES

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Abstract. We extend the theory of almost coherent modules that was introduced in "Almost Ring Theory" [GR03] by Gabber and Ramero. Then we globalize it by developing a new theory of almost coherent sheaves on schemes and on a class of "nice" formal schemes. We show that these sheaves satisfy many properties similar to usual coherent sheaves, i.e. the Almost Proper Mapping Theorem, the Formal GAGA, etc. We also construct an almost version of the Grothendieck twisted image functor $f^!$ and verify its properties. Lastly, we study sheaves of p-adic nearby cycles on admissible formal models of rigid-analytic varieties and show that these sheaves provide examples of almost coherent sheaves. This gives a new proof of the finiteness result for étale cohomology of proper rigid-analytic varieties obtained before in the work of Peter Scholze "p-adic Hodge Theory For Rigid-Analytic Varieties" [Sch13].

Contents

1. Introduction

1.1. **Motivation.** The purpose of this paper is threefold. The first goal is to develop a sufficiently rich theory of almost coherent sheaves on schemes and a class of formal schemes. The second goal is to provide the reader with one interesting source of examples of almost coherent sheaves. Namely, we show that the complex of p-adic nearby cycles $\mathbf{R}\nu_*(\mathcal{E})$ has quasi-coherent, almost coherent cohomology sheaves for any admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ and $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ (see Definition 6.1.1).

Before we discuss the content of each chapter in detail, we explain the motivation behind the work done in this manuscript.

The first source motivation comes from the work of P. Scholze on the finiteness of $\mathbf{F}_p$-cohomology groups of proper rigid-analytic varieties over p-adic fields (see [Sch13]). The second source of motivation (clearly related to the first one) is the desire to set up a robust enough theory of almost coherent sheaves that is crucially used in our proof of Poincaré Duality for $\mathbf{F}_p$-local systems on smooth and proper rigid-analytic varieties over p-adic fields in [Zav21a].

We start with the work of P. Scholze. In [Sch13], he showed that there is an almost isomorphism

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p \simeq^a \mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$$

for any proper rigid-analytic variety X over a p-adic algebraically closed field C. This almost isomorphism allows us to reduce studying certain properties of $\mathrm{H}^i(X, \mathbf{F}_p)$ for a p-adic proper rigid-analytic space X to studying almost properties of the cohomology groups $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$, or the full complex $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$. For instance, Scholze shows that $\mathrm{H}^i(X, \mathbf{F}_p)$ are finite groups by deducing it from almost coherence of $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ over $\mathcal{O}_C/p$.

Scholze's argument does not involve any choice of an admissible formal model for X and is performed entirely on the generic fiber via an elaborate study of cancellations in certain spectral sequences. A different natural approach to studying $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ is to rewrite this complex as

$$\mathbf{R}\Gamma\left(X, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

for a choice of an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ and the natural morphism of ringed sites

$$t\colon (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p) \to (\mathfrak{X}_{0,\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}_0})$$

with $\mathfrak{X}_0$ the mod-p fiber of $\mathfrak{X}$. Then we can separately study the complex $\mathbf{R}t_*\left(\mathcal{O}^+_{X_{\text{ét}}}/p\right)$ and the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)$. In order to make this strategy work, we develop the notion of almost coherent sheaves on $\mathfrak{X}$ and $\mathfrak{X}_0$ and show its various properties similar to the properties of coherent sheaves. This occupies Chapters 2-5. Chapter 6 is devoted to showing that the complex $\mathbf{R}t_*\left(\mathcal{O}^+_{X_{\text{ét}}}/p\right)$ (and, more generally, "nearby cycles" of any $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle) has almost coherent cohomology groups. Combining it with the Almost Proper Mapping Theorem 1.2.9, we reprove [Sch13, Lemma 5.8 and Theorem 5.1] in a slightly greater generality.

Theorem 1.1.1. (Lemma 6.3.4, Lemma 6.3.7, and Lemma C.5.10) Let C be a p-adic algebraically closed non-archimedean field, X a proper rigid-analytic variety over C, and $\mathcal{F}$ a Zariski-constructible sheaf of $\mathbf{F}_p$-modules (see Definition 6.1.7). Then

(1) $\mathrm{H}^i(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p)$ is an almost finitely generated $\mathcal{O}_C/p$-module for $i \geq 0$;

(2) the natural morphism
$$\mathrm{H}^i\left(X, \mathcal{F}\right) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \to \mathrm{H}^i\left(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p\right)$$
is an almost isomorphism for $i \geq 0$;

(3) $\mathrm{H}^i(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p)$ is almost zero for $i > 2\dim X$.

Theorem 1.1.2. (Corollary 6.3.8)[1] In the notation of Theorem 1.1.1. Then

(1) $\mathrm{H}^i(X, \mathcal{F})$ is a finite group for $i \geq 0$;

(2) $\mathrm{H}^i(X, \mathcal{F}) \simeq 0$ for $i > 2\dim X$.

Now we discuss the role this paper plays in our proof of Poincaré Duality in [Zav21a]. We start with a precise formulation of this result.

Theorem 1.1.3. [Zav21a] Let C be a p-adic algebraically closed non-archimedean field, X a rigid-analytic variety over C of pure dimension d, and $\mathbf{L}$ an $\mathbf{F}_p$-local system on $X_{\text{ét}}$. Then there is a canonical trace map
$$t_X\colon \mathrm{H}^{2d}\left(X, \mathbf{F}_p(d)\right) \to \mathbf{F}_p$$
such that the induced pairing
$$\mathrm{H}^i\left(X, \mathbf{L}\right) \otimes \mathrm{H}^{2d-i}(X, \mathbf{L}^\vee(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}(X, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p$$
is perfect.

The essential idea of the proof (at least for $\mathbf{L} = \underline{\mathbf{F}}_p$) is to use Theorem 1.1.1 to reduce Poincare Duality to the almost duality on the complex $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$. This complex is studied via the isomorphism
$$\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p).$$
Roughly, we separately show almost duality for the "nearby cycles functor" $\mathbf{R}t_*$ and the almost version of Grothendieck Duality for the $\mathcal{O}_C/p$-scheme $\mathfrak{X}_0$. In order to even formulate these things precisely, one needs to have a good way to globalize almost (coherent) modules to almost (coherent) sheaves in a way that almost coherent sheaves share many properties similar to coherent sheaves *and* the "nearby cycles" $\mathbf{R}t_*\left(\mathcal{O}^+_{X_{\text{ét}}}/p\right)$ (and its integral counterpart) fit into this theory.

The main content of Sections 2-5 is to develop this general theory, and the main content of Section 6 is to verify that "nearby cycles" are almost coherent.

That being said, we now discuss content and the main results of each section in more detail.

1.2. **Foundations of Almost Mathematics (Sections 2-5).** Section 2.1 is devoted to defining the category of almost modules and studying its main properties. This section is very motivated by [GR03]. However, it seems that some results that we need later in the paper are not present in [GR03], so we give an (almost) self-contained introduction to almost commutative algebra. We define the notions of the category of almost modules (see the discussion after Corollary 2.1.4), their tensor products (see Proposition 2.2.1(1)), almost Hom functor $\mathrm{alHom}_{R^a}(-,-)$ (see Proposition 2.2.1(3)), almost finitely generated (see Definitions 2.5.1), almost finitely presented (see Definition 2.5.2), and almost coherent modules (see Definition 2.6.1). We show that almost coherent modules satisfy most natural properties similar to the properties of classical coherent modules. We summarize some of them in the theorem below:

[1]Theorem 1.1.2 can also be easily deduced from the results of [BH21].

Theorem 1.2.1. (Lemma 2.6.8, Propositions 2.6.18, 2.6.19, 2.6.20, Theorem 2.10.3, and Lemma 2.10.5) Let R be a ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$.

(1) Almost coherent R^a-modules form a Weak Serre subcategory of $\mathbf{Mod}_R^a$.

(2) If R is an almost coherent ring (i.e. free rank-1 R-module is almost coherent), and M^a, N^a two objects in $\mathbf{D}^-_{acoh}(R)^a$. Then $M^a \otimes^L_{R^a} N^a \in \mathbf{D}^-_{acoh}(R)^a$.

(3) If R is an almost coherent ring, and $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N^a \in \mathbf{D}^+_{acoh}(R)^a$. Then

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \in \mathbf{D}^+_{acoh}(R)^a.$$

(4) If R is an almost coherent ring, $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$, and P^a an almost flat R^a-module. Then the natural map $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \otimes_{R^a} P^a \to \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$ is an almost isomorphism.

(5) Descent of almost modules along an almost faithfully flat morphism $R \to S$ is always effective.

(6) Let $R \to S$ be an almost faithfully flat map, and let M^a be an R^a-module. Suppose that $M^a \otimes_{R^a} S^a$ is almost finitely generated (resp. almost finitely presented, resp. almost coherent) S^a-module. Then so is M^a.

In case, R is I-adically adhesive for some finitely generated ideal I (see Definition 2.12.1), we can show that almost finitely generated R-modules satisfy a (weak) version of the Artin-Rees Lemma, and behave nicely with respect to the completion functor. These results will be crucial for globalizing the theory of almost coherent modules on formal schemes.

Lemma 1.2.2. (Lemma 2.12.6 and Lemma 2.12.7) Let R be an I-adically adhesive ring with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$, and $\mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat (see Set-up 2.12.3). Let M be an almost finitely generated R-module. Then

(1) for any R-submodule $N \subset M$, the induced topology on N coincides with the I-adic topology;

(2) The natural morphism $M \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism. In particular, if R is I-adically complete, then any almost finitely generated R-modules is also I-adically complete.

In case R is a (topologically) finitely generated algebra over a perfectoid valuation ring K^+ (see Definition B.2), we can say even more. In this case, it turns out that R is almost noetherian (see Definition 2.7.1), so the theory simplifies significantly. Another useful result that we can obtain in this situation is that it suffices to check that a derived complete complex is almost coherent after a taking the derived quotient by a pseudo-uniformizer ϖ. This is very handy in practise because it reduces many (subtle) integral question to the torsion case where there are no topological subtleties.

Theorem 1.2.3. (Theorem 2.11.4, Theorem 2.11.8, Theorem 2.13.2) Let K^+ be a perfectoid valuation ring with a pseudo-uniformizer ϖ as in Lemma B.5, and R a K^+-algebra. Then

(1) R is almost noetherian if R is (topologically) finite type over K^+;

(2) if R is a topologically finite type K^+-algebra and M is a derived ϖ-adically complete object in $\mathbf{D}(R)$ such that $[M/\varpi] \in \mathbf{D}^{[c,d]}_{acoh}(R/\varpi)$. Then $M \in \mathbf{D}^{[c,d]}_{acoh}(R)$.

We discuss the extension of almost mathematics to ringed sites in Section 3. The main goal is to generalize all constructions from almost mathematics to a general ringed site. We define a notion of almost $\mathcal{O}_X$-modules on a ringed site $(X, \mathcal{O}_X)$ (see Definition 3.1.9) and of $\mathcal{O}_X^a$-modules (see Definition 3.1.10) and show that they are equivalent:

Theorem 1.2.4. (Theorem 3.1.20) Let R be as in Theorem 1.2.1 and $(X, \mathcal{O}_X)$ a ringed R-site. Then the functor

$$(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}^a_X}$$

is an equivalence of categories.

We also define the functors $-\otimes-$, $\mathrm{Hom}_{\mathcal{O}^a_X}(-,-)$, $\mathrm{alHom}_{\mathcal{O}^a_X}(-,-)$, $\underline{\mathcal{H}om}_{\mathcal{O}^a_X}(-,-)$, $\underline{al\mathcal{H}om}_{\mathcal{O}^a_X}(-,-)$, f_*, and f^* on the category of $\mathcal{O}^a_X$-modules. We refer to Section 3.2 for an extensive discussion of these functors. Then we study the derived category of $\mathcal{O}^a_X$-modules and derived analogues of the functors mentioned above. This is done in Sections 3.4 and 3.5.

We develop the theory of almost finitely presented and almost (quasi-)coherent sheaves on schemes and a class of formal schemes in Section 4.1. The main goal is to show that these sheaves behave similarly to the classical coherent sheaves in many aspects.

We roughly define almost finitely presented $\mathcal{O}^a_X$-modules as modules such that, for any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, can be locally approximated by finitely presented $\mathcal{O}_X$-modules up to modules annihilated by $\mathfrak{m}_0$ (see Definition 4.1.4 for a precise definition). Sections 4.1-4.4 are mostly concerned with local properties of these sheaves. We summarize some of the main results below:

Theorem 1.2.5. (Corollary 4.1.12, Theorem 4.4.6, Lemmas 4.4.8, 4.4.7, 4.4.9, and 4.4.10) Let R be a ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$.

(1) For any R-scheme X, almost coherent $\mathcal{O}^a_X$-modules form a Weak Serre subcategory of $\mathbf{Mod}^a_{\mathcal{O}_X}$.

(2) The functor
$$\widetilde{(-)} \colon \mathbf{D}_*(R)^a \to \mathbf{D}_{aqc,*}(\operatorname{Spec} R)^a$$
is a t-exact equivalence of triangulated categories for $* \in \{\text{“ ”}, \mathrm{acoh}\}$. Its quasi-inverse is given by $\mathbf{R}\Gamma(\operatorname{Spec} R, -)$. In particular, an almost quasi-coherent $\mathcal{O}^a_{\operatorname{Spec} R}$-module $\mathcal{F}^a$ is almost coherent if and only if $\mathcal{F}^a(\operatorname{Spec} R)$ is an almost coherent R^a-module.

(3) The natural morphism $\widetilde{M^a \otimes^L_{R^a} N^a} \to \widetilde{M^a} \otimes^L_{\mathcal{O}^a_{\operatorname{Spec} R}} \widetilde{N^a}$ is an isomorphism for any $M^a, N^a \in \mathbf{D}(R)^a$.

(4) Let that $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ is an R-morphism of affine schemes. Then $\mathbf{L}f^*(\widetilde{M^a})$ is functorially isomorphic to $\widetilde{M^a \otimes^L_{A^a} B^a}$ for any $M^a \in \mathbf{D}(A)^a$.

(5) Let $f \colon X \to Y$ be a quasi-compact and quasi-separated morphism of R-schemes. Suppose that Y is quasi-compact. Then $\mathbf{R}f_*$ carries $\mathbf{D}^*_{aqc}(X)^a$ to $\mathbf{D}^*_{aqc}(Y)^a$ for any $* \in \{\text{“ ”}, -, +, b\}$.

(6) Suppose that R is almost coherent. Then the natural maps
$$\widetilde{\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a)} \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_{\operatorname{Spec} R}}(\widetilde{M^a}, \widetilde{N^a}),$$
$$\widetilde{\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)} \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}^a_{\operatorname{Spec} R}}(\widetilde{M^a}, \widetilde{N^a})$$
are almost isomorphisms for $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$.

We also establish one non-trivial global result on almost finitely presented $\mathcal{O}^a_X$-modules. Namely, we show that even though the definition of almost finitely presented $\mathcal{O}^a_X$-modules is local, we can find good approximations by finitely presented $\mathcal{O}_X$-modules globally under some mild assumption on X. This result is systematically used in Chapter 5 to get global properties of almost coherent $\mathcal{O}^a_X$-modules.

Theorem 1.2.6. (Corollary 4.3.5) Let X be a quasi-compact and quasi-separated R-scheme, and let $\mathcal{F}$ be an almost quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is almost finitely presented (resp. almost finitely generated) if and only if for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there is a morphism $f\colon \mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}$ is a quasi-coherent finitely presented (resp. finitely generated) $\mathcal{O}_X$-module , $\mathfrak{m}_0(\ker f) = 0$ and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.

We now discuss the content of Sections 4.5-4.9. The main goal there is to prove analogues of the results in Theorem 1.2.5 for a class of formal schemes. In order to achieve this we restrict our attention to the class of topologically finitely presented schemes over a topologically universally adhesive ring R (see Setup 4.5.1). This, in particular, includes admissible formal schemes over a mixed characteristic, p-adically complete rank-1 valuation ring $\mathcal{O}_C$ with algebraically closed fraction field C.

One of the main difficulties in developing a good theory of almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules on a formal scheme $\mathfrak{X}$ is that there is no good abelian theory of "quasi-coherent" on $\mathfrak{X}$. This was an important auxillary tool used in developing the theory of almost coherent sheaves on schemes that does not have an immediate counterpart in the world of formal schemes.

We overcome this issue in two different ways: we use the notion of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules introduced in [FK18] (see Definition 4.5.2) and the notion of derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules introduced in [Lur18] (see Definition 4.8.1). The first notion has the advantage that every adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module is an actual $\mathcal{O}_{\mathfrak{X}}$-module, but these modules do not form a Weak Serre subcategory inside $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}$, so they are not always very useful in practice. The latter definition has the advantage that derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules form a triangulated subcategory inside $\mathbf{D}(\mathfrak{X})$, it is quite convenient for certain purposes. However, derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules are merely objects of $\mathbf{D}(\mathfrak{X})$ and not actual $\mathcal{O}_{\mathfrak{X}}$-modules in the classical sense. Therefore, we usually use adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules when needed except for Section 4.8, where the notion of derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules seems to be more useful for our purposes. In particular, it allows us to define the functor

$$(-)^{L\Delta}\colon \mathbf{D}_{acoh}(A)^a \to \mathbf{D}_{acoh}(\mathrm{Spf}\ A)^a$$

for any topologically finitely presented R-algebra A in a way that "extends" the classical functor $(-)^{\Delta}\colon \mathbf{Mod}_A^{\mathrm{acoh}} \to \mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}$ (see Definition 4.8.7 and Lemma 4.8.13).

Theorem 1.2.7. (Lemma 4.5.23, Corollary 4.8.16, Lemmas 4.9.4, 4.9.3, 4.9.4) Let R be a ring with a finitely generated ideal I such that R is I-adically complete, I-adically topologically universally adhesive, I-torsion free with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat.

(1) For any topologically finitely presented formal R-scheme $\mathfrak{X}$, almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules form a Weak Serre subcategory of $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}^a$.

(2) The functor
$$\mathbf{R}\Gamma(\mathrm{Spf}\ R, -)\colon \mathbf{D}_{acoh}(\mathrm{Spf}\ R)^a \to \mathbf{D}_{acoh}(R)^a$$
is a t-exact equivalence of triangulated categories.

(3) The natural morphism $(M^a \otimes^L_{R^a} N^a)^{L\Delta} \to (M^a)^{L\Delta} \otimes^L_{\mathcal{O}^a_{\mathrm{Spf}\ R}} (N^a)^{L\Delta}$ is an isomorphism for any for any $M^a, N^a \in \mathbf{D}_{acoh}(R)^a$.

(4) Let $\mathfrak{f}\colon \mathrm{Spf}\ B \to \mathrm{Spf}\ A$ be a morphism of topologically finitely presented affine formal R-schemes. Then $\mathbf{L}\mathfrak{f}^*\left((M^a)^{L\Delta}\right)$ is functorially isomorphic to $(M^a \otimes^L_{A^a} B^a)^{L\Delta}$ for any $M^a \in \mathbf{D}_{acoh}(A)^a$.

(5) The natural map

$$\left(\mathbf{R}\mathrm{alHom}_{R^a}\left(M^a, N^a\right)\right)^{L\Delta} \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_{\mathrm{Spf}\ R}}\left(\left(M^a\right)^{L\Delta}, \left(N^a\right)^{L\Delta}\right),$$

$$\left(\mathbf{R}\mathrm{Hom}_{R^a}\left(M^a, N^a\right)\right)^{L\Delta} \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}^a_{\mathrm{Spf}\ R}}\left(\left(M^a\right)^{L\Delta}, \left(N^a\right)^{L\Delta}\right)$$

are almost isomorphisms for $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N^a \in \mathbf{D}^+_{acoh}(R)^a$.

Similarly to the case of schemes, almost coherent sheaves on formal schemes satisfy the global approximation property:

Theorem 1.2.8. (Theorem 4.7.6) Let R be as in Theorem 1.2.7, and $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is an adically quasi-coherent, finitely generated (resp. finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}$ and a map $\phi\colon \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\mathrm{Coker}\,\phi) = 0$ and $\mathfrak{m}_0(\ker\phi) = 0$.

We discuss global properties of almost coherent sheaves in Chapter 5. Namely, we generalize certain cohomological properties of classical coherent sheaves to the case of almost coherent sheaves. We start with the almost version of the Proper Mapping Theorem:

Theorem 1.2.9. (Theorem 5.1.3) Let R be a universally coherent[2] ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. And let $f\colon X \to Y$ be a proper morphism of finitely presented R-schemes with quasi-compact Y. Then $\mathbf{R}f_*$ carries $\mathbf{D}^*_{acoh}(X)^a$ to $\mathbf{D}^*_{acoh}(Y)^a$ for $* \in \{\text{“ ”}, -, +, b\}$.

The essential idea of the proof is to reduce Theorem 1.2.9 to the classical Proper Mapping Theorem over an universally coherent base [FK18, Theorem I.8.1.3]. The key input to make this reduction work is Theorem 1.2.6.

We also prove a version of the Almost Proper Mapping Theorem for a morphism of formal schemes:

Theorem 1.2.10. (Theorem 5.1.6) Let R be a ring with a finitely generated ideal I such that R is I-adically complete, I-adically topologically universally adhesive an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m} = \cup_{n=1}^{\infty}(\varpi^{1/n})$ for a non-zero divisor $\varpi \in R$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. And let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of finitely presented formal R-schemes with quasi-compact Y. Then $\mathbf{R}\mathfrak{f}_*$ carries $\mathbf{D}^*_{acoh}(\mathfrak{X})^a$ to $\mathbf{D}^*_{acoh}(\mathfrak{Y})^a$ for $* \in \{\text{“ ”}, -, +, b\}$.

Then we provide a characterization of quasi-coherent, almost coherent complexes on finitely presented, separated schemes over a universally coherent base ring R. This is an almost analogue of [Sta21, Tag 0CSI]. We follow the same proof strategy but adjust it in certain places to make it work in the almost setting. This result is important for us as it will later play a crucial role in the proof of the Formal GAGA Theorem for almost coherent sheaves.

Theorem 1.2.11. (Theorem 5.2.3) Let R be a universally coherent[3] ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. And let X be a separated, finitely presented R-scheme. Let $\mathcal{F} \in \mathbf{D}^-_{qc}(X)$ be an object such that

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{acoh}(R)$$

for any $\mathcal{P} \in \mathrm{Perf}(X)$. Then $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(X)$.

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[2] Any finitely presented R-algebra A is coherent

[3] Any finitely presented R-algebra A is coherent

Theorem 1.2.12. (Corollary 5.3.3) Let R be as in Theorem 1.2.10, and X a finitely presented R-scheme. Then the functor

$$\mathbf{L}c^*\colon \mathbf{D}^*_{acoh}(X)^a \to \mathbf{D}^*_{acoh}(\mathfrak{X})^a$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

We note that the standard proof of the classical formal GAGA theorem via projective methods has no chance to work in the almost coherent situation (due to a lack of “finiteness” for almost coherent sheaves). Instead, we “explicitly” construct a pseudo-inverse to $\mathbf{L}c^*$ in the derived world by adapting an argument from the paper of J. Hall [Hal18].

The last thing we discuss in Section 5 is the almost version of the Grothendieck Duality. This is an important technical tool in our proof of Poincaré Duality in [Zav21a]. So we develop some foundations of the $f^!$ functor in the almost world in this manuscript. We summarize the main properties of this functors below:

Theorem 1.2.13. (Theorem 5.5.8) Let R be as in Theorem 1.2.9, and FPS_R be the category of finitely presented, separated R-schemes. Then there is a well-defined functor $(-)^!$ from FPS_R into the 2-category of categories such that

- **(1)** $(X)^! = \mathbf{D}^+_{aqc}(X)^a$,
- **(2)** for a smooth morphism $f\colon X \to Y$ of pure relative dimension d, $f^! \simeq \mathbf{L}f^*(-) \otimes^L_{\mathcal{O}_X^a} \Omega^d_{X/Y}[d]$.
- **(3)** for a proper morphism $f\colon X \to Y$, $f^!$ is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{acoh}(X)^a \to \mathbf{D}^+_{acoh}(Y)^a$.

1.3. **p-adic Nearby Cycles Sheaves (Section 6).** The main goal of Section 6 is to give the main non-trivial example of almost coherent sheaves. These are the so-called p-adic nearby cycles sheaves.

We fix a p-adic perfectoid field K, and a rigid-analytic variety X over K with an admissible formal $\mathcal{O}_K$-model $\mathfrak{X}$.

The rigid-analytic variety X comes with a morphism of ringed sites

$$\nu\colon (X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}})$$

and

$$\nu\colon (X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}/p) \to (\mathfrak{X}_{0,\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}_0})$$

where $\mathfrak{X}_0$ is the mod-p fiber of $\mathfrak{X}$ and $X^\diamondsuit_v$ is the v-site of the associated diamond (see Appendix C.1) and $\mathcal{O}^+_{X^\diamondsuit}$ its integral “untilted” structure sheaf (see Definition C.3.1).

The main goal of Section 6 is to show that certain nearby cycles sheaves produce examples of almost coherent sheaves. More precisely, we show that, for any $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle $\mathcal{E}$ (see Definition 6.1.1), the complex $\mathbf{R}\nu_*\mathcal{E}$ has quasi-coherent and almost coherent cohomology sheaves. We also give a bound on its almost cohomological dimension.

Theorem 1.3.1. (Theorem 6.1.2) Let $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. Then

- **(1)** the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X}_0)^a$;
- **(2)** for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$ with the adic generic fiber X, the natural map

  $$\mathrm{H}^i\left(\widetilde{X^\diamondsuit_v, \mathcal{E}}\right) \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

  is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$;

(4) if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small (see Definition 6.1.1), then

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a;$$

(5) if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small.

In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$\left(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_{i,0})^a.$$

for each $i \in I$.

Remark 1.3.2. We note that Theorem 1.3.1 implies that the nearby cycles complex $\mathbf{R}\nu_*\mathcal{E}$ is quasi-coherent on the nose (as opposed to being almost quasi-coherent). This is quite unexpected to the author since all previous results on the cohomology groups of $\mathcal{O}^+/p$ were only available in the almost category.

Remark 1.3.3. We do not know if an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ in the formulation of Theorem 1.3.1 is really necessary or just an artefact of the proof. More importantly, we do not know if, for every $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$, there is an admissible formal model $\mathfrak{X}$ such that the "nearby cycles" sheaf $\mathbf{R}\nu_{\mathfrak{X},*}\mathcal{E}$ lies in $\mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a$.

In the proof of Theorem 1.3.1, we crucially use the following result that is essentially due to B. Heuer (see [Heu] for a similar result in a slightly different level of generality).

Theorem 1.3.4. (Corollary C.4.10) Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the categories $\mathrm{Vect}^{\text{ét}}_X$, $\mathrm{Vect}^{\mathrm{qp}}_X$, and Vect^{v}_X (see Definition C.4.1) are equivalent. Furthermore, if X is affinoid, and $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then there is

(1) a finite étale surjective morphism $X' \to X$;

(2) a finite covering by rational subdomains $\{X'_i \to X'\}_{i\in I}$;

(3) a finite étale surjective morphism $X''_i \to X$

such that $\mathcal{E}|_{X''_i}$ is a trivial $\mathcal{O}^+_{X^\diamond}/p$-vector bundle.

Another family of sheaves for which we can establish a good behaviour of "nearby cycles" is sheaves of the form $\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p$ for a Zariski-constructible étale sheaf of $\mathbf{F}_p$-modules (see Definition 6.1.7). Namely, in this case we can get a better cohomological bound, and also show that nearby cycles almost commute with proper base change as this happens in algebraic geometry.

Theorem 1.3.5. (Theorem 6.1.9 and Lemma 6.3.9) Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X;\mathbf{F}_p)$. Then

(1) there is an isomorphism $\mathbf{R}t_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)$;

(2) the nearby cycles $\mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$, and $\mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{acoh}(\mathfrak{X}_0)^a$;

(3) for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$, the natural map

$$\mathrm{H}^i\left(X_v^{\diamondsuit}, \widetilde{\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p}\right) \to \mathrm{R}^i\nu_*\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)$$

is an isomorphism for every $i \geq 0$;

(4) the formation of $\mathrm{R}^i\nu_*\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\right)$$

is an isomorphism for any $i \geq 0$;

(5) if $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is a proper morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, then the natural morphism

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)\right)$$

is an almost isomorphism.

We also show an integral version of Theorem 1.3.1:

Theorem 1.3.6. (Theorem 6.1.11) Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then

(1) the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X})^a$;

(2) for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$ with the adic generic fiber X, the natural map

$$\mathrm{H}^i\left(X_v^{\diamondsuit}, \mathcal{E}\right)^\Delta \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$;

(4) if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small (see Definition 6.1.10), then

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a;$$

(5) if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small.

In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_i)^a.$$

for each $i \in I$.

Theorem 1.3.6 has an interesting consequence saying that v-cohomology groups of any $\mathcal{O}^+_{X^\diamond}$-vector bundle are almost coherent and almost vanish in degrees larger than $2\dim X$. This (together with Theorem 1.1.1) indicates that there probably should be much stronger (almost) finiteness results for some class $\mathcal{O}^+_{X^\diamond}$-sheaves.

Theorem 1.3.7. (Theorem 6.3.3) Let K be a p-adic perfectoid field, X a proper rigid-analytic K-variety of dimension d, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle (resp. $\mathcal{O}^+_{X^\diamond}/p$-vector bundle). Then

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_K)^a.$$

We now explain the main steps of our proof of Theorems 1.3.1 and 1.3.6 for $\mathcal{E} = \mathcal{O}^+_{X^\diamond}/p$ and $\mathcal{E} = \mathcal{O}^+_{X^\diamond}$ respectively:

Proof Sketch. **(1)** We first show that the sheaves $\mathrm{R}^i\nu_*(\mathcal{O}^+_{X^\diamond}/p)$ are quasi-coherent. The main key input is that cohomology of $\mathcal{O}^+_{X^\diamond}/p$-vector bundles vanish on strictly totally disconnected spaces (see Definition C.2.1), and that each affinoid rigid-analytic variety admits a v-covering such that all terms of its Čech nerve are strictly totally disconnected.

(2) The same ideas can be used to show that the formation of $\mathrm{R}^i\nu_*(\mathcal{O}^+_{X^\diamond}/p)$ commutes with étale base change.

(3) We show that the $\mathcal{O}_{\mathfrak{X}_0}$-modules $\mathrm{R}^i\nu_*\left(\mathcal{O}^+_{X^\diamond}/p\right)$ are almost coherent for smooth X. This is done in three steps: we firstly find an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I} \mathfrak{U}_i$ such that each $\mathfrak{U}_i = \mathrm{Spf}\ A_i$ admits a finite rig-étale morphism to $\widehat{\mathbf{A}}^d_{\mathcal{O}_C}$, then we show that the cohomology groups $\mathrm{H}^i(\mathfrak{U}^\diamondsuit_{i,C,v}, \mathcal{O}^+_{X^\diamond}/p)$ are almost coherent over A_i/pA_i, and finally we conclude the almost coherence of $\mathrm{R}^i\nu_*\left(\mathcal{O}^+_{X^\diamond}/p\right)$.

The first step is the combination of [BLR95, Proposition 3.7] and Theorem D.4. The first result allows to choose an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ with an open affine covering $\mathfrak{X}' = \bigcup_{i\in I} \mathfrak{U}_i$ such that each $\mathfrak{U}_i$ admits a rig-étale morphism $\mathfrak{U}_i \to \widehat{\mathbf{A}}^d_{\mathcal{O}_C}$. Then Theorem D.4 guarantees that actually we can change these morphisms so that $\mathfrak{U}_i \to \widehat{\mathbf{A}}^d_{\mathcal{O}_C}$ are *finite* and rig-étale. This is the non-noetherian generalization of Achinger's result [Ach17, Proposition 6.6.1] proven over a discretely valued ring.

The second step follows the strategy presented in [Sch13]. We construct an explicit affinoid perfectoid cover of $\mathfrak{U}_i$ that is a $\mathbf{Z}_p(1)^d$-torsor. So we reduce studying of $\mathrm{H}^i(\mathfrak{U}^\diamondsuit_{i,C,v}, \mathcal{O}^+_{X^\diamond}/p)$ to studying cohomology groups of $\mathbf{Z}_p(1)^d$ that can be explicitly understood via the Koszul complex.

The last step is the consequence of the Almost Proper Mapping Theorem 1.2.9 and the already obtained results.

(4) The next step is to show that $\mathrm{R}^i\nu_*\left(\mathcal{O}^+_{X^\diamond}/p\right)$ is almost coherent for a general X. This is done by choosing a proper hypercovering by smooth spaces $X_\bullet$ and then use a version of cohomological v-descent to conclude almost coherence of the p-adic nearby cycles sheaves. As an important technical tool, we use the theory of diamonds developed in [Sch17].

(5) Next we show that $\mathbf{R}\nu_*\left(\mathcal{O}^+_{X^\diamond}/p\right)$ is almost concentrated in degrees $[0, d]$. This claim is quite subtle. The key input is the version of the purity theorem [BS22, Theorem 10.11] that implies that any *finite* (but not necessarily étale) adic space over an affinoid perfectoid has a diamond that is isomorphic to a diamond of an affinoid perfectoid. This allows us to reduce the question of cohomological bounds of $\mathbf{R}\nu_*\left(\mathcal{O}^+_{X^\diamond}/p\right)^a$ to the question of cohomological dimension of the pro-finite group $\mathbf{Z}_p(1)^d$. This can be understood quite explicitly via Koszul complexes.

(6) Finally, we show Theorem 1.3.6 by reducing it to Theorem 1.3.1. The key input is Theorem 1.2.3 that allows us to check finiteness mod-p.

□

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1.5. **Notation.** A *non-archimedean field* K is always assumed to be complete. A non-archimedean field K is called *p-adic* if its ring of powerbounded elements $\mathcal{O}_K = K^\circ$ is a ring of mixed characteristic $(0,p)$.

We follow [Sta21, Tag 02MN] for the definition of a (Weak) Serre subcategory of an abelian cateogry $\mathcal{A}$.

For an R-ringed site $(X, \mathcal{O}_X)$, an element of the derived category $\mathcal{F} \in \mathbf{D}(X)$, and an element $\varpi \in R$, we denote by $[\mathcal{F}/\varpi]$ the cone of the multiplication by ϖ-morphism, i.e.

$$[\mathcal{F}/\varpi] \coloneqq \mathrm{cone}(\mathcal{F} \xrightarrow{\varpi} \mathcal{F}).$$

Namely, we say that a non-empty full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if, for any exact sequence $A \to B \to C$ with $A, C \in \mathcal{C}$, we have $B \in \mathcal{C}$. We say that $\mathcal{C}$ is a *Weak Serre subcategory* if, for any exact sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$

with $A_0, A_1, A_3, A_4 \in \mathcal{C}$, we have $A_2 \in \mathcal{C}$. Look at [Sta21, Tag 02MP] and [Sta21, Tag 0754] for an alternative way to describe (Weak) Serre subcategories.

If $\mathcal{C}$ is a Serre subcategory of an abelian category $\mathcal{A}$ we define the *quotient category* as a pair $(\mathcal{A}/\mathcal{C}, F)$ of an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor

$$F \colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$$

such that, for any exact functor $G \colon \mathcal{A} \to \mathcal{B}$ to an abelian category $\mathcal{B}$ with $\mathcal{C} \subset \ker G$, there is a factorization $G = H \circ F$ for a unique exact functor $H \colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$. The quotient category always exists by [Sta21, Tag 02MS].

If $\mathcal{B}$ is a full triangulated subcategory of a triangulated category $\mathcal{D}$ we define the *Verdier quotient* as a pair $(\mathcal{D}/\mathcal{B}, F)$ of a triangulated category $\mathcal{D}/\mathcal{B}$ and an exact functor

$$F \colon \mathcal{D} \to \mathcal{D}/\mathcal{B}$$

such that, for any exact functor $G \colon \mathcal{D} \to \mathcal{D}'$ to a pre-triangulated category $\mathcal{D}'$ with $\mathcal{B} \subset \ker G$, there is a factorization $G = H \circ F$ for a unique exact functor $H \colon \mathcal{D}/\mathcal{B} \to \mathcal{D}'$. The Verdier quotient always exists by [Sta21, Tag 05RJ].

We say that a diagram of categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{B} \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "k"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{D} \end{tikzcd}$$

is $(2,1)$-*commutative* if $\alpha \colon k \circ h \Rightarrow g \circ f$ is a natural isomorphism of functors.

For an abelian group M and commuting endomorphisms $f_1, \dots, f_n$, we define the *Koszul complex*

$$K(M; f_1, \dots, f_n) \coloneqq M \to M \otimes_{\mathbf{Z}} \mathbf{Z}^n \to M \otimes_{\mathbf{Z}} \wedge^2 (\mathbf{Z}^n) \to \dots \to M \otimes_{\mathbf{Z}} \wedge^n (\mathbf{Z}^n)$$

viewed as a chain complex in cohomological degrees $0, \dots, n$. The differential

$$d^k \colon M \otimes_{\mathbf{Z}} \wedge^k (\mathbf{Z}^n) \simeq \bigoplus_{1 \le i_1 < \dots < i_k \le n} M \to M \otimes_{\mathbf{Z}} \wedge^{i+1} (\mathbf{Z}^n) \simeq \bigoplus_{1 \le j_1 < \dots < j_{k+1} \le n} M$$

from M in spot $i_1 < \dots < i_k$ to M in spot $j_1 < \dots < j_{k+1}$ is nonzero only if $\{i_1, \dots, i_k\} \subset \{j_1, \dots, j_{k+1}\}$, in which case it is given by $(-1)^{m-1} f_{j_m}$, where $m \in \{1, \dots, k+1\}$ is the unique integer such that $j_m \notin \{i_1, \dots, i_k\}$.

If M is an R-module and f_i are elements of R the complex $K(M; f_1, \dots, f_n)$ is a complex of R-modules and can be identified with

$$M \to M \otimes_R R^n \to M \otimes_R \wedge^2 (R^n) \to \dots \to M \otimes_R \wedge^n (R^n).$$

2. Almost Commutative Algebra

This chapter is devoted to the study of almost coherent modules. We recall some basic definitions of almost mathematics in Section 2.1. Then we discuss the main properties of almost finitely generated and almost finitely presented modules in Section 2.5. These two sections closely follow the discussion of almost mathematics in [GR03]. Section 2.6 is dedicated to almost coherent modules and almost coherent rings. We show that almost coherent modules from a Weak Serre subcategory of R-modules, and they coincide with almost finitely presented ones in the case of almost coherent rings. We discuss base change results in Section 2.8. Finally, we develop some topological aspects of almost finitely generated modules over "topologically universally adhesive rings" in Section 2.12.

2.1. The Category of Almost Modules.

We begin this section by recalling some basic definitions of almost mathematics from [GR03]. We fix some "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat. We always do almost mathematics with respect to $\mathfrak{m}$.

Lemma 2.1.1. Let M be an R-module. Then the following are equivalent:

- **(1)** The module $\mathfrak{m}M$ is the zero module.
- **(2)** The module $\mathfrak{m} \otimes_R M$ is the zero module.
- **(3)** The module $\widetilde{\mathfrak{m}} \otimes_R M$ is the zero module.
- **(4)** The module M is annihilated by ε for every $\varepsilon \in \mathfrak{m}$.

Proof. Note that the multiplication map $\mathfrak{m} \otimes_R \mathfrak{m} \to \mathfrak{m}$ is surjective as $\mathfrak{m}^2 = \mathfrak{m}$. This implies that we have surjections

$$\widetilde{\mathfrak{m}} \otimes_R M \twoheadrightarrow \mathfrak{m} \otimes_R M \twoheadrightarrow \mathfrak{m}M.$$

This shows that **(3)** implies **(2)**, and **(2)** implies **(1)**. It is clear that **(2)** implies **(3)**, and **(1)** is equivalent to **(4)**. So the only thing we are left to show is that **(1)** implies **(2)**.

Suppose that $\mathfrak{m}M \simeq 0$. Pick an arbitrary element $a \otimes m \in \mathfrak{m} \otimes_R M$ with $a \in \mathfrak{m}$, $m \in M$. Since $\mathfrak{m}^2 = \mathfrak{m}$ there is a finite number of elements $y_1, \dots, y_k, x_1, \dots, x_k \in \mathfrak{m}$ such that

$$a = \sum_{i=1}^{k} x_i y_i.$$

Then we have an equality

$$a \otimes m = \sum_{i=1}^{k} x_i y_i \otimes m = \sum_{i=1}^{k} x_i \otimes y_i m = 0.$$

□

Definition 2.1.2. An R-module M is *almost zero*, if any of the equivalent conditions of Lemma 2.1.1 is satisfied for M.

Lemma 2.1.3. Under the assumption as above, the "multiplication" morphism $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ is an isomorphism.

Proof. We consider a short exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Note that $(R/\mathfrak{m}) \otimes_R \mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2 = 0$, so we get a short exact sequence

$$0 \to \mathrm{Tor}_1^R(R/\mathfrak{m}, \mathfrak{m}) \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to 0.$$

Since $\mathrm{Tor}_1^R(R/\mathfrak{m}, \mathfrak{m})$ is almost zero, Lemma 2.1.1 says that after applying the functor $- \otimes_R \widetilde{\mathfrak{m}}$ we get an isomorphism

$$\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \mathfrak{m} \otimes_R \widetilde{\mathfrak{m}}.$$

Since $\widetilde{\mathfrak{m}}$ is R-flat, we also see that $\mathfrak{m} \otimes_R \widetilde{\mathfrak{m}}$ injects into $\widetilde{\mathfrak{m}}$. Moreover, it maps isomorphically onto its image $\mathfrak{m}\widetilde{\mathfrak{m}} = \widetilde{\mathfrak{m}}$ as $\mathfrak{m}^2 = \mathfrak{m}$. Altogether it shows that

$$\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \widetilde{\mathfrak{m}}.$$

It is straightforward to see that the constructed isomorphism is the "multiplication" map. □

We denote by Σ_R the category of almost zero R-modules considered as a full subcategory of $\mathbf{Mod}_R$.

Corollary 2.1.4. The category Σ_R is a Serre subcategory of $\mathbf{Mod}_R$[4].

Proof. This follows directly from criterion **(3)** from Lemma 2.1.1, flatness of $\widetilde{\mathfrak{m}}$ and [Sta21, Tag 02MP]. □

This corollary allows us to define the quotient category $\mathbf{Mod}_R^a := \mathbf{Mod}_R/\Sigma_R$ that we call as the category of almost R-modules[5]. Note that the localization functor

$$(-)^a : \mathbf{Mod}_R \to \mathbf{Mod}_R^a$$

is an exact and essentially surjective functor. We refer to elements of $\mathbf{Mod}_R^a$ as almost R-modules or R^a-modules. We will usually denote them by M^a in order to distinguish almost R-modules from R-modules.

To simplify some notations, we will use the notation $\mathbf{Mod}_R^a$ and $\mathbf{Mod}_{R^a}$ interchangeably.

Definition 2.1.5. A morphism $f : M \to N$ is called *an almost isomorphism* (resp. *almost injection*, resp. *almost surjection*) if the corresponding morphism $f^a : M^a \to N^a$ is an isomorphism (resp. injection, resp. surjection) in $\mathbf{Mod}_R^a$.

[4]We refer to [Sta21, Tag 02MN] for the discussion of (Weak) Serre categories.
[5]We refer to [Sta21, Tag 02MS] for the discussion of quotient categories.

Remark 2.1.6. For any R-module M, the natural morphism $\pi\colon \widetilde{\mathfrak{m}} \otimes_R M \to M$ is an almost isomorphism. Indeed, it suffices to show that

$$\widetilde{\mathfrak{m}} \otimes_R \ker \pi \simeq 0 \text{ and } \widetilde{\mathfrak{m}} \otimes_R \operatorname{Coker} \pi \simeq 0.$$

Using R-flatness of $\widetilde{\mathfrak{m}}$, we can reduce the question to showing that the the map

$$\widetilde{\mathfrak{m}} \otimes_R \pi\colon \widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R M$$

is an isomorphism. This follows from Lemma 2.1.3.

Definition 2.1.7. Two R-modules M and N are called *almost isomorphic* if M^a is isomorphic to N^a in $\mathbf{Mod}_R^a$.

Lemma 2.1.8. Let $f : M \to N$ be a morphism of R-modules, then

(1) The morphism f is an almost injection (resp. almost surjection, resp. almost isomorphism) if and only if $\ker(f)$ (resp. $\operatorname{Coker}(f)$, resp. $\ker(f)$ and $\operatorname{Coker}(f)$) is an almost zero module.

(2) We have a functorial bijection $\operatorname{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N) = \operatorname{Hom}_{\mathbf{Mod}_R^a}(M^a, N^a)$.

(3) Modules M and N are almost isomorphic (not necessary via a morphism f) if and only if $\widetilde{\mathfrak{m}} \otimes_R M \simeq \widetilde{\mathfrak{m}} \otimes_R N$.

Proof. **(1)** just follows from definition of the quotient category. **(2)** is discussed in detail in [GR03, page 12, (2.2.4)].

Now we show that **(3)** follows from **(1)** and **(2)**. Remark 2.1.6 implies that M and N are almost isomorphic if $\widetilde{\mathfrak{m}} \otimes_R M \simeq \widetilde{\mathfrak{m}} \otimes_R N$.

Now suppose that there is an almost isomorphism $\varphi : M^a \to N^a$. It has a representative $f : \widetilde{\mathfrak{m}} \otimes_R M \to N$ by **(2)**. Now **(1)** and R-flatness of $\widetilde{\mathfrak{m}}$ imply that $\widetilde{\mathfrak{m}} \otimes_R f : \widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R N$ is an isomorphism. Now $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \widetilde{\mathfrak{m}}$ by Lemma 2.1.3, so $\widetilde{\mathfrak{m}} \otimes_R f$ gives an isomorphism

$$\widetilde{\mathfrak{m}} \otimes_R f : \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R N.$$

□

We now define the *functor of almost sections*

$$(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$$

as

$$(M^a)_* \coloneqq \operatorname{Hom}_{\mathbf{Mod}_R^a}(R^a, M^a) = \operatorname{Hom}_R(\widetilde{\mathfrak{m}}, M)$$

for any R^a-module M^a with an R-module representative M. The construction is clearly functorial in M^a, so it does define the functor $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$.

The functor of almost sections is going to be the right adjoint to the almostification functor $(\text{-})^a$. Before we discuss why this is the case, we need to define the unit and counit transformations.

We start with the unit of the adjunction. For any R-module M, there is a functorial morphism

$$\eta_{M,*}\colon M \to \operatorname{Hom}_R(\widetilde{\mathfrak{m}}, M) = M^a_*$$

that can easily be seen to be an almost isomorphism.

This allows us to define a functorial morphism

$$\varepsilon_{N^a,*}\colon (N^a_*)^a \to N^a$$

for any R^a-module N^a. Namely, the map $\eta_{N,*}\colon N \to N^a_*$ is an almost isomorphism, so we can invert it in the almost category and define

$$\varepsilon_{N^a,*} \coloneqq (\eta^a_{N,*})^{-1}\colon (N^a_*)^a \to N^a$$

Now we define another functor

$$(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$$

that is going to be a left adjoint to the almostification functor $(-)^a$. Namely, we put

$$(M^a)_! := (M^a)_* \otimes_R \widetilde{\mathfrak{m}} \xleftarrow{\sim} M \otimes_R \widetilde{\mathfrak{m}}$$

for any R^a-module M^a with an R-module representative M. This construction is clearly functorial in M^a, so it does define a functor. Similarly to the discussion above, for any R-module M, we define the transformation

$$\varepsilon_{M,!}\colon (M^a)_! = \widetilde{\mathfrak{m}} \otimes_R M \to M$$

as the map induces by the the natural morphism $\widetilde{\mathfrak{m}} \to R$. Clearly, $\varepsilon_{M,!}$ is an almost isomorphism for any M. So, this actually allows us to define the morphism

$$\eta_{N^a,!}\colon N^a \to (\widetilde{\mathfrak{m}} \otimes_R N)^a \simeq (N^a_!)^a$$

as $\eta_{N^a,!} = (\varepsilon^a_{N,!})^{-1}$.

We summarize the main properties of these functors in the lemma below:

Lemma 2.1.9. Let R and $\mathfrak{m}$ be as above. Then

(1) The functor $(-)_*$ is the right adjoint to $(-)^a$. In particular, it is left exact.

(2) The unit of the adjunction is equal to $\eta_{M,*}$, the counit of the adjunction is equal to $\varepsilon_{N^a,*}$. In particular, both of the are isomorphisms.

(3) The functor $(-)_!$ is the left adjoint to the localization functor $(-)^a$.

(4) The functor $(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ is exact.

(5) The unit of the adjunction is equal to $\eta_{N^a,!}$, the counit of the adjunction is equal to $\varepsilon_{M,!}$. In particular, both are almost isomorphisms.

Proof. This is explained [GR03, Proposition 2.2.13 and Proposition 2.2.21]. □

Corollary 2.1.10. Let R and $\mathfrak{m}$ be as above. Then $(-)^a\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ commutes with limits and colimits. In particular, $\mathbf{Mod}_R^a$ is complete and cocomplete, and filtered colimits and (arbitrary) products are exact in $\mathbf{Mod}_R^a$.

Proof. The first claim follows from the fact that $(-)^a$ admits left and right adjoints. The second claim follows the first claim, exactness of $(-)^a$, and analogous exactness properties in $\mathbf{Mod}_R$. □

The last thing we need to address in this section is how almost mathematics interacts with base change. We want to be able to speak about preservation of various properties of modules under a base change along a map $R \to S$. The issue here is to define the corresponding ideal $\mathfrak{m}_S$ as in the definition of almost mathematics. It turns out that the most naive ideal $\mathfrak{m}_S := \mathfrak{m}S$ works well, but the reason is that the assumptions on the ideal defining almost mathematics are rather weak. More specifically, we could have required the flatness of the ideal $\mathfrak{m}$ (instead of $\widetilde{\mathfrak{m}}$), and then the ideal $\mathfrak{m}S$ would not serve well for defining almost mathematics on S. The next lemma shows that everything works well in the current setup.

Lemma 2.1.11. Let $f : R \to S$ be a ring homomorphism, and let $\mathfrak{m}_S$ be the ideal $\mathfrak{m}S \subset S$. Then we have an equality $\mathfrak{m}_S^2 = \mathfrak{m}_S$ and the S-module $\widetilde{\mathfrak{m}_S} := \mathfrak{m}_S \otimes_S \mathfrak{m}_S$ is S-flat.

Proof. The equality $\mathfrak{m}_S^2 = \mathfrak{m}_S$ follows from the analogous assumption on $\mathfrak{m}$ and the construction of $\mathfrak{m}_S$. As for the flatness issue, we claim that $\mathfrak{m}_S \otimes_S \mathfrak{m}_S \simeq (\mathfrak{m} \otimes_R S) \otimes_S (\mathfrak{m} \otimes_R S)$. That would certainly imply that desired flatness statement. In order to prove this claim, we look at a short exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$$

We apply $- \otimes_R S$ to get a short exact sequence

$$0 \to \mathrm{Tor}_1^R(R/\mathfrak{m}, S) \to \mathfrak{m} \otimes_R S \to \mathfrak{m}S \to 0.$$

We observe that $\mathrm{Tor}_1^R(R/\mathfrak{m}, S)$ is almost zero, so both $\mathrm{Tor}_1^R(R/\mathfrak{m}, S) \otimes_S \mathfrak{m}S$ and $\mathrm{Tor}_1^R(R/\mathfrak{m}, S) \otimes_S (\mathfrak{m} \otimes_R S)$ are zero modules by Lemma 2.1.1. So we use functors $- \otimes_S (\mathfrak{m} \otimes_R S)$ and $- \otimes_S \mathfrak{m}S$ to obtain isomorphisms

$$(\mathfrak{m} \otimes_R S) \otimes_S (\mathfrak{m} \otimes_R S) \simeq \mathfrak{m}S \otimes_R (\mathfrak{m} \otimes_R S) \simeq (\mathfrak{m}S) \otimes_S (\mathfrak{m}S).$$

Thus we get the desired equality. ☐

Lemma 2.1.12. Let $f\colon R \to S$ be a ring homomorphism, and $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ an R-linear functor (resp. $F\colon \mathbf{Mod}_R^{op} \to \mathbf{Mod}_S$ an R-linear functor). Then F sends almost zero R-modules to almost zero S-modules.

Proof. Suppose that M is an almost zero R-module, so $\varepsilon M = 0$ for any $\varepsilon \in \mathfrak{m}$. Then $\varepsilon F(M) = 0$ because F is R-linear, so $F(M)$ is almost zero by Lemma 2.1.1. ☐

Corollary 2.1.13. Let $f\colon R \to S$ be a ring homomorphism, and $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ a left or right exact R-linear functor (resp. $F\colon \mathbf{Mod}_R^{op} \to \mathbf{Mod}_S$ a left or right exact R-linear functor). Then F preserves almost isomorphisms.

Proof. We only show the case of a left exact functor $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$, all other cases are analogous to the this one.

Choose any almost isomorphism $f\colon M' \to M''$, we want to show that $F(f)$ is an almost isomorphism. Consider the following exact sequences:

$$0 \to K \to M' \to M \to 0,$$

$$0 \to M \to M'' \to Q \to 0.$$

We know that K and Q are almost zero by our assumption on f. Now, the above short exact sequences induce the following exact sequences:

$$0 \to F(K) \to F(M') \to F(M) \to \mathrm{R}^1F(K),$$

$$0 \to F(M) \to F(M'') \to F(Q).$$

Lemma 2.1.12 guarantees that $F(K)$, $\mathrm{R}^1F(K)$, and $F(Q)$ are almost zero S-modules. Therefore, the morphisms $F(M') \to F(M)$ and $F(M) \to F(M'')$ are both almost isomorphisms. In particular, the composition $F(M') \to F(M'')$ is an almost isomorphism as well. ☐

2.2. **Basic Functors on the Categories of Almost Modules.** The category of almost modules admits certain natural functors induced from the category of R-modules. It has two versions of the Hom-functor and the tensor product functor. We summarize properties of these functors in the following proposition:

Proposition 2.2.1. Let $R, \mathfrak{m}$ be as above. Then

(1) We define *tensor product* functor $-\otimes_{R^a}-\colon \mathbf{Mod}_R^a \times \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ as

$$(M^a, N^a) \mapsto (M_!^a \otimes_R N_!^a)^a \ .$$

Then there is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{Mod}_R \times \mathbf{Mod}_R \arrow[r, "-\otimes_R -"] \arrow[d, "(-)^a\times(-)^a"'] & \mathbf{Mod}_R \arrow[d, "(-)^a"] \\
\mathbf{Mod}_R^a \times \mathbf{Mod}_R^a \arrow[r, "-\otimes_{R^a}-"'] \arrow[ur, phantom, "\rho"] & \mathbf{Mod}_R^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(M \otimes_R N)^a \simeq M^a \otimes_{R^a} N^a$ for any $M, N \in \mathbf{Mod}_R$.

(2) There is a functorial isomorphism

$$\mathrm{Hom}_{R^a}(M^a, N^a) \simeq \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N) \ .$$

for any $M, N \in \mathbf{Mod}_R$. In particular, there is a canonical structure of an R-module on the group $\mathrm{Hom}_{R^a}(M^a, N^a)$; thus defines the functor

$$\mathrm{Hom}_{R^a}(-,-)\colon \mathbf{Mod}_{R^a}^{op} \times \mathbf{Mod}_{R^a} \to \mathbf{Mod}_R$$

(3) We define the functor $\mathrm{alHom}_{R^a}(-,-)\colon \mathbf{Mod}_{R^a}^{op} \times \mathbf{Mod}_{R^a} \to \mathbf{Mod}_{R^a}$ of *almost homomorphisms* as

$$(M^a, N^a) \mapsto \mathrm{Hom}_{R^a}(M^a, N^a)^a \ .$$

Then there is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{Mod}_R^{op} \times \mathbf{Mod}_R \arrow[r, "\mathrm{Hom}_R(-,-)"] \arrow[d, "(-)^a\times(-)^a"'] & \mathbf{Mod}_R \arrow[d, "(-)^a"] \arrow[dl, phantom, "\rho"] \\
\mathbf{Mod}_{R^a}^{op} \times \mathbf{Mod}_{R^a} \arrow[r, "\mathrm{alHom}_{R^a}(-,-)"] & \mathbf{Mod}_{R^a}
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, $\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathrm{Hom}_R(M,N)^a$ for any $M, N \in \mathbf{Mod}_R$.

Proof. **(1)**. We define

$$\rho_{M,N}\colon (M_!^a \otimes_R N_!^a)^a \to (M \otimes_R N)^a$$

to be the morphism induced by

$$M_!^a \simeq \widetilde{\mathfrak{m}} \otimes_R M \to M \text{ and } N_!^a \simeq \widetilde{\mathfrak{m}} \otimes_R N \to N.$$

It is clear that $\rho_{M,N}$ is functorial in both variables, so it defines a natural transformation of functors ρ. We also need to check that $\rho_{M,N}$ is an isomorphism for any M and N. This follows from the following two observations: $\rho_{M,N}$ is an isomorphism if and only if $\rho_{M,N} \otimes_R \widetilde{\mathfrak{m}}$ is an isomorphism; and $\rho_{M,N} \otimes_R \widetilde{\mathfrak{m}}$ is easily seen to be an isomorphism as $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ is an isomorphism.

(2) is just a reformulation of Lemma 2.1.8**(2)**.

In order to show **(3)**, we need to define a functorial morphism

$$\rho_{M,N}\colon \mathrm{Hom}_R(M,N)^a \to \mathrm{alHom}_{R^a}(M^a, N^a).$$

We start by using the functorial identification

$$\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N)^a$$

from (**2**). Namely, we define $\rho_{M,N}$ as the morphism $\mathrm{Hom}_R(M,N)^a \to \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N)^a$ induced by the map $\widetilde{\mathfrak{m}} \otimes M \to M$. This is clearly functorial in both variables, so it defines the natural transformation ρ.

We also need to check that $\rho_{M,N}$ is an isomorphism for any M and N. This boils down to the fact that $\mathrm{Hom}_R(-,N)$ sends almost isomorphisms to almost isomorphisms. This, in turn, follows from Corollary 2.1.13. □

Remark 2.2.2. It is straightforward to check that if N has a structure of an S^a-module for some R-algebra S, then the R^a-modules $M^a \otimes_{R^a} N^a$, $\mathrm{alHom}_{R^a}(M^a, N^a)$ have functorial-in-M^a structures of S^a-modules. This implies that the functors $-\otimes_{R^a} N^a$, $\mathrm{alHom}_{R^a}(-, N^a)$ naturally land in $\mathbf{Mod}_S^a$, i.e. define functors

$$- \otimes_{R^a} N^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a, \text{ and } \mathrm{alHom}_{R^a}(-, N^a) \colon \mathbf{Mod}_R^{a,op} \to \mathbf{Mod}_S^a$$

Similarly, $\mathrm{Hom}_{R^a}(-, N^a)$ defines a functor $\mathbf{Mod}_R^a \to \mathbf{Mod}_S$.

The functor of almost homomorphisms is quite important as it turns out to be the *inner Hom functor*, i.e. it is right adjoint to the tensor product.

Lemma 2.2.3. Let $f\colon R \to S$ be a ring homomorphism, and let M^a be an R^a-module and N^a, K^a be S^a-modules. Then there is a functorial S-linear isomorphism

$$\mathrm{Hom}_{S^a}(M^a \otimes_{R^a} N^a, K^a) \simeq \mathrm{Hom}_{R^a}(M^a, \mathrm{alHom}_{S^a}(N^a, K^a)) .$$

Proof. This is a consequence of the usual $\otimes$-Hom-adjunction, Proposition 2.2.1, and the fact that $\widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}}$. Indeed, we have the following sequence of functorial isomorphisms

$$\begin{aligned}\mathrm{Hom}_{S^a}(M^a \otimes_{R^a} N^a, K^a) &\simeq \mathrm{Hom}_S(\widetilde{\mathfrak{m}} \otimes_R M \otimes_R N, K)\\ &\simeq \mathrm{Hom}_S((\widetilde{\mathfrak{m}} \otimes_R M) \otimes_R (\widetilde{\mathfrak{m}} \otimes_R N), K)\\ &\simeq \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, \mathrm{Hom}_S(\widetilde{\mathfrak{m}} \otimes_R N, K))\\ &\simeq \mathrm{Hom}_{R^a}(M, \mathrm{alHom}_{S^a}(N^a, K^a)) .\end{aligned}$$

The first isomomorphism follows from Proposition 2.2.1(**1**), (**2**), the second isomorphism follows from the observation $\widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}}$, the third isomorphism is just the classical $\otimes$-Hom-adjunction, and the last isomorphism is a consequence of Proposition 2.2.1(**2**), (**3**). □

Corollary 2.2.4.
(1) Let N be an R^a-module, then the functor $- \otimes_{R^a} N^a$ is left adjoint to the functor $\mathrm{alHom}_{R^a}(N^a, -)$.
(2) Let $R \to S$ be a ring homomorphism. Then the functor $- \otimes_{R^a} S^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a$ is left adjoint to the forgetful functor.

Proof. Part (**1**) follows from Lemma 2.2.3 by taking S to be equal to R. Part (**2**) follows from Lemma 2.2.3 by taking N^a to be equal to S^a. □

We finish the section by introducing the certain types of R^a-modules that will be used throughout the paper.

Definition 2.2.5.
- An R^a-module M^a is *flat* if the functor $M^a \otimes_{R^a} - \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ is exact.
- An R^a-module M^a is *faithfully flat* if it is flat and $N^a \otimes_{R^a} M^a \simeq 0$ if and only if $N^a \simeq 0$.
- An R-module M is *almost flat* (resp. *almost faithfully flat*) if an R^a-module M^a is *flat* (resp. *faithfully flat*)
- An R^a-module I^a is *injective* if the functor $\mathrm{Hom}_{R^a}(-, I^a) \colon \mathbf{Mod}_R^{a,op} \to \mathbf{Mod}_R$ is exact.

- An R^a-module P^a is *almost projective* if the functor $\mathrm{alHom}_{R^a}(P^a, -) \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ is exact.

Lemma 2.2.6. The functor $(-)^a \colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ sends flat (resp. faithfully flat, resp. injective, resp. projective) R-modules to flat (resp. faithfully flat, resp. injective, resp. almost projective) R^a-modules.

Proof. The case of flat modules is clear from Lemma 2.2.1(**1**). Now suppose that M is a faithfully flat R-module. Recall that $M \otimes_R - \colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ is an exact and faithful functor. Therefore, if $M \otimes_R N$ is almost zero, it implies that so is N. Thus Lemma 2.2.1(**1**) ensures that M^a is almost faithfully flat.

The case of injective modules follows from the fact that $(-)^a$ admits an exact left adjoint functor $(-)_!$. The case of projective modules is clear from the definition. $\square$

Lemma 2.2.7. The functor $(-)_! \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ sends flat R^a-modules to flat R-modules.

Proof. This follows from the formula $M^a_! \otimes_R N \simeq (M^a \otimes_{R^a} N^a)_!$ for any R^a-module M^a and R-module N. $\square$

Warning 2.2.8. If M^a is a faithfully flat R^a-module, the R-module $M^a_!$ may not be faithfully flat. For instance, R^a is a faithfully flat R^a-module, but $R^a_! = \widetilde{\mathfrak{m}}$ is not. For example, $\widetilde{\mathfrak{m}} \otimes_R R/\mathfrak{m} \simeq 0$.

Corollary 2.2.9. Any bounded above complex $C^{\bullet,a} \in \mathbf{Comp}^-(R^a)$ admits a resolution $P^{\bullet,a} \to C^\bullet$ by a bounded above complex of almost projective modules.

Proof. We consider the complex $C^{\bullet,a}_! \in \mathbf{Comp}^-(R)$, this complex admits a resolution by complex of free modules $p \colon P^\bullet \to C^{\bullet,a}_!$. Now we apply $(-)^a$ to this morphism to get the map

$$P^{\bullet,a} \xrightarrow{p^a} (C^{\bullet,a}_!)^a \xleftarrow{\varepsilon} C^{\bullet,a} .$$

The map ε is an *isomorphism* in $\mathbf{Comp}(R^a)$ by Lemma 2.1.9, and p^a is a quasi-isomorphism. Thus $\varepsilon^{-1} \circ p^a \colon P^{\bullet,a} \to C^{\bullet,a}$ is a quasi-isomorphism in $\mathbf{Comp}(R^a)$. Now note that each term of $P^{\bullet,a}$ is almost projective by Lemma 2.2.6. $\square$

2.3. **Derived Category of Almost Modules.** We define the derived category of almost modules in two different ways and show that these definitions coincide. Later we define certain derived functors on the derived category of almost modules. We pay some extra attention to show that the functors in this section are well-defined on unbounded derived categories.

We start the section by introducing two different notions of the derived category of almost modules and then show that they are actually the same.

Definition 2.3.1. We define the *derived category of almost R-modules* as $\mathbf{D}(R^a) := \mathbf{D}(\mathbf{Mod}_R^a)$.

We define the bounded version of derived category of almost R-modules $\mathbf{D}^*(R^a)$ for $* \in \{+, -, b\}$ as the full subcategory consisting of bounded below (resp. bounded above, resp. bounded) complexes.

Definition 2.3.2. We define the *almost derived category of R-modules* as the Verdier quotient $\mathbf{D}(R)^a := \mathbf{D}(\mathbf{Mod}_R)/\mathbf{D}_{\Sigma_R}(\mathbf{Mod}_R)$.

We recall that Σ_R is the Serre subcategory of $\mathbf{Mod}_R$ that consists of almost zero modules, and $\mathbf{D}_{\Sigma_R}(\mathbf{Mod}_R)$ is the full triangulated category of elements in $\mathbf{D}(\mathbf{Mod}_R)$ with almost zero cohomology modules.

We note that the functor $(-)^a\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ is exact and additive. Thus it can be derived to the functor $(-)^a\colon \mathbf{D}(R) \to \mathbf{D}(R^a)$. Similarly, the functor $(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ is additive and exact, thus it can be derived to the functor $(-)_!\colon \mathbf{D}(R^a) \to \mathbf{D}(R)$. The standard argument shows that $(-)_!$ is a left adjoint functor to the functor $(-)^a$ as this already happens on the level of abelian categories. Now we also want to derive the functor $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. In order to do this on the level of unbounded derived categories, we need to show that $\mathbf{D}(R^a)$ has "enough K-injective objects".

Definition 2.3.3. We say that a complex of R^a-module $I^{\bullet,a}$ is *K-injective* if $\mathrm{Hom}_{K(R^a)}(C^{\bullet,a}, I^{\bullet,a}) = 0$ for any acyclic complex $C^{\bullet,a}$ of R^a-modules.

Remark 2.3.4. We remind the reader that $K(R^a)$ stands for the homotopy category of R^a-modules.

The first thing we need to show is that $\mathbf{Comp}(R^a)$ has "enough" K-injective objects. This will allow us derive many functors.

Lemma 2.3.5. The functor $(-)^a\colon \mathbf{Comp}(R) \to \mathbf{Comp}(R^a)$ sends K-injective R-complexes to K-injective R^a-complexes.

Proof. We note that $(-)^a$ admits an exact left adjoint $(-)_!$ thus [Sta21, Tag 08BJ] ensures that $(-)^a$ preserves K-injective complexes. $\square$

Corollary 2.3.6. Every object $M^{\bullet,a} \in \mathbf{Comp}(R^a)$ is quasi-isomorphic to a K-injective complex.

Proof. We know that the complex $M^\bullet \in \mathbf{Comp}(R)$ is quasi-isomorphic to a K-injective complex $I^\bullet$ by [Sta21, Tag 090Y] (or [Sta21, Tag 079P]). Now we use Lemma 2.3.5 to say that $I^{\bullet,a}$ is a K-injective complex that is quasi-isomorphic to $M^{\bullet,a}$. $\square$

Now as the first application of Corollary 2.3.6 we define the functor $(-)_*\colon \mathbf{D}(R^a) \to \mathbf{D}(R)$ as the derived functor of $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. This functor exists by [Sta21, Tag 070K].

Lemma 2.3.7.
(1) The functors $\mathbf{D}(R) \underset{(-)^a}{\overset{(-)_!}{\leftrightarrows}} \mathbf{D}(R^a)$ are adjoint. Moreover, the unit (resp. counit) morphism
$$(M^a)_! \to M \text{ (resp. } N \to (N_!)^a)$$
is an almost isomorphism (resp. isomorphism) for any $M \in \mathbf{D}(R), N \in \mathbf{D}(R^a)$. In particular, the functor $(-)^a$ is essentially surjective.

(2) The functors $\mathbf{D}(R) \underset{(-)_*}{\overset{(-)^a}{\rightleftarrows}} \mathbf{D}(R^a)$ are adjoint. Moreover, the unit (resp. counit) morphism
$$M \to (M^a)_* \text{ (resp. } (N_*)^a \to N)$$
is an almost isomorphism (resp. isomorphism) for any $M \in \mathbf{D}(R), N \in \mathbf{D}(R^a)$.

Proof. We start the proof by showing **(1)**. Firstly we note that the functors $(-)_!$ and $(-)^a$ are adjoint by the discussion above. Now we show the cone of the counit map is always in $\mathbf{D}_{\Sigma_R}(R)$. As both functors $(-)^a$ and $(-)_!$ are exact on the level of abelian categories, it suffices to show the claim for $M \in \mathbf{Mod}_R^a$. But then the statement follows from Lemma 2.1.9**(5)**. The same argument shows that the unit map $N \to (N_!)^a$ is an isomorphism for any $N \in \mathbf{D}(R^a)$.

Now we go to **(2)**. We define the functor $(-)_*\colon \mathbf{D}(R^a) \to \mathbf{D}(R)$ as the right derived functor of the left exact additive functor $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. This functor exists by [Sta21, Tag 070K] and Corollary 2.3.6. The functor $(-)_*$ is right adjoint to $(-)^a$ by [Sta21, Tag 0DVC].

We check that the natural map $M \to (M^a)_*$ is an almost isomorphism for any $M \in \mathbf{D}(R)$. We choose some K-injective resolution $M \xrightarrow{\sim} I^\bullet$. Then Lemma 2.3.5 guarantees that $M^a \to I^{\bullet,a}$ is a K-injective resolution of the complex M^a. The map $M \to (M^a)_*$ has a representative

$$I^\bullet \to (I^{\bullet,a})_* \ .$$

This map is an almost isomorphism of complexes by Lemma 2.1.9**(2)**. Thus the map $M \to (M^a)_*$ is an almost isomorphism. A similar argument shows that the counit map $(N_*)^a \to N$ is an (almost) isomorphism for any $N \in \mathbf{D}(R^a)$. □

Theorem 2.3.8. The functor $(-)^a\colon \mathbf{D}(R) \to \mathbf{D}(R^a)$ induces an equivalence of triangulated categories $(-)^a\colon \mathbf{D}(R)^a \to \mathbf{D}(R^a)$.

Proof. We recall that the Verdier quotient is constructed as the localization of $\mathbf{D}(R)$ along the morphisms f such that $\mathrm{cone}(f) \in \mathbf{D}_{\Sigma_R}(R)$. For instance, this is the definition of Verdier quotient at [Sta21, Tag 05RI]. Now we see that a morphism $f^a\colon C^a \to C'^a$ is invertible in $\mathbf{D}(R^a)$ if and only if $\mathrm{cone}(f) \in \mathbf{D}_{\Sigma_R}(R)$ by the definition of Σ_R and exactness of $(-)^a$. Moreover, $(-)^a$ admits a right adjoint such that $(-)^a \circ (-)_* \to \mathrm{Id}$ is an isomorphism of functors. Thus we can apply [GZ67, Proposition 1.3] to say that the induced functor $(-)^a\colon \mathbf{D}(R)^a \to \mathbf{D}(R^a)$ must be an equivalence. □

Remark 2.3.9. Theorem 2.3.8 shows that the two notions of the derived category of almost modules are the same. In what follows, we do not distinguish $\mathbf{D}(R^a)$ and $\mathbf{D}(R)^a$ anymore.

2.4. **Basic Functors on the Derived Categories of Almost Modules.** Now we can "derive" certain functors constructed in previous section. We start with defining the derived versions of different Hom functors, after that we move to the case of the derived tensor product functor.

Definition 2.4.1. We define the *derived Hom* functor

$$\mathbf{R}\mathrm{Hom}_{R^a}(-,-)\colon \mathbf{D}(R^a)^{op} \times \mathbf{D}(R^a) \to \mathbf{D}(R)$$

as it is done in [Sta21, Tag 0A5W] using the fact that $\mathbf{Comp}(R^a)$ has enough K-injective complexes.

We define the *Ext modules* as R-modules defined as

$$\mathrm{Ext}^i_{R^a}(M^a, N^a) \coloneqq \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a))$$

for $M^a, N^a \in \mathbf{Mod}_R^a$.

Explicitly, for any $M^a, N^a \in \mathbf{D}(R^a)$, the construction of the complex $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)$ goes as follows. We pick a representative $C^{\bullet,a} \to M^a$ and a K-injective resolution $N^a \to I^{\bullet,a}$. Then we set $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) = \mathrm{Hom}^\bullet_{R^a}(C^{\bullet,a}, I^{\bullet,a})$. This construction is independent of the choices and functorial in both variables. We are not going to review this theory here, but rather refer to [Sta21, Tag 0A5W] for the details.

Remark 2.4.2. We see that [Sta21, Tag 0A64] implies that there is a functorial isomorphism

$$\mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{R^a}\left(M^a, N^a\right)\right) \simeq \mathrm{Hom}_{\mathbf{D}(R)^a}\left(M^a, N^a[i]\right) \ .$$

Lemma 2.4.3. **(1)** There are functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{D}(R)^a}(M^a, N^a) \simeq \mathrm{Hom}_{\mathbf{D}(R)}(M^a_!, N) \text{ and } \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \simeq \mathbf{R}\mathrm{Hom}_R(M^a_!, N)$$

for any $M, N \in \mathbf{D}(R)$.

(2) For any chosen $M^a \in \mathbf{Mod}_R^a$, the functor $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, -)\colon \mathbf{D}(R)^a \to \mathbf{D}(R)$ is isomorphic to the (right) derived functor of $\mathrm{Hom}_{R^a}(M^a, -)$.

Proof. The first claim easily follows from the fact $(-)^a$ is a right adjoint to the exact functor $(-)_!$. We leave the details to the reader.

The second claim follows from [Sta21, Tag 070K] and Corollary 2.3.6. $\square$

Definition 2.4.4. We define the *derived functor of almost homomorphisms*

$$\mathbf{R}\mathrm{alHom}_{R^a}(-,-)\colon \mathbf{D}(R^a)^{op} \times \mathbf{D}(R^a) \to \mathbf{D}(R^a)$$

as

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \coloneqq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)^a = \mathbf{R}\mathrm{Hom}_R(M^a_!, N)^a \ .$$

We define the *almost Ext modules* as R^a-modules defined by

$$\mathrm{alExt}^i_{R^a}(M^a, N^a) \coloneqq \mathrm{H}^i(\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a))$$

for $M^a, N^a \in \mathbf{Mod}^a_R$.

Definition 2.4.5. We define the *the complex of almost homomorphisms* $\mathrm{alHom}^\bullet_{R^a}(K^{\bullet,a}, L^{\bullet,a})$ for $K^{\bullet,a}, L^{\bullet,a} \in \mathbf{Comp}(R^a)$ as follows:

$$\mathrm{alHom}^n_{R^a}(K^{\bullet,a}, L^{\bullet,a}) \coloneqq \prod_{n=p+q} \mathrm{alHom}_{R^a}(K^{-q,a}, L^{p,a})$$

with the differential

$$\mathrm{d}(f) = \mathrm{d}_{L^{\bullet,a}} \circ f - (-1)^n f \circ \mathrm{d}_{K^{\bullet,a}} \ .$$

Lemma 2.4.6. Let $P^{\bullet,a}$ be a bounded above complex of R^a-modules with almost projective cohomology modules. Suppose that $M^{\bullet,a} \to N^{\bullet,a}$ is an almost quasi-isomorphism of bounded below complex of R^a-modules. Then the natural morphism

$$\mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, M^{\bullet,a}) \to \mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, N^{\bullet,a})$$

is an almost quasi-isomorphism.

Proof. We note that as in the case of the usual Hom-complexes, there are convergent[6] spectral sequences

$$\mathrm{E}_1^{i,j} = \mathrm{H}^j\left(\mathrm{alHom}^\bullet_{R^a}(P^{-i,a}, M^{\bullet,a})\right) \Rightarrow \mathrm{H}^{i+j}\left(\mathrm{alHom}^\bullet_{R^a}\left(P^{\bullet,a}, M^{\bullet,a}\right)\right)$$

$$\mathrm{E'}_1^{i,j} = \mathrm{H}^j\left(\mathrm{alHom}^\bullet_{R^a}(P^{-i,a}, N^{\bullet,a})\right) \Rightarrow \mathrm{H}^{i+j}\left(\mathrm{alHom}^\bullet_{R^a}\left(P^{\bullet,a}, N^{\bullet,a}\right)\right)$$

Moreover, there is a natural morphism of spectral sequences $\mathrm{E}_1^{i,j} \to \mathrm{E'}_1^{i,j}$. Thus it suffices to show that the associated map on the first page is an almost isomorphism on each entry. Now we use the fact that $\mathrm{alHom}_{R^a}(P^{-i,a}, -)$ is exact to rewrite the first page of this spectral sequence as

$$\mathrm{E}_1^{i,j} = \mathrm{alHom}_{R^a}\left(P^{-i,a}, \mathrm{H}^j(M^{\bullet,a})\right)$$

and the same for $\mathrm{E'}_1^{i,j}$. So the question boils down to show that the natural morphisms

$$\mathrm{alHom}_{R^a}\left(P^{-i,a}, \mathrm{H}^j(M^{\bullet,a})\right) \to \mathrm{alHom}_{R^a}\left(P^{-i,a}, \mathrm{H}^j(N^{\bullet,a})\right)$$

are almost isomorphisms. But this is clear as $M^{\bullet,a} \to N^{\bullet,a}$ is an almost quasi-isomorphism. $\square$

[6] Here we use that $P^{\bullet,a}$ is bounded above, $M^{\bullet,a}$ and $N^{\bullet,a}$ are bounded below

Lemma 2.4.7. Let $P_1^{\bullet,a} \to P_2^{\bullet,a}$ be an almost quasi-isomorphism of bounded above complexes with almost projective cohomology modules. Suppose that $M^{\bullet,a}$ is a bounded below complex of R^a-modules. Then the natural morphism

$$\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, M^{\bullet,a}) \to \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a})$$

is an almost quasi-isomorphism.

Proof. We choose some injective resolution $M^{\bullet,a} \to I^{\bullet,a}$ of the bounded below complex $M^{\bullet,a}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, M^{\bullet,a}) \arrow[r] \arrow[d] & \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a}) \arrow[d] \\
\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, I^{\bullet,a}) \arrow[r] & \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a}).
\end{tikzcd}$$

The bottom horizontal arrow is an almost quasi-isomorphism by the standard categorical argument with injective resolutions. The vertical maps are almost quasi-isomorphism by Lemma 2.4.6. □

Proposition 2.4.8. **(1)** There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(R)^{op} \times \mathbf{D}(R) \arrow[r, "\mathbf{R}\mathrm{Hom}_R(-,-)"] \arrow[d, "(-)^a \times (-)^a"'] & \mathbf{D}(R) \arrow[d, "(-)^a"] \arrow[dl, Rightarrow, "\rho"] \\
\mathbf{D}(R^a)^{op} \times \mathbf{D}(R^a) \arrow[r, "\mathbf{R}\mathrm{alHom}_{R^a}(-,-)"] & \mathbf{D}(R^a)
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular,

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathbf{R}\mathrm{Hom}_R(M, N)^a$$

for any $M, N \in \mathbf{D}(R)$.

(2) For any chosen $M^a \in \mathbf{Mod}_R^a$, the functor $\mathbf{R}\mathrm{alHom}_{R^a}(M^a, -)\colon \mathbf{D}(R^a) \to \mathbf{D}(R^a)$ is isomorphic to the (right) derived functor of $\mathrm{alHom}_{R^a}(M^a, -)$.

(3) For any chosen $N^a \in \mathbf{Mod}_R^a$, the functor $\mathbf{R}\mathrm{alHom}_{R^a}(-, N^a)\colon \mathbf{D}^-(R^a)^{op} \to \mathbf{D}(R^a)$ is isomorphic to the (right) derived functor of $\mathrm{alHom}_{R^a}(-, N^a)$.

Proof. In order to show Part (1), we construct functorial morphisms

$$\rho_{M,N}\colon \mathbf{R}\mathrm{Hom}_R(M, N)^a \to \mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a)\ .$$

for any $M, N \in \mathbf{D}(R)$. We recall that there is a functorial identification

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathbf{R}\mathrm{Hom}_R(M^a_!, N)^a \cong^a \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N)^a.$$

So we define

$$\rho_{M,N}\colon \mathbf{R}\mathrm{Hom}_R(M, N)^a \to \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N)^a$$

as the morphism induced by the canonical map $\widetilde{\mathfrak{m}} \otimes_R M \to M$. This is clearly functorial, so it defines the natural transformation of functors. The only thing we are left to show is that $\rho_{M,N}$ is an almost isomorphism for any $M, N \in \mathbf{D}(R)$.

Let us recall that the way we compute $\mathbf{R}\mathrm{Hom}_R(M, N)$. It is isomorphic to $\mathrm{Hom}^\bullet_R(C^\bullet, I^\bullet)$ for any choice of a K-injective resolution of $N \xrightarrow{\sim} I^\bullet$ and any resolution $M \xrightarrow{\sim} C^\bullet$. Since $\widetilde{\mathfrak{m}} \otimes_R C^\bullet$ is a resolution of $\widetilde{\mathfrak{m}} \otimes_R M$ by R-flatness of $\widetilde{\mathfrak{m}}$, we reduce the question to show that the natural map

$$\mathrm{Hom}^\bullet_R(C^\bullet, I^\bullet) \to \mathrm{Hom}^\bullet_R(\widetilde{\mathfrak{m}} \otimes_R C^\bullet, I^\bullet)$$

is an almost quasi-isomorphism of complexes. We actually show more, we show that it is an almost isomorphism of complexes. Indeed, the degree n part of this map is the map

$$\prod_{p+q=n} \mathrm{Hom}_R(C^{-q}, I^p) \to \prod_{p+q=n} \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R C^{-q}, I^p) \ .$$

Since the (infinite) product is an exact functor in $\mathbf{Mod}_R^a$, and any (infinite) product of almost zero modules is almost zero, it is actually sufficient to show that each particular map $\mathrm{Hom}_R(C^{-q}, I^p) \to \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R C^{-q}, I^p)$ is an almost isomorphism. This follows from Proposition 2.2.1(**3**).

Part **(2)** is similar to that of Proposition 2.4.3.

Part **(3)** is also similar to Part **(2)** of Proposition 2.4.3, but there are some subtleties due to the fact that $\mathbf{Mod}_R^a$ does not have enough projective objects. We fix this issue by using instead [Sta21, Tag 06XN] of [Sta21, Tag 070K]. We apply it to the subset $\mathcal{P}$ being the set of bounded above complexes with almost projective terms. This result is indeed applicable in our situation due to Corollary 2.2.9 and Lemma 2.4.7. □

Now we deal with the case of the derived tensor product functor.

Definition 2.4.9. We say that a complex of R^a-module $K^{\bullet,a}$ is *almost K-flat* if the naive tensor product complex $C^{\bullet,a} \otimes^{\bullet}_{R^a} K^{\bullet,a}$ is acyclic for any acyclic complex $C^{\bullet,a}$ of R^a-modules

Lemma 2.4.10. The functor $(-)^a \colon \mathbf{Comp}(R) \to \mathbf{Comp}(R^a)$ sends K-flat R-complexes to almost K-flat R^a-complexes.

Proof. Suppose that $C^{\bullet,a}$ is an acyclic complex of R^a-modules and $K^\bullet$ is a K-flat compelx. Then we see that

$$C^{\bullet,a} \otimes^{\bullet}_{R^a} K^{\bullet,a} \cong^a (C^\bullet \otimes^{\bullet}_R K^\bullet)^a \cong^a (\widetilde{\mathfrak{m}} \otimes_R C^\bullet \otimes^{\bullet}_R K^\bullet)^a \cong^a ((\widetilde{\mathfrak{m}} \otimes_R C^\bullet) \otimes^{\bullet}_R K^\bullet)^a \ .$$

The latter complex is acyclic as $\widetilde{\mathfrak{m}} \otimes C^\bullet$ is acyclic and $K^\bullet$ is K-flat. □

Corollary 2.4.11. Every object $M^{\bullet,a} \in \mathbf{Comp}(R^a)$ is quasi-isomorphic to an almost K-flat complex.

Proof. We know that the complex $M^\bullet \in \mathbf{Comp}(R)$ is quasi-isomorphic to a K-flat complex $K^\bullet$ by [Sta21, Tag 06Y4]. Now we use Lemma 2.4.10 to say that $K^{\bullet,a}$ is almost K-flat complex that is quasi-isomorphic to $M^{\bullet,a}$. □

Definition 2.4.12. We define the *derived tensor product functor*

$$- \otimes^L_{R^a} - \colon \mathbf{D}(R)^a \times \mathbf{D}(R)^a \to \mathbf{D}(R)^a$$

by the rule $(M^a, N^a) \mapsto (M_! \otimes^L_R N_!)^a$ for any $M^a, N^a \in \mathbf{D}(R)^a$.

Proposition 2.4.13. **(1)** There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(R) \times \mathbf{D}(R) \arrow[r, "- \otimes^L_R -"] \arrow[d, "(-)^a \times (-)^a"'] & \mathbf{D}(R) \arrow[d, "(-)^a"] \\
\mathbf{D}(R)^a \times \mathbf{D}(R)^a \arrow[r, "- \otimes^L_{R^a} -"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{D}(R)^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(M \otimes^L_R N)^a \simeq M^a \otimes^L_{R^a} N^a$ for any $M, N \in \mathbf{D}(R)$.

(2) For any chosen $M^a \in \mathbf{Mod}^a_R$, the functor $M^a \otimes^L_{R^a} -\colon \mathbf{D}(R)^a \to \mathbf{D}(R)^a$ is isomorphic to the (left) derived functor of $M^a \otimes_{R^a} -$.

Proof. The proof of Part **(1)** is similar to that of Lemma 2.2.1**(1)**. We leave details to the reader.

The proof of Part **(2)** is similar to that of Proposition 2.4.8**(2)**. The claim follows by applying [Sta21, Tag 06XN] with $\mathcal{P}$ being the subset of almost K-flat complexes. This result is indeed applicable in our situation due to Corollary 2.4.11 and the almost version of [Sta21, Tag 064L]. $\square$

Lemma 2.4.14. Let $M^a, N^a, K^a \in \mathbf{D}(R)^a$, then we have a functorial isomorphism

$$\mathbf{R}\mathrm{Hom}_{R^a}(M^a \otimes^L_{R^a} N^a, K^a) \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{alHom}_{R^a}(N^a, K^a)) \ .$$

In particular, the functors $\mathbf{R}\mathrm{alHom}_{R^a}(N^a, -)\colon \mathbf{D}(R)^a \xleftarrow{\quad}\!\!\xrightarrow{\quad} \mathbf{D}(R)^a \colon - \otimes^L_{R^a} N^a$ are adjoint.

Proof. The claim follows from the following sequence of canonical identifications:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{R^a}(M^a \otimes^L_{R^a} N^a, K^a) &\simeq \mathbf{R}\mathrm{Hom}_R((\widetilde{\mathfrak{m}} \otimes_R M) \otimes^L_R (\widetilde{\mathfrak{m}} \otimes_R N), K) && \text{Lemma 2.4.3(1)}\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R N, K)) && \text{[Sta21, Tag 0A5W]}\\
&\simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R N, K)^a) && \text{Lemma 2.4.3(1)}\\
&\simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{alHom}_{R^a}(N^a, K^a)) \ . && \text{Definition 2.4.4}
\end{aligned}$$

$\square$

Definition 2.4.15. Let $f\colon R \to S$ be a ring homomorphism. We define the *base change functor*

$$- \otimes^L_{R^a} S^a \colon \mathbf{D}(R)^a \to \mathbf{D}(S)^a$$

by the rule $M^a \mapsto (M_! \otimes^L_R S)^a$ for any $M^a \in \mathbf{D}(R)^a$.

Proposition 2.4.16. **(1)** There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(R) \arrow[r, "- \otimes^L_R S"] \arrow[d, "(-)^a"'] & \mathbf{D}(S) \arrow[d, "(-)^a"] \\
\mathbf{D}(R)^a \arrow[r, "- \otimes^L_{R^a} S^a"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{D}(S)^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(M \otimes^L_R S)^a \simeq M^a \otimes^L_{R^a} S^a$ for any $M \in \mathbf{D}(R)$.

(2) The functor $- \otimes^L_{R^a} S^a \colon \mathbf{D}(R)^a \to \mathbf{D}(S)^a$ is isomorphic to the (left) derived functor of $- \otimes^L_{R^a} S^a$.

Proof. The proof is identical to Proposition 2.4.13. $\square$

Lemma 2.4.17. Let $R \to S$ be a ring homomorphism, and let $M^a \in \mathbf{D}(R)^a$, $N^a \in \mathbf{D}(S)^a$. Then we have a functorial isomorphism

$$\mathbf{R}\mathrm{Hom}_{S^a}(M^a \otimes^L_{R^a} S^a, N^a) \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \ .$$

In particular, the functors $\mathrm{Forget}\colon \mathbf{D}(S)^a \xleftarrow{\quad}\!\!\xrightarrow{\quad} \mathbf{D}(R)^a \colon - \otimes^L_{R^a} S^a$ are adjoint.

Proof. The proof is similar to that of Lemma 2.4.14. $\square$

2.5. **Almost Finitely Generated and Almost Finitely Presented Modules.** We discuss the notions of almost finitely generated and almost finitely presented modules in section. The discussion follows [GR03] closely. The main difference is that we avoid any use of "uniform structures" in our treatment, we think that it simplifies the exposition. We recall that we fixed some "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and we always do almost mathematics with respect to this ideal.

Definition 2.5.1. An R-module M is called *almost finitely generated*, if for any $\varepsilon \in \mathfrak{m}$ there is an integer n_ε and an R-homomorphism

$$R^{n_\varepsilon} \xrightarrow{f} M$$

such that $\mathrm{Coker}(f)$ is killed by ε.

Definition 2.5.2. An R-module M is called *almost finitely presented*, if for any $\varepsilon, \delta \in \mathfrak{m}$ there are integers $n_{\varepsilon,\delta}$, $m_{\varepsilon,\delta}$ and a complex

$$R^{m_{\varepsilon,\delta}} \xrightarrow{g} R^{n_{\varepsilon,\delta}} \xrightarrow{f} M$$

such that $\mathrm{Coker}(f)$ is killed by ε and $\delta(\ker f) \subset \mathrm{Im}\, g$.

Remark 2.5.3. Clearly, any almost finitely presented R-module is almost finitely generated.

Remark 2.5.4. A typical example of an almost finitely presented module that is not finitely generated is $M = \oplus_{n\geq 1} \mathcal{O}_C/p^{1/n}\mathcal{O}_C$ for an algebraically closed non-archimedean field C of mixed characteristic $(0, p)$.

The next few lemmas discuss the most basic properties of almost finitely generated and almost finitely presented modules. For example, it is not entirely obvious that these notions transfer across almost isomorphisms. We show that this is actually the case, so these notions descend to $\mathbf{Mod}^a_R$. We also show that almost finitely generated and almost finitely presented modules have many good properties that we have for the usual finitely generated and finitely presented modules. Although all proofs below are elementary, they require some accuracy to rigorously prove them.

Our first main goal is to get some other useful criteria for a module to be almost finitely generated (resp. almost finitely presented) and finally show that this notion does not depend on a class of almost isomorphism.

Lemma 2.5.5. Let M be an R-module, then M is almost finitely generated if and only if for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there a morphism $R^n \xrightarrow{f} M$ such that $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.

Proof. The "if" part is clear, so we only need to deal with the "only if" part. We choose a set of generators $(\varepsilon_0, \dots, \varepsilon_n)$ for an ideal $\mathfrak{m}_0$. Then we have R-morphisms

$$f_i \colon R^{n_{\varepsilon_i}} \to M$$

such that $\varepsilon_i(\mathrm{Coker}\, f_i) = 0$ for all i. Then the sum of these morphisms

$$f := \bigoplus_{i=1}^{n} f_i \colon R^{\sum n_{\varepsilon_i}} \to M$$

defines a map such that $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$. Since $\mathfrak{m}_0$ was an arbitrary morphism, this finishes the proof. ∎

Lemma 2.5.6. Let M be an almost finitely presented R-module, and let $\varphi : R^n \to M$ be an R-homomorphism such that $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$ for some ideal $\mathfrak{m}_1 \subset \mathfrak{m}$. Then for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m}$ there is morphism $\psi : R^m \to M$ such that

$$R^m \xrightarrow{\psi} R^n \xrightarrow{\varphi} M$$

is a three-term complex and $\mathfrak{m}_0(\operatorname{Ker}\varphi) \subset \operatorname{Im}(\psi)$.

Proof. Since M is almost finitely presented, for any two elements $\varepsilon_1, \varepsilon_2 \in \mathfrak{m}$, we can find a complex

$$R^{m_2} \xrightarrow{g} R^{m_1} \xrightarrow{f} M$$

such that $\varepsilon_1(\operatorname{Coker} f) = 0$ and $\varepsilon_2(\ker f) \subset \operatorname{Im} g$. Now we choose some element $\delta \in \mathfrak{m}_1$, and we shall define morphisms

$$\alpha : R^{m_1} \to R^n \text{ and } \beta : R^n \to R^{m_1}$$

such that $\varphi \circ \alpha = \delta f$ and $f \circ \beta = \varepsilon_1\varphi$. Here is the corresponding picture:

$$\begin{tikzcd}
R^{m_2} \arrow[r, "g"] & R^{m_1} \arrow[r, "f"] \arrow[d, shift left, "\alpha"] & M \\
& R^n \arrow[u, shift left, "\beta"] \arrow[ur, "\varphi"] &
\end{tikzcd}$$

We define α and β in the following way: we fix a basis $e_1, \dots, e_{m_1}$ of R^{m_1} and a basis $e'_1, \dots, e'_n$ of R^n, then we define

$$\alpha(e_i) = y_i \in R^n \text{ for some } y_i \text{ such that } \varphi(y_i) = \delta f(e_i),$$
$$\beta(e'_j) = x_j \in R^{m_1} \text{ for some } x_j \text{ such that } f(x_j) = \varepsilon_1\varphi(e'_j)$$

and then extend these maps by linearity. It is clear that $\varphi \circ \alpha = \delta f$ and $f \circ \beta = \varepsilon_1\varphi$ as it holds on basis elements.

Now we can define a morphism $\psi : R^n \oplus R^{m_2} \to R^n$ by the rule

$$\psi(x, y) = \alpha \circ \beta(x) - (\varepsilon_1\delta)x + \alpha \circ g(y).$$

We now show that

$$\varphi \circ \psi = 0 \text{ and } \varepsilon_1\varepsilon_2\delta \operatorname{Ker}\varphi \subset \operatorname{Im}\psi.$$

We start by showing that $\varphi \circ \psi = 0$: it suffices to prove that

$$(\alpha \circ g)(y) \in \operatorname{Ker}\varphi \text{ for } y \in R^{m_2}, \text{ and } (\alpha \circ \beta)(x) - (\varepsilon_1\delta)x \in \operatorname{Ker}\varphi \text{ for } x \in R^n$$

We note that we have an equality

$$(\varphi \circ \alpha \circ g)(y) = \delta(f \circ g)(y) = \delta 0 = 0,$$

so $(\alpha \circ g)(y) \in \operatorname{Ker}(\varphi)$. We also have an equality

$$\begin{aligned}
(\varphi \circ (\alpha \circ \beta - \varepsilon_1\delta))(x) &= (\varphi \circ \alpha \circ \beta)(x) - \varepsilon_1\delta\varphi(x) \\
&= \delta(f \circ \beta)(x) - \varepsilon_1\delta\varphi(x) \\
&= \delta\varepsilon_1\varphi(x) - \varepsilon_1\delta\varphi(x) \\
&= 0.
\end{aligned}$$

this shows that $(\alpha \circ \beta)(x) - (\varepsilon_1\delta)x \in \operatorname{Ker}(\varphi)$ as well.

We show that $(\varepsilon_1\varepsilon_2\delta)\operatorname{Ker}\varphi \subset \operatorname{Im}(\psi)$: we observe that for any $x \in \operatorname{Ker}\varphi$ we have $\beta(x) \subset \operatorname{Ker} f$ as $f \circ \beta = \varepsilon_1\varphi$. This implies that $\varepsilon_2\beta(x) \in \operatorname{Im} g$ since $\varepsilon_2 \operatorname{Ker} f \subset \operatorname{Im} g$. Thus there is $y \in R^{m_2}$ such that $g(y) = \varepsilon_2\beta(x)$, so $(\alpha \circ g)(y) = \varepsilon_2\alpha \circ \beta(x)$. This shows that

$$\psi(-\varepsilon_2 x, y) = -\varepsilon_2(\alpha \circ \beta)(x) + \varepsilon_1\varepsilon_2\delta x + (\alpha \circ g)(y) = \\ -\varepsilon_2(\alpha \circ \beta)(x) + \varepsilon_1\varepsilon_2\delta x + \varepsilon_2(\alpha \circ \beta)(x) = \varepsilon_1\varepsilon_2\delta x$$

We conclude that $\varepsilon_1\varepsilon_2\delta x \in \operatorname{Im}(\psi)$ for any $x \in \operatorname{Ker}(\varphi)$.

Finally, we recall that $\mathfrak{m}_0$ is a finitely generated ideal, and that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}^2 \subset \mathfrak{m}_1$. This means that we can find a finite set I, and a finite set of elements $\varepsilon_{i,1}, \varepsilon_{i,2} \in \mathfrak{m}, \delta_i \in \mathfrak{m}_1$ such that $\mathfrak{m}_0$ is contained in the ideal $J := (\varepsilon_{i,1}\varepsilon_{i,2}\delta_i)_{i\in I}$ (the ideal generated by all the products $\varepsilon_{i,1}\varepsilon_{i,2}\delta_i$). The previous discussion implies that for each $i \in I$, we have a map $\psi_i : R^{k_i} \to R^n$ such that $\varphi \circ \psi_i = 0$ and $(\varepsilon_{i,1}\varepsilon_{i,2}\delta_i)(\operatorname{Ker}\varphi) \subset \operatorname{Im}\psi_i$. By passing to the homomorphism

$$\psi \coloneqq \bigoplus_{i\in I} \psi_i \colon R^{\sum k_i} \to R^n$$

we get a map ψ such that $\varphi \circ \psi = 0$ and $\mathfrak{m}_0(\operatorname{Ker}\varphi) \subset \operatorname{Im}(\psi)$. Therefore ψ does the job. $\square$

Lemma 2.5.7. Let M be an R-module. Then the following conditions are equivalent:

- **(1)** The R-module M is almost finitely presented.
- **(2)** For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist a finitely presented R-module N and a homomorphism $f : N \to M$ such that $\mathfrak{m}_0(\ker f) = 0$ and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.
- **(3)** For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist integers n, m and a three-term complex

$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$

  such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f) \subset \operatorname{Im} g$.

Proof. It is clear that the condition **(3)** implies both conditions **(1)** and **(2)**.

We show that **(1)** implies **(3)**. Since M is an almost finitely generated R-module, Lemma 2.5.5 guarantees that for any finitely generated ideal $\mathfrak{m}' \subset \mathfrak{m}$ there is a morphism $R^n \xrightarrow{f} M$ such that $\mathfrak{m}'(\operatorname{Coker} f) = 0$.

We know that $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^2$, this easily implies that there is a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m} \subset \mathfrak{m}_1$. So, using $\mathfrak{m}' = \mathfrak{m}_1$, we can find a homomorphism $R^n \xrightarrow{\varphi} M$ such that $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$. Lemma 2.5.6 claims that we can also find a homomorphism $\psi \colon R^m \to R^n$ such that

$$R^m \xrightarrow{\psi} R^n \xrightarrow{\varphi} M$$

is a three-term complex and $\mathfrak{m}_0(\ker\varphi) \subset \operatorname{Im}\psi$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1$ and $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$, we get that $\mathfrak{m}_0(\operatorname{Coker}\varphi) = 0$ as well. This finishes the proof since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal of $\mathfrak{m}$.

Now we show that **(2)** implies **(3)**. We pick an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, and we try to find a three-term complex

$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$

such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\ker f) \subset \operatorname{Im}(g)$. In order to achieve this we use the assumption **(2)** to find a morphism $h : N \to M$ such that N is a finitely presented R-module, $\mathfrak{m}_0(\operatorname{Coker} h) = 0$, and $\mathfrak{m}_0(\ker h) = 0$. Since N is finitely presented we can find a short exact sequence

$$R^m \xrightarrow{g} R^n \xrightarrow{f'} N \to 0$$

It is straightforward to see that a three-term complex

$$R^m \xrightarrow{g} R^n \xrightarrow{f := h \circ f'} M$$

satisfies the condition that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\ker f) \subset \operatorname{Im}(g)$. □

Lemma 2.5.8. Let M be an R-module, and suppose that for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a morphism $f\colon N \to M$ such that $\mathfrak{m}_0(\ker f) = 0$, $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and N is almost finitely generated (resp. almost finitely presented). Then M is also almost finitely generated (resp. almost finitely presented).

Proof. We give a proof only in the almost finitely presented case; the other case is easier. We pick an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Then we use the assumption to get a morphism

$$f : N \to M$$

such that $\mathfrak{m}_1(\operatorname{Ker} f) = 0, \mathfrak{m}_1(\operatorname{Coker} f) = 0$ and N is an almost finitely presented R-module. Lemma 2.5.7 guarantees that there is a three-term complex

$$R^m \xrightarrow{h} R^n \xrightarrow{g} N$$

such that $\mathfrak{m}_1(\operatorname{Coker} g) = 0$ and $\mathfrak{m}_1(\operatorname{Ker} g) \subset \operatorname{Im} h$. Then we can consider a three-term complex

$$R^m \xrightarrow{h} R^n \xrightarrow{f' := f \circ g} M,$$

it is easily seen that $\mathfrak{m}_1^2(\operatorname{Coker} f') = 0$ and $\mathfrak{m}_1^2(\ker f') \subset \operatorname{Im}(h)$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ we conclude that $\mathfrak{m}_0(\operatorname{Coker} f') = 0$ and $\mathfrak{m}_0(\ker f') \subset \operatorname{Im}(h)$. This shows that M is almost finitely presented. □

Lemma 2.5.9. Let M be an R-module, and $\{N_i\}_{i \in I}$ is a filtered diagram of R-modules. Then

(1) The natural morphism
$$\gamma^0_M\colon \operatorname{colim}_I \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \operatorname{colim}_I N_i)$$
is almost injective for an almost finitely generated M;

(2) The natural morphism
$$\gamma^0_M\colon \operatorname{colim}_I \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \operatorname{colim}_I N_i)$$
is an almost isomorphism and
$$\gamma^1_M\colon \operatorname{colim} \operatorname{Ext}^1_R(M, N_i) \to \operatorname{Ext}^1_R(M, \operatorname{colim} N_i)$$
is almost injective for an almost finitely presented M.

Proof. We give a proof for an almost finitely presented M, the case of an almost finitely generated M is similar.

Step 1*: The case of finitely presented M.* If M is finitely presented, γ^0_M is an isomorphism and γ^1_M is injective. This follows from [Sta21, Tag 064T] and [Sta21, Tag 0G8W].

Step 2*: General case.* We fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Since $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^4$, there is a finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^4$. Now we use Lemma 2.5.7**(2)** to find a finitely presented module M' and a morphism $f\colon M' \to M$ such that $\ker(f)$ and $\mathrm{Coker}(f)$ are annihilated by $\mathfrak{m}_1$. We denote the image of f by M'' and consider the short exact sequences

$$0 \to K \to M' \to M'' \to 0\ ,$$

$$0 \to M'' \to M \to Q \to 0$$

with K and Q being annihilated by $\mathfrak{m}_1$. After applying the functors $\mathrm{colim}_I \mathrm{Hom}_R(-, N_i)$ and $\mathrm{Hom}_R(-, \mathrm{colim}_I N_i)$ and considering the associated long exact sequences, we see that

$$b_i\colon\ \mathrm{colim}_I \mathrm{Ext}^i_R(M, N_i) \to \mathrm{colim}_I \mathrm{Ext}^i_R(M', N_i)$$

and

$$c_i\colon \mathrm{Ext}^i_R(M, \mathrm{colim}_I N_i) \to \mathrm{Ext}^i_R(M', \mathrm{colim}_I N_i)$$

have kernels and cokernels annihilated by $\mathfrak{m}_1^2$ for any $i \geq 0$. Now we consider a commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_I \mathrm{Ext}^i_R(M', N_i) \arrow[r, "\gamma^i_{M'}"] & \mathrm{Ext}^i_R(M', \mathrm{colim}_I N_i) \\
\mathrm{colim}_I \mathrm{Ext}^i_R(M, N_i) \arrow[r, "\gamma^i_M"] \arrow[u, "b_i"] & \mathrm{Ext}^i_R(M, \mathrm{colim}_I N_i) \arrow[u, "c_i"]
\end{tikzcd}$$

By Step 1, we know that $\gamma^i_{M'}$ is an isomorphism for $i = 0$ and injective for $i = 1$. Moreover, we know that b_i and c_i have kernels and cokernels annihilated by $\mathfrak{m}_1^2$. Then it is easy to see that $\mathrm{Coker}(\gamma^0_M)$, $\ker(\gamma^0_M)$, and $\ker(\gamma^1_M)$ are annihilated by $\mathfrak{m}_1^4$. In particular, they are annihilated by $\mathfrak{m}_0 \subset \mathfrak{m}_1^4$. Since $\mathfrak{m}_0$ was arbitrary finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, we conclude that γ^0_M is an almost isomorphism and γ^1_M is almost injective. □

Lemma 2.5.10. Let M be an R-module.

(1) If, for any filtered diagram of R-modules $\{N_i\}_{i \in I}$, the natural morphism

$$\mathrm{colim}_I \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \mathrm{colim}_I N_i)$$

is almost injective, then M is almost finitely generated.

(2) If, for any filtered system of R-modules $\{N_i\}$, the natural morphism

$$\mathrm{colim}_I \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \mathrm{colim}_I N_i)$$

is an almost isomorphism, then M is almost finitely presented.

Proof. **(1)** : Note that $M \simeq \mathrm{colim}_I M_i$ is a filtered colimit of its finitely generated *submodules*. Therefore, we see that

$$\mathrm{colim}_I \mathrm{Hom}_R(M, M/M_i) \simeq^a \mathrm{Hom}_R\left(M, \mathrm{colim}_I(M/M_i)\right) \simeq 0.$$

Consider an element α of $\mathrm{colim}_I \mathrm{Hom}_R(M, M/M_i)$ that has a representative the quotient morphism $M \to M/M_i$ (for some choice of $i \in I$). Then, for every $\varepsilon \in \mathfrak{m}$, $\varepsilon\alpha = 0$ in $\mathrm{colim}_I \mathrm{Hom}_R(M, M/M_i)$. Explicitly this means that there is $j \geq i$ such that $\varepsilon M \subset M_j$. Now we choose a surjection $R^{n_j} \to M_j$ to see that the composition $f\colon R^{n_j} \to M$ gives a map with $\varepsilon(\mathrm{Coker}\, f) = 0$. Now note that this property is preserved by choose any $j' > j$. Therefore, for any $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n)$, we can find a finitely generated submodule $M_i \subset M$ such that $\mathfrak{m}_0 M \subset M_i$. Therefore, M is almost finitely generated.

(2) : Fix any finitely generated sub-ideal $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n) \subset \mathfrak{m}$. We use [Sta21, Tag 00HA] to write $M \simeq \mathrm{colim}_\Lambda M_\lambda$ as a filtered colimit of *finitely presented* R-modules. By assumption, the natural morphism

$$\mathrm{colim}_\Lambda \mathrm{Hom}_R(M, M_\lambda) \to \mathrm{Hom}_R(M, \mathrm{colim}_\Lambda M_\lambda) = \mathrm{Hom}_R(M, M)$$

is an almost isomorphism. In particular, $\varepsilon_i \mathrm{Id}_M$ is in the image of this map for every $i = 1, \dots, n$. This means that, for every ε_i, there is $\lambda_i \in \Lambda$ and a morphism $g_i \colon M \to M_{\lambda_i}$ such that the composition

$$f_{\lambda_i} \circ g_i = \varepsilon_i \mathrm{Id}_M,$$

where $f_{\lambda_i} \colon M_{\lambda_i} \to M$. Note that existence of such g_i is preserved by replacing λ_i by any $\lambda_i' \geq \lambda_i$. Therefore, using that $\{M_\lambda\}$ is a filtered diagram, we can find one index λ with maps

$$g_i \colon M \to M_\lambda$$

such that $f_\lambda \circ g_i = \varepsilon_i \mathrm{Id}_M$. Now we consider a morphism

$$F_i \coloneqq g_i \circ f_\lambda - \varepsilon_i \mathrm{Id}_{M_\lambda} \colon M_\lambda \to M_\lambda.$$

Note that $\mathrm{Im}(F_i) \subset \ker(f_\lambda)$ because

$$f_\lambda \circ g_i \circ f_\lambda - f_\lambda \varepsilon_i \mathrm{Id}_{M_i} = \varepsilon_i f_\lambda - \varepsilon_i f_\lambda = 0.$$

We also have that $\varepsilon_i \ker(f_\lambda) \subset \mathrm{Im}(F_i)$ because $F_i|_{\ker(f_\lambda)} = \varepsilon_i \mathrm{Id}$. Therefore, $\sum_i \mathrm{Im}(F_i)$ is a finite R-module such that

$$\mathfrak{m}_0(\ker f_\lambda) \subset \sum_i \mathrm{Im}(F_i) \subset \ker(f_\lambda).$$

Therefore, $f \colon M' \coloneqq M_\lambda / (\sum_i \mathrm{Im}(F_i)) \to M$ is morphism such that M' is finitely presented, $\mathfrak{m}_0(\ker f) = 0$, and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}$ was an arbitrary finitely generated sub-ideal, we conclude that M is almost finitely presented. □

Corollary 2.5.11. Let M be an R-module. Then

(1) M is almost finitely generated if and only if, for every filtered diagram $\{N_i^a\}_{i \in I}$ of R^a-modules, the natural morphism

$$\mathrm{colim}_I \, \mathrm{alHom}_R(M^a, N_i^a) \to \mathrm{alHom}_R(M^a, \mathrm{colim}_I N_i^a)$$

is injective in $\mathbf{Mod}_R^a$;

(2) M is almost finitely presented if and only if, for every filtered diagram $\{N_i^a\}_{i \in I}$ of R^a-modules, the natural morphism

$$\mathrm{colim}_I \, \mathrm{alHom}_R(M^a, N_i^a) \to \mathrm{alHom}_R(M^a, \mathrm{colim}_I N_i^a)$$

is an isomorphism in $\mathbf{Mod}_R^a$;

Proof. It formally follows from Lemma 2.5.9, Lemma 2.5.10, Proposition 2.2.1 **(3)**, and Corollary 2.1.10. □

Corollary 2.5.12. Let M and N be two almost isomorphic R-modules (see Definition 2.1.7). Then M is almost finitely generated (resp. almost finitely presented) if and only if so is N.

Proof. Corollary 2.5.11 implies that M is almost finitely generated (resp. almost finitely presented) if and only if $M_!^a$ is. Since $M_!^a \simeq N_!^a$, we get the desired result. □

Corollary 2.5.13. Let $R \to S$ be an almost isomorphism of rings. Then the forgetful functor $\mathbf{Mod}_{S^a}^* \to \mathbf{Mod}_{R^a}^*$ is an equivalence for $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}\}$.

Proof. Corollary 2.5.11 ensures that it suffices to prove the claim for $* =$ " " as the property of being almost finitely generated (resp. almost finitely presented) depends only on the category $\mathbf{Mod}_{R^a}$ and not on the ring R itself.

Corollary 2.2.4 **(2)** guarantee that the forgetful functor admits a right adjoint $-\otimes_{R^a} S^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a$. Therefore, it suffices to show that the natural morphisms

$$M^a \otimes M^a \otimes_{R^a} S^a$$

and

$$N^a \otimes_{R^a} S^a \to N^a$$

are isomorphisms for any $M \in \mathbf{Mod}_R^a$ and $N \in \mathbf{Mod}_S^a$. This is obvious from the fact that $R^a \to S^a$ is an isomorphism of R^a-modules. □

Definition 2.5.14. We say that an R^a-module $M^a \in \mathbf{Mod}_R^a$ is *almost finitely generated* (resp. *almost finitely presented*) if its representative $M \in \mathbf{Mod}_R$ is almost finitely generated (resp. almost finitely presented). This definition does not depend on a choice of representative by Lemma 2.5.12

We now want to establish certain good properties of almost finitely presented modules in short exact sequences. This will be crucial later to develop a good theory of almost coherent modules.

Lemma 2.5.15. Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be an exact sequence of R-modules, then

(1) If M is almost finitely generated, then so is M''.
(2) If M' and M'' are almost finitely generated (resp. finitely presented), then so is M.
(3) If M is almost finitely generated and M'' is almost finitely presented, then M' is almost finitely generated.
(4) If M is almost finitely presented and M' is almost finitely generated, then M'' is almost finitely presented.

Proof. The previous version of this manuscript contained a direct (but very tedious) proof of this claim. However, now we only note that it can be easily deduced from Lemma 2.5.9 and Lemma 2.5.10 via the five lemma (or diagram chase). We only note that the Ext^1 part of Lemma 2.5.9 **(2)** is crucial to make the argument work. □

Corollary 2.5.16. Let $0 \to M'^a \xrightarrow{\varphi} M^a \xrightarrow{\psi} M''^a \to 0$ be an exact sequence of R^a-modules. Then all the conclusions of Lemma 2.5.15 still hold.

Proof. We use Lemma 2.1.9**(4)**,**(5)** to see that the sequence

$$0 \to (M'^a)_! \xrightarrow{\varphi_!} (M^a)_! \xrightarrow{\psi_!} (M''^a)_! \to 0$$

is exact and almost isomorphic to the original one. Moreover, Corollary 2.5.12 says that each of those modules $N^a_!$ is almost finitely generated (resp. almost finitely presented) if and only if so is the corresponding N^a. Thus the problem is reduced to Lemma 2.5.15. □

Lemma 2.5.17. Let M^a, N^a be two almost finitely generated (resp. almost finitely presented) R^a-modules, then so is $M^a \otimes_{R^a} N^a$. Similarly, $M \otimes_R N$ is almost finitely generated (resp. almost finitely presented) for any almost finitely generated (resp. almost finitely presented) R-modules M and N.

Proof. We show the claim only in the case of almost finitely presented modules, the case of almost finitely generated modules is significantly easier. Moreover, we use Proposition 2.2.1(**1**) to reduce the question to show that the tensor product of two almost finitely presented R-modules is almost finitely presented.

Step 1. The case of finitely presented modules: If both M and N are finitely presented, then this is a standard fact proven in [Bou98, II, §3.6, Proposition 6].

Step 2. The case of M being finitely presented: Now we deal with the case of a finitely presented R-module M and merely almost finitely presented N. We fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and a finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Now we use Lemma 2.5.7(**2**) to find a finitely presented module N' and a morphism $f\colon N' \to N$ such that $\ker(f)$ and $\mathrm{Coker}(f)$ are annihilated by $\mathfrak{m}_0$. We denote the image of f by N'' and consider the short exact sequences

$$0 \to K \to N' \to N'' \to 0 \ ,$$

$$0 \to N'' \to N \to Q \to 0$$

with K and Q being annihilated by $\mathfrak{m}_0$. After applying the functor $M \otimes_R -$, we get the following exact sequences:

$$M \otimes_R K \to M \otimes_R N' \to M \otimes_R N'' \to 0 \ ,$$

$$\mathrm{Tor}_1^R(M, Q) \to M \otimes_R N'' \to M \otimes_R N \to M \otimes_R Q \to 0 \ .$$

We note that $M \otimes_R K, \mathrm{Tor}_1^R(M, Q)$, and $M \otimes_R Q$ are annihilated by $\mathfrak{m}_0$. Now it is straightforward to conclude that the map

$$M \otimes_R f\colon M \otimes N' \to M \otimes N$$

has kernel and cokernel annihilated by $\mathfrak{m}_1 \subset \mathfrak{m}_0^2$. Moreover, $M \otimes N'$ is a finitely presented module by Step 1. Since $\mathfrak{m}_1$ was an arbitrary finitely generated subideal of $\mathfrak{m}$, we conclude that $M \otimes N$ is almost finitely presented by Lemma 2.5.7(**2**).

Step 3. The general case: Repeat the argument of Step 2 once again using Step 2 in place of Step 1 at the end, and Lemma 2.5.8 in place of Lemma 2.5.7(**2**). ☐

Lemma 2.5.18. Let M be an almost finitely presented R-module, let N be any R-module, and let P be an almost flat R-module. Then the natural map $\mathrm{Hom}_R(M, N) \otimes_R P \to \mathrm{Hom}_R(M, N \otimes_R P)$ is an almost isomorphism.

Similarly, $\mathrm{Hom}_{R^a}(M^a, N^a) \otimes_{R^a} P^a \to \mathrm{Hom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$ is an almost isomorphism for any almost finitely presented R^a-module M^a, any R^a-module N^a, and an almost flat R^a-module P^a.

Proof. Proposition 2.2.1(**1**) and (**3**) ensure that it suffices to prove the claim for the case of honest R-modules M, N, and P.

Step 1. The case of a finitely presented module M: We choose a presentation of M:

$$R^n \to R^m \to M \to 0$$

Then we use that P is almost flat to get a morphism of almost exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, N) \otimes_R P \arrow[r] \arrow[d] & \mathrm{Hom}_R(R^m, N) \otimes_R P \arrow[r] \arrow[d] & \mathrm{Hom}_R(R^n, N) \otimes_R P \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(M, N \otimes_R P) \arrow[r] & \mathrm{Hom}_R(R^m, \otimes_R P) \arrow[r] & \mathrm{Hom}_R(R^n, N \otimes_R P).
\end{tikzcd}$$

Clearly, the second and third vertical arrows are (almost) isomorphisms, so the first vertical arrow is an almost isomorphism as well.

Step 2. The General Case: The case of almost finitely presented module M follows from the finitely presented case by approximating it by finitely presented ones. This is similar to the strategy used in Lemma 2.5.17, we leave the details to the reader. $\square$

The last thing that we will need is the interaction between properties of an R-module M and its "reduction" M/I for some finitely generated ideal $I \subset \mathfrak{m}$. For example, we know that for an ideal $I \subset \mathrm{rad}(R)$ and a finite module M, Nakayama's lemma states that $M/I = 0$ if and only if $M = 0$. Another thing is that an I-adically complete module M is R-finite if and only if M/I is R/I-finite. It turns out that both facts have their "almost" analogues.

Lemma 2.5.19. Let $I \subset \mathfrak{m} \cap \mathrm{rad}(R)$ be a finitely generated ideal. If M is an almost finitely generated R-module such that $M/IM \simeq 0$. Then $M \simeq 0$. If $M/IM \cong^a 0$, then $M \cong^a 0$.

Proof. We use a definition of an almost finitely generated module to find a finite submodule N that contains IM. If M/IM is isomorphic to the zero module, then the containment $IM \subset N \subset M$ implies that $N = M$. Thus M is actually finitely generated, now we use the usual Nakayama's Lemma to finish the proof.

If M/IM is merely almost isomorphic to the zero module, then we see that the inclusion $IM \subset M$ is an almost isomorphism. In particular, $\mathfrak{m}M$ is contained in IM. Using that $\mathfrak{m}^2 = \mathfrak{m}$, we obtain an *equality*

$$\mathfrak{m}M = \mathfrak{m}^2 M = \mathfrak{m}(IM) = I(\mathfrak{m}M)$$

Thus we can apply the argument from above to conclude that $\mathfrak{m}M = 0$. This finishes the proof as $\mathfrak{m}M \cong^a M$. $\square$

Lemma 2.5.20. Let R be I-adically complete for some finitely generated $I \subset \mathfrak{m}$. Then an I-adically complete R-module M is almost finitely generated if and only if M/IM is almost finitely generated.

Proof. [GR03, Lemma 5.3.18] $\square$

2.6. **Almost Coherent Modules and Almost Coherent Rings.** This section is devoted to the study of "almost coherent" modules. They are supposed to be "almost" analogues of usual coherent modules. We show that they always form a Weak Serre subcategory in $\mathbf{Mod}_R$. Then we study the special case of almost coherent modules over an almost coherent ring, and show that in this case almost coherent modules are the same as almost finitely presented modules. We recall that we fixed some "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and always we do almost mathematics with respect to this ideal.

Definition 2.6.1. We say that an (almost) R-module M is *almost coherent* if it is almost finitely generated and every almost finitely generated almost submodule $N^a \subset M^a$ is almost finitely presented.

Remark 2.6.2. An almost submodule $f\colon N^a \hookrightarrow M^a$ does not necessarily give rise to a submodule $N' \subset M$ for some $(N')^a \simeq N$. The most we can say is that there is an injection $f_!\colon (N^a)_! \hookrightarrow (M^a)_!$ whose almostification is equal to the the morphism f (this follows from Lemma 2.1.8(**2**)).

Lemma 2.6.3. Let $R \to S$ be an almost isomorphism of rings. Then the forgetful functor $\mathbf{Mod}^{\mathrm{acoh}}_{S^a} \to \mathbf{Mod}^{\mathrm{acoh}}_{R^a}$ is an equivalence.

Proof. It directly follows from Corollary 2.5.13 and Definition 2.6.1. $\square$

Lemma 2.6.4. Let M^a be an almost R-module with a representative $M \in \mathbf{Mod}_R$. Then the following are equivalent

(1) The almost module M^a is almost coherent.

(2) The actual R-module $(M^a)_*$ is almost finitely generated, and any almost finitely generated R-submodule of $(M^a)_*$ is almost finitely presented.

(3) The actual R-module $(M^a)_!$ is almost finitely generated, and any almost finitely generated R-submodule of $(M^a)_!$ is almost finitely presented.

Proof. First of all, we note that Corollary 2.5.12 guarantees that M is almost finitely generated if and only if so is $(M^a)_*$. Secondly, Lemma 2.1.9 implies that the functor $(-)_*$ is left exact. Therefore, any almost submodule $N^a \subset M^a$ gives rise to an actual submodule $(N^a)_* \subset (M^a)_*$ that is almost isomorphic to N. In reverse, any submodule $N \subset (M^a)_*$ gives rise to an almost submodule of M^a. Hence, we see that all almost finitely generated almost submodules of M^a are almost finitely presented if and only if all actual almost finitely generated submodules of M_* are almost finitely presented (here we again use Corollary 2.5.12). This shows the equivalence of **(1)** and **(2)**. The same argument shows that **(1)** is equivalent to **(3)**. □

Note that it is not that clear whether a coherent R-module is almost coherent. The issue is that in the definition of almost coherent modules we need to be able to handle all almost finitely generated almost submodules and not only finitely generated. The lemma below is a useful tool to deal with such problems, in particular, it turns out (Corollary 2.6.7) that all coherent modules are indeed almost coherent, but we do not know a direct way to see it.

Lemma 2.6.5. Let M be an R-module. Then M is an almost coherent module if one of the following holds:

(1) For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a coherent R-module N and morphism $f\colon N \to M$ such that $\mathfrak{m}_0(\ker f) = 0$ and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.

(2) For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists an *almost* coherent R-module N and morphism $f\colon N \to M$ such that $\mathfrak{m}_0(\ker f) = 0$ and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.

Proof. We start the proof by noting that M comes with the natural almost isomorphism $M \to M^a_*$, so both of the assumptions on M pass through this almost isomorphism. Thus, Lemma 2.6.4 implies that it suffices to show that $M_* := M^a_*$ is almost coherent.

Lemma 2.5.7 guarantees that M_* is almost finitely generated. Thus we only need to check the second condition from Definition 2.6.1. So we pick an arbitrary almost finitely generated R-submodule $M_1 \subset M_*$, we want to show that it is almost finitely presented. We choose an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

We use Lemma 2.5.8 to find a morphism $\varphi : R^n \to M_1$ such that $\mathfrak{m}_1(\operatorname{Coker} \varphi) = 0$. Let $e_1, \dots, e_n$ be the standard basis in R^n, and define $x_i := \varphi(e_i)$ to be their images. We also choose some set of generators $(\varepsilon_1, \dots, \varepsilon_m)$ for the ideal $\mathfrak{m}_1$.

Now we recall that by our assumption there is a morphism $f\colon N \to M_*$ with a(n) (almost) coherent R-module N such that $\mathfrak{m}_1(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_1(\ker f) = 0$. This implies that $\varepsilon_i x_j$ is in the image of f for any $i = 1, \dots, m, j = 1, \cdots n$. Let us choose some $y_{i,j} \in N$ such that $f(y_{i,j}) = \varepsilon_i x_j$, and we define an R-module N' as the submodule of N generated by all $y_{i,j}$, this is a finite R-module by the construction. Since N is a (almost) coherent module, we conclude that N' is actually (almost) finitely presented.

We observe that $f' := f|_{N'}$ naturally lands in M_1, and we have $\mathfrak{m}_1(\ker f') = 0$ and $\mathfrak{m}_1^2(\operatorname{Coker} f') = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ this shows that the morphism

$$N' \xrightarrow{f'} M_1$$

has a kernel and cokernel killed by $\mathfrak{m}_0$. Lemma 2.5.8 shows that M_1 is almost finitely presented. $\square$

Question 2.6.6. Does the converse of this Lemma hold?

Corollary 2.6.7. Any coherent R-module M is almost coherent.

The next thing that we want to show is that almost coherent modules from a Weak Serre subcategory of $\mathbf{Mod}_R$. This is an almost analogue of the corresponding statement in the classical case.

Lemma 2.6.8. Let R and $\mathfrak{m}$ as above. Then

(1) An almost finitely generated almost submodule of an almost coherent module is almost coherent.

(2) Let $\varphi : N^a \to M^a$ be an almost homomorphism from an almost finitely generated R^a-module to an almost coherent R^a-module, then $\ker \varphi$ is almost finitely generated R^a-module.

(3) Let $\varphi : N^a \to M^a$ be an injective almost homomorphism of almost coherent R^a-modules, then $\operatorname{Coker} \varphi$ is almost coherent R^a-module.

(4) Let $\varphi : N^a \to M^a$ be an almost homomorphism of almost coherent R^a-modules, then $\ker \varphi$ and $\operatorname{Coker} \varphi$ are almost coherent R^a-modules.

(5) Given a short exact sequence of R^a-modules $0 \to M'^a \to M^a \to M''^a \to 0$ if two out of three are almost coherent so is the third.

Proof. ***(1):*** This is evident from the definition of an almost coherent almost module.

(2): Let us define $N''^a := \operatorname{Im} \varphi$ and $N'^a := \ker \varphi$, then Corollary 2.5.16 implies that N''^a is an almost finitely generated almost submodule of M^a. It is actually almost finitely presented since M^a is almost coherent, we use Corollary 2.5.16 to get that N' is almost finitely generated as well.

(3): We denote $\operatorname{Coker} \varphi$ by M''^a, then we have a short exact sequence

$$0 \to N^a \to M^a \to M''^a \to 0.$$

Corollary 2.5.16 implies that M''^a is almost finitely generated. Let us choose any almost finitely generated almost submodule $M_1''^a \subset M''^a$ and denote its pre-image in M^a by M_1^a. Then we have a short exact sequence

$$0 \to N^a \to M_1^a \to M_1''^a \to 0.$$

Corollary 2.5.16 guarantees that M_1^a is an almost finitely generated almost submodule of M^a. Since M^a is almost coherent, we see that M_1^a is an almost finitely presented R^a-module. Therefore, Corollary 2.5.16 implies that $M_1''^a$ is also almost finitely presented. Hence, the R^a-module M''^a is almost coherent.

(4): We know that $N'^a := \ker \varphi$ is almost finitely generated by **(2)**. Since N^a is almost coherent, we conclude that N'^a is almost coherent by **(1)**. We define $N''^a := \operatorname{Im} \varphi$ and $M''^a := \operatorname{Coker} \varphi$, then we note that we have two short exact sequences

$$0 \to N'^a \to N^a \to N''^a \to 0,$$
$$0 \to N''^a \to M^a \to M''^a \to 0.$$

We observe that **(3)** shows that N''^a is almost coherent, then we use **(3)** once more to conclude that M''^a is also almost coherent.

***(5)**:* The only thing that we are left to show is that if M'^a and M''^a are almost coherent so is M^a. It is almost finitely generated by Corollary 2.5.16. Now to check the second condition from Definition 2.6.1, we choose an almost finitely generated almost submodule $M_1^a \subset M^a$. Let us denote by $M_1''^a$ its image in M''^a, and by $M_1'^a$ the kernel of this map. So we have a short exact sequence

$$0 \to M_1'^a \to M_1^a \to M_1''^a \to 0.$$

Corollary 2.5.16 guarantees that $M_1''^a$ is an almost finitely generated almost submodule of the almost coherent R^a-module M''^a. Hence, **(1)** implies that $M_1''^a$ is almost coherent, in particular, it is almost finitely presented. Moreover, we can now use **(2)** to get that $M_1'^a$ is an almost finitely generated almost submodule of M'^a. Since M'^a is almost coherent, we conclude that $M_1'^a$ is actually almost finitely presented. Finally, Corollary 2.5.16 shows that M_1^a is almost finitely presented as well. This finishes the proof of almost coherence of the R^a-module M. □

Corollary 2.6.9. Let M^a be an almost finitely presented R^a-modules and let N^a be an almost coherent R^a-module. Then $M^a \otimes_{R^a} N^a$ and $\mathrm{alHom}_{R^a}(M^a, N^a)$ are almost coherent.

Proof. We use Proposition 2.2.1**(1)**,**(3)** to reduce the question to show that $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ are almost coherent R-modules for any almost finitely presented R-module M and almost coherent R-module N.

Step 1. The case of finitely presented module M: In this case we pick a presentation of M as the quotient

$$R^n \to R^m \to M \to 0\ .$$

Then we have short exact sequences

$$N^n \to N^m \to M \otimes_R N \to 0$$

and

$$0 \to \mathrm{Hom}_R(M, N) \to N^m \to N^n\ .$$

We note that Lemma 2.6.8**(5)** implies that N^m and N^n are almost coherent. Thus Lemma 2.6.8**(5)** guarantees that both $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ are almost coherent as well.

Step 2. The General Case: The argument is similar to the one used in Step 2 of the proof of Lemma 2.5.17. We approximate M with finitely presented R-modules. This gives us an approximations of $M^a \otimes_{R^a} N^a$ and $\mathrm{alHom}_{R^a}(M^a, N^a)$ by almost coherent modules. Now Lemma 2.6.5 guarantees that these modules are actually almost coherent. We leave details to the interested reader. □

We define $\mathbf{Mod}_R^{acoh}$ (resp. $\mathbf{Mod}_{R^a}^{acoh}$) to be the strictly full[7] subcategoty of $\mathbf{Mod}_R$ (resp. $\mathbf{Mod}_{R^a}$) consisting of almost coherent R-modules (resp. R^a-modules).

Corollary 2.6.10. The category $\mathbf{Mod}_R^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_{R^a}^{\mathrm{acoh}}$) is a Weak Serre subcategory of $\mathbf{Mod}_R$ (resp. $\mathbf{Mod}_{R^a}$).

[7] i.e. full subcategory that is closed under isomorphisms.

Corollary 2.6.10 and the discussion in [Sta21, Tag 06UP] ensure that $\mathbf{D}_{acoh}(R)$ and $\mathbf{D}_{acoh}(R)^a$[8] are strictly full saturated[9] triangulated subcategories of $\mathbf{D}(R)$ and $\mathbf{D}(R)^a$ respectively. We define $\mathbf{D}^+_{acoh}(R) \coloneqq \mathbf{D}_{acoh}(R) \cap \mathbf{D}^+(R)$ and similarly for all other bounded versions.

Lemma 2.6.11. Let $M \in \mathbf{D}(R)$ be a complex of R-modules. Then $M \in \mathbf{D}_{acoh}(R)$ if one of the following holds:

(1) For every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is $N \in \mathbf{D}_{coh}(R)$ and a morphism $f\colon N \to M$ such that $\mathfrak{m}_0 \left(\mathrm{H}^i\left(\mathrm{cone}\left(f\right)\right)\right) = 0$ for every $i \in \mathbf{Z}$,

(2) For every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is $N \in \mathbf{D}_{acoh}(R)$ and a morphism $f\colon N \to M$ such that $\mathfrak{m}_0 \left(\mathrm{H}^i\left(\mathrm{cone}\left(f\right)\right)\right) = 0$ for every $i \in \mathbf{Z}$.

Proof. This is an easy consequence of Lemma 2.6.5 and the definition of $\mathbf{D}_{acoh}(R)$. $\square$

The last part of this subsection is dedicated to the study of almost coherent rings and almost coherent modules over almost coherent rings. Recall that coherent modules over a coherent ring coincide with finitely presented ones. Similarly, we will show that almost coherent modules over an almost coherent ring turn out to be the same as almost finitely presented ones.

Definition 2.6.12. We say that a ring R is almost coherent if the rank-1 free module R is almost coherent as an R-module.

Lemma 2.6.13. A coherent ring R is almost coherent.

Proof. Apply Corrollary 2.6.7 to a rank-1 free module R. $\square$

Lemma 2.6.14. If R is an almost coherent ring, then any almost finitely presented R-module M is almost coherent.

Proof. Step 1: If M is finitely presented over R, then we can write it as a cokernel of a map between free finite rank modules. A free finite rank module over an almost coherent ring is almost coherent by Lemma 2.6.8**(5)**. A cokernel of a map of almost coherent modules is almost coherent by Lemma 2.6.8**(4)**. Therefore, any finitely presented M is almost coherent.

Step 2: Suppose that M is merely almost finitely presented. Lemma 2.5.7 guarantees that, for any finitely generated $\mathfrak{m}_0 \subset \mathfrak{m}$, we can find a finitely presented module N and a map $f\colon N \to M$ such that $\ker f$ and $\mathrm{Coker}\, f$ are annihilated by $\mathfrak{m}_0$. N is almost coherent by Step 1. Therefore, Lemma 2.6.5**(2)** implies that M is almost coherent as well. $\square$

Corollary 2.6.15. Let R be an almost coherent ring. Then an R-module M is almost coherent if and only if it is almost finitely presented.

Proof. The "only if" part is clear from the definition, the "if" part follows from Lemma 2.6.14. $\square$

Our next big goal is to show that bounded above almost coherent complexes over an almost coherent ring are exactly "almost pseudo-coherent complexes" in some precise way. More precisely, that any element $M \in \mathbf{D}^-_{acoh}(R)$ can be "approximated" up to any small torsion by complexes of finite free modules.

[8] These are full subcategories of $\mathbf{D}(R)$ and $\mathbf{D}(R)^a$ of complexes with almost coherent cohomology modules, respectively.

[9] A strictly full subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is saturated if $X \oplus Y \in \mathcal{D}'$ implies $X, Y \in \mathcal{D}'$.

Proposition 2.6.16. Let R be an almost coherent ring and $M \in \mathbf{D}^-(R)$. Then $M \in \mathbf{D}^-_{acoh}(R)$ if and only if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is a complex $F^\bullet$ of finite free R-modules, and a morphism

$$f\colon F^\bullet \to M$$

such that $\mathfrak{m}_0\left(\mathrm{H}^i(\mathrm{cone}(f))\right) = 0$ for every $i \in \mathbf{Z}$. Moreover, if $M \in \mathbf{D}^{\leq 0}_{coh}(R)$ one can choose $F^\bullet \in \mathbf{Comp}^{\leq 0}(R)$.

Proof. The "if" direction is Lemma 2.6.11. So we only need to prove the "only if" direction. For this direction, we fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

Without loss of generality, we may and do assume that $M \in \mathbf{D}^{\leq 0}(R)$, and we choose a complex $M^\bullet \in \mathbf{Comp}^{\leq 0}(R)$ that represents M. Now we prove a slightly more precise claim:

Claim: For every $n \in \mathbf{Z}$, there is a complex of finite free modules $F^\bullet_n$ with a morphism $f_n\colon F^\bullet_n \to M^\bullet$ such that

1. $F^\bullet_n \in \mathbf{Comp}^{[-n,0]}(R)$;
2. $\sigma^{\geq n-1}F^\bullet_n = F^\bullet_{n-1}$ *and* $\sigma^{\geq n-1}f_n = f_{n-1}$, *where* $\sigma^{\geq n-1}$ *is the naive truncation;*
3. *kernels and cokernels of* $\mathrm{H}^i(f_n)$ *are annihilated by* $\mathfrak{m}_1$ *for* $i \geq n+1$*;*
4. *the cokernel of* $\mathrm{H}^n(f_n)$ *is annihilated by* $\mathfrak{m}_1$*;*

Proof of the claim: We argue by descending induction on n. If $n \geq 1$, $F^\bullet = 0$ works. Now we suppose that we can construct $F^\bullet_n$, and wish to construct $F^\bullet_{n-1}$. Consider the morphism f_n presented as a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F^n_n \arrow[r, "\mathrm{d}^n_F"] \arrow[d, "f^n_n"] & F^{n+1}_n \arrow[r, "\mathrm{d}^{n+1}_F"] \arrow[d, "f^{n+1}_n"] & \dots \arrow[d] \\
M^{n-2} \arrow[r, "\mathrm{d}^{n-2}_M"] & M^{n-1} \arrow[r, "\mathrm{d}^{n-1}_M"] & M^n \arrow[r, "\mathrm{d}^n_M"] & M^{n+1} \arrow[r, "\mathrm{d}^{n+1}_M"] & \dots
\end{tikzcd}$$

Firstly, $\ker(\mathrm{d}^n_F)$ is almost coherent as a kernel between finitely presented modules over an almost coherent ring. Secondly, the R-module

$$B^n := \ker\left(\ker\left(\mathrm{d}^n_F\right) \to \mathrm{H}^n\left(M\right)\right),$$

is also almost coherent as a kernel between almost coherent modules. Therefore, there is a finite free R-module F'^{n-1} and a morphism

$$\mathrm{d}'\colon F'^{n-1} \to B^n$$

such that $\mathfrak{m}_1(\mathrm{Coker}\,\mathrm{d}') = 0$. Since $\mathrm{H}^{n-1}(M)$ is almost coherent, we can find a finite free R-module F''^{n-1} and a morphism

$$\lambda\colon F''^{n-1} \to \mathrm{H}^{n-1}(M)$$

such that $\mathfrak{m}_1(\mathrm{Coker}\,\lambda) = 0$. Let $\nu\colon F''^{n-1} \to Z^{n-1}(M^\bullet)$ be any lift of λ to the module of closed elements $Z^{n-1}(M^\bullet) = \ker(\mathrm{d}^{n-1}_M)$. We define

$$f''^{n-1}\colon F''^{n-1} \to M^{n-1}$$

be the composition of ν with the inclusion $Z^{n-1}(M^\bullet) \to M^{n-1}$.

Now we wish to define $F^\bullet_{n-1}$ and f_{n-1}. We start with $F^\bullet_{n-1}$; we put $F^m_{n-1} = F^m_n$ if $m \geq n$, $F^m_{n-1} = 0$ if $m < n-1$, $F^{n-1}_{n-1} = F'^{n-1} \oplus F''^{n-1}$, and define the only non-evident differential

$$\mathrm{d}^{n-1}_F\colon F^{n-1}_{n-1} = F'^{n-1} \oplus F''^{n-1} \to F^n_n$$

to be zero on F''^{n-1} and equal to d' on F'^{n-1}. It is evident that $\mathrm{d}_F^n \circ \mathrm{d}_F^{n-1} = 0$, so this structure defines us a complex $F^\bullet_{n-1}$ of finite free R-modules.

We are only left to define f_{n-1}. We must put $f_{n-1}^m = f_n^m$ if $m > n-1$ and $f_{n-1}^m = 0$ if $m < n-1$, so the only question is to define f_{n-1}^{n-1}. By construction $f_n^n(\mathrm{d}'F'^{n-1}) \subset \mathrm{d}_M^{n-1}M^{n-1}$, so we can find

$$f'_{n-1}\colon F'^{n-1} \to M^{n-1}$$

such that $\mathrm{d}^{n-1} \circ f'_{n-1} = f_n^n \circ \mathrm{d}'$. Thus we define

$$f_{n-1}^{n-1}\colon F_{n-1}^{n-1} = F'^{n-1} \oplus F''^{n-1} \to M^{n-1}$$

to be f'_{n-1} on F'^{n-1} and f''_{n-1} on F''^{n-1}. Then it is evident from the construction that $f^\bullet_{n-1}$ is a morphism of complexes, i.e. the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_{n-1}^{n-1} \arrow[r, "\mathrm{d}_F^{n-1}"] \arrow[d, "f_{n-1}^{n-1}"] & F_{n-1}^{n} \arrow[r, "\mathrm{d}_F^{n}"] \arrow[d, "f_{n-1}^{n}"] & F_{n-1}^{n+1} \arrow[r, "\mathrm{d}_F^{n+1}"] \arrow[d, "f_{n-1}^{n+1}"] & \dots \arrow[d] \\
M^{n-2} \arrow[r, "\mathrm{d}_M^{n-2}"] & M^{n-1} \arrow[r, "\mathrm{d}_M^{n-1}"] & M^n \arrow[r, "\mathrm{d}_M^{n}"] & M^{n+1} \arrow[r, "\mathrm{d}_M^{n+1}"] & \dots
\end{tikzcd}$$

By construction, kernel and cokernel of $\mathrm{H}^n(f_{n-1})$ are annihilated by $\mathfrak{m}_1$, and cokernel of $\mathrm{H}^{n-1}(f_{n-1})$ is annihilated by $\mathfrak{m}_1$. So this finishes the proof of the claim.

Now the morphism $f\colon F^\bullet \to M^\bullet$ simply comes as the colimit of f_n, i.e.

$$f = \mathrm{colim}\, f_n\colon F^\bullet \coloneqq \mathrm{colim}\, F^\bullet_n \to M^\bullet.$$

It is easy to see that cohomology groups of $\mathrm{cone}(f)$ are annihilated by $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. $\square$

Corollary 2.6.17. Let R be a coherent ring and $M \in \mathbf{D}^b(R)$. Then $M \in \mathbf{D}^b_{acoh}(R)$ if and only if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is a complex $N \in \mathbf{D}^b_{coh}(R)$ and a morphism $f\colon N \to M$ such that $\mathfrak{m}_0(\mathrm{H}^i(\mathrm{cone}(f))) = 0$ for all i.

Proof. The "if" direction is Lemma 2.6.11. So we only need to deal with the "only if" direction. Assume that $M \in \mathbf{D}^b(R)$. Then Proposition 2.6.16 implies that there is $F \in \mathbf{D}^-_{coh}(R)$ and a morphism $f\colon F \to M$ such that $\mathfrak{m}_0(\mathrm{H}^i(\mathrm{cone}(f))) = 0$ for all i. Now replace F by $F' \coloneqq \tau^{\geq a}F$ to get the desired approximation with $F' \in \mathbf{D}^b_{coh}(R)$. $\square$

Proposition 2.6.18. Let R be an almost coherent ring, and let M^a, N^a be two objects in $\mathbf{D}^-_{acoh}(R)^a$. Then $M^a \otimes^L_{R^a} N^a \in \mathbf{D}^-_{acoh}(R)^a$.

Proof. Proposition 2.4.13 ensures that it suffices to show that $M \otimes^L_R N \in \mathbf{D}^-_{acoh}(R)$ for M, $N \in \mathbf{D}^-_{coh}(R)$. Clearly, we can cohomologically shift both M and N to assume that they lie $\mathbf{D}^{\leq 0}_{coh}(R)$.

Now we fix a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ and use Proposition 2.6.16 to find an exact triangle

$$F^\bullet \to M \to Q$$

where $F^\bullet \in \mathbf{D}^{\leq 0}(R)$ a complex of finite free modules and $\mathrm{H}^i(Q)$ are all annihilated by $\mathfrak{m}_1$. Then it is easy to see that kernel and cokernel of the map

$$\mathrm{H}^{-i}(F^\bullet \otimes^L_R N) \to \mathrm{H}^{-i}(M \otimes^L_R N)$$

are annihilated by $\mathfrak{m}_1^{i+1}$. Now we note that, clearly,

$$F^\bullet \otimes^L_R N \simeq F^\bullet \otimes^\bullet_R N$$

lies in $\mathbf{D}^-_{coh}(R)$ because $F^\bullet$ is a complex of finite free modules. For each pair of an integer $i \geq 0$ and a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^{i+1}$, we can find another finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^{i+1}$. Therefore, the map

$$\mathrm{H}^{-i}(F^\bullet \otimes^L_R N) \to \mathrm{H}^{-i}(M \otimes^L_R N)$$

is a morphism with an almost coherent source and $\mathfrak{m}_0$-torsion kernel and cokernel. Therefore, Lemma 2.6.5 **(2)** implies the claim. □

Proposition 2.6.19. Let R be an almost coherent ring, and let $M^a \in \mathbf{D}^-_{acoh}(R)^a, N^a \in \mathbf{D}^+_{acoh}(R)^a$. Then $\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \in \mathbf{D}^+_{acoh}(R)^a$.

Proof. The proof is similar to that of Proposition 2.6.18. We use Proposition 2.4.8 and the same approximation argument to reduce to the case $M = F^\bullet$ is a bounded above complex of a finite free modules. In this case the claim is essentially obvious due to the explicit construction of the Hom-complex $\mathrm{Hom}^\bullet_R(F^\bullet, N)$. □

Proposition 2.6.20. Let R be an almost coherent ring, let $M \in \mathbf{D}^-_{acoh}(R)$, $N \in \mathbf{D}^+(R)$, and let P be an almost flat R-module. Then the natural map $\mathbf{R}\mathrm{Hom}_R(M, N) \otimes_R P \to \mathbf{R}\mathrm{Hom}_R(M, N \otimes_R P)$ is an almost isomorphism.

Similarly, $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \otimes^L_{R^a} P^a \to \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a \otimes^L_{R^a} P^a)$ is an almost isomorphism for any $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$, and let P^a a flat R^a-module.

Proof. The proof is similar to that of the lemmas above. □

Corollary 2.6.21. Let R be an almost coherent ring, let $M^a \in \mathbf{D}^-_{acoh}(R)^a$, $N \in \mathbf{D}^+(R)^a$, and let P^a be an almost flat R^a-module. Then the natural map

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \otimes^L_{R^a} P^a \to \mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$$

is an isomorphism in $\mathbf{D}(R^a)$.

2.7. **Almost Noetherian Rings.** The main goal of this section is to define the almost analogue of the noetherianness property and verify some of its basic properties. However, we want to emphasize that Hilbert's Nullstellensatz seems much more subtle in the almost world (see Warning 2.7.9). We are able to establish it only in a very particular situation in Section 2.11.

As in the previous sections, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and always we do almost mathematics with respect to this ideal.

Definition 2.7.1. A ring R is *almost noetherian* if every ideal $I \subset R$ is almost finitely generated.

The main goal is to show that every almost finitely generated module over an almost noetherian ring is almost finitely presented. In particular, an almost noetherian ring is almost coherent.

Lemma 2.7.2. Let R be an almost noetherian ring, and $M \subset R^n$ an R-submodule. Then M is almost finitely generated.

Proof. We argue by induction on n. The base of induction is $n = 1$, where the claim follows from the definition of an almost noetherian ring.

Suppose we know the claim for $n-1$, so we deduce the claim for n. Denote by $R^{n-1} \subset R^n$ a free R-module spanned by first $n-1$ standard basis elements of R^n, and denote by $M' := M \cap R^{n-1}$ the intersection of M with R^{n-1}. Then we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

where M'' is naturally an R-submodule of $R \simeq R^n/R^{n-1}$. By the induction hypothesis, M' is almost finitely generated. M'' is almost finitely generated by almost noetherianness of R. Therefore, M is almost finitely generated by Lemma 2.5.15 **(2)**. $\square$

Lemma 2.7.3. Let R be an almost noetherian ring. Then any almost finitely generated R-module M is almost finitely presented.

Proof. Pick any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. By Lemma 2.5.5, there is an R-linear homomorphism

$$f\colon R^n \to M$$

such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$. Consider $N := \ker(f)$. Lemma 2.7.2 ensures that N is also almost finitely generated, so there is an R-linear homomorphism

$$g'\colon R^m \to N$$

such that $\mathfrak{m}_0(\operatorname{Coker} g') = 0$. Therefore, the composition

$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$

is a three-term complex with $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\ker f) \subset \operatorname{Im}(g)$. Since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal in $\mathfrak{m}$, we conclude that M is almost finitely presented by Lemma 2.5.7 **(3)**. $\square$

Corollary 2.7.4. A ring R is almost noetherian if and only if any almost finitely generated R-module M is almost finitely presented.

Proof. If R is almost noetherian, then any almost finitely generated R-module is almost finitely presented due to Lemma 2.7.3.

Now we suppose that every almost finitely generated R-module is almost finitely presented, and we wish to show that R is almost noetherian. Consider an ideal $I \subset R$. Then R/I is clearly a finitely generated R-module, in particular, it is almost finitely generated. Therefore, it is almost finitely presented by our assumption on R. Now the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and Lemma 2.5.15 **(3)** imply that I is almost finitely generated. $\square$

Corollary 2.7.5. Let $R \to R'$ be an almost isomorphism of rings. Then R is almost noetherian if and only if R' is.

Corollary 2.7.6. Let R be an almost noetherian ring, and M an almost finitely generated R-module. Then any submodule $N \subset M$ is almost finitely generated.

Proof. Consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

By construction, M/N is almost finitely generated and, therefore, almost finitely presented by Lemma 2.7.3. So Lemma 2.5.15 **(3)** implies that N is almost finitely generated. $\square$

Corollary 2.7.7. Let R be an almost noetherian ring. Then R is almost coherent.

Proof. Lemma 2.6.4 guarantees that it suffices to show that every finitely generated sub-module of $R_!$ is almost finitely presented $R_! \simeq \widetilde{\mathfrak{m}}$ is almost finitely generated and every finitely generated sub-module of $R_!$ is almost finitely presented. The first property is trivial since $R_!$ is almost isomorphic to R, and the second one follows from Lemma 2.7.3. $\square$

Corollary 2.7.8. Let R be an almost noetherian ring. Then an R-module M (resp. an R^a-module M^a) is almost coherent if and only if it is almost finitely generated.

Proof. It suffices to prove the claim for an honest R-module M. Corollary 2.7.7 and Corollary 2.6.15 imply that M is almost coherent if and only if it is almost finitely presented. Now Lemma 2.7.3 says that M is almost finitely presented if and only if it is almost finitely generated. This finishes the proof. $\square$

Warning 2.7.9. Unlike the case of usual noetherian rings, Hilbert's Nullstellensatz seems as a much more subtle problem in the almost world. In particular, we do not know if a polynomial algebra in a finite number of variables over an almost noetherian ring is almost noetherian. However, we show that Hilbert's Nullstellensatz holds for perfectoid valuation rings in Section 2.11.

2.8. **Base Change for Almost Modules.** The last topic that we want to discuss about almost modules over general rings is their behavior with respect to base change. Recall that given a ring homomorphism $\varphi\colon R \to S$ we always do almost mathematics on S-modules with respect to the ideal $\mathfrak{m}_S := \mathfrak{m}S$; look at Lemma 2.1.11 to see why $\widetilde{\mathfrak{m}_S}$ is flat.

Lemma 2.8.1. Let $\varphi : R \to S$ be a ring homomorphism, and let M^a be an almost finitely generated (resp. almost finitely presented) R^a-module. Then the module $M^a_S := M^a \otimes_{R^a} S^a$ is almost finitely generated (resp. almost finitely presented).

Proof. The claim follows from Lemma 2.5.7(**2**) and the fact that for any finitely generated ideal $\mathfrak{m}_0' \subset \mathfrak{m}_S$ there is a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $\mathfrak{m}_0' \subset \mathfrak{m}_0 S$. We only give a complete proof in the case of finitely presented modules as the other case is an easier version of the same.

Firstly, we note that it suffices to show that $M \otimes_R S$ is almost finitely presented. Now the observation above implies that it suffices to check the condition of Lemma 2.5.7(**2**) only for ideals of the form $\mathfrak{m}_0 S$ for a finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Then we choose some finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ and we use Lemma 2.5.7(**2**) to find a finitely presented module N and a map $f\colon N \to M$ such that $\mathfrak{m}_1(\operatorname{Ker} f) = \mathfrak{m}_1(\operatorname{Coker} f) = 0$. Consider an exact sequence

$$0 \to K \to N \xrightarrow{f} M \to Q \to 0$$

and denote the image f by M'. Then we have the following exact sequences:

$$K \otimes_R S \to N \otimes_R S \to M' \otimes_R S \to 0$$

$$\operatorname{Tor}_1^R(Q, S) \to M' \otimes_R S \to M \otimes_R S \to Q \otimes_R S$$

Since $K \otimes_R S$, $\operatorname{Tor}_1^R(Q, S)$ and $Q \otimes_R S$ are killed by $\mathfrak{m}_1 S$, we conclude that $\operatorname{Coker}(f \otimes_R S)$ and $\ker(f \otimes_R S)$ are annihilated by $\mathfrak{m}_1^2 S$. In particular, they are killed by $\mathfrak{m}_0 S$. Since $N \otimes_R S$ is finitely presented over S, Lemma 2.5.7 finishes the proof. $\square$

Corollary 2.8.2. Let $R \to S$ be a ring homomorphism of almost coherent rings, and let M^a be an object of $\mathbf{D}^-_{acoh}(R)^a$. Then $M^a \otimes^L_{R^a} S^a \in \mathbf{D}^-_{acoh}(S)^a$.

Proof. The proof is similar to that of Proposition 2.6.18. We use Proposition 2.4.16 and a similar approximation argument based on Proposition 2.6.16 to reduce to the case $M \simeq F^\bullet$, where $F^\bullet$ is a bounded above complex of finite free modules. In this case, the claim is essentially obvious. $\square$

Lemma 2.8.3. Let S be a R-algebra that is finite (resp. finitely presented) as an R-module, and let M^a be an S^a-module. Then M^a is almost finitely generated (resp. almost finitely presented) over R^a if and only if it is almost finitely generated (resp. almost finitely presented) over S^a.

Proof. As always, we firstly reduce the question to the case of an honest S-module M. Now we use the observation that it suffices to check the condition of Lemma 2.5.7**(2)** only for the ideals of the form $\mathfrak{m}_0 S$ for some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m} \subset R$. Then the only non-trivial direction is to show that M is almost finitely presented over S if it is almost finitely presented over R. This is proven in a more general situation in Lemma 2.8.4 $\square$

Lemma 2.8.4. Let S be a possibly non-commutative R-algebra that is finite as a left (resp. right) R-module, and let M be a left (resp. right) S-module that is almost finitely presented over R. Then M is almost finitely presented over S (i.e. for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exists a finitely presented left (resp. right) S-module N and a map $N \to M$ such that $\ker f$ and $\operatorname{Coker} f$ are annihilated by $\mathfrak{m}_0$).

Remark 2.8.5. This lemma will actually be used for a non-commutative ring S in the proof of Theorem 5.2.1 that, in turn, will be used in the proof of formal GAGA for almost coherent sheaves Theorem 5.3.2. Namely, we will apply to result to $S = \operatorname{End}_{\mathbf{P}^N}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \dots \mathcal{O}(N))$.

Besides this application, Lemma 2.8.4 will be mostly used for almost coherent commutative rings R and S, where the proof can be significantly simplified.

Proof. We give a proof for left S-modules, the proof for right S-modules is the same. We start the proof by choosing some generators $x_1, \dots, x_n$ of S as an R-module. Then we pick a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. And we also choose some generators $(\varepsilon_1, \dots, \varepsilon_k) = \mathfrak{m}_1$ and find a three-term complex

$$R^t \xrightarrow{g} R^m \xrightarrow{f} M$$

such that $\mathfrak{m}_1(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_1(\ker f) \subset \operatorname{Im} g$. We consider the images $y_i \coloneqq f(e_i) \in M$ of the standard basis elements in R^m. Then we can find some $\beta_{i,j,s,r} \in R$ such that

$$\varepsilon_s x_i y_j = \sum_{r=1}^{m} \beta_{i,j,s,r} \cdot y_r \text{ with } \beta_{i,j,s,r} \in R$$

for any $s = 1, \dots k;\ i = 1, \dots, n;\ j = 1, \dots, m$. Moreover, we have t "relations"

$$\sum_{j=1}^{m} \alpha_{i,j} y_j = 0 \text{ with } \alpha_{i,j} \in R$$

such that for any relation $\sum_{i=1}^{m} b_i y_i = 0$ with $b_i \in R$ and any $\varepsilon \in \mathfrak{m}_1$, we have that the vector $\{\varepsilon b_i\}_{i=1}^{m} \in R^m$ lives in the R-subspace generated by vectors $\{\alpha_{i,j}\}_{i=1}^{m}$ for $j = 1, \dots, t$. Or, in other words, if $\sum_{j=1}^{m} \alpha_{i,j} y_j = 0$ then $\varepsilon(\sum_{j=1}^{m} \alpha_{i,j} e_j) \in \operatorname{Im}(g)$ for any $\varepsilon \in \mathfrak{m}_1$.

Now we are finally ready to define a three-term complex

$$S^{nmk+t} \xrightarrow{\psi} S^m \xrightarrow{\varphi} M$$

We define the map φ as the unique S-linear homomorphism such that $\varphi(e_i) = y_i$ for the standard basis in S^m. We define ψ as the unique S-linear homomorphism such that

$$\psi(f_{i,j,s}) = \varepsilon_s x_i e_j - \sum_{r=1}^{m} \beta_{i,j,s,r} \cdot e_r \text{ and } \psi(f'_l) = \sum_{j=1}^{m} \alpha_{l,j} e_j$$

for the standard basis

$$\{f_{i,j,s}, f'_l\}_{i \le n, j \le m, s \le k, l \le t} \in S^{nmk+t}$$

Then we clearly have that $\varphi \circ \psi = 0$ and that $\mathfrak{m}_1(\operatorname{Coker} \varphi) = 0$. We claim that $\mathfrak{m}_1^2(\ker \varphi) \subset \operatorname{Im} \psi$.

Let $\varphi(\sum_{i=1}^m c_ie_i) = 0$ for some elements $c_i \in S$. We can write each

$$c_i = \sum_{j=1}^n r_{i,j}x_j \text{ with } r_{i,j} \in R \tag{2.1}$$

because $x_1, \dots, x_n$ are R-module generators of S. Thus, the condition that $\varphi(\sum_{i=1}^m c_ie_i) = 0$ is equivalent to $\sum_{i,j} r_{i,j}x_jy_i = 0$. Now recall that for any $s = 1, \dots k$ we have

$$\varepsilon_s x_j y_i = \sum_{r=1}^m \beta_{j,i,s,r} \cdot y_r.$$

Therefore, multiplying equation 2.1 by ε_s, we get an equality

$$0 = \varepsilon_s \left(\sum_{i,j} r_{i,j}x_jy_i\right) = \sum_{i,j} r_{i,j}\left(\sum_{r=1}^m \beta_{j,i,s,r}\cdot y_r\right) = \sum_{r=1}^m \left(\sum_{i,j} r_{i,j}\beta_{j,i,s,r}\right) y_r$$

This means that for any $s' = 1, \dots, k$ the vector $\{\varepsilon_{s'}(\sum_{i,j} r_{i,j}\beta_{j,i,s,r})\}_{r=1}^m \in R^m$ lives in an R-subspace generated by vectors $\{\alpha_{i,j}\}_{i=1}^m$. In particular, for any r and s', $\varepsilon_{s'}(\sum_{i,j} r_{i,j}\beta_{j,i,s,r}e_r)$ is equal to ψ (some sum of f'_l) by definition of ψ.

After unwinding all the definitions we get the following:

$$\begin{aligned}
\varepsilon_{s'}\varepsilon_s\left(\sum_{i=1}^m c_ie_i\right) &= \varepsilon_{s'}\varepsilon_s\left(\sum_{i,j} r_{i,j}x_je_i\right) \\
&= \varepsilon_{s'}\left(\sum_{i,j} r_{i,j}\left(\varepsilon_s x_je_i - \sum_r \beta_{j,i,s,r}e_r + \sum_r \beta_{j,i,s,r}e_r\right)\right) \\
&= \varepsilon_{s'}\left(\sum_{i,j} r_{i,j}\left(\varepsilon_s x_je_i - \sum_r \beta_{j,i,s,r}e_r\right)\right) + \varepsilon_{s'}\left(\sum_r\left(\sum_{i,j} r_{i,j}\beta_{j,i,s,r}\right)e_r\right) \\
&= \psi\left(\varepsilon_{s'}\sum_{i,j} r_{i,j}f_{j,i,s}\right) + \psi\left(\text{some sum of } f'_l\right)
\end{aligned}$$

So we see that $\mathfrak{m}_1^2 \ker(\varphi) \subset \operatorname{Im}\psi$. In particular, we have $\mathfrak{m}_0 \ker(\varphi) \subset \operatorname{Im}\psi$. Now we replace the map $\varphi\colon S^n \to M$ with the induced map

$$\overline{\varphi}\colon \operatorname{Coker}(\psi) \to M$$

to get a map from a finitely presented left S-module such that $\ker(\overline{\varphi})$ and $\operatorname{Coker}(\overline{\varphi})$ are annihillated by $\mathfrak{m}_0$. ☐

2.9. **Almost Faithfully Flat Algebras.** This section is devoted to the notion of almost faithfully flat morphism of *algebras*. This notion is a bit subtle in the almost context. Similar to the case of the usual commutative algebra, one defines $R \to S$ to be almost (faithfully) flat if S^a is a (faithfully) flat R^a-module. Note that this implies that $S^a_!$ is a flat R-module, but $S^a_!$ is not necessarily faithfully flat as an R-module if S^a is faithfully flat as an R^a-module (see Warning C.1.8).

Another subtlety of this definition is that $S^a_!$ is not longer an R-algebra. So it seems difficult to relate almost faithful flatness of an R-algebra to some actual faithful flatness from this point of view. However, it turns out that things get better if we change the definition of the $(-)_!$-functor.

We introduce a different functor $(-)_{!!}\colon \mathbf{Alg}_R \to \mathbf{Alg}_R$. However, this functor will not in general send almost flat morphisms to flat morphisms, but it will send almost faithfully flat morphisms to faithfully flat morphisms. So it will be very usefull to deduce certain properties of almost faithfully flat morphisms from the analogous properties of classically faithfully flat morphisms.

We follows the exposition in [GR03] pretty closely here.

For the rest of the section, we fix a ring R with an ideal of almost mathematics $\mathfrak{m}$.

Definition 2.9.1. A homomorphism of R-algebras $A \to B$ is *almost flat* (resp. *almost faithfully flat*) if B^a is a flat (resp. faithfully flat) A^a-module (see Definition 2.2.5).

Lemma 2.9.2. Any (faithfully) flat A-algebra B is almost (faithfully) flat.

Proof. It follows directly from Lemma 2.2.6. □

Lemma 2.9.3. Let A be an R-algebra and $f\colon A \to B$ a morphism of R-algebras. Then B is almost faithfully flat over A if and only if B^a is a flat A^a-module and $A^a \to B^a$ is universally injective, i.e., for any A^a-module M^a, the natural morphism $M^a \to M^a \otimes_{A^a} B^a$ is injective in $\mathbf{Mod}_A^a$.

Proof. Suppose that B is almost faithfully flat. Then B^a is a flat A^a-module by definition. So we only need to show that $A^a \to B^a$ is universally injective. Pick any $M^a \in \mathbf{Mod}_A^a$ and consider an A^a-module

$$N^a := \ker(M^a \to M^a \otimes_{A^a} B^a).$$

It comes with a short exact sequence

$$0 \to N^a \to M^a \to M'^a \to 0,$$

where $M'^a = M^a/N^a$. Flatness of B^a implies that we have a short exact sequence

$$0 \to N^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a \to M'^a \otimes_{A^a} B^a \to 0.$$

Now we see that the morphism

$$N^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a$$

is equal to zero by our choice of N^a. But it is also injective (in $\mathbf{Mod}_A^a$), so $N^a \otimes_{A^a} B^a \simeq 0$. Since B^a is faithfully flat over A^a, we conclude that $N^a \simeq 0$.

Now we suppose that B^a is a flat A^a-module and $A^a \to B^a$ is universally injective. Thus, for any A^a-module M^a, we have an injection $M^a \to M^a \otimes_{A^a} B^a$. So if $M^a \otimes_{A^a} B^a \simeq 0$, we conclude that $M^a \simeq 0$. Thus B^a is faithfully flat over A^a. □

Corollary 2.9.4. Let A be an R-algebra and $f\colon A \to B$ is a morphism of R-algebras. Then B is almost faithfully flat over A if and only if B^a and $\mathrm{Coker}(f^a)$ are flat A^a-modules.

Proof. By Lemma 2.9.3, it suffices to show that f^a is universally injective if and only if $\mathrm{Coker}(f^a)$ is A^a-flat. We note that, for any A^a-module M^a, $\ker(M^a \to M^a \otimes_{A^a} B^a) \simeq \mathrm{H}^{-1}(M^a \otimes^L_{A^a} \mathrm{Coker}(f^a))$. In particular,

$$\mathrm{H}^{-1}\left(M^a \otimes^L_{A^a} \mathrm{Coker}(f^a)\right) \simeq 0$$

for any A^a-module M^a if and only if the functor $- \otimes_{A^a} \mathrm{Coker}(f^a)\colon \mathbf{Mod}_A^a \to \mathbf{Mod}_A^a$ is exact. In other words, $A^a \to B^a$ is universally injective if and only if $\mathrm{Coker}(f^a)$ is flat over A^a. □

Now we define the functor $(-)_{!!}\colon \mathbf{Alg}_R \to \mathbf{Alg}_R$. We start by constructing an R-algebra structure on $R \oplus A^a_! = R \oplus (\widetilde{\mathfrak{m}} \otimes_R A)$ by defining the multiplication as

$$(r \oplus a) \cdot (r' \oplus a') = (rr') \oplus (ra' + r'a + aa')$$

and the summation law is the usual one. One easily checks that this is a well-defined (unital, commutative) R-algebra structure on $R \oplus A_!^a$. We consider the R-submodule I_A of $R \oplus A_!$ generated by elements of the form $(mn, -m \otimes n \otimes 1_A)$ for $m, n \in \mathfrak{m}$.

Lemma 2.9.5. The R-module $I_A \subset R \oplus A_!^a$ is an ideal.

Proof. It suffices to show that, for any element $(r, x \otimes y \otimes a)$ in $R \oplus A_!^a$, the product

$$(r \oplus x \otimes y \otimes a) \cdot (mn \oplus -m \otimes n \otimes 1_A)$$

lies in I_A for any $m, n \in \mathfrak{m}$. By definition,

$$\begin{aligned}(r \oplus x \otimes y \otimes a) \cdot (mn \oplus -m \otimes n \otimes 1_A) &= (rmn) \oplus (-rm \otimes n \otimes 1_A + xm \otimes yn \otimes a - xm \otimes yn \otimes a) \\ &= r(mn \oplus -m \otimes n \otimes 1_A) \in I_A.\end{aligned}$$

□

Definition 2.9.6. The functor $(-)_{!!} \colon \mathbf{Alg}_R \to \mathbf{Alg}_R$ is defined as

$$A \mapsto (R \oplus A_!^a)/I_A$$

with the induced R-algebra structure.

For any R-algebra A, there is a functorial R-algebra homomorphism $R \oplus A_!^a \to A$ defined by

$$r \oplus (m \otimes n \otimes a) \mapsto r + mna.$$

Clearly, this homomorphism is zero on I_A so it descends to an R-algebra homomorphism $\eta \colon A_{!!} \to A$.

Lemma 2.9.7. **(1)** For any R-algebra A, the natural morphism $\eta \colon A_{!!} \to A$ is an almost isomorphism.

(2) A morphism of R-algebras $f \colon A \to B$ is almost injective (as a morphism of R-modules) if and only if $f_{!!} \colon A_{!!} \to B_{!!}$ is injective.

(3) For any morphism of R-algebras $f \colon A \to B$, there is a canonical isomorphism of $A_{!!}$-modules $\mathrm{Coker}(f_{!!}) \simeq \mathrm{Coker}(f)_!$.

(4) The functor $(-)_{!!} \colon \mathbf{Alg}_R \to \mathbf{Alg}_R$ commutes with tensor products.

Proof. **(1)** : We recall that the morphism $A_! \to A$ is almost isomorphism. In particular, it is almost surjective. Thus $A_{!!} \to A$ is also almost surjective. Now we check almost injectivity. Suppose $\eta(\overline{a}) = 0$ where $a = r \oplus \sum_{i=1}^k m_i \otimes n_i \otimes a_i \in R \oplus \widetilde{\mathfrak{m}} \otimes A$ and $\overline{a} \in A_{!!}$ is the class of a in $A_{!!}$. Then the condition $\eta(\overline{a}) = 0$ implies that there is an equality

$$r + \sum_{i=1}^k m_i n_i a_i = 0$$

in A. In particular, for every $\varepsilon \in \mathfrak{m}$, we have $\varepsilon r = \sum_{i=1}^k (-m_i)(\varepsilon n_i a_i)$ in A. Thus, we see that

$$\begin{aligned}\varepsilon a =& \varepsilon r \oplus \sum_{i=1}^k m_i \otimes n_i \otimes \varepsilon a_i \\ &= \sum_{i=1}^k (-m_i)(\varepsilon n_i a_i) \oplus \sum_{i=1}^k m_i \otimes n_i \varepsilon a_i \otimes 1_A \\ &= \sum_{i=1}^k \left((-m_i) \left(\varepsilon n_i a_i \right) \oplus m_i \otimes \varepsilon n_i a_i \otimes 1_A \right) \in I_A.\end{aligned}$$

Therefore, $\varepsilon\overline{a} = 0$ for every $\varepsilon \in \mathfrak{m}$. In particular, η is almost injective.

(**2**) and (**3**) : Consider a commutative diagram

$$\begin{tikzcd} A_{!!} \arrow[r, "f_{!!}"] \arrow[d, "\eta_A"] & B_{!!} \arrow[d, "\eta_B"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Since η_A and η_B are almost isomorphism, we see that f is almost injective if and only if $f_{!!}$ is almost injective. So we are left to show that $f_{!!}$ is injective if f is almost injective, and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$. For this, we consider a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & I_A \arrow[r] \arrow[d, "\alpha"] & R \oplus A_! \arrow[r] \arrow[d, "\mathrm{Id}\oplus f_!"] & A_{!!} \arrow[r] \arrow[d, "f_{!!}"] & 0 \\ 0 \arrow[r] & I_B \arrow[r] & R \oplus B_! \arrow[r] & B_{!!} \arrow[r] & 0. \end{tikzcd}$$

Clearly, α is surjective, $\ker(\mathrm{Id}\oplus f_!) = \ker(f_!) = \ker(f)_!$, and $\mathrm{Coker}(\mathrm{Id}\oplus f_!) = \mathrm{Coker}(f_!) = \mathrm{Coker}(f)_!$. Thus, the Snake Lemma implies that

$$\ker(f)_! \to \ker(f_{!!})$$

is surjective and

$$\mathrm{Coker}(f_{!!}) \to \mathrm{Coker}(f)_!$$

is an isomorphism. Thus $f_{!!}$ is injective if f is almost injective, and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$.

(**4**) : This is an elementary but pretty tedious computation. We leave it to the interested reader. $\square$

Corollary 2.9.8. For any R-algebra A, the forgetful functor $\mathbf{Mod}^*_{A^a} \to \mathbf{Mod}^*_{A^a_{!!}}$ is an equivalence for $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$.

Proof. For $* =$ “ ”,the claim follows from Lemma 2.9.7 **(1)**, Corollary 2.5.13, and Lemma 2.6.3. $\square$

Corollary 2.9.9. Let $f\colon A \to B$ be an almost faithfully flat morphism of R-algebras. Then $f_{!!}\colon A_{!!} \to B_{!!}$ is faithfully flat.

Proof. Denote by Q the cokernel f as an A-module. Then Lemma 2.9.3 and Lemma 2.9.7 **(2)**, **(3)** ensure that $f_{!!}\colon A_{!!} \to B_{!!}$ is injective and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$. Now Corollary 2.9.4 and Lemma 2.2.7 applied to $A^a_{!!} \simeq A^a$ imply that $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$ is a flat $A_{!!}$-module. This already implies that B is a flat $A_{!!}$-module as an extension of two flat $A_{!!}$-modules. To see that it is faithfully flat, we note that flatness of $\mathrm{Coker}(f_{!!})$ implies that

$$M \to M \otimes_{A_{!!}} B_{!!}$$

is injective for any $A_{!!}$-module M. So $M \otimes_{A_{!!}} B_{!!} \simeq 0$ if and onlyif $M \simeq 0$. In other words, $B_{!!}$ is a faithfully flat $A_{!!}$-module. $\square$

Warning 2.9.10. The functor $(-)_{!!}$ does not send flat A-algebras to flat $A_{!!}$-algebras. See [GR03, Remark 3.1.3].

For the future reference, we also show that the base change functor interacts especially well with the Hom-functor in the almost flat situation.

Lemma 2.9.11. Let $R \to S$ be an almost flat morphism, M an almost finitely presented R-module, and N an R-module. Then the natural map

$$\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an almost isomorphism.

Proof. This follows from the classical $\otimes$-Hom adjunction and Lemma 2.5.18. $\square$

Lemma 2.9.12. Let R be an almost coherent ring, $R \to S$ be an almost flat map, and $M \in \mathbf{D}^-_{acoh}(R)$, $N \in \mathbf{D}^+(R)$. Then the natural map

$$\mathbf{R}\mathrm{Hom}_R(M, N) \otimes^L_R S \to \mathbf{R}\mathrm{Hom}_S(M \otimes^L_R S, N \otimes^L_R S)$$

is an almost isomorphism.

Proof. We recall that we always have a canonical isomorphism $\mathbf{R}\mathrm{Hom}_R(K, L) \simeq \mathbf{R}\mathrm{Hom}_S(K \otimes^L_R S, L)$ for any $K \in \mathbf{D}^-(R)$ and any $L \in \mathbf{D}^+(S)$. This implies that it suffices to show that the natural map

$$\mathbf{R}\mathrm{Hom}_R(M, N) \otimes^L_R S \to \mathbf{R}\mathrm{Hom}_R(M, N \otimes^L_R S)$$

is an almost isomorphism. This follows from Proposition 2.6.20. $\square$

2.10. **Almost Faithfully Flat Descent.** The main goal of this section is to show almost faithfully flat descent for almost modules.

For the rest of the section, we fix a ring R with an ideal of almost mathematics $\mathfrak{m}$.

In this section, for any morphism $A \to B$ of R-algebras, we denote the tensor product functor $- \otimes_{A^a} B^a$ simply by

$$f^* \colon \mathbf{Mod}^a_A \to \mathbf{Mod}^a_B.$$

In particular, if $A \to B$ is a morphism of R-algebras, the canonical "co-projection" morphisms $p_i \colon B \to B \otimes_A B$ induce morphisms

$$p_i^* \colon \mathbf{Mod}^a_B \to \mathbf{Mod}^a_{B \otimes_A B}$$

for $i \in \{1, 2\}$. The same applies to the "co-projections"

$$p_{i,j}^* \colon \mathbf{Mod}^a_{B \otimes_A B} \to \mathbf{Mod}^a_{B \otimes_A B \otimes_A B}$$

for $i \neq j \in \{1, 2\}$.

Definition 2.10.1. An *almost descent category* $\mathbf{Desc}^a_{B/A}$ for a morphism of R-algebras $A \to B$ is a category whose objects are pairs (M^a, ϕ), where $M^a \in \mathbf{Mod}^a_B$ and

$$\phi \colon p_1^*(M^a) \to p_2^*(M^a)$$

in an isomorphism of $(B \otimes_A B)^a$-modules such that $p_{1,3}^*(\phi) = p_{2,3}^*(\phi) \circ p_{1,2}^*(\phi)$. Morphisms between (M^a, ϕ_M) and (N^a, ϕ_N) are defined to be B^a-linear homomorphisms $f \colon M^a \to N^a$ such that the diagram

$$\begin{tikzcd} p_1^*(M^a) \arrow[r, "\phi_M"] \arrow[d, "p_1^*(f)"] & p_2^*(M^a) \arrow[d, "p_2^*(f)"] \\ p_1^*(N^a) \arrow[r, "\phi_N"] & p_2^*(N^a) \end{tikzcd}$$

commutes.

Remark 2.10.2. Explicitly, an object of the descent category $\mathbf{Desc}^a_{B/A}$ is a B^a-module M^a with a $(B \otimes_A B)^a$-linear homomorphism $\phi \colon M^a \otimes_{A^a} B^a \to B^a \otimes_{A^a} M^a$ satisfying the "cocycle condition".

There is a natural functor

$$\mathrm{Ind}\colon \mathbf{Mod}_A^a \to \mathbf{Desc}_{B/A}^a$$

that sends M^a to $f^*(M^a) = M^a \otimes_{A^a} B^a$ with a canonical identification $\phi\colon p_1^* f^*(M^a) \simeq p_2^* f^*(M^a)$ coming from the equality $f \circ p_1 = f \circ p_2$.

To define a functor in the other direction, we note that we have natural B^a-module morphisms $\iota_i\colon M^a \to p_i^*(M^a)$ for $i \in \{1,2\}$. Explicitly, they are defined as morphisms induced by $\iota_1(m) = m \otimes 1$ and $\iota_2(m) = 1 \otimes m$. Therefore, given a descent data $(M^a, \phi) \in \mathbf{Desc}_{B/A}^a$, we can define

$$\ker(M^a, \phi) \coloneqq \ker(M^a \xrightarrow{i_1 - \phi^{-1} i_2} M^a \otimes_{A^a} B^a).$$

This is an A^a-module, and it is not difficult to check that this association is functorial in $\mathbf{Desc}_{B/A}^a$. Therefore, it defines a functor

$$\ker\colon \mathbf{Desc}_{B/A}^a \to \mathbf{Mod}_A^a.$$

We show that ker and Ind are quasi-inverse to each other and induce an equivalence between $\mathbf{Desc}_{B/A}^a$ and $\mathbf{Mod}_A^a$ for an almost faithfully flat morphism $f\colon A \to B$.

Theorem 2.10.3. Let $f\colon A \to B$ be an almost faithfully flat morphism. Then

$$\mathrm{Ind}\colon \mathbf{Mod}_A^a \to \mathbf{Desc}_{B/A}^a$$

is an equivalence, and its quasi-inverse is given by the functor $\ker\colon \mathbf{Desc}_{B/A}^a \to \mathbf{Mod}_A^a$.

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is faithfully flat. Then the claim follows from the classical faithfully flat descent (see [BLR90, Theorem 6.1/4]) and the observation that the non-almost versions of Ind and ker carry almost isomorphisms to almost isomorphisms. $\square$

On a similar note, we show that the Amitsur complex for an almost faithfully flat morphism is acyclic.

Lemma 2.10.4. Let $f\colon A \to B$ be an almost faithfully flat morphism of R-algebras, and $M \in \mathbf{Mod}_B^a$. Then the Amitsur complex

$$0 \to M^a \to M^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a \otimes_{A^a} B^a \to$$

is an exact complex of $\mathbf{Mod}_B^a$-modules (see the discussion around [Sta21, Tag 023K] for the precise definition of differentials in this complex).

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is faithfully flat. Then the claim follows from [Sta21, Tag 023M]. $\square$

Now we show that some properties of A^a-modules can be verified after a faithfully flat base change.

Lemma 2.10.5. Let $f\colon A \to B$ be an almost faithfully flat morphism of R-algebras, and let M^a be an A^a-module. Then M^a is an almost finitely generated (resp. almost finitely presented) A^a-module if and only if $M^a \otimes_{A^a} B^a$ is an almost finitely generated (resp. almost finitely presented) B^a-module.

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is a faithfully flat morphism. Then a standard argument reduces the questions to the case of an honest A-module M, i.e. we show that an A-module M is almost finitely generated (resp. almost finitely presented) if so is the B-module $M \otimes_A B$.

We start with the almost finitely generated case. So we suppose that $M \otimes_A B$ is almost finitely generated, thus given any $\varepsilon \in \mathfrak{m}$ we can choose a morphism $g: B^n \to M \otimes_A B$ such that $\varepsilon(\mathrm{Coker}\, g) = 0$. Let us consider the standard basis $e_1, \dots, e_n$ of B^n, and we write

$$g(e_i) = \sum_j m_{i,j} \otimes b_{i,j} \text{ with } m_{i,j} \in M, b_{i,j} \in B.$$

We define an A-module F to be a finite free A-module with a basis $e_{i,j}$. Then we define morphism

$$h\colon F \to M$$

as a unique A-linear homomorphism with $h(e_{i,j}) = m_{i,j}$. It is easy to see that $\varepsilon(\mathrm{Coker}(h \otimes_A B)) = 0$. Since f is faithfully flat, this implies that $\varepsilon(\mathrm{Coker}\, h) = 0$. We conclude that M is almost finitely generated as ε was an arbitrary element of $\mathfrak{m}$.

Now we deal with the almost finitely presented case. We pick some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m}$. We try to find a three-term complex

$$A^m \xrightarrow{g} A^n \xrightarrow{f} M$$

such that $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$ and $\mathfrak{m}_0(\ker f) \subset \mathrm{Im}\, g$.

The settled almost finitely generated case implies that M is at least almost finitely generated. In particular, we have some morphism

$$A^n \xrightarrow{f} M$$

such that $\mathfrak{m}_1(\mathrm{Coker}\, f) = 0$, thus $\mathfrak{m}_1(\mathrm{Coker}(f \otimes_A B)) = 0$ as well. Therefore, we can apply Lemma 2.5.6 to find a homomorphism $g'\colon B^m \to B^n$ such that $\mathfrak{m}_0(\ker(f \otimes_A B)) \subset \mathrm{Im}(g')$ and $(f \otimes_A B) \circ g' = 0$. This implies that g' actually lands inside $\ker(f \otimes_A B) = \ker(f) \otimes_A B$ by A-flatness of B.

Now we do the same trick as above: we write

$$g(e_i) = \sum_j m_{i,j} \otimes b_{i,j} \text{ with } m_{i,j} \in \ker(f), b_{i,j} \in B.$$

We define an R-module F to be a finite free A-module with a basis $e_{i,j}$. Then we define a morphism

$$g\colon F \to \ker(f)$$

as the unique A-linear morphism such that $g(e_{i,j}) = m_{i,j}$. Then we see that $\mathfrak{m}_0(\ker(f \otimes_A B)) \subset \mathrm{Im}(g \otimes_A B)$. Since B is faithfully flat we conclude that $\mathfrak{m}_0(\ker f) \subset \mathrm{Im}(g)$ as well. This shows that a three-term complex

$$F \xrightarrow{g} A^n \xrightarrow{f} M$$

does the job. Therefore, M is an almost finitely presented A-module. □

Corollary 2.10.6. Let $f\colon A \to B$ be an almost faithfully flat morphism of R-algebras, let M^a be an A^a-module. Suppose that $M^a \otimes_{A^a} B^a$ is an almost coherent B^a-module. Then so is M^a.

Proof. This follows directly from Lemma 2.6.3 and Lemma 2.10.5. □

Lemma 2.10.7. Let $f\colon A \to B$ be an almost faithfully flat morphism of R-algebras, and let M^a be an A^a-module. Then M^a is a flat (resp. faithfully flat) A^a-module if and only if $M^a \otimes_{A^a} B^a$ is a flat (resp. faithfully flat) B^a-module.

Proof. The classical proof works verbatim in the almost world. We leave details to the reader. □

2.11. **(Topologically) Finite Type K^+-Algebras.** This section is devoted to the proof that (topologically) finite type algebras over a perfectoid valuation ring K^+ are almost noetherian. We refer to Appendix B.1 for the relevant background on perfectoid valuation rings.

For the rest of the section we fix a perfectoid valuation ring K^+ (see Definition B.2) with perfectoid fraction field K, associated rank-1 valuation ring $\mathcal{O}_K = K^\circ$ (see Remark B.3), and ideal of topologically nilpotent elements $\mathfrak{m} = K^{\circ\circ} \subset K^+$. Lemma B.6 ensures that $\mathfrak{m}$ is flat over K^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$. Therefore, it makes sense to do almost mathematics with respect to the pair $(K^+, \mathfrak{m})$. In what follows, we always do almost mathematics on K^+-modules with respect to this ideal.

Warning 2.1. The ideal $\mathfrak{m} \subset K^+$ is not the maximal ideal of K^+. Instead, it is the maximal ideal of the associated rank-1 valuation ring $\mathcal{O}_K$.

Lemma 2.11.1. Let K^+ be a perfectoid valuation ring. Then the natural inclusion $\iota\colon K^+ \to \mathcal{O}_K$ is an almost isomorphism.

Proof. Clearly, the map $\iota\colon K^+ \to \mathcal{O}_K$ is injective, so it suffices to show that its cokernel is almost zero, i.e. annihilated by any $\varepsilon \in \mathfrak{m}$. Pick an element $x \in \mathcal{O}_K$, then $\varepsilon x \in \mathfrak{m} \subset K^+$. Therefore we conclude that $\varepsilon(\operatorname{Coker} \iota) = 0$ finishing the proof. □

The first main result of this section is that any (topologically) finite type algebra over K^+ is almost noetherian.

Lemma 2.11.2. Let K^+ be a perfectoid valuation ring, and $n \geq 0$ an integer. Then the Tate algebra $K^+\langle T_1, \dots, T_n\rangle$ is almost noetherian.

Proof. Firstly, we note that $\mathcal{O}_K\langle T_1, \dots, T_n\rangle \simeq K^+\langle T_1, \dots, T_n\rangle \otimes_{K^+} \mathcal{O}_K$. Therefore, Lemma 2.11.1 implies that the natural morphism

$$K^+\langle T_1, \dots, T_n\rangle \to \mathcal{O}_K\langle T_1, \dots, T_n\rangle$$

is an almost isomorphism. So Corollary 2.7.5 ensures that it suffices to show that $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$ is almost noetherian.

Pick any ideal $I \subset \mathcal{O}_K\langle T_1, \dots, T_n\rangle = K\langle T_1, \dots, T_n\rangle^\circ$ and $0 \neq \varepsilon \in \mathfrak{m}$. Now [Bos14, Lemma 6.4/5] applied to $B = K\langle T_1, \dots, T_n\rangle$, $E = \mathcal{O}_K\langle T_1, \dots, T_n\rangle$, $E' = I$, and $\alpha = |\varepsilon|_K$ guarantees that there is a finite submodule $E'' \subset I$ such that $\varepsilon I \subset E''$. Since ε was an arbitrary element of $\mathfrak{m}$, we conclude that I is indeed almost finitely generated. □

Corollary 2.11.3. Let K^+ be a perfectoid valuation ring, $\varpi \in \mathfrak{m}$, and $n \geq 0$ an integer. Then the polynomial algebra $(K^+/\varpi^m)[T_1, \dots, T_n]$ is almost noetherian for any $m \geq 1$.

Proof. It easily follows from Lemma 2.11.2, Corollary 2.7.4, and Lemma 2.8.3. □

Theorem 2.11.4. Let K^+ be a perfectoid valuation ring, and A a topologically finite type K^+-algebra. Then A is almost noetherian.

Proof. Since A is topologically finite type over K^+, there exists a surjection

$$f\colon K^+\langle T_1, \dots, T_n\rangle \to A \to 0.$$

Pick an ideal $I \subset A$ and consider its preimage $J = f^{-1}(I)$. Then J is almost finitely generated over $K^+\langle T_1, \dots, T_n\rangle$ by Lemma 2.11.2. Therefore, Lemma 2.5.15 (1) ensures that I is almost finitely generated over $K^+\langle T_1, \dots, T_n\rangle$. Finally, Lemma 2.8.3 ensures that I is therefore also almost finitely generated over A. □

Now we are going to show that any finite type K^+-algebra is almost noetherian. Before doing this, we need a couple of preliminary lemmas.

Lemma 2.11.5. Let R be a rank-1 valuation ring with a non-zero topologically nilpotent element $\varpi \in R$, and M a finite $R[T_1, \dots, T_n]$-module. Then $M[\varpi^\infty] = M[\varpi^c]$ for some $c \geq 0$.

Proof. The $R[T_1, \dots, T_n]$-module $M' \coloneqq M/M[\varpi^\infty]$ is finitely generated. Moreover M' is R-flat because it is torsion-free (and R is a valuation ring). Therefore, [Sta21, Tag 053E] ensures that M' is finitely presented over $R[T_1, \dots, T_n]$. Thus we conclude that $M[\varpi^\infty]$ is finitely generated. In particular, $M[\varpi^\infty] = M[\varpi^c]$ for some N. □

Lemma 2.11.6. Let R be a rank-1 valuation ring with a non-zero topologically nilpotent element $\varpi \in R$, M a finite $R[T_1, \dots, T_n]$-module, and $N \subset M$ an $R[T_1, \dots, T_n]$-submodule. Then there is c such that

$$N \cap \varpi^{m+c} M = \varpi^m (N \cap \varpi^c M)$$

for every $m \geq 0$.

Proof. Lemma 2.11.5 ensures that there c such that $(M/N)[\varpi^\infty] = (M/N)[\varpi^c]$. Therefore, [FK18, Lemma 0.8.2.14] guarantees that, indeed,

$$N \cap \varpi^{m+c} M = \varpi^m (N \cap \varpi^c M)$$

for every $m \geq 0$. □

Lemma 2.11.7. Let K^+ be a perfectoid valuation ring, and $n \geq 0$ an integer. Then the polynomial algebra $K^+[T_1, \dots, T_n]$ is almost noetherian.

Proof. Similar to the proof of Lemma 2.11.2, it suffices to treat the case $K^+ = \mathcal{O}_K$ a perfectoid valuation ring of rank-1 with a pseudo-uniformizer ϖ.

Now we fix an ideal $I \subset A \coloneqq \mathcal{O}_K[T_1, \dots, T_n]$ and wish to show that I is almost finitely generated. Recall that the polynomial algebra $K[T_1, \dots, T_n]$ is noetherian by Hilbert's Nullstellensatz. Therefore, the ideal

$$I\left[\frac{1}{\varpi}\right] \subset K[T_1, \dots, T_n]$$

is finitely generated. So we can choose a finitely generated sub-ideal $J \subset I$ such that any element of I/J is annihilated by a power of ϖ, i.e. $(I/J)[\varpi^\infty] = I/J$. Clearly I/J is a submodule of a finite A-module A/J, so Lemma 2.11.5 easily implies that

$$I/J = (I/J)[\varpi^\infty] = (I/J)[\varpi^c]$$

for some $c \geq 0$. In other words, $\varpi^c I \subset J$. Now we use Lemma 2.11.6 to get an integer c' such that

$$I \cap \varpi^{c'} A \subset \varpi^c I \subset J.$$

We note that $I/(I \cap \varpi^{c'} A)$ is an ideal in $A/\varpi^{c'} A$, and therefore it is almost finitely generated over $A/\varpi^{c'} A$ by Corollary 2.11.3. Lemma 2.8.3 guarantees that it is also almost finitely generated over A.

The inclusion $I \cap \varpi^{c'} A \subset J$ implies that I/J is a quotient of an almost finitely generated $A/(I \cap \varpi^{c'} A)$, and so is also almost finitely generated. Finally, the short exact sequence

$$0 \to J \to I \to I/J \to 0$$

and Lemma 2.5.15 **(2)** imply that I is almost finitely generated as well. □

Theorem 2.11.8. Let K^+ be a perfectoid valuation ring, and A a finite type K^+-algebra. Then A is almost noetherian.

Proof. It follows from Lemma 2.11.7 similar to how Theorem 2.11.4 follows from Lemma 2.11.2. □

2.12. **Almost Finitely Generated Modules over Adhesive Rings.** This section discusses some basic aspects of almost finitely generated modules over adhesive rings. The motivation for this discussion will be the notion of almost coherent sheaves on formal schemes that we develop in Section 4.5. The results of this Section would be crucial in verifying certain good properties of adically quasi-coherent, almost coherent sheaves on "good" formal schemes. One of the main ingredients that we would need is the "Weak" version of the Artin-Rees Lemma (Lemma 2.12.6) and Lemma 2.12.7. Recall that these properties are already known for finite modules over the so-called "adhesive" rings. This is explained in a beautiful paper [FGK11]. The main goal of this section is to extend these result to the case of almost finitely generated modules.

That being said, let us introduce the Setup for this section. We start with the definition of an adhesive ring:

Definition 2.12.1. [FGK11, Definition 7.1.1] An adically topologized ring R endowed with the adic topology defined by a finitely generated ideal $I \subset R$ is said to be *(I-adically) adhesive* if it is Noetherian outside[10] I and satisfies the following condition: for any finitely generated R-module M, its I^∞-torsion part $M[I^\infty]$ is finitely generated.

Remark 2.12.2. Following the convention of [FGK11] we do not require a ring R with adic topology to be either I-adically complete or separated.

Set-up 2.12.3. We fix an I-adically adhesive ring R with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is flat. We always do almost mathematics with respect to the ideal $\mathfrak{m}$.

The main example of an adhesive ring is a (topologically) finitely presented algebra over a complete microbial valuation ring. This follows from [FGK11, Proposition 7.2.2] and [FGK11, Theorem 7.3.2]. For example, any topologically finitely presented algebra over a complete rank-1 valuation ring is adhesive.

Lemma 2.12.4. Let R be as in the Setup 2.12.3, and let M be an I-torsionfree almost finitely generated module. Then M is almost finitely presented. Similarly, any saturated submodule[11] of an almost finitely generated R-module is almost finitely generated.

Proof. As M is almost finitely generated, we can find a finitely generated submodule $N \subset M$ that contains $\mathfrak{m}_0 M$ for a choice of a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Since N is a submodule of M, it is itself I-torsionfree. Then [FGK11, Proposition 7.1.2] shows that N is finitely presented. Then Lemma 2.5.7**(2)** implies that M is almost finitely presented.

Now let M be an almost finitely generated R-module, and let $M' \subset M$ be a saturated submodule. Then M/M' is almost finitely generated by Lemma 2.5.15**(1)** and it is I-torsionfree. Therefore, it is almost finitely presented by the argument above. Then Lemma 2.5.15**(3)** guarantees that M' is almost finitely generated. □

Lemma 2.12.5. Let R be as in the Setup 2.12.3, and let M be an almost finitely generated R-module. Then the I^∞-torsion module $M[I^\infty]$ is bounded (i.e. there is an integer n such that $M[I^n] = M[I^\infty]$).

[10]By definition, this means that the scheme $\operatorname{Spec} A \setminus V(I)$ is noetherian.

[11]A submodule $N \subset M$ is *saturated* if $M/N[I^\infty] = 0$.

Proof. Since M is almost finitely generated and the ideal $I \subset \mathfrak{m}$ is finitely generated, we conclude that there exists a finitely generated submodule $N \subset M$ such that $IM \subset N$. Then $I(M[I^\infty]) \subset N[I^\infty]$, and $N[I^\infty]$ is finitely generated by adhesiveness of the ring R. In particular, there is an integer n such that $N[I^\infty]$ is annihilated by I^n. This implies that any element of $M[I^\infty]$ is annihilated by $n+1$. □

Lemma 2.12.6. Let R be as in the Setup 2.12.3, and let M be an almost finitely generated R-module. Suppose that $N \subset M$ is a submodule of M. For any integer n, there is an integer m such that $N \cap I^m M \subset I^n N$. In particular, the induced topology on the module N coincides with the I-adic one.

Proof. If M is finitely generated, then this is [FGK11, Theorem 4.2.2]. In general we use the definition of almost finitely generated module to find a submodule $M' \subset M$ such that M' is finitely generated and $IM \subset M'$. We define $N' := N \cap M'$ as the intersection of those modules. Then the established "weak" form of the Artin-Rees Lemma for finitely generated R-modules provides us with an integer m such that $N' \cap I^m M' \subset I^n N'$. In particular, we have

$$I^{m+1}M \cap N' \subset I^m M' \cap N' \subset I^n N' \subset I^n N.$$

Then we conclude that

$$I^{m+2}M \cap N \subset I^{m+1}M \cap M' \cap N \subset I^{m+1}M \cap N' \subset I^n N.$$

Since n was arbitrary, we conclude the claim. □

Lemma 2.12.7. Let R be as in the Setup 2.12.3, and let M be an almost finitely generated R-module. Then the natural morphism $M \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism. In particular, any almost finitely generated module over a complete adhesive ring is complete.

Proof. We know that the claim holds for finitely generated modules by [FGK11, Proposition 4.3.4]. Now we deal with the almost finitely generated case. We choose a finitely generated submodule $N \subset M$ such that $IM \subset N$. Lemma 2.12.6 implies that the induced topology on N coincides with the I-adic topology on N. Thus the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

remains exact after completion. Since $R \to \widehat{R}$ is flat by [FGK11, Proposition 4.3.4], we conclude that we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_R \widehat{R} \arrow[r] \arrow[d, "\varphi_N"] & M \otimes_R \widehat{R} \arrow[r] \arrow[d, "\varphi_M"] & (M/N) \otimes_R \widehat{R} \arrow[r] \arrow[d, "\varphi_{M/N}"] & 0 \\
0 \arrow[r] & \widehat{N} \arrow[r] & \widehat{M} \arrow[r] & \widehat{M/N} \arrow[r] & 0
\end{tikzcd}$$

Note that φ_N is an isomorphism as N is finitely generated, and $\varphi_{M/N}$ is isomorphism since it is an I-torsion module so $M/N \simeq (M/N) \otimes_R \widehat{R} \simeq \widehat{M/N}$. The five-lemma implies that φ_M is an isomorphism as well. □

Corollary 2.12.8. Let R be as in the Setup 2.12.3, and let $M \in \mathbf{D}_{acoh}(R)$. Suppose that R is I-adically complete. Then M is I-adically derived complete[12].

[12] Look at [Sta21, Tag 091N] for the definition of derived completeness (or Definition A.1 in case of a principal ideal I).

Proof. First of all, we note that [Sta21, Tag 091P] implies that M is derived complete if and only if so are $\mathrm{H}^i(M)$ for any integer i. So it suffices to show that any almost coherent R-module is derived complete. Lemma 2.12.7 gives that any such module is classically complete, and [Sta21, Tag 091T] ensures that any classically complete module is derived complete. □

2.13. **Modules Over Topologically Finite Type K^+-Algebras.** The main goal of this section is to show that almost coherentness of derived complete modules over a topologically finite type K^+-algebras can be checked modulo the pseudo-uniformizer.

For the rest of the section we fix a valuation perfectoid ring K^+ (see Definition B.2) with perfectoid fraction field K, associated rank-1 valuation ring $\mathcal{O}_K = K^\circ$ (see Remark B.3), and ideal of topologically nilpotent elements $\mathfrak{m} = K^{\circ\circ} \subset K^+$ with a pseudo-uniformizer $\varpi \in \mathfrak{m}$ as in Lemma B.5 (in particular, $\mathfrak{m} = \bigcup_n \varpi^{1/p^n} K^+$). Lemma B.6 ensures that $\mathfrak{m}$ is flat over K^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$. Therefore, it makes sense to do almost mathematics with respect to the pair $(K^+, \mathfrak{m})$. In what follows, we always do almost mathematics on K^+-modules with respect to this ideal.

Lemma 2.13.1. Let R be a topologically finite type K^+-algebra, and M an R-module that is ϖ-adically derived complete. Suppose that $M/\varpi M$ is almost coherent, then M is almost coherent as well.

Proof. Theorem 2.11.4 ensures that R is almost noetherian, and so Corollary 2.7.8 implies that it suffices to check that M is almost finitely generated. Recall that $\mathfrak{m} = \bigcup_n \varpi^{1/p^n} K^+$ for a pseudo-uniformizer ϖ as in Lemma B.5.

The assumption on M says that $M/\varpi M$ is almost coherent. Therefore, there is a morphism

$$\overline{g}\colon (R/\varpi R)^c \to M/\varpi M$$

such that $\varpi^{1/p}(\operatorname{Coker} \overline{g}) = 0$. We denote its cokernel by $\overline{Q} \coloneqq \operatorname{Coker}(\overline{g})$. Now we lift $\overline{g}$ to a morphism

$$g\colon R^c \to M$$

and denote is cokernel by $Q \coloneqq \operatorname{Coker}(g)$.

Step 1: Q is annihilated by $\varpi^{1/p}$. Suppose that $\varpi^{1/p} Q \neq 0$, so there is $x_0 \in Q$ such that $\varpi^{1/p} x_0 \neq 0$. Firstly, we note that $Q/\varpi \simeq \overline{Q}$ is annihilated by $\varpi^{1/p}$, so

$$\varpi^{1/p} x_0 = \varpi x_1 = \varpi^{1-1/p}(\varpi^{1/p} x_1)$$

Now we apply the same thing to x_1 to get

$$\varpi^{1/p} x_0 = \varpi^{1-1/p}(\varpi^{1/p} x_1) = (\varpi^{1-1/p})^2(\varpi^{1/p} x_2).$$

Keep going, to get a sequence of elements $x_n \in Q$ such that

$$\varpi^{1-1/p}(\varpi^{1/p} x_n) = \varpi^{1/p} x_{n-1}.$$

The sequence $\{\varpi^{1/p} x_i\}$ gives an element of

$$T^0(Q, \varpi^{1-1/p}) \coloneqq \lim_n (\dots \xrightarrow{\varpi^{1-1/p}} Q \xrightarrow{\varpi^{1-1/p}} Q)$$

that is non-trivial because $\varpi^{1/p} x_0 \neq 0$. Now we note that R^c is derived ϖ-adically complete since R is classically ϖ-adically complete by [Bos14, Corollary 7.3/9] and any classically complete module is derived complete by [Sta21, Tag 091T]. Therefore, Q is ϖ-adically derived complete derived complete as a cokernel of derived complete modules (see [Sta21, Tag 091U]). Now [Sta21, Tag 091S], Remark A.2, and [Sta21, Tag 091Q] imply that $T^0(Q, \varpi^{1-1/p})$ must be zero leading to the contradiction.

Step 2: M is almost coherent. Note that $\overline{Q} \simeq Q/\varpi Q$ and Q is $\varpi^{1/p}$-torsion, so $\overline{Q} \simeq Q$. We know that $\overline{Q}$ is almost finitely generated over $R/\varpi R$ because it is a quotient of an almost finitely generated module $M/\varpi M$. Therefore, $Q \simeq \overline{Q}$ is almost finitely generated over R by Lemma 2.8.3. Now M is an extension of a finite R-module $\mathrm{Im}(g)$ by an almost finitely generated R-module Q, so it is also almost finitely generated by Lemma 2.5.15 **(2)**. In particular, it is almost coherent since R is almost noetherian. □

Theorem 2.13.2. Let R be a topologically finite type K^+-algebra, and $M \in \mathbf{D}(R)$ a ϖ-adically derived complete complex. Suppose that $[M/\varpi] \in \mathbf{D}^{[c,d]}_{acoh}(R/\varpi)$, then $M \in \mathbf{D}^{[c,d]}_{acoh}(R)$.

Proof. Lemma A.3 guarantees that $M \in \mathbf{D}^{[c,d]}(R)$, so we only need to show that cohomology groups of M are almost coherent over R.

We argue by induction on $d-c$. If $c = d$, then $\mathrm{H}^d(M)/\varpi \simeq \mathrm{H}^d([M/\varpi])$ is almost coherent. Therefore, $M \simeq \mathrm{H}^d(M)[-d]$ is almost coherent by Lemma 2.13.1.

If $d > c$, we consider an exact triangle

$$\tau^{\leq d-1}M \to M \to \mathrm{H}^d(M)[-d].$$

We see that both $\tau^{\leq d-1}M$ and $\mathrm{H}^d(M)$ are derived complete by [Sta21, Tag 091P] and [Sta21, Tag 091S]. Moreover, we know that $\mathrm{H}^d(M)/\varpi \simeq \mathrm{H}^d([M/\varpi])$ is almost coherent. Therefore, $\mathrm{H}^d(M)$ is almost coherent by Lemma 2.13.1. Finally,

$$[\tau^{\leq d-1}M/\varpi] \simeq \mathrm{cone}\left([M/\varpi] \to [\mathrm{H}^d(M)/\varpi][-d]\right)[1]$$

is a (shifted) cone of a morphism in $\mathbf{D}^b_{acoh}(R/\varpi)$, therefore, $[\tau^{\leq d-1}M/\varpi]$ also lies in $\mathbf{D}^b_{acoh}(R/\varpi)$. By the induction hypothesis, we conclude that $\tau^{\leq d-1}M \in \mathbf{D}^{[c,d-1]}_{acoh}(R)$. So $M \in \mathbf{D}^{[c,d]}_{acoh}(R)$. □

Corollary 2.13.3. Let R be a topologically finite type K^+-algebra, and $M \in \mathbf{D}(R)$ a ϖ-adically derived complete complex. Suppose that $[M^a/\varpi] \in \mathbf{D}^{[c,d]}_{acoh}(R/\varpi)^a$, then $M^a \in \mathbf{D}^{[c,d]}_{acoh}(R)^a$.

Proof. Note that $\mathfrak{m} \otimes M$ is derived complete by Lemma A.4. So the claim follows from Theorem 2.13.2 applied to $\mathfrak{m} \otimes M$. □

3. Almost Mathematics on Ringed Sites

The main goal of this Chapter is to "globalize" results from Chapter 2. The two main cases of interest are almost coherent sheaves on schemes and "good" formal schemes. In order to treat those case somewhat uniformly we define some notions in the most general set-up of locally ringed spaces and check their basic properties. This is the content of Section 3.1. Sections 4.1 are 4.5 are devoted to the setting up foundations of almost coherent sheaves on schemes and formal schemes, respectively. In particular, we show that the notion of almost finitely generated (resp. presented, resp. coherent) module globalizes well on schemes and some "good" formal schemes. We prove the Proper Mapping Theorems in Section 5.1 both in the algebraic and formal Setups. Finally, we show the formal GAGA Theorem for adically quasi-coherent, almost coherent sheaves in Section 5.3. This is perhaps the most surprising result in this chapters as almost coherent sheaves are usually not finite type sheaves, so the "classical" proofs of Formal GAGA Theorem cannot really work in that situation.

3.1. **The Category of $\mathcal{O}_X^a$-modules.** We start this section by fixing a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal. The main goal of this section is to globalize the notion of almost mathematics on ringed R-sites.

The main object of our study in this Section will be a ringed site $(X, \mathcal{O}_X)$ with $\mathcal{O}_X$ being a sheaf of R-algebras. We call such sites as *ringed R-sites*. Note that any ringed site $(X, \mathcal{O}_X)$ is, in particular, a ringed $\mathcal{O}_X(X)$-site. On each open U, it makes sense to speak about almost mathematics on $\mathcal{O}_X(U)$-modules with respect to the ideal $\mathfrak{m}\mathcal{O}_X(U)$[13]. Usually definitions of many notions in almost mathematics involve tensoring against the module $\widetilde{\mathfrak{m}}$. We globalize this procedure in the following definition:

Definition 3.1.1. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be any $\mathcal{O}_X$-module. Then we define the sheaf $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ as the sheafification of the the presheaf that is defined as

$$U \mapsto \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U)$$

Remark 3.1.2. We note that this definition coincides with the tensor product $\underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}$ is the constant sheaf associated with the R-module $\mathfrak{m}$. Using flatness of the R-module $\widetilde{\mathfrak{m}}$, it is easy to see that the functor $- \otimes \widetilde{\mathfrak{m}}$ is exact and descends to a functor on the derived categories:

$$- \otimes \widetilde{\mathfrak{m}} \colon \mathbf{D}(X) \to \mathbf{D}(X)$$

where we denote by $\mathbf{D}(X)$ the derived category of $\mathcal{O}_X$-modules. Another way to think about it is to introduce the sheaf $\underline{\widetilde{\mathfrak{m}}}_X \coloneqq \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_X$. Then one easily see that there is a functorial isomorphism $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \simeq \underline{\widetilde{\mathfrak{m}}}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

Definition 3.1.3. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *almost zero* if $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is zero. We denote the category of almost zero $\mathcal{O}_X$-modules by Σ_X.

Remark 3.1.4. Since $\widetilde{\mathfrak{m}}$ is an R-flat module, we easily see that the category of almost zero $\mathcal{O}_X$-modules form a Serre subcategory of $\mathbf{Mod}_{\mathcal{O}_X} = \mathbf{Mod}_X$.

Lemma 3.1.5. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Suppose that $\mathcal{U}$ is a base of topology on X. Then the following conditions are equivalent:

(1) $\mathcal{F} \otimes \widetilde{\mathfrak{m}}$ is the zero sheaf.

(2) For any $\varepsilon \in \mathfrak{m}$, $\varepsilon\mathcal{F} = 0$.

(3) For any $U \in \mathcal{U}$, the module $\widetilde{\mathfrak{m}} \otimes \mathcal{F}(U)$ is zero.

(4) For any $U \in \mathcal{U}$, the module $\mathfrak{m} \otimes \mathcal{F}(U)$ is zero.

(5) For any $U \in \mathcal{U}$, the module $\mathfrak{m}\,(\mathcal{F}(U))$ is zero.

Proof. We firstly show that (**1**) implies (**2**). We pick an element $\varepsilon \in \mathfrak{m} = \mathfrak{m}^2$ and write it as $\varepsilon = \sum x_i \cdot y_i$ for some $x_i, y_i \in \mathfrak{m}$. So the multiplication by ε map can be decomposed as

$$\mathcal{F} \xrightarrow{s \mapsto s \otimes \sum x_i \otimes y_i} \mathcal{F} \otimes \widetilde{\mathfrak{m}} \xrightarrow{m} \mathcal{F}$$

where the last map is induced by the multiplication by $\widetilde{\mathfrak{m}} \to R$. Then if $\mathcal{F} \otimes \widetilde{\mathfrak{m}} = 0$, then the multiplication by ε map is zero for any $\varepsilon \in \mathfrak{m}$. Now (**2**) easily implies (**5**). Lemma 2.1.1 ensures that (**3**), (**4**), and (**5**) are equivalent. Finally, (**3**) clearly implies (**1**). ☐

Lemma 3.1.6. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be an almost zero $\mathcal{O}_X$-module. Then $\mathrm{H}^i(U, \mathcal{F}) \cong^a 0$ for any open $U \in X$[14] and any $i \geq 0$.

[13]look at Lemma 2.1.11 for the reason why this makes sense.

[14]An open $U \in X$ is by definition an object $U \in \mathrm{Ob}(X)$ of the category underlying the site X.

Proof. If $\mathcal{F}$ is almost zero, then $\varepsilon\mathcal{F} = 0$ for any $\varepsilon \in \mathfrak{m}$ by Lemma 3.1.5. Since the functors $\mathrm{H}^i(X, -)$ are R-linear, we conclude that $\varepsilon\mathrm{H}^i(U, \mathcal{F}) = 0$ for any open U and any $\varepsilon \in \mathfrak{m}, i \geq 0$. Thus Lemma 2.1.1 ensures that $\mathrm{H}^i(U, \mathcal{F}) \cong^a 0$. □

Definition 3.1.7. We say that a homomorphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is an *almost isomorphism* if $\ker(\varphi)$ and $\mathrm{Coker}(\varphi)$ are almost zero.

Lemma 3.1.8. A homomorphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is an almost isomorphism if and only if $\varphi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an almost isomorphism of $\mathcal{O}_X(U)$-modules for any open $U \in X$.

Proof. The $\Leftarrow$ implication is clear from the definitions. We give a proof of the $\Rightarrow$ implication.

Suppose that φ is an almost isomorphism. We define the auxillary $\mathcal{O}_X$-modules: $\mathcal{K} \coloneqq \ker(\varphi), \mathcal{F}' \coloneqq \mathrm{Im}(\varphi), \mathcal{Q} \coloneqq \mathrm{Coker}(\varphi)$. Lemma 3.1.6 implies that the maps

$$\mathcal{F}(U) \to \mathcal{F}'(U) \text{ and } \mathcal{F}'(U) \to \mathcal{G}(U)$$

are almost isomorphisms. In particular, the composition $\mathcal{F}(U) \to \mathcal{G}(U)$ must also be an almost isomorphism. □

Now we discuss the notion of almost $\mathcal{O}_X$-modules on a ringed R-site $(X, \mathcal{O}_X)$. This notion can be defined in two different ways: either as the quotient of the category of $\mathcal{O}_X$-modules by the Serre subcategory of almost zero modules or as modules over the almost structure sheaf $\mathcal{O}_X^a$. We need to explain these two notions in more detail now.

Definition 3.1.9. We define the *category of almost $\mathcal{O}_X$-modules* as the quotient category

$$\mathbf{Mod}^a_{\mathcal{O}_X} \coloneqq \mathbf{Mod}_{\mathcal{O}_X}/\Sigma_X \ .$$

Now we want to define the category $\mathbf{Mod}_{\mathcal{O}_X^a}$ of $\mathcal{O}_X^a$-modules that we will show to be equivalent to $\mathbf{Mod}^a_{\mathcal{O}_X}$. We recall that the almostification functor $(-)^a$ is exact on the level of modules and commutes with arbitrary products. This allows us to define the almost structure sheaf:

Definition 3.1.10. The *almost structure sheaf* $\mathcal{O}_X^a$ is the sheaf[15] of R^a-modules $\mathcal{O}_X^a \colon (\mathrm{Ob}(X))^{op} \to \mathbf{Mod}^a_R$ defined as $U \mapsto \mathcal{O}_X(U)^a$.

Definition 3.1.11. We define the *category of $\mathcal{O}_X^a$-modules* $\mathbf{Mod}_{\mathcal{O}_X^a}$ as the category of the modules over $\mathcal{O}_X^a \in \mathbf{Shv}(X, R^a)$ in the categorical sense. More precisely, the objects are sheaves of R^a-modules $\mathcal{F}$ with a map $\mathcal{F} \otimes_{R^a} \mathcal{O}_X^a \to \mathcal{F}$ over R^a satisfying the usual axioms for a module. Morphisms are defined in the evident way.

We now define the functor

$$(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

that sends a sheaf to its "almostification", i.e. it applies the functor $(-)^a \colon \mathbf{Mod}_R \to \mathbf{Mod}^a_R$ section-wise. Since the almostification functor $(-)^a$ is exact and commutes with arbitrary product, it is evident that $\mathcal{F}^a$ is actually a sheaf for any $\mathcal{O}_X$-module $\mathcal{F}$. Moreover, it is clear that $\mathcal{F}^a \simeq 0$ for any almost zero $\mathcal{O}_X$-module $\mathcal{F}$. Thus, it induces the functor

$$(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a} \ .$$

The claim is that this functor induces the equivalence of categories. The first step towards the proof is to construct the right adjoint to $(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. Our construction of the right adjoint functor will use the existence of the left adjoint functor. So we slightly postpone the proof of the mentioned above equivalence and discuss adjoints to $(-)^a$.

[15] It is a sheaf exactly because $(-)^a$ is exact and commutes with arbitrary products.

We start with the definition of the left adjoint functor. The idea is to apply the functor $(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ section-wise, though this strategy does not quite work as $(-)_!$ does not commute with infinite products.

Definition 3.1.12.
- We define the functor $(-)_!^p\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}^p$ [16] as
$$\mathcal{F} \mapsto (U \mapsto \mathcal{F}(U)_!)$$
- We define the functor $(-)_!\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ as the composition $(-)_! := (-)^\# \circ (-)_!^p$, where $(-)^\#$ is the sheafification functor.

Lemma 3.1.13. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then

(1) The functor
$$(-)_!\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$$
is the left adjoint to the localization functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, we have a functorial isomorphism
$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!, \mathcal{G})$$
for any $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X^a}, \mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X}$.

(2) The functor $(-)_!\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is exact.

(3) The counit morphism $(\mathcal{F}^a)_! \to \mathcal{F}$ is an almost isomorphism for any $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. The unit morphism $\mathcal{G} \to (\mathcal{G}_!)^a$ is an isomorphism for any $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, the functor $(-)^a$ is essentially surjective.

Proof. **(1)** follows from Lemma 2.1.9**(3)** and the adjunction between sheafication and the forgetful functor. More precisely, we have the following functorial isomorphisms
$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}^p}(\mathcal{F}_!^p, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!, \mathcal{G}) \ .$$

We show **(2)**. It is easy to see that $(-)_!$ is left exact from Lemma 2.1.9**(4)** and the exactness of the sheafification functor. It is also right exact since it is a left adjoint functor to $(-)^a$.

Now we show **(3)**. Lemma 2.1.9**(5)** ensures that the kernel and cokernel of the counit map of presheaves $(\mathcal{F}^a)_!^p \to \mathcal{F}$ are annihilated by any $\varepsilon \in \mathfrak{m}$. Then the same holds after sheafification, proving the $(\mathcal{F}^a)_!^p \to \mathcal{F}$ is an almost isomorphism by Lemma 3.1.5.

We consider the unit map $\mathcal{G} \to (\mathcal{G}_!)^a$, we note that using the adjuction $((-)_!, (-)^a)$ section-wise, we can refine this map
$$\mathcal{G} \to (\mathcal{G}_!^p)^a \to (\mathcal{G}_!)^a \ .$$
It suffices to show that both maps are isomorphisms, the first map is an isomorphism by Lemma 2.1.9**(5)**. In particular, this implies that $(\mathcal{G}_!^p)^a$ is already a sheaf of almost R^a-modules, but then we see that the natural map $(\mathcal{G}_!^p)^a \to (\mathcal{G}_!)^a$ must also be an isomorphism as it coincides with the sheafification in the category of presheaves of R^a-modules. $\square$

Remark 3.1.14. In what follows, we denote the objects of $\mathrm{Mod}_{\mathcal{O}_X^a}$ by $\mathcal{F}^a$ to distinguish $\mathcal{O}_X$ and $\mathcal{O}_X^a$-modules. This notation does not cause any confusion as $(-)^a$ is indeed essentially surjective.

Now we construct the right adjoint functor to $(-)^a$. The naive idea of applying $(-)_*$ section-wise works well in this case. The only thing we emphasize here is that essential surjectivity of $(-)^a$ is used in our definition of $(-)_*$.

[16] $\mathbf{Mod}_{\mathcal{O}_X}^p$ stands for the category of modules over $\mathcal{O}_X$ in the category of presheaves

Definition 3.1.15. The functor of *almost sections* $(-)_*\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is defined as

$$\mathcal{F}^a \mapsto (U \mapsto \mathrm{Hom}_R(\widetilde{\mathfrak{m}}, \mathcal{F}(U)))$$

with the structure of $\mathcal{O}_X$-module coming from the structure of $\mathcal{O}_X$-module on $\mathcal{F}$.

Remark 3.1.16. The functor $(-)_*$ is well-defined, i.e. is independent of a choice of $\mathcal{F}$ and defines a *sheaf* of $\mathcal{O}_X$-modules. The first claim follows from Lemma 2.1.8**(2)** and Lemma 3.1.8, the second claim follows from the fact that $\mathrm{Hom}_R(\widetilde{\mathfrak{m}}, -)$ commutes with arbitrary products.

Lemma 3.1.17. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then

(1) The functor $(-)_*\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is the right adjoint to the exact localization functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, it is left exact.

(2) The unit morphism $\mathcal{F} \to (\mathcal{F}^a)_*$ is an almost isomorphism for any $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. The counit morphism $(\mathcal{G}^a_*)^a \to \mathcal{G}^a$ is an isomorphism for any $\mathcal{G}^a \in \mathbf{Mod}_{\mathcal{O}_X^a}$.

Proof. It is sufficient to check both claims section-wise. This, in turn, follow from Lemma 2.1.9**(1)** and Lemma 2.1.9**(2)** respectively. $\square$

Corollary 3.1.18. The functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$ commutes with limits and colimits. In particular, $\mathbf{Mod}_{\mathcal{O}_X^a}$ is complete and cocomplete, and filtered colimits and (finite) products are exact in $\mathbf{Mod}_{\mathcal{O}_X^a}$.

Proof. The first claim follows from the fact that $(-)^a$ admits left and right adjoints. The second claim follows the first claim, exactness of $(-)^a$, and analogous exactness properties in $\mathbf{Mod}_R$. $\square$

Corollary 3.1.19. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then the functor

$$(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

is exact.

Proof. The functor $(-)^a$ is exact as it has both left and right adjoints. $\square$

Theorem 3.1.20. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then the functor

$$(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X}^a \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

is an equivalence of categories.

Proof. Lemma 3.1.17 implies that the functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$ has right adjoint functor $(-)_*$ such that the counit morphism $(-)^a \circ (-)_* \to \mathrm{Id}$ is an isomorphism of functors. Moreover, exactness of $(-)^a$ implies that a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an almost isomorphism if and only if $\varphi^a\colon \mathcal{F}^a \to \mathcal{G}^a$ is an isomorphism. Thus [GZ67, Proposition 1.3] guarantees that the induced functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X}^a \to \mathbf{Mod}_{\mathcal{O}_X^a}$ is an equivalence. $\square$

Remark 3.1.21. In what follows, we do not distinguish $\mathbf{Mod}_{\mathcal{O}_X^a}$ and $\mathbf{Mod}_{\mathcal{O}_X}^a$. Moreover, we sometimes denote both categories by $\mathbf{Mod}_X^a$ or $\mathbf{Mod}_{X^a}$ to simplify the notation.

3.2. Basic Functors on the Category Of $\mathcal{O}_X^a$-Modules.

We discuss how to define certain basic functors on $\mathbf{Mod}_X^a$. Our main functors of interest are Hom, alHom, $\otimes$, f^*, and f_*. We define their almost analogues and their relation with the original functors. As a by-product we give a slightly more intrinsic definition of $(-)_*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_X$ along the lines of the definition of the $\mathbf{Mod}_R^a$-version of this functor.

For the rest of the section we fix a ringed site $(X, \mathcal{O}_X)$ that we consider as a ringed $\mathcal{O}_X(X)$-site.

Definition 3.2.1.
- The *global Hom functor*
$$\mathrm{Hom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{op} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_{\mathcal{O}_X(X)}$$
is defined as $(\mathcal{F}^a, \mathcal{G}^a) \mapsto \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$.
- The *local Hom functor*
$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{op} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_X$$
is defined as $(\mathcal{F}^a, \mathcal{G}^a) \mapsto \Big(U \mapsto \mathrm{Hom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)\Big)$. The standard argument shows that this functor is well-defined, i.e. $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G})$ is indeed a sheaf of $\mathcal{O}_X$-modules.

Lemma 3.2.2. Let U be an open in X, and let $\mathcal{F}^a, \mathcal{G}^a$ be $\mathcal{O}_X^a$-modules. Then the natural map
$$\Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right) \to \mathrm{Hom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)$$
is an isomorphism of $\mathcal{O}_X(U)$-modules.

Proof. This is evident from the definition. □

Lemma 3.2.3. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then there is a functorial isomorphism of $\mathcal{O}_X$-modules
$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \xrightarrow{\sim} \underline{\mathcal{H}om}_{\mathcal{O}_X}((\mathcal{F}^a)_!, \mathcal{G})$$
for $\mathcal{F}^a \in \mathbf{Mod}_X^a$ and $\mathcal{G} \in \mathbf{Mod}_X$.

Proof. Lemma 3.2.2 and Lemma 3.1.13 ensure that the desired isomorphism exists section-wise. It glues to a global isomorphism of sheaves since these section-wise isomorphisms are functorial in U. □

Now we move on to show a promised more intrinsic definition of the functor $(-)_*$. As a warm-up we need the following result:

Lemma 3.2.4. Suppose that the ringed R-site $(X, \mathcal{O}_X)$ has a final object. By slightly abusing the notation, we also denote the final object by X. Then the evaluation map
$$\mathrm{ev}_X\colon \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{O}_X^a, \mathcal{G}^a) \to \mathrm{Hom}_{\mathcal{O}_X(X)^a}(\mathcal{O}_X^a(X), \mathcal{G}^a(X))$$
$$\varphi \mapsto \varphi(X)$$
is an isomorphism of $\mathcal{O}_X(X)$-modules for any $\mathcal{G}^a \in \mathbf{Mod}_X^a$.

Proof. As $(-)^a$ is essentially surjective by Lemma 3.1.13**(3)**, there actually exists some $\mathcal{O}_X$-module $\mathcal{G}$ with almostification being equal to $\mathcal{G}^a$. Now we recall that the data of an $\mathcal{O}_X^a$-linear homomorphism $\varphi\colon \mathcal{O}_X^a \to \mathcal{G}^a$ is equivalent to the data of $\mathcal{O}_X(U)^a$-linear homomorphisms $\varphi_U \in \mathrm{Hom}_{\mathcal{O}_X(U)^a}(\mathcal{O}_X^a(U), \mathcal{G}^a(U))$ for each open U in X such the diagram
$$\begin{tikzcd} \mathcal{O}_X(U)^a \arrow[r, "\varphi_U"] \arrow[d, "r_{\mathcal{O}_X^a}|_V^U"] & \mathcal{G}(U)^a \arrow[d, "r_{\mathcal{G}^a}|_V^U"] \\ \mathcal{O}_X(V)^a \arrow[r, "\varphi_V"] & \mathcal{G}(V)^a \end{tikzcd}$$

commutes for any $V \subset U$. Now we note that an $\mathcal{O}_X(U)^a$-linear homomorphism φ_U uniquely determines an $\mathcal{O}_X(V)^a$-linear homomorphism φ_V in such a diagram. Indeed, this follows from the equality

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X(V)^a}\left(\mathcal{O}_X(V)^a, \mathcal{G}(V)^a\right) &= \mathrm{Hom}_{\mathcal{O}_X(V)}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(V), \mathcal{G}(V)\right) \\ &= \mathrm{Hom}_{\mathcal{O}_X(V)}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V), \mathcal{G}(V)\right) \\ &= \mathrm{Hom}_{\mathcal{O}_X(U)}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(U), \mathcal{G}(V)\right) \\ &= \mathrm{Hom}_{\mathcal{O}_X(U)^a}\left(\mathcal{O}_X(U)^a, \mathcal{G}(V)^a\right).\end{aligned}$$

Now we use the assumption that X is the final object to conclude that any homomorphism $\varphi\colon \mathcal{O}_X^a \to \mathcal{G}^a$ is uniquely defined by $\varphi(X)$. □

Corollary 3.2.5. Let $(X, \mathcal{O}_X)$ be an R-ringed site, and let $U \in X$ be an open. Then the evaluation map

$$\begin{gathered}\mathrm{ev}_U\colon \mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{O}_U^a, \mathcal{G}|_U^a\right) \to \mathrm{Hom}_{\mathcal{O}_U(U)^a}\left(\mathcal{O}_U^a(U), \mathcal{G}^a(U)\right) \\ \varphi \mapsto \varphi(U)\end{gathered}$$

is an isomorphism of $\mathcal{O}_X(U)$-modules for any $\mathcal{G}^a \in \mathbf{Mod}_X^a$.

Proof. For the purpose of the proof, we can change the site X by the slicing site X/U of objects over U. Then U automatically becomes the final object in X/U, so we can just apply Lemma 3.2.4 to finish the proof. □

Now we are ready to prove a new description of the sheaf version of the functor $(-)_*$.

Lemma 3.2.6. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then there is a functorial isomorphism of $\mathcal{O}_X$-modules

$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{O}_X^a, \mathcal{F}^a\right) \to \mathcal{F}_*^a$$

for $\mathcal{F}^a \in \mathbf{Mod}_X^a$.

Proof. Lemma 3.2.2 and Corollary 3.2.5 imply that there is an isomorphism of $\mathcal{O}_X(U)$-modules

$$\Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{O}_X^a, \mathcal{F}^a\right)\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U(U)^a}\left(\mathcal{O}_U^a(U), \mathcal{F}^a(U)\right)$$

that is functorial in both U and $\mathcal{F}^a$. Now we use the functorial isomorphism of $\mathcal{O}_X(U)$

$$\mathrm{Hom}_{\mathcal{O}_U(U)^a}\left(\mathcal{O}_U(U)^a, \mathcal{F}^a(U)\right) \simeq \mathrm{Hom}_{R^a}\left(R^a, \mathcal{F}^a(U)\right) = (\mathcal{F}^a)_*(U)$$

to construct a functorial isomorphism

$$\Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{O}_X^a, \mathcal{F}^a\right)\right) \xrightarrow{\sim} (\mathcal{F}^a)_*(U)\ .$$

Functoriality in U ensures that it glues to the global isomorphism of $\mathcal{O}_X$-modules

$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{O}_X^a, \mathcal{F}^a\right) \xrightarrow{\sim} \mathcal{F}_*^a\ .$$

□

Now we discuss the functor of almost homomorphisms.

Definition 3.2.7.
- The *global alHom functor*

$$\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{op} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_{R^a}$$

is defined as

$$(\mathcal{F}^a, \mathcal{G}^a) \mapsto \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)^a \simeq \mathrm{Hom}_{\mathcal{O}_X}\left((\mathcal{F}^a)_!, \mathcal{G}\right)^a\ .$$

- The *local alHom functor*
$$\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{op}\times \mathbf{Mod}_{X^a}\to \mathbf{Mod}_{X^a}$$
is defined as
$$(\mathcal{F}^a,\mathcal{G}^a)\mapsto \left(U\mapsto \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U,\mathcal{G}^a|_U)^a\right) .$$

Remark 3.2.8. At this point we have not checked that $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)$ is actually a sheaf. However, this follows from the following lemma.

Lemma 3.2.9. The natural map
$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}}\otimes\mathcal{F},\mathcal{G})^a\to \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)$$
is an almost isomorphism of $\mathcal{O}_X^a$-modules for any $\mathcal{F}^a,\mathcal{G}^a\in\mathbf{Mod}_X^a$. In particular, $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)$ is a sheaf of $\mathcal{O}_X^a$-modules.

Proof. This follows from the sequence of functorial in U isomorphisms:
$$\begin{aligned}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}}\otimes\mathcal{F},\mathcal{G})(U)^a &\simeq^a \mathrm{Hom}_{\mathcal{O}_U}(\widetilde{\mathfrak{m}}\otimes\mathcal{F}|_U,\mathcal{G}|_U)^a\\ &\simeq^a \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U,\mathcal{G}^a|_U)\\ &\simeq^a \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)(U)\end{aligned}$$
□

In order to make Definition 3.2.7, we need to show that these functors can actually be computed by using any representative for $\mathcal{F}^a$ and $\mathcal{G}^a$.

Proposition 3.2.10. Let $(X,\mathcal{O}_X)$ be a ringed R-site. Then:

(1) There is a natural transformation of functors
$$\begin{tikzcd} \mathbf{Mod}_X^{op}\times\mathbf{Mod}_X \arrow[r, "\mathrm{Hom}_{\mathcal{O}_X}(-,-)"] \arrow[d, "(-)^a\times(-)^a"'] & \mathbf{Mod}_X \arrow[d, "(-)^a"] \arrow[dl, Rightarrow, "\rho"'] \\ \mathbf{Mod}_{X^a}^{op}\times\mathbf{Mod}_{X^a} \arrow[r, "\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)"'] & \mathbf{Mod}_{X^a} \end{tikzcd}$$
that makes the diagram $(2,1)$-commutative. In particular, $\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)\simeq\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^a$ for any $\mathcal{F},\mathcal{G}\in\mathbf{Mod}_X$.

(2) Then there is a natural transformation of functors
$$\begin{tikzcd} \mathbf{Mod}_X^{op}\times\mathbf{Mod}_X \arrow[r, "\underline{\mathcal{H}om}_{\mathcal{O}_X}(-,-)"] \arrow[d, "(-)^a\times(-)^a"'] & \mathbf{Mod}_X \arrow[d, "(-)^a"] \arrow[dl, Rightarrow, "\rho"'] \\ \mathbf{Mod}_{X^a}^{op}\times\mathbf{Mod}_{X^a} \arrow[r, "\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)"'] & \mathbf{Mod}_{X^a} \end{tikzcd}$$
that makes the diagram $(2,1)$-commutative. In particular, $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)\simeq\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^a$ for any $\mathcal{F},\mathcal{G}\in\mathbf{Mod}_X$.

Proof. The proof is similar to the proof of Proposition 2.2.1**(3)**. The only new thing is that we need to prove an analogue of Corollary 2.1.13, i.e. that the functors $\mathrm{alHom}_{\mathcal{O}_X}(-,\mathcal{G})$, $\underline{al\mathcal{H}om}_{\mathcal{O}_X}(-,\mathcal{G})$ preserve almost isomorphisms. It essentially boils down to showing that $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})\cong^a 0$ and $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})\cong^a 0$ for any $\mathcal{K}\in\Sigma_X,\mathcal{G}\in\mathbf{Mod}_X$, and an integer $i\geq 0$.

Now Lemma 3.1.5 implies that $\varepsilon\mathcal{K} = 0$ for any $\varepsilon \in \mathfrak{m}$. Thus we see that $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{G})$ and $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{G})$ are also annihilated by any $\varepsilon \in \mathfrak{m}$ since the functors $\mathrm{Ext}^i_{\mathcal{O}_X}(-, \mathcal{G})$, $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X}(-, \mathcal{G})$ are R-linear. Thus $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{G})$ and $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{G})$ are almost zero by Lemma 2.1.1 and Lemma 3.1.5 respectively. □

Definition 3.2.11. The *tensor product functor* $-\otimes_{\mathcal{O}_X^a} -\colon \mathbf{Mod}_X^a \times \mathbf{Mod}_X^a \to \mathbf{Mod}_X^a$ is defined as

$$(\mathcal{F}^a, \mathcal{G}^a) \mapsto \mathcal{F}^a_! \otimes_{\mathcal{O}_X} \mathcal{G}^a_! \ .$$

Proposition 3.2.12. There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{Mod}_X \times \mathbf{Mod}_X \arrow[r, "- \otimes_{\mathcal{O}_X} -"] \arrow[d, "(-)^a \times (-)^a"'] & \mathbf{Mod}_X \arrow[d, "(-)^a"] \\
\mathbf{Mod}_X^a \times \mathbf{Mod}_X^a \arrow[r, "- \otimes_{\mathcal{O}_X^a} -"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{Mod}_X^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^a \simeq \mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$$

for any $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}_X$.

Proof. The proof is absolutely analogous to that of Propisition 2.2.1**(1)**. □

The tensor product is adjoint to $\underline{\mathcal{H}om}$ as it happens in the case of R^a-modules. We give a proof of the local version of this statement.

Lemma 3.2.13. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}^a, \mathcal{G}^a, \mathcal{H}^a$ be $\mathcal{O}_X^a$-modules. Then there is a functorial isomorphism

$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

After passing to the global sections, this gives the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

And after passing to the almostifications, it gives an isomorphism

$$\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

Proof. We compute $\Gamma(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a))$ by using Lemma 3.2.2 and the standard $\otimes$-$\underline{\mathcal{H}om}$ adjunction. Namely,

$$\begin{aligned}
\Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\right)\right) &\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{F}^a|_U \otimes_{\mathcal{O}_U^a} \mathcal{G}^a|_U, \mathcal{H}^a|_U\right) && \text{Lemma 3.2.2} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\left((\mathcal{F}|_U \otimes_{\mathcal{O}_U} \mathcal{G}|_U)^a, \mathcal{H}^a|_U\right) && \text{Proposition 3.2.12} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U}\left(\widetilde{\mathfrak{m}} \otimes (\mathcal{F}|_U \otimes_{\mathcal{O}_U} \mathcal{G}|_U), \mathcal{H}|_U\right) && \text{Lemma 3.1.13} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U}\left((\widetilde{\mathfrak{m}} \otimes \mathcal{F}|_U) \otimes_{\mathcal{O}_U} (\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U), \mathcal{H}|_U\right) && \widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}|_U, \underline{\mathcal{H}om}_{\mathcal{O}_U}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U, \mathcal{H}|_U\right)\right) && \otimes - \mathcal{H}om \text{ adjunction} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{F}^a|_U, \underline{al\mathcal{H}om}_{\mathcal{O}_U}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U, \mathcal{H}|_U\right)\right) && \text{Lemma 3.1.13} \\
&\simeq \Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}\left(\mathcal{G}^a, \mathcal{H}^a\right)\right)\right) && \text{Lemma 3.2.2}
\end{aligned}$$

Since these identifications are functorial in U, we can glue them to a global isomorphism

$$\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

This finishes the proof. $\square$

Corollary 3.2.14. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}^a$ be an $\mathcal{O}_X^a$-module. Then the functor $-\otimes_{\mathcal{O}_X^a} \mathcal{F}^a$ is left adjoint to $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.

For what follows, we fix a map $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed R-sites. We are going to define the almost version of the pullback and pushforward functors.

Definition 3.2.15. The *pullback functor* $f_a^*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_Y^a$ is defined as

$$\mathcal{F}^a \mapsto \left(f^*\left(\mathcal{F}_!^a\right)\right)^a \ .$$

In what follows, we will often abuse notation and simply write f^* instead of f_a^*. This is "allowed" by Proposition 3.2.19.

As always, we want to show that this functor can be actually computed by applying f^* to *any* representative of $\mathcal{F}^a$. The main ingredient is to show that f^* sends almost isomorphisms to almost isomorphisms. The following lemma shows slightly more, and will be quite useful later on.

Lemma 3.2.16. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism $\varphi_f(\mathcal{F})\colon f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes f^*\mathcal{F}$ functorial in $\mathcal{F}$.

Proof. We use Remark 3.1.2 to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is functorially isomorphic to $\widetilde{\mathfrak{m}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}_Y \coloneqq \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_Y$. Now we note that $f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq \underline{\widetilde{\mathfrak{m}}}_X$ as can be easily seen (using the $\widetilde{\mathfrak{m}}$ is R-flat) from the very definitions. Therefore, $\varphi_f(\mathcal{F})$ comes from the fact that the pullback functor commutes with the tensor product. More precisely, we define it as the composition

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \xrightarrow{\sim} f^*(\underline{\widetilde{\mathfrak{m}}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}) \xrightarrow{\sim} f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \otimes_{\mathcal{O}_X} f^*(\mathcal{F}) \xrightarrow{\sim} \underline{\widetilde{\mathfrak{m}}}_X \otimes_{\mathcal{O}_X} f^*(\mathcal{F}) \ .$$

$\square$

We now also show a derived version of Lemma 3.2.16 that will be used later in the text.

Lemma 3.2.17. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then for any $\mathcal{F} \in \mathbf{D}(X)$, there is a natural isomorphism

$$\varphi_f(\mathcal{F})\colon \mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*\mathcal{F}$$

functorial in $\mathcal{F}$.

Proof. Similarly, we use Remark 3.1.2 to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is functorially isomorphic to $\widetilde{\mathfrak{m}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}_Y \coloneqq \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_Y$. Now we note that $\mathbf{L}f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq \underline{\widetilde{\mathfrak{m}}}_X$ as $\widetilde{\mathfrak{m}}$ is R-flat. The rest of the proof is the same using the $\mathbf{L}f^*$ functorially commutes with the derived tensor product. $\square$

Corollary 3.2.18. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an almost isomorphism of $\mathcal{O}_Y$-modules. Then the homomorphism $f^*(\varphi)\colon f^*(\mathcal{F}) \to f^*(\mathcal{G})$ is an almost isomorphism.

Proof. The question boils down to show that the homomorphism

$$\widetilde{\mathfrak{m}} \otimes f^*(\mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes f^*(\mathcal{G})$$

is an isomorphism. Lemma 3.2.16 ensures that it is sufficient to prove that the map

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{G})$$

is an isomorphism. But this is clear as the map $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \widetilde{\mathfrak{m}} \otimes \mathcal{G}$ is already an isomorphism. $\square$

Proposition 3.2.19. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors

$$\begin{tikzcd} \mathbf{Mod}_Y \arrow[r, "f^*"] \arrow[d, "(-)^a"] & \mathbf{Mod}_X \arrow[d, "(-)^a"] \\ \mathbf{Mod}_Y^a \arrow[r, "f_a^*"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{Mod}_X^a \end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(f^*\mathcal{F})^a \simeq f_a^*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{Mod}_X$.

Proof. The proof is similar to Proposition 2.2.1. We define $\rho_{\mathcal{F}}\colon f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})^a \to f^*(\mathcal{F})^a$ as the map induced by the natural homomorphism $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \mathcal{F}$. It is clearly functorial in $\mathcal{F}$, and it is an isomorphism by Corollary 3.2.18. $\square$

Definition 3.2.20. The *pushforward functor* $f_*^a\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_Y^a$ is defined as

$$\mathcal{F}^a \mapsto \left(f_*\left(\mathcal{F}^a_!\right)\right)^a \ .$$

In what follows, we will often abuse the notations and simply write f_* instead of f_*^a. This is "allowed" by Proposition 3.2.24.

Definition 3.2.21. The *global sections functor* $\Gamma^a(X, -)\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_R^a$ is defined as

$$\mathcal{F}^a \mapsto \Gamma(X, \mathcal{F}^a_!)^a \ .$$

In what follows, we will often abuse the notations and simply write Γ instead of Γ^a. This is also "allowed" by Proposition 3.2.24.

Remark 3.2.22. The global section functor can be realized as the pushforward along the map $(X, \mathcal{O}_X) \to (*, R)$.

Lemma 3.2.23. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an almost isomorphism. Then the morphism $f_*(\varphi)\colon f_*(\mathcal{F}) \to f_*(\mathcal{G})$ is an almost isomorphism.

Proof. The standard argument with considering the kernel and cokernel of φ shows that it is sufficient to prove that $f_*\mathcal{K} \cong^a 0$, $\mathrm{R}^1 f_*\mathcal{K} \cong^a 0$ for any almost zero $\mathcal{O}_X$-module $\mathcal{K}$. This follows from R-linearity of f_* and Lemma 3.1.5. $\square$

Proposition 3.2.24. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-spaces. Then there is a natural transformation of functors

$$\begin{tikzcd} \mathbf{Mod}_X \arrow[r, "f_*"] \arrow[d, "(-)^a"] & \mathbf{Mod}_Y \arrow[d, "(-)^a"] \\ \mathbf{Mod}_X^a \arrow[r, "f_*^a"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{Mod}_Y^a \end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(f_*\mathcal{F})^a \simeq f_*^a(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{Mod}_X$. The same results hold true for $\Gamma^a(X, -)$.

Proof. We define $\rho_{\mathcal{F}}\colon f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})^a \to f_*(\mathcal{F})^a$ as the map induced by the natural homomorphism $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \mathcal{F}$. It is clearly functorial in $\mathcal{F}$, and it is an isomorphism by Lemma 3.2.23. $\square$

Lemma 3.2.25. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X^a$-modules. Then there is a natural morphism

$$\Gamma\left(U, \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right) \to \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)$$

is an isomorphism of R^a-modules for any open $U \subset X$.

Proof. The claim easily follows from Lemma 3.2.2, Proposition 3.2.10**(2)**, and Proposition 3.2.24 $\square$

Lemma 3.2.26. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-site, and let $\mathcal{F}^a \in \mathbf{Mod}_Y^a$, and $\mathcal{G}^a \in \mathbf{Mod}_X^a$. Then there is a functorial isomorphism of $\mathcal{O}_Y$-modules

$$f_*\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, f_*(\mathcal{G}^a)) \ .$$

After passing to the global sections, this gives the isomorphism of $\mathcal{O}_Y(Y)$-modules

$$\mathrm{Hom}_{\mathcal{O}_X^a}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathcal{O}_Y^a}(\mathcal{F}^a, f_*(\mathcal{G}^a)) \ .$$

And after passing to the almostifications, it gives the isomorphism of $\mathcal{O}_Y^a$-modules

$$f_*\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(f^*(\mathcal{F}^a), \mathcal{G}^a) \cong^a \underline{al\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, f_*(\mathcal{G}^a)) \ .$$

Proof. This is a combination of the classical (f^*, f_*)-adjunction, Lemma 3.1.13, Lemma 3.2.16, Proposition 3.2.19, and Proposition 3.2.24. Indeed, we choose an open $U \subset Y$ and denote its preimage by $V := f^{-1}(U)$. We also define $\mathcal{F}_U^a := \mathcal{F}^a|_U$ and $\mathcal{G}_V^a := \mathcal{G}^a|_V$. The claim follows from the sequence of functorial isomorphisms

$$\begin{aligned}
\Gamma\left(U, \underline{\mathcal{H}om}_{\mathcal{O}_Y^a}\left(\mathcal{F}^a, f_*\left(\mathcal{G}^a\right)\right)\right) &\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{F}_U^a, f_*\left(\mathcal{G}_V^a\right)\right) && \text{Lemma 3.2.2} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{F}_U^a, f_*\left(\mathcal{G}_V\right)^a\right) && \text{Proposition 3.2.24} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_U}\left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}_U, f_*\left(\mathcal{G}_V\right)\right) && \text{Lemma 3.1.13} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_V}\left(f^*\left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}_U\right), \mathcal{G}_V\right) && (f^*, f_*)\text{-adjunction} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_V}\left(\widetilde{\mathfrak{m}} \otimes f^*\left(\mathcal{F}_U\right), \mathcal{G}_V\right) && \text{Lemma 3.2.16} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_V^a}\left(f^*\left(\mathcal{F}_U\right)^a, \mathcal{G}_V^a\right) && \text{Lemma 3.1.13} \\
&\simeq \mathrm{Hom}_{\mathcal{O}_V^a}\left(f^*\left(\mathcal{F}_U^a\right), \mathcal{G}_V^a\right) && \text{Proposition 3.2.19} \\
&\simeq \Gamma\left(U, f_*\underline{\mathcal{H}om}_{\mathcal{O}_X^a}\left(f^*\left(\mathcal{F}^a\right), \mathcal{G}^a\right)\right) \ . && \text{Lemma 3.2.2}
\end{aligned}$$

Since these identifications are functorial in U, we can glue them to a global isomorphism

$$f_*\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, f_*(\mathcal{G}^a)) \ .$$

$\square$

Corollary 3.2.27. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-site. Then the functors $\mathbf{Mod}_X^a \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Mod}_Y^a$ are adjoint.

3.3. The Projection Formula.

The definition of $\mathcal{O}_X$-modules behaves especially nicely on locally spectral spaces[17]. For instance, we show that we can explicitly describe sections of $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ on a basis of opens for such spaces. Moreover, we show that the projection formula holds for spectral morphisms of locally spectral spaces.

Remark 3.3.1. We mention one problem of working with locally spectral spaces that we deliberately avoid in all of our proofs. Suppose that X is a locally ringed space and $U \subset X$ is an open spectral subspace then the natural map $U \to X$ need not be quasi-compact. In particular, an intersection of two open spectral subspaces in X need not be spectral itself.

In order to get such examples, one can consider X to a scheme that is not quasi-separated and U an open affine subscheme. Then the inclusion map $U \to X$ is usually not quasi-compact.

[17]We refer to [Sta21, Tag 08YF] and [Wed19, §3] for a comprehensive discussion of (locally) spectral spaces

Lemma 3.3.2. Let $(X, \mathcal{O}_X)$ be a locally spectral, locally ringed R-space. Then for any spectral[18] open subset $U \subset X$ the natural morphism

$$\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U) \to (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

Proof. As spectral subspaces form a basis of topology on X, it suffices to show that the functor

$$U \to \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U)$$

restricted to spectral open subsets satisfy the sheaf condition. In particular, we can assume that X itself is spectral.

As any open spectral U is quasi-compact, we conclude that any open covering $U = \bigcup_{i \in I} U_i$ admits a refinement by a finite one. Thus, it is sufficient to check the sheaf condition for finite coverings of a spectral spaces by spectral open subspaces. Thus, we need to show that, for any finite covering $U = \bigcup_{i \in I} U_i$, the sequence

$$0 \to \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U) \to \prod_{i=1}^{n} (\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U_i)) \to \prod_{i,j=1}^{n} (\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U_i \cap U_j)).$$

is exact. But this follows from flatness of $\widetilde{\mathfrak{m}}$ and the fact that tensor product commutes with *finite* direct products. $\square$

Now we want to show a version of the projection formula for the functor $\widetilde{\mathfrak{m}} \otimes -$, it will take some time to rigorously prove it. We recall that a map of locally spectral spaces is called *spectral*, if the pre-image of any spectral open subset is spectral.

Lemma 3.3.3. Let $(X, \mathcal{O}_X)$ be a spectral locally ringed R-space. Then for any injective $\mathcal{O}_X$-module $\mathcal{I}$ the $\mathcal{O}_X$-module $\widetilde{\mathfrak{m}} \otimes \mathcal{I}$ is an $\mathrm{H}^0(X, -)$-acyclic.

Proof. We start the proof by noting that [Sta21, Tag 01EV] guarantees that it suffices to show that the Čech cohomology groups $\check{\mathrm{H}}^i(U, \widetilde{\mathfrak{m}} \otimes \mathcal{I})$ vanish for all open subsets $U \subset X$ and $i > 0$. Since any open subset of a locally spectral space is locally spectral, it suffices to show that $\check{\mathrm{H}}^i(U, \widetilde{\mathfrak{m}} \otimes \mathcal{I}) = 0$ for $i > 0$.

We note that quasi-compact opens form a basis for the topology on X. Since X is quasi-compact, finite coverings by quasi-compact opens form a cofinal subsystem in the system of coverings of X. Thus it is enough to check vanishing of higher $\check{\mathrm{H}}^i(\mathcal{U}, \widetilde{\mathfrak{m}} \otimes \mathcal{I})$ for any such coverings $\mathcal{U}$ of X.

We pick such a covering $\mathcal{U} : X = \cup_{i=1}^n U_i$ and observe that all the intersections $U_{i_1, \dots, i_m} = \cap_{k=1}^m U_{i_k}$ are again quasi-compact by spectrality of X. In particular, they are spectral. Now we invoke [Sta21, Tag 0A36] to say that it suffices to show that

$$(\widetilde{\mathfrak{m}} \otimes \mathcal{I})(V) \xrightarrow{r_{\widetilde{\mathfrak{m}} \otimes \mathcal{I}}|_U^V} (\widetilde{\mathfrak{m}} \otimes \mathcal{I})(U)$$

is surjective for any inclusion of any *spectral* open subsets $U \hookrightarrow V$. Lemma 3.3.2 says that this map $r_{\widetilde{\mathfrak{m}} \otimes \mathcal{I}}|_U^V$ is identified with the map

$$\widetilde{\mathfrak{m}} \otimes_R \mathcal{I}(V) \xrightarrow{\widetilde{\mathfrak{m}} \otimes_R r_{\mathcal{I}}|_U^V} \widetilde{\mathfrak{m}} \otimes_R \mathcal{I}(U).$$

But now we note that $r_{\mathcal{I}}|_U^V$ is surjective since any injective $\mathcal{O}_X$-module is flasque by [Sta21, Tag 01EA], and therefore the map $\widetilde{\mathfrak{m}} \otimes_R r_{\mathcal{I}}|_U^V$ is surjective as well. $\square$

[18]We remind the reader that actually any quasi-compact quasi-separated open subset of a locally spectral space is spectral. This can be easily seen from the definitions.

Corollary 3.3.4. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces, and let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then $\widetilde{\mathfrak{m}} \otimes \mathcal{I}$ is an $f_*(-)$-acyclic

Proof. It suffices to show that for any open spectral $U \subset Y$ the higher cohomology groups

$$\mathrm{H}^i(X_U, (\widetilde{\mathfrak{m}} \otimes \mathcal{I})|_{X_U})$$

vanish. This follows from Lemma 3.3.3 since X_U is spectral by the assumption on f. □

Lemma 3.3.5. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then there is an isomorphism

$$\beta : \widetilde{\mathfrak{m}} \otimes f_*\mathcal{F} \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$$

functorial in $\mathcal{F}$.

Proof. It suffices to define a morphism on a basis of spectral open subspaces $U \subset Y$. For any such $U \subset Y$ we define

$$\beta_U : (\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F})(U) \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)$$

as the composition of isomorphisms

$$(\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F})(U) \xrightarrow{\alpha_U^{-1}} \widetilde{\mathfrak{m}} \otimes_R (f_*\mathcal{F})(U) = \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(X_U) \xrightarrow{\alpha_{X_U}} (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(X_U) = f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)$$

with α_U and α_{X_U} isomorphisms from Lemma 3.3.2. Since the construction of α was functorial in U we conclude that β defines a morphism of sheaves. It is an isomorphism because we constructed β_U to be isomorphism an a basis of Y. □

Lemma 3.3.6. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces. Then for any $\mathcal{F} \in \mathbf{D}(X)$, there is a morphism

$$\rho_f(\mathcal{F}) : \widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F} \to \mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$$

functorial in $\mathcal{F}$. This map is an isomorphism in either of the following cases:

- The complex $\mathcal{F}$ is bounded below, i.e. $\mathcal{F} \in \mathbf{D}^+(X)$, or
- The space X is locally of uniformly bounded Krull dimension and $\mathcal{F} \in \mathbf{D}(X)$.

Proof. We start the proof by constructing the map $\rho_f(\mathcal{F})$. Note that by the adjunction, it suffices to construct a map

$$\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes \mathcal{F}$$

We also denote the counit of the adjunction between $\mathbf{L}f^*$ and $\mathbf{R}f_*$ by

$$\eta_{\mathcal{F}}\colon \mathbf{L}f^*\mathbf{R}f_*\mathcal{F} \to \mathcal{F}$$

Then we define the map

$$\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes \mathcal{F}$$

as the composition

$$\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}) \xrightarrow{\varphi_f(\mathbf{R}f_*\mathcal{F})} \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*\mathbf{R}f_*\mathcal{F} \xrightarrow{\widetilde{\mathfrak{m}}\otimes\eta_{\mathcal{F}}} \widetilde{\mathfrak{m}} \otimes \mathcal{F}$$

where the first map is the isomorphism coming from Lemma 3.2.17 and the second map comes from the adjunction morphism $\varepsilon_{\mathcal{F}}$.

Now we show that $\rho_f(\mathcal{F})$ is an isomorphism for $\mathcal{F} \in \mathbf{D}^+(X)$. We choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. In this case we use Corollary 3.3.4 to note that β is the natural map

$$\widetilde{\mathfrak{m}} \otimes f_*(\mathcal{I}^\bullet) \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{I}^\bullet)$$

that is an isomorphism by Lemma 3.3.5.

The last thing we need to show is that $\rho_f(\mathcal{F})$ is an isomorphism for any $\mathcal{F} \in \mathbf{D}(X)$ if X is locally of uniformly bounded Krull dimension. The claim is local, so we may and do assume that both X and Y are spectral spaces. As X is quasi-compact (as it is spectral now) and locally of finite Krull dimension, we conclude that X has finite Krull dimension, say $N := \dim X$. Then [Sch92, Corollary 4.6] (another reference is [Sta21, Tag 0A3G]) implies that $\mathrm{H}^i(U, \mathcal{G}) = 0$ for any open spectral $U \subset X$, $\mathcal{G} \in \mathbf{Mod}_X$, and $i > N$. In particular, $\mathrm{R}^i f_* \mathcal{G} = 0$ for any $\mathcal{G} \in \mathbf{Mod}_X$, and $i > N$. Thus we see that the assumptions of [Sta21, Tag 0D6U] are verified in this case (with $\mathcal{A} = \mathbf{Mod}_X$ and $\mathcal{A}' = \mathbf{Mod}_Y$), so the natural map

$$\mathcal{H}^j\left(\mathbf{R}f_*\mathcal{F}\right) \to \mathcal{H}^j\left(\mathbf{R}f_*\left(\tau^{\geq -n}\mathcal{F}\right)\right)$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}(X)$, $j \geq N - n$. As $\widetilde{\mathfrak{m}}$ is R-flat, we get the commutative diagram

$$\begin{tikzcd}
\mathcal{H}^j\left(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}\right) \arrow[r, "\mathcal{H}^j(\rho_{\mathcal{F}})"] \arrow[d, "\sim"] & \mathcal{H}^j\left(\mathbf{R}f_*\left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}\right)\right) \arrow[d, "\sim"] \\
\mathcal{H}^j\left(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\left(\tau^{\geq -n}\mathcal{F}\right)\right) \arrow[r, "\mathcal{H}^j(\rho_{\tau^{\geq -n}\mathcal{F}})"] & \mathcal{H}^j\left(\mathbf{R}f_*\left(\widetilde{\mathfrak{m}} \otimes \tau^{\geq -n}\mathcal{F}\right)\right)
\end{tikzcd}$$

with the vertical arrows being isomorphisms for $j \geq N - n$, and the bottom horizontal map is an isomorphism as $\tau^{\geq -n}\mathcal{F} \in \mathbf{D}^+(X)$. Thus, by choosing an appropriate $n \geq 0$, we see that $\mathcal{H}^j(\rho_{\mathcal{F}})$ is an isomorphism for any j; so $\rho_{\mathcal{F}}$ is an isomorphism itself. $\square$

3.4. **Derived Category of $\mathcal{O}_X^a$-Modules.** This section is a global analogue of Section 2.3. We give two different definitions of the derived category of almost $\mathcal{O}_X$-modules and show that they coincide.

Definition 3.4.1. We define the *derived category of $\mathcal{O}_X^a$-modules* as $\mathbf{D}(X^a) := \mathbf{D}(\mathbf{Mod}_X^a)$.

We define the bounded version of derived category of almost R-modules $\mathbf{D}^*(X^a)$ for $* \in \{+, -, b\}$ as the full subcategory of $\mathbf{D}(X^a)$ consisting of bounded below (resp. bounded above, resp. bounded) complexes.

Definition 3.4.2. We define the *almost derived category of $\mathcal{O}_X$-modules* as the Verdier quotient[19] $\mathbf{D}(X)^a := \mathbf{D}(\mathbf{Mod}_X)/\mathbf{D}_{\Sigma_X}(\mathbf{Mod}_X)$.

Remark 3.4.3. We recall that Σ_X is the Serre subcategory of $\mathbf{Mod}_X$ that consists of almost zero $\mathcal{O}_X$-modules.

We note that the functor $(-)^a \colon \mathbf{Mod}_X \to \mathbf{Mod}_X^a$ is exact and additive. Thus it can be derived to the functor $(-)^a \colon \mathbf{D}(X) \to \mathbf{D}(X^a)$. Similarly, the functor $(-)_! \colon \mathbf{Mod}_X^a \to \mathbf{Mod}_X$ can be derived to the functor $(-)_! \colon \mathbf{D}(X^a) \to \mathbf{D}(X)$. The standard argument shows that $(-)_!$ is a left adjoint functor to the functor $(-)^a$ as this already happens on the level of abelian categories.

We also want to establish a derived version of the functor $(-)_*$. But since functor is only left exact, we do need to do some work to derive it. Namely, we need to ensure that $\mathcal{O}_X^a$-modules admit enough K-injective complexes.

Definition 3.4.4. We say that a complex of $\mathcal{O}_X^a$-module $I^{\bullet,a}$ is *K-injective* if $\mathrm{Hom}_{K(\mathcal{O}_X^a)}(C^{\bullet,a}, I^{\bullet,a}) = 0$ for any acyclic complex $C^{\bullet,a}$ of R^a-modules.

[19] We refer to [Sta21, Tag 05RA] for an extensive discussion of Verdier quotients of triangulated categories.

Remark 3.4.5. We remind the reader that $K(\mathcal{O}_X^a)$ stands for the homotopy category of $\mathcal{O}_X^a$-modules.

Lemma 3.4.6. The functor $(-)^a\colon \mathbf{Comp}(\mathcal{O}_X) \to \mathbf{Comp}(\mathcal{O}_X^a)$ sends K-injective $\mathcal{O}_X^a$-complexes to K-injective $\mathcal{O}_X^a$-complexes.

Proof. We note that $(-)^a$ admits an exact left adjoint $(-)_!$ thus [Sta21, Tag 08BJ] ensures that $(-)^a$ preserves K-injective complexes. □

Corollary 3.4.7. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then every object $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_X^a)$ is quasi-isomorphic to a K-injective complex.

Proof. The proof of Corollary 2.3.6 works verbatim with the only exception that one needs to use [Sta21, Tag 079P] instead of [Sta21, Tag 090Y]. □

Now, similarly to the case of R^a-modules, we define the functor $(-)_*\colon \mathbf{D}(X^a) \to \mathbf{D}(X)$ as the derived functor of $(-)_*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_X$. This functor exists by [Sta21, Tag 070K].

Lemma 3.4.8. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then

(1) The functors $\mathbf{D}(X) \underset{(-)^a}{\overset{(-)_!}{\rightleftarrows}} \mathbf{D}(X^a)$ are adjoint. Moreover, the counit (resp. unit) morphism
$$(\mathcal{F}^a)_! \to \mathcal{F} \text{ (resp. } \mathcal{G} \to (\mathcal{G}_!)^a)$$
is an almost isomorphism (resp. isomorphism) for any $\mathcal{F} \in \mathbf{D}(X), \mathcal{G} \in \mathbf{D}(X^a)$. In particular, the functor $(-)^a$ is essentially surjective.

(2) The functor $(-)^a\colon \mathbf{D}(X) \to \mathbf{D}(X^a)$ also admits a right adjoint functor $(-)_*\colon \mathbf{D}(X^a) \to \mathbf{D}(X)$. Moreover, the unit (resp. counit) morphism
$$\mathcal{F} \to (\mathcal{F}^a)_* \text{ (resp. } (\mathcal{G}_*)^a \to \mathcal{G})$$
is an almost isomorphism (resp. isomorphism) for any $\mathcal{F} \in \mathbf{D}(X), \mathcal{G} \in \mathbf{D}(X^a)$.

Proof. The proof is absolutely similar to Lemma 2.3.7. □

Theorem 3.4.9. The functor $(-)^a\colon \mathbf{D}(X) \to \mathbf{D}(X^a)$ induces an equivalence of triangulated categories $(-)^a\colon \mathbf{D}(X)^a \to \mathbf{D}(X^a)$.

Proof. The proof is similar to that of Theorem 2.3.8. □

Remark 3.4.10. Theorem 3.4.9 shows that the two notions of the derived category of almost modules are the same. In what follows, we do not distinguish $\mathbf{D}(X^a)$ and $\mathbf{D}(X)^a$ anymore.

3.5. **Basic Functors on the Derived Categories of $\mathcal{O}_X^a$-modules.** Now we can "derive" certain functors constructed in section 3.2. For the rest of the section we fix a ringed R-site $(X, \mathcal{O}_X)$. The section follows the exposition of section 2.4 very closely.

Definition 3.5.1. We define the *derived Hom* functors
$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{op} \times \mathbf{D}(X^a) \to \mathbf{D}(X^a), \text{ and}$$
$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{op} \times \mathbf{D}(X^a) \to \mathbf{D}(R)$$
as it is done in [Sta21, Tag 08DH] and [Sta21, Tag 0B6A], respectively.

Definition 3.5.2. We define the *global Ext-modules* as the R-modules

$$\mathrm{Ext}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$$

for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}^a_X$.

We define the *local Ext-sheaves* as the $\mathcal{O}_X$-modules $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathcal{H}^i(\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$ for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}^a_X$.

Remark 3.5.3. We see that [Sta21, Tag 0A64] implies that there is a functorial isomorphism

$$\mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right) \simeq \mathrm{Hom}_{\mathbf{D}(R)^a}(\mathcal{F}^a, \mathcal{G}^a[i])$$

for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Remark 3.5.4. The standard argument shows that there is a functorial isomorphism

$$\mathbf{R}\Gamma(U, \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)$$

for any open $U \in X$, $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Now we give show a local version of the $((-)_!, (-)^a)$-adjunction, and relate $\mathbf{R}\underline{\mathcal{H}om}$ (resp. $\mathbf{R}\mathrm{Hom}$) to the certain derived functor. This goes in complete analogy with the situation in the usual (not almost) world.

Lemma 3.5.5. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then

(1) There is a functorial isomorphism

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})$$

for any $\mathcal{F}^a \in \mathbf{D}(X)^a$ and $\mathcal{G} \in \mathbf{D}(X)$. In particular, this isomorphism induces functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G}) \text{ and } \mathrm{Hom}_{\mathbf{D}(X)^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{F}^a_!, \mathcal{G})\ .$$

(2) For any chosen $\mathcal{F}^a \in \mathbf{Mod}^a_X$, the functor $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(R)$ is isomorphic to the (right) derived functor of $\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.

(3) For any chosen $\mathcal{F}^a \in \mathbf{Mod}^a_X$, the functor $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(X)$ is isomorphic to the (right) derived functor of $\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.

Proof. We prove Part (1). We firstly define the map

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})\ .$$

We choose some representation $\mathcal{F}^{\bullet,a}$ of $\mathcal{F}^a$ and a quasi-isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{I}^\bullet$ of $\mathcal{G}$ to a K-injective complex $\mathcal{I}^\bullet$. Then we know that $\mathcal{I}^{\bullet,a}$ is a K-injective resolution of $\mathcal{G}^a$ by Lemma 3.4.6. Therefore, the construction of the derived hom says that we have isomorphisms

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a})$$

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G}) \simeq \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^{\bullet,a}_!, \mathcal{I}^\bullet)$$

Now we recall that term-wise we have the following equalities:

$$\underline{\mathcal{H}om}^n_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a}) = \prod_{p+q=n} \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^{-q,a}, \mathcal{I}^{p,a})$$

$$\underline{\mathcal{H}om}^n_{\mathcal{O}_X}(\mathcal{F}^{\bullet,a}_!, \mathcal{I}^\bullet) = \prod_{p+q=n} \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^{-q,a}_!, \mathcal{I}^p)$$

Thus we can apply Lemma 3.2.3 term-wise to produce an isomorphism

$$\kappa_n \colon \underline{\mathcal{H}om}^n_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a}) \to \underline{\mathcal{H}om}^n_{\mathcal{O}_X}(\mathcal{F}^{\bullet,a}_!, \mathcal{I}^\bullet)$$

for each n. It is then straightforward to see that κ_n commute with the differential, and thus induce the isomorphism of complexes

$$\kappa \colon \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a}) \xrightarrow{\sim} \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^{\bullet,a}_!, \mathcal{I}^\bullet) .$$

In particular, it produces the desired isomorphism $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \xrightarrow{\sim} \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$. The construction is clearly functorial in both $\mathcal{F}^a$ and $\mathcal{G}$.

Parts **(2)** and **(3)** are identical to Lemma 2.4.3**(2)**. $\square$

Definition 3.5.6. We define the *derived almost Hom* functors

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(-,-) \colon \mathbf{D}(X^a)^{op} \times \mathbf{D}(X^a) \to \mathbf{D}(X^a)$$

$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(-,-) \colon \mathbf{D}(X^a)^{op} \times \mathbf{D}(X^a) \to \mathbf{D}(R^a)$$

as

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)^a = \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})^a$$

$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)^a = \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})^a$$

Definition 3.5.7. We define the *global almost Ext modules* as the R^a-modules $\mathrm{alExt}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathrm{H}^i(\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$ for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}^a_X$.

We define the *local almost Ext sheaves* as the $\mathcal{O}_X^a$-modules $\underline{al\mathcal{E}xt}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \coloneqq \mathcal{H}^i(\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$ for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}^a_X$.

Proposition 3.5.8. Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then:

(1) There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(X)^{op} \times \mathbf{D}(X) \arrow[r, "\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(-{,}-)"] \arrow[d, "(-)^a \times (-)^a"'] & \mathbf{D}(X) \arrow[d, "(-)^a"] \arrow[dl, Rightarrow, "\rho"] \\
\mathbf{D}(X^a)^{op} \times \mathbf{D}(X^a) \arrow[r, "\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(-{,}-)"] & \mathbf{D}(X^a)
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a$ for any $\mathcal{F}, \mathcal{G} \in \mathbf{D}(X)$.

(2) For any chosen $\mathcal{F}^a \in \mathbf{Mod}^a_R$, the functor $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, -) \colon \mathbf{D}(X)^a \to \mathbf{D}(X)^a$ is isomorphic to the (right) derived functor of $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.

(3) The analogous results hold true for the functor $\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)$.

Proof. The proof is identical to that of Proposition 2.4.8. One only needs to use Proposition 3.2.10 in place of Proposition 2.2.1**(3)**. $\square$

Now we deal with the case of the derived tensor product functor. We will show that our definition of the derived tensor product functor makes $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)$ into the inner Hom functor.

Definition 3.5.9. We say that a complex of $\mathcal{O}_X^a$-module $\mathcal{F}^{\bullet,a}$ is *almost K-flat* if the naive tensor product complex $\mathcal{C}^{\bullet,a} \otimes^{\bullet}_{\mathcal{O}_X^a} \mathcal{F}^{\bullet,a}$ is acyclic for any acyclic complex $\mathcal{C}^{\bullet,a}$ of $\mathcal{O}_X^a$-modules.

Lemma 3.5.10. The functor $(-)^a\colon \mathbf{Comp}(\mathcal{O}_X) \to \mathbf{Comp}(\mathcal{O}_X^a)$ sends K-flat $\mathcal{O}_X$-complexes to almost K-flat $\mathcal{O}_X^a$-complexes.

Proof. The proof Lemma 2.4.10 applies verbatim. ☐

Lemma 3.5.11. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_Y^a)$ be an almost K-flat complex. Then $f^*(\mathcal{F}^{\bullet,a}) \in \mathbf{Comp}(\mathcal{O}_X^a)$ is almost K-flat.

Proof. The proof of [Sta21, Tag 06YW] works verbatim in this situation. ☐

Corollary 3.5.12. Every object $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_X^a)$ is quasi-isomorphic to an almost K-flat complex.

Proof. The proof of Corollary 2.4.11 applies verbatim with the only difference that one needs to use [Sta21, Tag 06YF] in place of [Sta21, Tag 06Y4]. ☐

Definition 3.5.13. We define the *derived tensor product functor*

$$- \otimes^L_{\mathcal{O}_X^a} -\colon \mathbf{D}(X)^a \times \mathbf{D}(X)^a \to \mathbf{D}(X)^a$$

by the rule $(\mathcal{F}^a, \mathcal{G}^a) \mapsto (\mathcal{G}_! \otimes^L_{\mathcal{O}_X} \mathcal{G}_!)^a$ for any $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Proposition 3.5.14. **(1)** There is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(X) \times \mathbf{D}(X) \arrow[r, "- \otimes^L_{\mathcal{O}_X} -"] \arrow[d, "(-)^a \times (-)^a"'] & \mathbf{D}(X) \arrow[d, "(-)^a"] \\
\mathbf{D}(X)^a \times \mathbf{D}(X)^a \arrow[r, "- \otimes^L_{\mathcal{O}_X^a} -"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{D}(X)^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G})^a \simeq \mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a$ for any $\mathcal{F}, \mathcal{G} \in \mathbf{D}(X)$.

(2) For any chosen $\mathcal{F}^a \in \mathbf{Mod}_X^a$, the functor $\mathcal{F}^a \otimes^L_{R^a} -\colon \mathbf{D}(X)^a \to \mathbf{D}(X)^a$ is isomorphic to the (left) derived functor of $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} -$.

Proof. Again, the proof is identical to that of Proposition 3.5.14. The only non-trivial input that we need is existence of sufficiently many K-flat complexes of $\mathcal{O}_X^a$-modules. But this is guaranteed by Lemma 3.5.12. ☐

Remark 3.5.15. For any $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$, there is a canonical morphism

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \otimes^L_{\mathcal{O}_X} \mathcal{F}^a \to \mathcal{G}^a$$

that, after the identifications from Proposition 3.5.8 and Proposition 3.5.14, is the almostification of the canonical morphism

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G}^a_!) \otimes^L_{\mathcal{O}_X} \mathcal{F}^a_! \to \mathcal{G}^a_!$$

from [Sta21, Tag 0A8V].

Lemma 3.5.16. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}^a, \mathcal{G}^a, \mathcal{H}^a \in \mathbf{D}(X)^a$. Then we have a functorial isomorphism

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathbf{R}\underline{al\mathcal{H}om}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

This induces functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathbf{R}\underline{al\mathcal{H}om}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)) \ ,$$

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathbf{R}\underline{al\mathcal{H}om}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)) \ ,$$

$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathbf{R}\underline{al\mathcal{H}om}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)) \ .$$

Proof. The proof of the first isomorphism is very similar to that of Lemma 2.4.14. We leave the details to the interested reader. The second isomorphism comes from the fist one by applying the functor $\mathbf{R}\Gamma(X, -)$. The third and the fourth isomorphisms are obtained by applying $(-)^a$ to the first and the second isomorphisms respectively. Here we implicitly use Proposition 3.5.8. □

Corollary 3.5.17. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{G}^a \in \mathbf{D}(X)^a$. Then the functors

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{G}^a, -) \colon \mathbf{D}(X)^a \rightleftarrows \mathbf{D}(X)^a \colon - \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a$$

are adjoint.

The next two functors we deal with are the derived pullback and derived pushforward. We start with the derived pullback.

Definition 3.5.18. Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. We define the *derived pullback functor*

$$\mathbf{L}f^* \colon \mathbf{D}(Y)^a \to \mathbf{D}(X)^a$$

as the derived functor of the right exact, additive functor $f^* \colon \mathbf{Mod}_Y^a \to \mathbf{Mod}_X^a$.

Remark 3.5.19. We need to explain why the desired derived functor exists and how it can be computed. It turns out that it can be constructed by choosing K-flat resolutions, the argument for this is identical to [Sta21, Tag 06YY]. We only emphasize that three main inputs are Lemma 3.5.11, Lemma 3.5.10 and an almost analogue of [Sta21, Tag 06YG].

Proposition 3.5.20. Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors

$$\begin{tikzcd}
\mathbf{D}(Y) \arrow[r, "\mathbf{L}f^*"] \arrow[d, "(-)^a"'] & \mathbf{D}(X) \arrow[d, "(-)^a"] \\
\mathbf{D}(Y)^a \arrow[r, "\mathbf{L}f^*"'] \arrow[ur, Rightarrow, "\rho"] & \mathbf{D}(X)^a
\end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(\mathbf{L}f^*\mathcal{F})^a \simeq \mathbf{L}f^*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{D}(Y)$.

Proof. We construct the natural tranformation $\rho \colon \mathbf{L}f^* \circ (-)^a \Rightarrow (-)^a \circ \mathbf{L}f^*$ as follows. Pick any object $\mathcal{F} \in \mathbf{D}(Y)$ and its K-flat representative $\mathcal{K}^\bullet$, then $\mathcal{K}^\bullet$ is adapted to compute the usual derived pullback $\mathbf{L}f^*$. Lemma 3.5.11 ensures $\mathcal{K}^{\bullet,a}$ is also adapted to compute the almost version of the derived pullback $\mathbf{L}f^*$. So we define the morphism

$$\rho_{\mathcal{F}} \colon (f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet))^a \to f^*(\mathcal{K}^\bullet)^a$$

as the natural morphism induced by $\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet \to \mathcal{K}^\bullet$. This map is clearly functorial, so it defines a transformation of functors ρ. In order to show that it is an isomorphism of functors, it suffices to show that the map

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet) \to f^*(\mathcal{K}^\bullet)$$

is an almost isomorphism of complexes for any K-flat complex $K^\bullet$. But this is clear as $\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet \to \mathcal{K}^\bullet$ is an almost isomorphism, and Corollary 3.2.18 ensures that f^* preserves almost isomorphisms. □

Definition 3.5.21. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. We define the *derived pushforward functor*

$$\mathbf{R}f_*\colon \mathbf{D}(X)^a \to \mathbf{D}(Y)^a$$

as the derived functor of the left exact, additive functor $f_*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_Y^a$.

We define the *derived global sections functor* $\mathbf{R}\Gamma(U, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(R)^a$ in a similar way for any open $U \subset X$.

Remark 3.5.22. This functor exists by abstract nonsense (i.e. [Sta21, Tag 070K]) as the category $\mathbf{Mod}_X^a$ has enough K-injective complexes by Lemma 3.4.7.

Proposition 3.5.23. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors

$$\begin{tikzcd} \mathbf{D}(X) \arrow[r, "\mathbf{R}f_*"] \arrow[d, "(-)^a"'] & \mathbf{D}(Y) \arrow[d, "(-)^a"] \\ \mathbf{D}(X)^a \arrow[r, "\mathbf{R}f_*"] \arrow[ur, Rightarrow, "\rho"] & \mathbf{D}(Y)^a \end{tikzcd}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(\mathbf{R}f_*\mathcal{F})^a \simeq \mathbf{R}f_*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{D}(X)$. The analogous results hold for the functor $\mathbf{R}\Gamma(U, -)$.

Proof. The proof is very similar to that of Proposition 3.5.20. The main essential ingredients are: $(-)^a$ sends K-injective complexes to K-injective complexes, and f_* preserves almost isomorphisms. These two results were shown in Lemma 3.4.6 and Lemma 3.2.23. □

Lemma 3.5.24. Let $(X, \mathcal{O}_X)$ be a ringed R-site, let $\mathcal{F}$ be an $\mathcal{O}_X^a$-module, and let $U \in X$ be an open object. Then we have a canonical isomorphism

$$\mathbf{R}\Gamma(U, \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)$$

Proof. This easily follows from Remark 3.5.4, Proposition 3.5.8, and Proposition 3.5.23. □

Lemma 3.5.25. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a locally ringed morphism of R-spaces. Then there is a functorial isomorphism

$$\mathbf{R}f_*\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a)$$

for $\mathcal{F}^a \in \mathbf{D}(Y)^a$, $\mathcal{G}^a \in \mathbf{D}(X)^a$. This isomorphism induces isomorphisms

$$\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a) ,$$

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a) ,$$

$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a) .$$

Proof. It is a standard exercise to show that the first isomorphism implies all other isomorphisms by applying certain functors to it, so we deal only with the first one. The proof of the first one is also quite standard and similar to Lemma 3.2.26, but we spell it out for the reader's convenience. The desired isomorphism comes from a sequence of canonical identifications:

$$\begin{aligned}
\mathbf{R}f_*\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathbf{L}f^*(\mathcal{F}^a), \mathcal{G}^a) &\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathbf{L}f^*(\mathcal{F})^a, \mathcal{G}^a) && \text{Proposition 3.5.20} \\
&\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F}), \mathcal{G}) && \text{Lemma 3.5.5(1)} \\
&\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}), \mathcal{G}) && \text{Lemma 3.2.17} \\
&\simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_Y}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \mathbf{R}f_*(\mathcal{G})) && \text{Classical} \\
&\simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*(\mathcal{G})^a) && \text{Lemma 3.5.5(1)} \\
&\simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*(\mathcal{G}^a)) && \text{Proposition 3.5.23.}
\end{aligned}$$

□

Corollary 3.5.26. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a locally ringed morphism of locally ringed R-spaces. Then the functors $\mathbf{R}f_*(-)\colon \mathbf{D}(X)^a \rightleftarrows \mathbf{D}(Y)^a \colon \mathbf{L}f^*(-)$ are adjoint.

Now we discuss the projection formula in the world of almost sheaves. Suppose $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a locally ringed morphism of locally ringed R-spaces, $\mathcal{F}^a \in \mathbf{D}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}(Y)^a$. We wish to construct the projection morphism

$$\rho\colon \mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a \to \mathbf{R}f_*(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a)).$$

By Corollary 3.5.26, data of this morphism is equivalent to the data of a morphism

$$\mathbf{L}f^*(\mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a) \to \mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a).$$

This morphism is defined as the composition of natural isomorphism

$$\mathbf{L}f^*(\mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a) \simeq \mathbf{L}f^*(\mathbf{R}f_*(\mathcal{F}^a)) \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a)$$

and the morphism

$$\mathbf{L}f^*(\mathbf{R}f_*(\mathcal{F}^a)) \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a) \xrightarrow{\varepsilon_{\mathcal{F}^a} \otimes \mathrm{Id}} \mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a)$$

induced by the co-unit of the $(\mathbf{L}f^*, \mathbf{R}f_*)$-adjunction.

Proposition 3.5.27. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a locally ringed morphism of locally ringed R-spaces, $\mathcal{F}^a \in \mathbf{D}(X)^a$, and $\mathcal{G} \in \mathbf{D}(Y)$ a perfect complex. Then the projection morphism

$$\rho\colon \mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a \to \mathbf{R}f_*(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a))$$

is an isomorphism in $\mathbf{D}(Y)^a$.

Proof. The claim is local on Y, so we may assume that $\mathcal{G}$ is isomorphic to a bounded complex of finite free $\mathcal{O}_Y$-modules. Then an easy argument with stupid filtrations reduce the question to the case $\mathcal{G} = \mathcal{O}_Y^n$. This case is essentially obvious. □

4. Almost Coherent Sheaves on Schemes and Formal Schemes

4.1. **Schemes. The Category of Almost Coherent $\mathcal{O}_X^a$-modules.** In this Section we discuss the notion of almost quasi-coherent, almost finite type, almost finitely presented and almost coherent sheaves on an arbitrary scheme. The main content of this Section is to make sure that almost coherent sheaves form a Weak Serre subcategory in $\mathcal{O}_X$-modules. Another important statement is the "approximation" Corollary 4.3.5 that is the key fact to reduce many statements about almost finitely presented $\mathcal{O}_X$-modules to the "classical" case of finitely presented $\mathcal{O}_X$-modules. In particular, we follow this approach in our proof of the Almost Proper Mapping Theorem in Section 5.1.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal. In what follows X will always denote an R-scheme. Note that this implies that X is a locally spectral, ringed R-site, so the results of the previous sections apply.

We begin with some definitions:

Definition 4.1.1. We say that an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *almost quasi-coherent* if $\mathcal{F}^a_! \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *almost quasi-coherent* if $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_X^a$-module.

Remark 4.1.2. Any quasi-coherent $\mathcal{O}_X$-module is almost quasi-coherent.

Remark 4.1.3. We denote by $\mathbf{Mod}_{X^a}^{\mathrm{aqc}} \subset \mathbf{Mod}_{X^a}$ the full subcategory consisting of almost quasi-coherent $\mathcal{O}_X^a$-modules. It is straightforward[20] to see that the "almostification" functor induces an equivalence

$$\mathbf{Mod}_{X^a}^{\mathrm{aqc}} \simeq \mathbf{Mod}_X^{\mathrm{qc}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc}}),$$

i.e. $\mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ is equivalent to the quotient category of quasi-coherent $\mathcal{O}_X$-modules by the full subcategory of almost zero, quasi-coherent $\mathcal{O}_X$-modules.

Definition 4.1.4. We say that an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *of almost finite type* (resp. *almost finitely presented*) if $\mathcal{F}^a$ is almost quasi-coherent, and there is a covering of X by open affines $\{U_i\}_{i\in I}$ such that $\mathcal{F}^a(U_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_X^a(U_i)$-module.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *of almost finite type* (resp. *almost finitely presented*) if so is $\mathcal{F}^a$.

Remark 4.1.5. We denote by $\mathbf{Mod}_X^{\mathrm{qc,aft}}$ (resp. $\mathbf{Mod}_X^{\mathrm{qc,afp}}$) the full subcategory of $\mathbf{Mod}_X$ consisting of almost finite type (resp. almost finitely presented) quasi-coherent $\mathcal{O}_X$-modules. Similarly, we denote by $\mathbf{Mod}_{X^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{X^a}^{\mathrm{afp}}$) the full subcategory of $\mathbf{Mod}_{X^a}$ consisting of almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-modules. Again, it is straightforward to see that the "almostification" functors induce equivalences

$$\mathbf{Mod}_{X^a}^{\mathrm{aft}} \simeq \mathbf{Mod}_X^{\mathrm{qc,aft}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,aft}}),\ \mathbf{Mod}_{X^a}^{\mathrm{afp}} \simeq \mathbf{Mod}_X^{\mathrm{qc,afp}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,afp}}).$$

Remark 4.1.6. In the usual theory of $\mathcal{O}_X$-modules, finite type $\mathcal{O}_X$-modules are usually not required to be quasi-coherent. However, it is much more convenient for our purposes to put almost quasi-coherence in the definition of almost finite type modules.

The first thing that we need to check is that these notions do not depend on a choice of an affine covering.

[20] The proof is completely similar to the proof of Theorem 3.1.20 or Theorem 3.4.9.

Lemma 4.1.7. Let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-module on an R-scheme X. Then $\mathcal{F}^a(U)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_X^a(U)$-module for any open affine $U \subset X$.

Proof. First of all, Corollary 2.5.12 and Lemma 3.3.2 imply that for any open affine U, $\mathcal{F}^a(U)$ is almost finitely generated (resp. almost finitely presented) if and only if so is $(\widetilde{\mathfrak{m}} \otimes \mathcal{F}^a)(U)$. Thus we can replace $\mathcal{F}^a$ by $\mathcal{F}^a_! \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to assume that $\mathcal{F}$ is an honest quasi-coherent $\mathcal{O}_X$-module.

Now Lemma 2.8.1 guarantees that the problem is local on X. So we can assume that $X = U$ is an affine scheme and we need to show that $\mathcal{F}(X)$ is almost finitely generated (resp. almost finitely presented).

We pick some covering $X = \cup_{i=1}^n U_i$ by open affines U_i such that $\mathcal{F}(U_i)$ is almost finitely generated (resp. almost finitely presented) as an $\mathcal{O}_X(U_i)$-module. We note that since $\mathcal{F}$ is quasi-coherent we have an isomorphism

$$\mathcal{F}(U_i) \simeq \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U_i).$$

Now we see that a map $\mathcal{O}_X(X) \to \prod_{i=1}^n \mathcal{O}_X(U_i)$ is faithfully flat, and the module

$$\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \left(\prod_{i=1}^n \mathcal{O}_X(U_i)\right) \simeq \left(\prod_{i=1}^n \mathcal{O}_X(U_i)\right) \otimes_{\mathcal{O}_X(X)} \mathcal{F}(X)$$

is almost finitely generated (resp. almost finitely presented) over $\prod_{i=1}^n \mathcal{O}_X(U_i)$. Therefore, Lemma 2.10.5 guarantees that $\mathcal{F}(X)$ is almost finitely generated (resp. almost finitely presented) as an $\mathcal{O}_X(X)$-module. $\square$

Corollary 4.1.8. Let $X = \operatorname{Spec} A$ be an affine R-scheme, and let $\mathcal{F}^a$ be an almost quasi-coherent $\mathcal{O}_X^a$-module. Then $\mathcal{F}^a$ is almost finite type (resp. almost finitely presented) if and only if $\Gamma(X, \mathcal{F}^a)$ is almost finitely generated (resp. almost finitely presented) A-module.

Now we check that almost finite type and almost finitely presented $\mathcal{O}_X^a$ behave nicely in short exact sequences.

Lemma 4.1.9. Let $0 \to \mathcal{F}'^a \xrightarrow{\varphi} \mathcal{F}^a \xrightarrow{\psi} \mathcal{F}''^a \to 0$ be an exact sequence of $\mathcal{O}_X^a$-modules. Then

- **(1)** If $\mathcal{F}^a$ is almost of finite type and $\mathcal{F}''^a$ is almost quasi-coherent, then $\mathcal{F}''^a$ is almost finite type.
- **(2)** If $\mathcal{F}'^a$ and $\mathcal{F}''^a$ are of almost finite type (resp. finitely presented), then so is $\mathcal{F}^a$.
- **(3)** If $\mathcal{F}^a$ is of almost finite type and $\mathcal{F}''^a$ is almost finitely presented, then $\mathcal{F}'^a$ is of almost finite type.
- **(4)** If $\mathcal{F}^a$ is almost finitely presented and $\mathcal{F}'^a$ is of almost finite type, then $\mathcal{F}''^a$ is almost finitely presented.

Proof. First of all, we apply the exact functor $(-)_!$ to all $\mathcal{O}_X^a$-modules in question to assume the short sequence is an exact sequence of $\mathcal{O}_X$-modules and all $\mathcal{O}_X$-modules in this sequence are quasi-coherent. Note that we implicitly use here that quasi-coherent modules form a Serre subcategory of all $\mathcal{O}_X$-modules by [Sta21, Tag 01IE]. Then we check the statement on the level of global sections on all open affine subschemes $U \subset X$ using that quasi-coherent sheaves have vanishing higher cohomology on affine schemes. And that is done in Lemma 2.5.15. $\square$

Definition 4.1.10. We say that an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *almost coherent* if $\mathcal{F}^a$ is almost finite type, and for any open set U any almost finite type $\mathcal{O}_U^a$-submodule $\mathcal{G}^a \subset (\mathcal{F}^a|_U)$ is an almost finitely presented $\mathcal{O}_U^a$-module.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *almost coherent* if $\mathcal{F}^a$ is an almost coherent $\mathcal{O}_X^a$-module.

Lemma 4.1.11. Let X be an R-scheme, and let $\mathcal{F}^a$ be an $\mathcal{O}_X^a$-module. Then the following are equivalent:

- **(1)** $\mathcal{F}^a$ is almost coherent.
- **(2)** $\mathcal{F}^a$ is almost quasi-coherent, and the $\mathcal{O}_X^a(U)$-module $\mathcal{F}^a(U)$ is almost coherent for any open affine subscheme $U \subset X$.
- **(3)** $\mathcal{F}^a$ is almost quasi-coherent, and there is a covering of X by open affine subschemes $(U_i)_{i\in I}$ such that $\mathcal{F}^a(U_i)$ is almost coherent for each i.

In particular, if $X = \operatorname{Spec} A$ is an affine R-scheme and $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_X^a$-module. Then $\mathcal{F}^a$ is almost coherent if and only if $\mathcal{F}^a(X)$ is almost coherent as an A-module.

Proof. We start the proof by noting that we can replace $\mathcal{F}^a$ by $\mathcal{F}^a_!$ to assume that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.

Firstly, we check that (**1**) implies (**2**). Given any affine open $U \subset X$ and any almost finitely generated almost submodule $M^a \subset \mathcal{F}(U)^a$, we define an almost subsheaf $\widetilde{(M^a)_!} \subset (\mathcal{F}|_U)^a$. We see that $\widetilde{(M^a)_!}$ must be an almost finitely presented $\mathcal{O}_U$-module, so Lemma 4.1.7 guarantees that $M^a_!$ is almost finitely presented $\mathcal{O}_X(U)$-module. Therefore, any almost finitely generated submodule $M^a \subset \mathcal{F}(U)^a$ is almost finitely presented. This shows that $\mathcal{F}(U)$ is almost coherent.

Now we show that (**2**) implies (**1**). Suppose that $\mathcal{F}$ is almost quasi-coherent and $\mathcal{F}(U)$ is almost coherent for any open affine $U \subset X$. First of all, it shows that $\mathcal{F}$ is of almost finite type, since this notion is local by definition. Now suppose that we have an almost finite type almost $\mathcal{O}_X$-submodule $\mathcal{G} \subset (\mathcal{F}|_U)^a$ for some open U. It is represented by a homomorphism

$$\widetilde{\mathfrak{m}} \otimes \mathcal{G} \xrightarrow{g} \mathcal{F}$$

with $\mathcal{G}$ being an $\mathcal{O}_X$-module of almost finite type, and $\widetilde{\mathfrak{m}} \otimes \ker g \simeq 0$. We want to show that $\mathcal{G}$ is almost finitely presented as $\mathcal{O}_X$-module. This is a local question, so we can assume that U is affine. Then Lemma 3.3.2 implies that the natural morphism

$$g(U) : \widetilde{\mathfrak{m}} \otimes_R \mathcal{G}(U) \to \mathcal{F}(U)$$

defines an almost submodule of $\mathcal{F}(U)$. We conclude that $\widetilde{\mathfrak{m}} \otimes_R \mathcal{G}(U)$ is almost finitely presented by the assumption on $\mathcal{F}(U)$. Since the notion of almost finitely presented $\mathcal{O}_X$-module is local, we see that $\mathcal{G}$ is almost finitely presented.

Clearly, (**2**) implies (**3**). And it is easy to see that Lemma 2.10.6 guarantees that (**3**) implies (**2**). ☐

Corollary 4.1.12. Let X be an R-scheme, then:

- **(1)** Any almost finite type $\mathcal{O}_X^a$-submodule of an almost coherent $\mathcal{O}_X^a$-module is almost coherent.
- **(2)** Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism from an almost finite type $\mathcal{O}_X^a$-module to an almost coherent $\mathcal{O}_X^a$-module, then $\ker(\varphi)$ is an almost finite type $\mathcal{O}_X^a$-module.
- **(3)** Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism of almost coherent $\mathcal{O}_X^a$-modules, then $\ker(\varphi)$ and $\operatorname{Coker}(\varphi)$ are almost coherent $\mathcal{O}_X^a$-modules.
- **(4)** Given a short exact sequence of $\mathcal{O}_X^a$-modules

  $$0 \to \mathcal{F}'^a \to \mathcal{F}^a \to \mathcal{F}''^a \to 0$$

  if two out of three are almost coherent so is the third.

Proof. The proofs **(1)**, **(2)** and **(3)** are quite straightforward. As usually, we apply $(-)_!$ to assume that all sheaves in the question are quasi-coherent $\mathcal{O}_X$-modules. Then the question is local and it is sufficient to work on global sections over all affine open subschemes $U \subset X$. So the problem is reduced to Lemma 2.6.8.

The proof of part **(4)** is similar, but we only need to invoke that given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}'^a_! \to \mathcal{F}^a_! \to \mathcal{F}''^a_! \to 0$$

if two of these sheaves are quasi-coherent, so is the third one. This is proven in the affine case in [Sta21, Tag 01IE], the general case reduces to the affine one. The rest of the argument is the same. □

Definition 4.1.13. We define the categories $\mathbf{Mod}_X^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_X^{\mathrm{qc,acoh}}$, resp. $\mathbf{Mod}_{X^a}^{\mathrm{acoh}}$) as the full subcategory of $\mathbf{Mod}_X$ (resp. $\mathbf{Mod}_X$, resp. $\mathbf{Mod}_{X^a}$) consisting of almost coherent $\mathcal{O}_X$-modules (resp. quasi-coherent almost coherent modules, resp. almost coherent almost $\mathcal{O}_X$-modules). As always, it is straightforward to see that the "almostification" functor induces the equivalence

$$\mathbf{Mod}_{X^a}^{\mathrm{acoh}} \simeq \mathbf{Mod}_X^{\mathrm{qc,acoh}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,acoh}}).$$

Moreover, Corollary 4.1.12 ensures that $\mathbf{Mod}_X^{\mathrm{acoh}} \subset \mathbf{Mod}_X$, $\mathbf{Mod}_X^{\mathrm{qc,acoh}} \subset \mathbf{Mod}_X$, and $\mathbf{Mod}_{X^a}^{\mathrm{acoh}} \subset \mathbf{Mod}_{X^a}$ are weak Serre subcategories.

The last thing that we discuss here is the notion of almost coherent schemes.

Definition 4.1.14. We say that an R-scheme X is *almost coherent* if the sheaf $\mathcal{O}_X$ is an almost coherent $\mathcal{O}_X$-module.

Lemma 4.1.15. Let X be a coherent R-scheme. Then X is also almost coherent.

Proof. The structure sheaf $\mathcal{O}_X$ is quasi-coherent by definition. Lemma 4.1.11 says that it suffices to show that $\mathcal{O}_X(U)$ is an almost coherent $\mathcal{O}_X(U)$-module for any open affine $U \subset X$. Since X is coherent, we conclude that $\mathcal{O}_X(U)$ is coherent as an $\mathcal{O}_X(U)$-module. Then Lemma 2.6.7 implies that it is actually almost coherent. □

Lemma 4.1.16. Let X be an almost coherent R-scheme. Then an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is almost coherent if and only if it is of almost finite presentation.

Proof. The "only if" part is easy since any almost coherent $\mathcal{O}_X^a$-module is of almost finite presentation by the definition. The "if" part is a local question, so we can assume that X is affine, then the claim follows from Lemma 2.6.14. □

4.2. **Schemes. Basic Functors on Almost Coherent $\mathcal{O}_X^a$-modules.** This section is devoted to study how certain functors defined in Section 3.2 interact with the notions of almost (quasi-) coherent $\mathcal{O}_X^a$-modules defined in the previous section.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal.

We start with the affine situation, i.e. $X = \operatorname{Spec} A$. In this case, we note that the functor $\widetilde{(-)} \colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ sends almost zero A-modules to almost zero $\mathcal{O}_X$-modules. Thus it induces the functor

$$\widetilde{(-)} \colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}.$$

Lemma 4.2.1. Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the functor $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ induces equivalences $\widetilde{(-)}\colon \mathbf{Mod}_A^* \to \mathbf{Mod}_X^{\mathrm{qc},*}$ for any $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$. The quasi-inverse functor is given by $\Gamma(X, -)$.

Proof. We note that the functor $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ is an equivalence with the quasi-inverse $\Gamma(X, -)$. Now we note Lemma 4.1.8 and Lemma 4.1.11 guarantee that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is almost finite type (resp. almost finitely presented, resp. almost coherent) if $\Gamma(X, \mathcal{F})$ is almost finitely generated (resp. almost finitely presented, resp. almost coherent) as an A-module. ☐

Lemma 4.2.2. Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the functors $\widetilde{(-)}\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ induces an equivalence $\widetilde{(-)}^a\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ and restricts to the equivalences $\widetilde{(-)}^a\colon \mathbf{Mod}_{A^a}^* \to \mathbf{Mod}_{X^a}^*$ for any $* \in \{\mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$. The quasi-inverse functor is given by $\Gamma(X, -)$.

Proof. We note that $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ induces an equivalence between almost zero A-modules and almost zero, quasi-coherent $\mathcal{O}_X$-modules. Thus the claim follows from Lemma 4.2.1, Remark 4.1.3, Remark 4.1.5, Definition 4.1.13 and the analogous presentations of $\mathbf{Mod}_{A^a}^*$ as quotients of $\mathbf{Mod}_{A^a}$ for any $* \in \{\mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$. ☐

Now we show that the pullback functor preserves almost finite type and almost finitely presented $\mathcal{O}_X^a$-modules.

Lemma 4.2.3. Let $f\colon X \to Y$ be a morphism of R-scheme.

(1) Suppose that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine R-schemes. Then $f^*(\widetilde{M^a})$ is functorially isomorphic to $\widetilde{M^a \otimes_{A^a} B^a}$ for any $M^a \in \mathbf{Mod}_A^a$.

(2) The functor f^* preserves almost quasi-coherence (resp. almost finite type, resp. almost finitely presented) for $\mathcal{O}$-modules.

(3) The functor f^* preserves almost quasi-coherence (resp. almost finite type, resp. almost finitely presented) for $\mathcal{O}^a$-modules.

Proof. **(1)** follows from Proposition 3.2.19 and the analogous result for quasi-coherent $\mathcal{O}_Y$-modules. More precisely, Proposition 3.2.19 provides with the functorial isomorphism

$$f^*\left(\widetilde{M^a}\right) \simeq \left(f^*(\widetilde{M})\right)^a \simeq \left(\widetilde{M \otimes_A B}\right)^a \simeq \widetilde{M^a \otimes_{A^a} B^a}\ .$$

(2) and **(3)** are local on X and Y, so we may and do assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine R-schemes. In this case, Lemma 4.2.2 guarantees that any almost quasi-coherent $\mathcal{O}_X^a$-module is isomorphic to $\widetilde{M^a}$ for some A^a-module M^a. Now **(1)** ensures that $f^*(\widetilde{M^a}) \simeq \widetilde{M^a \otimes_{A^a} B^a}$ as almost quasi-coherent $\mathcal{O}_X^a$-modules. The other claims from **(2)** and **(3)** are proven similarly using Lemma 4.2.2 and Lemma 2.8.1. ☐

The next thing we discuss is how the finiteness properties interact with tensor products.

Lemma 4.2.4. Let X be an R-scheme.

(1) Suppose that $X = \operatorname{Spec} A$ is an affine R-scheme. Then $\widetilde{M^a} \otimes_{\mathcal{O}_X^a} \widetilde{N^a}$ is functorially isomorphic to $\widetilde{M^a \otimes_{A^a} N^a}$ for any $M^a, N^a \in \mathbf{Mod}_A^a$.

(2) Let $\mathcal{F}^a, \mathcal{G}^a$ be two almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-modules. Then the $\mathcal{O}_X^a$-module $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$ is almost finite type (resp. almost finitely presented). The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.

(3) Let $\mathcal{F}^a$ be an almost coherent $\mathcal{O}_X^a$-module, and $\mathcal{G}^a$ be an almost finitely presented $\mathcal{O}_X^a$-module. Then $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$ is an almost coherent $\mathcal{O}_X^a$-module. The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.

Proof. The proof is similar to the proof of Lemma 4.2.3. The only difference is that one needs to use Proposition 3.2.12 in place of Proposition 3.2.19 to prove Part **(1)**. Part **(2)** follows from Lemma 2.5.17, and Part **(3)** follows from Corollary 2.6.9. □

We show f_* preserves almost quasi-coherence of $\mathcal{O}^a$-modules for a quasi-compact and quasi-separated morphism f. Later on, we will be able to show that f_* also preserves almost coherence of $\mathcal{O}^a$-modules for certain proper morphisms.

Lemma 4.2.5. Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of R-schemes. Then

(1) The $\mathcal{O}_Y$-module $f_*(\mathcal{F})$ is almost quasi-coherent for any almost quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

(2) The $\mathcal{O}_Y^a$-module $f_*(\mathcal{F}^a)$ is almost quasi-coherent for any almost quasi-coherent $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

Proof. Since $\mathcal{F}$ is almost quasi-coherent, we conclude that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Thus $f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-module by [Sta21, Tag 01LC]. Recall that the projection formula (Lemma 3.3.5) ensures that

$$f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes f_*\mathcal{F} \ .$$

Thus, we see that $\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F} \simeq f_*(\mathcal{F}^a)_!$ is a quasi-coherent $\mathcal{O}_Y$-module. This shows that both $f_*(\mathcal{F})$ and $f_*(\mathcal{F}^a)$ are almost quasi-coherent over $\mathcal{O}_Y$ and $\mathcal{O}_Y^a$ respectively. This finishes the proof of the both parts. □

Finally, we deal with the $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)$ functor. This is probably the most subtle functor considered in this section. We start with the following preparatory lemma:

Lemma 4.2.6. Let X be an R-scheme.

(1) Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. Then the canonical map

$$\widetilde{\operatorname{Hom}_A(M, N)} \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \tag{4.1}$$

is an almost isomorphism of $\mathcal{O}_X$-modules for any almost finitely presented A-module M and any A-module N.

(2) Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. Then there is a functorial isomorphism

$$\widetilde{\operatorname{alHom}_{A^a}(M^a, N^a)} \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \tag{4.2}$$

of $\mathcal{O}_X^a$-modules for any almost finitely presented A^a-module M^a, and any A^a-module N^a. We also get a functorial almost isomorphism

$$\widetilde{\operatorname{Hom}_A(M, N)} \simeq^a \underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \tag{4.3}$$

of $\mathcal{O}_X$-modules for any almost finitely presented A-module M, and any A-module N.

(3) Suppose $\mathcal{F}$ is an almost finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ an almost quasi-coherent $\mathcal{O}_X$-module, then $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an almost quasi-coherent $\mathcal{O}_X$-module.

(4) Suppose $\mathcal{F}^a$ is an almost finitely presented $\mathcal{O}_X^a$-module and $\mathcal{G}^a$ an almost quasi-coherent $\mathcal{O}_X^a$-module, then $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ (resp. $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$) is an almost quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X^a$-module).

Proof. **(1)**: We note that we have a canonical isomorphism $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ for any A-modules M, N. This induces a morphism

$$\widetilde{\mathrm{Hom}_A(M,N)} \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \ .$$

In order to show that it is an almost isomorphism for an almost finitely presented M, it suffices to show that the natural map

$$\mathrm{Hom}_A(M,N) \otimes_A A_f \to \mathrm{Hom}_{A_f}(M \otimes_A A_f, N \otimes_A A_f)$$

is an almost isomorphism for any $f \in A$. This follows from Lemma 2.9.11.

(2) follows easily from **(1)**. Indeed, we apply the functorial isomorphism

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$$

from Proposition 3.2.10**(2)** to the almost isomorphism in Part **(1)** to get the functorial isomorphism

$$\widetilde{\mathrm{Hom}_A(M,N)}^a \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \ .$$

Now we use Proposition 2.2.1**(3)** to get the functorial isomorphism

$$\mathrm{alHom}_{A^a}(M^a, N^a) \simeq \mathrm{Hom}_A(M,N)^a.$$

Applying the functor $\widetilde{(-)}$ to it and composing with the isomorphism above, we get the functorial isomorphism

$$\widetilde{\mathrm{alHom}_{A^a}(M^a, N^a)} \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \ .$$

The construction of the isomorphism (4.3) is similar and even easier.

(3) is a local question, so we can assume that $X = \mathrm{Spec}\, A$. We note that

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq^a \underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \widetilde{\mathfrak{m}} \otimes \mathcal{G})$$

by Proposition 3.2.10**(2)**. Thus, we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent. Then the claim follows from **(1)** and Lemma 4.2.1.

(4) is similarly just a consequence of **(2)** and Lemma 4.2.2. ∎

Corollary 4.2.7. Let X be an R-scheme.

(1) Let $\mathcal{F}$ be an almost finitely presented $\mathcal{O}_X$-module, and let $\mathcal{G}$ be an almost coherent $\mathcal{O}_X$-module. Then $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an almost coherent $\mathcal{O}_X$-module.

(2) Let $\mathcal{F}^a$ be an almost finitely presented $\mathcal{O}_X^a$-module, and let $\mathcal{G}^a$ be an almost coherent $\mathcal{O}_X^a$-module. Then $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ (resp. $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$) is an almost coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X^a$-module).

Proof. We start the proof by observing that $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq^a \underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \widetilde{\mathfrak{m}} \otimes \mathcal{G})$ by Proposition 3.2.10**(2)**. Thus we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are actually quasi-coherent. In that case we use Lemma 4.2.6**(1)** and Lemma 4.1.11 to reduce the question to showing that $\underline{\mathcal{H}om}_A(M,N)$ is almost coherent for any almost finitely presented M and almost coherent N. However, this has already been done in Corollary 2.6.9.

Part **(2)** follows from Part **(1)** as $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})$ and $\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a$. ∎

4.3. **Schemes. Approximation of Almost Finitely Presented $\mathcal{O}_X^a$-modules.** One of the defects of our definition of almost finitely presented $\mathcal{O}_X$-modules is that we get an approximation by finitely presented $\mathcal{O}_X$-modules only (Zariski)-locally on X. So it is not quite well adapted to proving global statements such as the Almost Proper Mapping Theorem. We resolve this issue by show that (on a quasi-compact quasi-separated scheme) any almost finitely presented $\mathcal{O}_X^a$-module can be *globally* approximated by finitely presented $\mathcal{O}_X$-modules.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal.

Lemma 4.3.1. Let X be an R-scheme, and $\{\mathcal{G}_i^a\}_{i\in I}$ a filtered diagram of almost quasi-coherent $\mathcal{O}_X^a$-modules.

(1) The natural morphism
$$\gamma^0_{\mathcal{F}} \colon \operatorname{colim}_I \underline{al\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \underline{al\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a, \operatorname{colim}_I \mathcal{G}_i^a)$$
is injective for an almost finitely generated $\mathcal{O}_X^a$-module $\mathcal{F}^a$;

(2) The natural morphism
$$\gamma^0_{\mathcal{F}} \colon \operatorname{colim}_I \underline{al\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \underline{al\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a, \operatorname{colim}_I \mathcal{G}_i^a)$$
is an almost isomorphism for an almost finitely presented $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

Proof. The statement is local, so we can assume that $X = \operatorname{Spec} A$ is an affine scheme. Then Lemma 4.2.2 implies that $\mathcal{F}^a \simeq M^a$ and $\mathcal{G}_i^a \simeq N_i^a$ for an almost finitely generated (resp. almost finitely presented) A-module M. Then [Sta21, Tag 009F] and Lemma 4.2.6 imply that it suffices to show that
$$\gamma^0_M \colon \operatorname{colim}_i \operatorname{alHom}_{A^a}(M^a, N_i^a) \to \operatorname{alHom}_{A^a}(M^a, \operatorname{colim} N_i^a)$$
is injective (resp. an isomorphism) in $\mathbf{Mod}_R^a$. But this is exactly Corollary 2.5.11. $\square$

Corollary 4.3.2. Let X be an R-scheme, and $\{\mathcal{G}_i^a\}_I$ a filtered diagram of almost quasi-coherent $\mathcal{O}_X^a$-modules.

(1) The natural morphism
$$\gamma^0_{\mathcal{F}} \colon \operatorname{colim}_I \operatorname{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \operatorname{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \operatorname{colim}_I \mathcal{G}_i^a)$$
is injective for an almost finitely generated $\mathcal{O}_X^a$-module $\mathcal{F}^a$;

(2) The natural morphism
$$\gamma^0_{\mathcal{F}} \colon \operatorname{colim}_I \operatorname{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \operatorname{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \operatorname{colim}_I \mathcal{G}_i^a)$$
is an almost isomorphism for an almost finitely presented $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

Proof. It formally follows from Lemma 3.2.25, Lemma 4.3.1, and [Sta21, Tag 009F] (and Corollary 3.1.18). $\square$

Definition 4.3.3. An $\mathcal{O}_X$-module $\mathcal{F}$ is *globally almost finitely generated* (resp. *globally almost finitely presented*) if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is a quasi-coherent finitely generated (resp. finitely presented) $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism $f \colon \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\ker f) = 0$, $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.

Lemma 4.3.4. Let X be a quasi-compact quasi-separated R-scheme, and $\mathcal{F}$ an almost adically quasi-coherent $\mathcal{O}_X$-module.

(1) If, for any filtered diagram of adically quasi-coherent $\mathcal{O}_X$-modules $\{\mathcal{G}_i\}_{i\in I}$, the natural morphism
$$\mathrm{colim}_I \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_i) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{colim}_I \mathcal{G}_i)$$
is almost injective, then $\mathcal{F}$ is globally almost finitely generated.

(2) If, for any filtered system of adically quasi-coherent $\mathcal{O}_X$-modules $\{\mathcal{G}_i\}_{i\in I}$, the natural morphism
$$\mathrm{colim}_I \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_i) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{colim}_I \mathcal{G}_i)$$
is an almost isomorphism, then $\mathcal{F}$ is globally almost finitely presented.

Proof. Lemma 3.2.25 and Corollary 3.1.18 ensure that we can replace $\mathcal{F}$ with $\mathcal{F}_!^a$ without loss of generality. So we may and do assume that $\mathcal{F}$ is are quasi-coherent. Then the proof of Lemma 2.5.10 works essentially verbatim. We repeat it for the reader's convenience.

(1) : Note that $\mathcal{F} \simeq \mathrm{colim}_I \mathcal{F}_i$ is a filtered colimit of its finitely generated $\mathcal{O}_X$*-submodules* (see [Sta21, Tag 01PG]). Therefore, we see that
$$\mathrm{colim}_I \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\mathcal{F}_i) \simeq^a \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{colim}_I(\mathcal{F}/\mathcal{F}_i)) \simeq 0.$$
Consider an element α of the colimit that has a representative the quotient morphism $\mathcal{F} \to \mathcal{F}/\mathcal{F}_i$ (for some choice of i). Then, for every $\varepsilon \in \mathfrak{m}$, $\varepsilon\alpha = 0$ in $\mathrm{colim}_I \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\mathcal{F}_i)$. Explicitly this means that there is $j \geq i$ such that $\varepsilon\mathcal{F} \subset \mathcal{F}_j$. Now note that this property is preserved by choose any $j' > j$. Therefore, for any $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n)$, we can find a finitely generated $\mathcal{O}_X$-submodule $\mathcal{F}_i \subset \mathcal{F}$ such that $\mathfrak{m}_0\mathcal{F} \subset \mathcal{F}_i$. Therefore, $\mathcal{F}$ is almost finitely generated.

(2) : Fix any finitely generated sub-ideal $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n) \subset \mathfrak{m}$. We use [Sta21, Tag 01PJ] to write $\mathcal{F} \simeq \mathrm{colim}_\Lambda \mathcal{F}_\lambda$ as a filtered colimit of *finitely presented* $\mathcal{O}_X$-modules. By assumption, the natural morphism
$$\mathrm{colim}_\Lambda \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_\lambda) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{colim}_\Lambda \mathcal{F}_\lambda) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$
is an almost isomorphism. In particular, $\varepsilon_i \mathrm{Id}_{\mathcal{F}}$ is in the image of this map for every $i = 1, \dots, n$. This means that, for every ε_i, there is $\lambda_i \in \Lambda$ and a morphism $g_i \colon \mathcal{F} \to \mathcal{F}_{\lambda_i}$ such that the composition
$$f_{\lambda_i} \circ g_i = \varepsilon_i \mathrm{Id}_{\mathcal{F}},$$
where $f_{\lambda_i} \colon \mathcal{F}_{\lambda_i} \to \mathcal{F}$ is the morphism to the colimit. Note that existence of such g_i is preserved by replacing λ_i by any $\lambda'_i \geq \lambda_i$. Therefore, using that $\{\mathcal{F}_\lambda\}$ is a filtered diagram, we can find one index λ with maps
$$g_i \colon \mathcal{F} \to \mathcal{F}_\lambda$$
such that $f_\lambda \circ g_i = \varepsilon_i \mathrm{Id}_{\mathcal{F}}$. Now we consider a morphism
$$G_i := g_i \circ f_\lambda - \varepsilon_i \mathrm{Id}_{\mathcal{F}_\lambda} \colon \mathcal{F}_\lambda \to \mathcal{F}_\lambda.$$
Note that $\mathrm{Im}(G_i) \subset \ker(f_\lambda)$ because
$$f_\lambda \circ g_i \circ f_\lambda - f_\lambda \varepsilon_i \mathrm{Id}_{\mathcal{F}_\lambda} = \varepsilon_i f_\lambda - \varepsilon_i f_\lambda = 0.$$
We also have that $\varepsilon_i \ker(f_\lambda) \subset \mathrm{Im}(G_i)$ because $G_i|_{\ker(f_\lambda)} = \varepsilon_i \mathrm{Id}$. Therefore, $\sum_i \mathrm{Im}(G_i)$ is a quasi-coherent finitely generated $\mathcal{O}_X$-module such that
$$\mathfrak{m}_0(\ker f_\lambda) \subset \sum_i \mathrm{Im}(G_i) \subset \ker(f_\lambda).$$
Therefore, $f \colon \mathcal{F}' := \mathcal{F}_\lambda/(\sum_i \mathrm{Im}(G_i)) \to \mathcal{F}$ is a morphism such that $\mathcal{F}'$ is finitely presented, $\mathfrak{m}_0(\ker f) = 0$, and $\mathfrak{m}_0(\mathrm{Coker} f) = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}$ was an arbitrary finitely generated sub-ideal, we conclude that $\mathcal{F}$ is globally almost finitely presented. $\square$

Corollary 4.3.5. Let X be a quasi-compact and quasi-separated R-scheme, and let $\mathcal{F}$ be an almost quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is almost finitely presented (resp. almost finitely generated) if and only if for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there is a morphism $f\colon \mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}$ is a quasi-coherent finitely presented (resp. finitely generated) $\mathcal{O}_X$-module , $\mathfrak{m}_0(\ker f) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.

Proof. Corollary 4.3.2 ensures that $\mathcal{F}$ satisfies the conditions of Lemma 4.3.4. Lemma 4.3.4 now gives the desired result. ☐

Corollary 4.3.6. Let X be a quasi-compact, quasi-separated R-schme, and $\mathcal{F}^a$ an almost quasi-coherent $\mathcal{O}_X^a$-module. Then

(1) $\mathcal{F}^a$ is almost finitely generated if and only if, for every filtered diagram $\{\mathcal{G}_i^a\}_{i\in I}$ of almost quasi-coherent $\mathcal{O}_X^a$-modules, the natural morphism

$$\mathrm{colim}_I\, \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathrm{colim}_I\, \mathcal{G}_i^a)$$

is injective in $\mathbf{Mod}_R^a$;

(2) $\mathcal{F}^a$ is almost finitely presented if and only if, for every filtered diagram $\{\mathcal{G}_i^a\}$ of almost quasi-coherent $\mathcal{O}_X^a$-modules, the natural morphism

$$\mathrm{colim}_I\, \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathrm{colim}_I\, \mathcal{G}_i^a)$$

is an isomorphism in $\mathbf{Mod}_R^a$;

4.4. **Schemes. Derived Category of Almost Coherent $\mathcal{O}_X^a$-modules.** The goal of this section is to define different categories that can be called "derived category of almost coherent sheaves". Namely, we define the categories $\mathbf{D}_{acoh}(X)$, $\mathbf{D}_{qc,acoh}(X)$, and $\mathbf{D}_{acoh}(X)^a$. Then we show that many derived functors of interest preserve almost coherence in an appropriate sense.

Definition 4.4.1. We define $\mathbf{D}_{aqc}(X)$ (resp. $\mathbf{D}_{aqc}(X)^a$) to be the full triangulated subcategory of $\mathbf{D}(X)$ (resp. $\mathbf{D}(X)^a$) consisting of complexes with almost quasi-coherent cohomology sheaves.

Definition 4.4.2. We define $\mathbf{D}_{acoh}(X)$ (resp. $\mathbf{D}_{qc,acoh}(X)$, resp. $\mathbf{D}_{acoh}(X)^a$) to be the full triangulated subcategory of $\mathbf{D}(X)$ (resp. $\mathbf{D}(X)$, resp. $\mathbf{D}(X)^a$) consisting of complexes with almost coherent (resp. quasi-coherent and almost coherent, resp. almost coherent) cohomology sheaves.

Remark 4.4.3. The definition above makes sense as the categories $\mathbf{Mod}_X^{\mathrm{acoh}}$, $\mathbf{Mod}_X^{\mathrm{qc,acoh}}$, and $\mathbf{Mod}_{X^a}^{\mathrm{acoh}}$ are weak Serre subcategories of $\mathbf{Mod}_X$, $\mathbf{Mod}_X$, and $\mathbf{Mod}_X^a$ respectively.

Now suppose that $X = \mathrm{Spec}\, A$ is an affine R-scheme. Then we note that the functor

$$\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X$$

is additive and exact, thus can be easily derived to the functor

$$\widetilde{(-)}\colon \mathbf{D}(A) \to \mathbf{D}_{qc}(X)\ .$$

Lemma 4.4.4. Let $X = \mathrm{Spec}\, A$ be an affine R-scheme. Then the functor

$$\widetilde{(-)}\colon \mathbf{D}(A) \to \mathbf{D}_{qc}(X)$$

is a t-exact equivalence of triangulated categories[21] with quasi-inverse given by $\mathbf{R}\Gamma(X, -)$. Moreover, these two functors induce the equivalence

$$\widetilde{(-)}\colon \mathbf{D}^*_{acoh}(A) \leftrightarrows \mathbf{D}^*_{qc,acoh}(X) \colon \mathbf{R}\Gamma(X, -)$$

[21]with their standard t-structures

for any $* \in \{``\ ", +, -, b\}$.

Proof. The first part is just [Sta21, Tag 06Z0]. In particular, it shows that $\mathrm{H}^i(\mathbf{R}\Gamma(X, \mathcal{F})) \simeq \mathrm{H}^0(X, \mathcal{H}^i(\mathcal{F}))$ for any $\mathcal{F} \in \mathbf{D}_{qc}(X)$. Now Lemma 4.1.11 implies that $\mathcal{H}^i(\mathcal{F})$ is almost coherent if and only if so is $\mathrm{H}^0(X, \mathcal{H}^i(\mathcal{F}))$. So the functor $\mathbf{R}\Gamma(X, -)$ sends $\mathbf{D}^*_{qc,acoh}(X)$ to $\mathbf{D}^*_{acoh}(A)$.

We also observe that the functor $\widetilde{(-)}$ clearly sends $\mathbf{D}_{acoh}(A)$ to $\mathbf{D}_{qc,acoh}(X)$. Thus we conclude that $\widetilde{(-)}$ and $\mathbf{R}\Gamma(X, -)$ induce an equivalence of $\mathbf{D}_{acoh}(A)$ and $\mathbf{D}_{qc,acoh}(X)$. The bounded versions follow from t-exactness of both functors. $\square$

Lemma 4.4.5. Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the almostification functor

$$(-)^a \colon \mathbf{D}^*_{qc}(X) \to \mathbf{D}^*_{aqc}(X)^a$$

induces an equivalence $\mathbf{D}^*_{qc}(X)/\mathbf{D}^*_{qc,\Sigma_X}(X) \xrightarrow{\sim} \mathbf{D}^*_{aqc}(X)^a$ for any $* \in \{``\ ", +, -, b\}$. Similarly, the induced functor

$$\mathbf{D}^*_{qc,acoh}(X)/\mathbf{D}^*_{qc,\Sigma_X}(X) \xrightarrow{\sim} \mathbf{D}^*_{acoh}(X)^a$$

is an equivalence for any $* \in \{``\ ", +, -, b\}$.

Proof. The functor $(-)_! \colon \mathbf{D}^*_{aqc}(X)^a \to \mathbf{D}^*_{qc}(X)$ gives the left adjoint to $(-)^a$ such that $\mathrm{Id} \to (-)_! \circ (-)^a$ is an isomorphism and the kernel of $(-)^a$ consists exactly of the morphisms f such that $\operatorname{cone}(f) \in \mathbf{D}_{qc,\Sigma_X}(X)$. Thus the dual version of [GZ67, Proposition 1.3] finishes the proof of the first claim. The proof of the second claim is similar once one notices that $\widetilde{M^a}$ is almost coherent for any almost coherent A^a-module M^a. The latter fact follows from Lemma 4.1.11. $\square$

Lemma 4.1.11 ensures that $\mathbf{D}(A)^a \simeq \mathbf{D}(A)/\mathbf{D}_{\Sigma_A}(A)$. As $\widetilde{(-)}$ clearly sends $\mathbf{D}_{\Sigma_A}(A)$ into $\mathbf{D}^*_{qc,\Sigma_X}(X)$, we conclude that it induces the functor

$$\widetilde{(-)} \colon \mathbf{D}^*(A)^a \to \mathbf{D}^*_{aqc}(X)^a \ .$$

Theorem 4.4.6. Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the functor

$$\widetilde{(-)} \colon \mathbf{D}(A)^a \to \mathbf{D}_{aqc}(X)^a$$

is a t-exact equivalence of triangulated categories with quasi-inverse given by $\mathbf{R}\Gamma(X, -)$. Moreover, these two functors induce equivalences

$$\widetilde{(-)} \colon \mathbf{D}^*_{acoh}(A)^a \underset{\longrightarrow}{\longleftarrow} \mathbf{D}^*_{acoh}(X)^a \colon \mathbf{R}\Gamma(X, -)$$

for any $* \in \{``\ ", +, -, b\}$.

Proof. We note that Lemma 4.4.4 ensures that $\widetilde{(-)} \colon \mathbf{D}^*_{qc,acoh}(X) \to \mathbf{D}^*_{acoh}(X)^a$ is an equivalence with quasi-inverse $\mathbf{R}\Gamma(X, -)$. Moreover, $(-)^a$ induces an equivalence between $\mathbf{D}_{\Sigma_A}(A)$ and $\mathbf{D}_{qc,\Sigma_X}(X)$; we leave the verification to the interested reader. Thus, Lemma 4.4.5 ensures that $\widetilde{(-)}$ gives an equivalence

$$\mathbf{D}(A)^a \simeq \mathbf{D}(A)/\mathbf{D}_{\Sigma_A}(A) \xrightarrow{\sim} \mathbf{D}_{qc}(X)/\mathbf{D}_{qc,\Sigma_X}(X) \simeq \mathbf{D}_{aqc}(X)^a \ .$$

Its inverse is given by the functor $\mathbf{D}_{aqc}(X)^a \to \mathbf{D}(A)^a$ induced by $\mathbf{R}\Gamma(X, -)$ on $\mathbf{D}_{qc}(X)$ that exactly coincides with $\mathbf{R}\Gamma(X, -) \colon \mathbf{D}_{aqc}(X)^a \to \mathbf{D}(A)^a$ by Proposition 3.5.23.

The version with almost coherent cohomology sheaves is similar to the analogous statement from Lemma 4.4.4. $\square$

Lemma 4.4.7. Let $f \colon X \to Y$ be a morphism of R-schemes.

(1) Suppose that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine R-schemes. Then $\mathbf{L}f^*(\widetilde{M^a})$ is functorially isomorphic to $\widetilde{M^a \otimes^L_{A^a} B^a}$ for any $M^a \in \mathbf{D}(A)^a$.

(2) The functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^*_{aqc}(Y)$ to an object of $\mathbf{D}^*_{aqc}(X)$ for $* \in \{\text{“ ”}, -\}$.

(3) The functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^*_{aqc}(Y)^a$ to an object of $\mathbf{D}^*_{aqc}(X)^a$ for $* \in \{\text{“ ”}, -\}$.

(4) Suppose that X and Y are almost coherent R-schemes. Then the functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^-_{qc,acoh}(Y)$ (resp. $\mathbf{D}^-_{acoh}(Y)$) to an object of $\mathbf{D}^-_{qc,acoh}(X)$ (resp. $\mathbf{D}^-_{acoh}(X)$).

(5) Suppose that X and Y are almost coherent R-schemes. Then the functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^-_{acoh}(Y)^a$ to an object of $\mathbf{D}^-_{acoh}(X)^a$.

Proof. We start with **(1)**. We use Proposition 3.5.20 to see that $\mathbf{L}f^*(\widetilde{M^a}) \simeq (\mathbf{L}f^*(\widetilde{M}))^a$. Thus it suffices to show that $\mathbf{L}f^*(\widetilde{M}) \simeq \widetilde{M \otimes^L_A B}$ as $(\widetilde{M \otimes^L_A B})^a \simeq \widetilde{M^a \otimes^L_{A^a} B^a}$ by Proposition 2.4.16. But the result for quasi-coherent complexes is classical.

Now we show **(2)**. We note that Lemma 3.2.17 implies that $\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F})$ for any $\mathcal{F} \in \mathbf{D}(Y)$. Thus we can replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to assume that it is quasi-coherent. Then it is a standard fact that $\mathbf{L}f^*$ sends $\mathbf{D}^*_{qc}(Y)$ to $\mathbf{D}^*_{qc}(X)$ for $* \in \{\text{“ ”}, -\}$.

(3) follows from **(2)** by noting that $\mathbf{L}f^*(\mathcal{F}^a) \simeq (\mathbf{L}f^*(\mathcal{F}^a_!))^a$ according to Proposition 3.5.20.

To prove **(4)**, we again use the isomorphism $\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F})$ to assume that $\mathcal{F}$ is in $\mathbf{D}^-_{qc,acoh}(X)$. Lemma 4.4.4 guarantees that there is $M \in \mathbf{D}^-_{coh}(A)$ such that $\widetilde{M} \simeq \mathcal{F}$. Thus Part **(1)** and Lemma 4.1.11 ensure that it is sufficient to show that $M^a \otimes^L_{A^a} B^a \simeq (M \otimes^L_A B)^a$ has almost finitely presented cohomology modules. This is exactly the content of Corollary 2.8.2.

(5) follows from **(4)** as $\mathbf{L}f^*(\mathcal{F}^a) \simeq (\mathbf{L}f^*(\mathcal{F}^a_!))^a$. $\square$

Lemma 4.4.8. Let X be an R-scheme.

(1) Suppose that $X = \operatorname{Spec} A$ is an affine R-scheme. Then $\widetilde{M^a} \otimes^L_{\mathcal{O}^a_X} \widetilde{N^a}$ is functorially isomorphic to $\widetilde{M^a \otimes^L_{A^a} N^a}$ for any $M^a, N^a \in \mathbf{D}(A)^a$.

(2) Let $\mathcal{F}, \mathcal{G} \in \mathbf{D}^*_{aqc}(X)$, then $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} \in \mathbf{D}_{aqc}(X)$ for $* \in \{\text{“ ”}, -\}$.

(3) Let $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}^*_{aqc}(X)^a$, then $\mathcal{F}^a \otimes^L_{\mathcal{O}^a_X} \mathcal{G}^a \in \mathbf{D}_{aqc}(X)^a$ for $* \in \{\text{“ ”}, -\}$.

(4) Suppose that X is an almost coherent R-scheme, and let $\mathcal{F}, \mathcal{G} \in \mathbf{D}^-_{qc,acoh}(X)$ (resp. $\mathbf{D}^-_{acoh}(X)$). Then $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} \in \mathbf{D}^-_{qc,acoh}(X)$ (resp. $\mathbf{D}^-_{acoh}(X)$).

(5) Suppose that X is an almost coherent R-scheme, and let $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}^-_{acoh}(X)^a$. Then $\mathcal{F}^a \otimes^L_{\mathcal{O}^a_X} \mathcal{G}^a \in \mathbf{D}^-_{acoh}(X)^a$.

Proof. The proof is basically identical to that of Lemma 4.4.7 and left to the reader. We only mention that one has to use Proposition 2.6.18 in place of Corollary 2.8.2. $\square$

Lemma 4.4.9. Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of R-schemes. Suppose that Y is quasi-compact. Then

(1) The functor $\mathbf{R}f_*$ carries $\mathbf{D}^*_{aqc}(X)$ to $\mathbf{D}^*_{aqc}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$.

(2) The functor $\mathbf{R}f_*$ carries $\mathbf{D}^*_{aqc}(X)^a$ to $\mathbf{D}^*_{aqc}(Y)^a$ for any $* \in \{\text{“ ”}, -, +, b\}$.

Proof. Proposition 3.5.23 guarantees that $(\mathbf{R}f_*\mathcal{F})^a \simeq \mathbf{R}f_*\mathcal{F}^a$. Since $(\widetilde{\mathfrak{m}}\otimes\mathcal{F})^a \simeq \mathcal{F}^a$, we see that it suffices to show that the functor

$$\mathbf{R}f_*(\widetilde{\mathfrak{m}}\otimes -)$$

carries $\mathbf{D}^*_{aqc}(X)$ to $\mathbf{D}^*_{aqc}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$. Since $\widetilde{\mathfrak{m}}\otimes\mathcal{F}$ is in $\mathbf{D}_{qc}(X)$, we conclude that it is enough to show that $\mathbf{R}f_*(-)$ carries $\mathbf{D}^*_{qc}(X)$ to $\mathbf{D}^*_{qc}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$. This is proven in [Sta21, Tag 08D5]. $\square$

Before going to the case of the derived Hom-functors, we recall the construction of the functorial map

$$\psi\colon \widetilde{\mathbf{R}\mathrm{Hom}_A(M,N)} \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

for any $M \in \mathbf{D}^-(A)$, $N \in \mathbf{D}^+(A)$ on an affine scheme $X = \operatorname{Spec} A$. Indeed, the functor $\widetilde{(-)}$ is left adjoint to the global section functor $\Gamma(X,-)$ on the abelian level. Thus after deriving these functors, we get that $\widetilde{-}$ is left adjoint to $\mathbf{R}\Gamma(X,-)$. Therefore, for any $\mathcal{F} \in \mathbf{D}(X)$, there is a canonical morphism $\widetilde{\mathbf{R}\Gamma(X,\mathcal{F})} \to \mathcal{F}$. We apply it to $\mathcal{F} = \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ to get the desired morphism

$$\psi\colon \widetilde{\mathbf{R}\mathrm{Hom}_A(M,N)} \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})\ .$$

Lemma 4.4.10. Let X be an almost coherent R-scheme.

(1) Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. The canonical map

$$\psi\colon \widetilde{\mathbf{R}\mathrm{Hom}_A(M,N)} \to \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

is an almost isomorphism for $M \in \mathbf{D}^-_{acoh}(A)$, $N \in \mathbf{D}^+(A)$.

(2) Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. There is a functorial isomorphism

$$\widetilde{\mathbf{R}\mathrm{alHom}_{A^a}(M^a,N^a)} \simeq \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a},\widetilde{N^a})$$

for $M^a \in \mathbf{D}^-_{acoh}(A)^a$, $N^a \in \mathbf{D}^+(A)^a$. We also get a functorial almost isomorphism

$$\widetilde{\mathbf{R}\mathrm{Hom}_{A^a}(M^a,N^a)} \simeq^a \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\widetilde{M^a},\widetilde{N^a})$$

for $M \in \mathbf{D}^-_{acoh}(A)$, $N \in \mathbf{D}^+(A)$.

(3) Suppose $\mathcal{F} \in \mathbf{D}^-_{acoh}(X)$ and $\mathcal{G} \in \mathbf{D}^+_{aqc}(X)$. Then $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \in \mathbf{D}^+_{aqc}(X)$.

(4) Suppose $\mathcal{F}^a \in \mathbf{D}^-_{acoh}(X)^a$ and $\mathcal{G}^a \in \mathbf{D}^+_{aqc}(X)^a$. Then $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a) \in \mathbf{D}^+_{aqc}(X)$ and $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a) \in \mathbf{D}^+_{aqc}(X)^a$.

Proof. We start with **(1)**. We use the convergent compatible spectral sequences

$$\mathrm{E}_2^{p,q} = \widetilde{\mathrm{Ext}^p_A(\mathrm{H}^{-q}(M),N)} \Rightarrow \widetilde{\mathrm{Ext}^{p+q}_A(M,N)}$$

$$\mathrm{E'}_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_X}\left(\widetilde{\mathrm{H}^{-q}(M)},\widetilde{N}\right) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}\left(\widetilde{M},\widetilde{N}\right)$$

to see that we may assume that $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$ is just a module in degree 0. Similarly, we use the compatible spectral sequences

$$\mathrm{E}_2^{p,q} = \widetilde{\mathrm{Ext}^q_A(M,\mathrm{H}^p(N))} \Rightarrow \widetilde{\mathrm{Ext}^{p+q}_A(M,N)}$$

$$\mathrm{E'}_2^{p,q} = \mathcal{E}xt^q_{\mathcal{O}_X}(\widetilde{M},\widetilde{\mathrm{H}^p(N)}) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

to also assume that $N \in \mathbf{Mod}_A$. So the claim boils down to showing that the natural maps

$$\widetilde{\mathrm{Ext}^p_A(M,N)} \to \mathcal{E}xt^p_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

are almost isomorphisms for any $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$, $N \in \mathbf{Mod}_A$, and $p \geq 0$. Lemma 3.1.5 says that it is sufficient to say that kernel and cokernel are annihilated by any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$.

Recall that, for any $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, the sheaf $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is canonically isomorphic to sheafification of the presheaf

$$U \mapsto \mathrm{Ext}^p_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \ .$$

Thus, in order to show that the map $\widetilde{\mathrm{Ext}^p_A(M,N)} \to \mathcal{E}xt^p_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is an almost isomorphism, it suffices to show that

$$\mathrm{Ext}^p_A(M,N) \otimes_A A_f \to \mathrm{Ext}^p_{\mathcal{O}_{X_f}}(\widetilde{M_f}, \widetilde{N_f})$$

is an almost isomorphism. Now we use canonical isomorphisms

$$\begin{aligned}\mathrm{Ext}^p_{\mathcal{O}_{X_f}}(\widetilde{M_f}, \widetilde{N_f}) &\simeq \mathrm{Hom}_{\mathbf{D}(X_f)}(\widetilde{M_f}, \widetilde{N_f}[p]) \\ &\simeq \mathrm{Hom}_{\mathbf{D}(A_f)}(M_f, N_f[p]) \\ &\simeq \mathrm{Ext}^p_{A_f}(M_f, N_f),\end{aligned}$$

where the second isomorphism uses that $(\widetilde{-})$ induces a t-exact equivalence $(\widetilde{-})\colon \mathbf{D}(A_f) \to \mathbf{D}_{qc}(\operatorname{Spec} A_f)$. Thus, the question boils down to showing that the natural map

$$\mathrm{Ext}^p_A(M,N) \otimes_A A_f \to \mathrm{Ext}^p_{A_f}(M_f, N_f)$$

is an almost isomorphism. This follows from Proposition 2.9.12.

(2) formally follows from **(1)** by using Proposition 3.5.8**(1)**.

(3) is also a basic consequence of **(2)**. Indeed, the claim is local, so we can assume that $X = \operatorname{Spec} A$ is an affine R-scheme. In that case we use Theorem 4.4.6 to say that $\mathcal{F} \simeq \widetilde{M}$, $\mathcal{G} \simeq \widetilde{N}$ for some $M \in \mathbf{D}^-_{acoh}(A)$, $N \in \mathbf{D}^+(A)$. Then $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \widetilde{\mathbf{R}\mathrm{Hom}_A(M,N)}$ by **(2)**, and the latter complex has almost quasi-coherent cohomology sheaves by design.

(4) easily follows from **(3)** and the isomorphisms

$$\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})$$

$$\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})^a$$

that come from Lemma **(1)** and Definition 3.5.6. $\square$

Corollary 4.4.11. Let X be an almost coherent R-scheme.

(1) Let $\mathcal{F} \in \mathbf{D}^-_{aqc,acoh}(X)$, $\mathcal{G} \in \mathbf{D}^+_{aqc,acoh}(X)$. Then $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^+_{aqc,acoh}(X)$.

(2) Let $\mathcal{F}^a \in \mathbf{D}^-_{acoh}(X)^a$, $\mathcal{G}^a \in \mathbf{D}^+_{acoh}(X)^a$. Then $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \in \mathbf{D}^+_{acoh}(X)^a$.

Proof. The question is local on X, so we can suppose that $X = \operatorname{Spec} A$ is affine. Then Lemma 4.4.10, Theorem 4.4.6, and Lemma 4.1.11 reduce both question to showing that $\mathbf{R}\mathrm{Hom}_A(M,N) \in \mathbf{D}^+_{acoh}(A)$ for $M \in \mathbf{D}^-_{acoh}(A)$ and $N \in \mathbf{D}^+_{acoh}(A)$. This is the content of Proposition 2.6.19. $\square$

Proposition 4.4.12. Let $f\colon X \to Y$ be a quasi-compact quasi-separated morphism of R-schemes, $\mathcal{F}^a \in \mathbf{D}_{aqc}(X)^a$, and $\mathcal{G} \in \mathbf{D}_{aqc}(Y)^a$. Then the projection morphism (see the discussion before Proposition 3.5.27)

$$\rho\colon \mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a \to \mathbf{R}f_*(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a))$$

is an isomorphism in $\mathbf{D}(Y)^a$.

Proof. Proposition 3.5.14, Proposition 3.5.20, and Proposition 3.5.23 imply that we can replace $\mathcal{F}^a$ (resp. $\mathcal{G}^a$) with $\mathcal{F}^a_! \in \mathbf{D}_{qc}(X)^a$ (resp. $\mathcal{G}^a_! \in \mathbf{D}_{qc}(Y)^a$). So it suffices to show the analogous result for modules with *quasi-coherent* cohomology sheaves. This is proven in [Sta21, Tag 08EU]. ☐

4.5. **Formal Schemes. The Category of Almost Coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.** In this Section we discuss the notion of almost coherent sheaves on "good" formal schemes. One of the main complications is that there is no good notion of a "quasi-coherent" sheaf on a formal scheme. Namely, even though there is a notion of adically quasi-coherent sheaves on a big class of formal schemes due to [FK18, §I.3], this notion does not really behave well. For example, the category of adically quasi-coherent sheaves usually is not abelian. One of the main difficulties in working with adically quasi-coherent sheaves is the lack of the Artin-Rees lemma beyond the case of finitely generated modules. More precisely, many operations with adically quasi-coherent sheaves require taking completions, but it is difficult to control the effect of it without the use of the Artin-Rees lemma.

The way we deal with this problem is to use a version of the Artin-Rees lemma (Lemma 2.12.6) for almost finitely generated modules over "good" rings. The presence of the Artin-Rees lemma suggests that it is reasonable to expect that we might have a good notion of adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules on some "good" class of formal schemes.

We start by giving the Setup in which we can develop the theory of almost coherent sheaves

Set-up 4.5.1. We fix a ring R with a finitely generated ideal I such that R is I-adically complete, I-adically topologically universally adhesive[22], and I-torsion free with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat.

The basic example of such a ring is a complete microbial[23] valuation ring R with algebraically closed fraction field K. We pick a pseudo-uniformizer ϖ and define $I \coloneqq (\varpi)$, $\mathfrak{m} \coloneqq \cup_{i=1}^{\infty}(\varpi^{1/n})$ for some compatible choice of roots of ϖ. We note that R is topologically universally adhesive by [FGK11, Theorem 7.3.2].

We note that the assumptions in Setup 4.5.1 imply that any finitely presented algebra over a topologically finitely presented R-algebra is coherent and I-adically adhesive. Coherence follows from [FGK11, Proposition 7.2.2] and adhesiveness basically follows from the definition. In what follows, we will use those facts without saying.

In what follows, $\mathfrak{X}$ will always mean a topologically finitely presented formal R-scheme. We will denote by $\mathfrak{X}_k \coloneqq \mathfrak{X} \times_{\mathrm{Spf}\ R} \operatorname{Spec} R/I^{k+1}$ the "reduction" schemes. They come together with the closed immersion $i_k\colon \mathfrak{X}_k \to \mathfrak{X}$. Also, given any $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$, we will always denote its "reduction" $i_k^*\mathcal{F}$ by $\mathcal{F}_k$.

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[22]This means that the algebra $R\langle X_1, \dots, X_n\rangle[T_1, \dots, T_m]$ is I-adically adhesive for any n and m

[23]A valuation ring R is microbial if there is a non-zero topologically nilpotent element $\varpi \in R$. Any such element is called a pseudo-uniformizer.

Definition 4.5.2. [FK18, Definition I.3.1.3] An $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ on a formal scheme $\mathfrak{X}$ of finite ideal type is called *adically quasi-coherent* if $\mathcal{F} \to \lim_n \mathcal{F}_n$ is an isomorphism and, for any open formal subscheme $\mathfrak{U} \subset \mathcal{F}$ and any ideal of definition $\mathcal{I}$ of finite type, the sheaf $\mathcal{F}/\mathcal{I}\mathcal{F}$ is a quasi-coherent sheaf on the scheme $(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}}/\mathcal{I})$.

We denote by $\mathbf{Mod}_{\mathfrak{X}}^{qc}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}}$ consisting of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Definition 4.5.3. We say that an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *almost adically quasi-coherent* if $\mathcal{F}_!^a \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module. We denote by $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aqc}}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}^a}$ consisting of almost adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *almost adically quasi-coherent* if $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. We denote by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aqc}}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}}$ consisting of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Remark 4.5.4. In general, we can not say that an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is almost adically quasi-coherent. The problem is that the sheaf $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ might not be complete, i.e. the map $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \lim_k \widetilde{\mathfrak{m}} \otimes \mathcal{F}_k$ is a priori only an almost isomorphism.

Lemma 4.5.5. Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\mathcal{F}^a$ be an almost adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. Then $\mathcal{F}_k^a$ is almost quasi-coherent for all k. Moreover, if an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{G}^a$ is annihilated by some I^{n+1}. Then $\mathcal{G}^a$ is almost adically quasi-coherent if and only if so is $\mathcal{G}_n^a$.

Proof. In order to prove the first claim, it is sufficient to show that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}_k$ is quasi-coherent provided that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. We use Corollary 3.2.18 to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}_k \simeq (\widetilde{\mathfrak{m}} \otimes \mathcal{F})_k$ and the reduction of an adically quasi-coherent module is quasi-coherent. Therefore, each $\mathcal{F}_k^a$ is almost adically quasi-coherent.

Now if $\mathcal{G}$ is annihilated by I^{n+1} then $\mathcal{G} = i_{n,*}\mathcal{G}_n$. We use the Projection Formula (Corollary 3.3.6) to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{G} \simeq i_{n,*}(\mathcal{G}_n \otimes \widetilde{\mathfrak{m}})$. Clearly, $i_{n,*}$ sends quasi-coherent sheaves to adically quasi-coherent sheaves. So $\mathcal{G}^a$ is almost adically quasi-coherent if so is $\mathcal{G}_n^a$. ☐

Definition 4.5.6. We say that an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *of almost finite type* (resp. *almost finitely presented*) if $\mathcal{F}^a$ is almost adically quasi-coherent, and there is a covering of $\mathfrak{X}$ by open affines $\{\mathfrak{U}_i\}_{i\in I}$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U}_i)$-module. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aft}}$ and $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{afp}}$ respectively.

We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *of almost finite type* (resp. *almost finitely presented*) if so is $\mathcal{F}^a$. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aft}}$ and $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{afp}}$ respectively.

Definition 4.5.7. We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *adically quasi-coherent of almost finite type* (resp. *adically quasi-coherent almost finitely presented*) if it is adically quasi-coherent and there is a covering of $\mathfrak{X}$ by open affines $\{\mathfrak{U}_i\}_{i\in I}$ such that $\mathcal{F}(\mathfrak{U}_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_i)$-module. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,aft}}$ and $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,afp}}$ respectively.

Remark 4.5.8. If $\mathcal{F}^a$ is a finite type (resp. finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module, then $(\mathcal{F}^a)_!$ is adically quasi-coherent of almost finite type (resp. almost finite presentation).

Remark 4.5.9. We note that, a priori, it is not clear if $\mathcal{F}^a$ is an almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module for an adically quasi-coherent almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. The problem is that our definition of adically quasi-coherent almost finite type (resp. almost finitely presented) module does not require $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to be adically quasi-coherent. However, we will show in Lemma 4.5.10 that it is automatic in this case.

Lemma 4.5.10. Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\mathcal{F}$ be an adically quasi-coherent of almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. In particular, $\mathcal{F}$ is almost of finite type (resp. almost finitely presented).

Proof. Corollary 2.5.12 and Lemma 3.3.2 imply that the only condition we really need to check is that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. Therefore, it suffices to prove the result for an adically quasi-coherent almost finite type $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$.

Since the question is local on $\mathfrak{X}$, we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine and $M := \mathcal{F}(X)$ is almost finitely generated over A. Then we use [FK18, Theorem I.3.2.8] to say that $\mathcal{F}$ is isomorphic to M^{Δ}. We claim that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is isomorphic to $(\widetilde{\mathfrak{m}} \otimes_A M)^{\Delta}$ as that would imply that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent by [FK18, Proposition I.3.2.2]. In order to show that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is isomorphic to $(\widetilde{\mathfrak{m}} \otimes_R M)^{\Delta}$ we need to check two things: for any open affine $\operatorname{Spf} B = \mathfrak{U} \subset \mathfrak{X}$ the B-module $(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is I-adically complete, and then the natural map $(\widetilde{\mathfrak{m}} \otimes_R M)\widehat{\otimes}_A B \to (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is an isomorphism.

We start with the first claim. Lemma 3.3.2 says that $(\widetilde{\mathfrak{m}}\otimes\mathcal{F})(\mathfrak{U})$ is isomorphic to $\widetilde{\mathfrak{m}}\otimes_R\mathcal{F}(\mathfrak{U})$. Since $\mathcal{F}$ is adically quasi-coherent, $\mathcal{F}(\mathfrak{U}) \simeq M\widehat{\otimes}_A B$, so $(\widetilde{\mathfrak{m}}\otimes\mathcal{F})(\mathfrak{U}) \simeq \widetilde{\mathfrak{m}}\otimes_R(M\widehat{\otimes}_A B)$. Lemma 2.8.1 says that $M\otimes_A B$ is almost finitely generated over B, so it is already I-adically complete by Lemma 2.12.7. Therefore, we see that $\widetilde{\mathfrak{m}}\otimes_R\mathcal{F}(\mathfrak{U}) \simeq \widetilde{\mathfrak{m}}\otimes_R (M\otimes_A B)$, and the latter is almost finitely generated over B by Corollary 2.5.12. Thus we use Lemma 2.12.7 once more to show its completeness.

Now we show that the natural morphism $(\widetilde{\mathfrak{m}}\otimes_R M)\widehat{\otimes}_A B \to (\widetilde{\mathfrak{m}}\otimes\mathcal{F})(\mathfrak{U})$ is an isomorphism. Again, using the same results as above we can get rid of any completions and identify this map with the "identity" map

$$(\widetilde{\mathfrak{m}} \otimes_R M)\otimes_A B \to \widetilde{\mathfrak{m}} \otimes_R (M\otimes_A B)$$

This finishes the proof. ∎

Lemma 4.5.11. Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module. Then the $\mathcal{O}_{\mathfrak{X}_k}^a$-module $\mathcal{F}_k^a$ is almost finite type (resp. almost finitely presented) for any integer k.

Proof. Lemma 4.5.5 implies that each $\mathcal{F}_k^a$ is an almost quasi-coherent $\mathcal{O}_{X_k}$-module. So it is sufficient to find a covering of $\mathfrak{X}_k$ by open affines $\mathfrak{U}_{i,k}$ such that $\mathcal{F}_k^a(\mathfrak{U}_{i,k})$ is almost finitely generated (resp. almost finitely presented) over $\mathcal{O}_{\mathfrak{X}_k}^a(\mathfrak{U}_{i,k})$. We note that underlying topological spaces of $\mathfrak{X}$ and $\mathfrak{X}_k$ are the same, so we can choose some covering of $\mathfrak{X}$ by open affines $\mathfrak{U}_i$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ are almost finitely generated (resp. almost finitely presented) over $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U}_i)$, and define $\mathfrak{U}_{i,k}$ as the "reductions" of $\mathfrak{U}_i$. Then using the vanishing result for higher cohomology groups of adically quas-coherent sheaves on affine formal schemes of finite type [FK18, Theorem I.7.1.1] and Lemma 3.3.2, we deduce that

$$\mathcal{F}_k^a(\mathfrak{U}_{i,k}) \simeq (\widetilde{\mathfrak{m}} \otimes \mathcal{F}_k^a)(\mathfrak{U}_{i,k})^a \simeq \left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}\left(\mathfrak{U}_i\right)/I^{k+1}\right)^a$$

is almost finitely generated (resp. almost finitely presented) over $\mathcal{O}_{\mathfrak{X}_k}(\mathfrak{U}_{i,k})$. ∎

Lemma 4.5.12. Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module. Then $\mathcal{F}^a(\mathfrak{U})$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U})$-module for any open affine $\mathfrak{U} \subset \mathfrak{X}$.

Proof. Corollary 2.5.12 and Lemma 3.3.2 guarantee that we can replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ for the purpose of the proof. Thus we may and do assume that $\mathcal{F}$ is an adically quasi-coherent almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then using Lemma 2.8.1 and Lemma 2.12.7 we can use the argument as in the proof of Lemma 4.5.10 show that the restriction of $\mathcal{F}$ to any open formal subscheme is still adically quasi-coherent of almost finite type (resp. finitely presented), so we may and do assume that $\mathfrak{X} = \operatorname{Spf} A$ is an affine formal R-scheme. Since now $\mathfrak{X}$ is quasi-compact, we can choose a finite refinement of the covering $\mathfrak{X} = \cup \mathfrak{U}_i$ such that $\mathcal{F}(\mathfrak{U}_i)$ is almost finitely generated (resp. almost finitely presented) over $\mathcal{O}(\mathfrak{U}_i)$. Thus we may and do assume that a covering $(\mathfrak{U}_i)$ is finite.

Now we have an affine topologically finitely presented formal R-scheme $\mathfrak{X} = \operatorname{Spf} A$, a finite covering of $\mathfrak{X}$ by affines $\mathfrak{U}_i = \operatorname{Spf} A_i$, and an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ such that $\mathcal{F}(\mathfrak{U}_i)$ is almost finitely generated (resp. almost finitely presented) A_i-module. We want to show that $\mathcal{F}(\mathfrak{X})$ is an almost finitely generated (resp. almost finitely presented) A-module.

We firstly deal with the *almost finitely generated case.* We note that Lemma 4.1.7, Lemma 4.5.11, and [FK18, Theorem I.7.1.1] imply that $\mathcal{F}(\mathfrak{X})/I$ is almost finitely generated. We know that $\mathcal{F}$ is adically quasi-coherent so $\mathcal{F}(\mathfrak{X})$ must be an I-adically complete A-module. Therefore, Corollary 2.5.20 guarantees that $\mathcal{F}(X)$ is an almost finitely generated A-module.

Now we move to the *almost finitely presented case.* We already now that $\mathcal{F}(\mathfrak{X})$ is almost finitely generated over A. Thus the standard argument with Lemma 2.12.7 implies that $\mathcal{F}(\mathfrak{U}_i) = \mathcal{F}(\mathfrak{X}) \otimes_A A_i$ for any i. Recall that [FK18, Proposition I.4.8.1] implies[24] that each $A \to A_i$ is flat. Since $\operatorname{Spf} A_i$ form a covering of $\operatorname{Spf} A$, we conclude that $A \to \prod_{i=1}^n A_i$ is faithfully flat. Now the result follows from faithfully flat descent for almost finitely presented modules that is proven in Lemma 2.10.5. □

Corollary 4.5.13. Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in the Setup 4.5.1, and let $\mathcal{F}^a$ be an almost adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. Then $\mathcal{F}^a$ is almost finite type (resp. almost finitely presented) if and only if $\mathcal{F}^a(\mathfrak{X})$ is almost finitely generated (resp. almost finitely presented) A^a-module.

Similarly, an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is is almost finite type (resp. almost finitely presented) if and only if $\mathcal{F}(\mathfrak{X})$ is almost finitely generated (resp. almost finitely presented) A-module.

Lemma 4.5.14. Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in the Setup 4.5.1, let $\varphi \colon N \to M$ be a homomorphism of almost finitely generated A-modules. Then the following sequence

$$0 \to (\ker \phi)^\Delta \to N^\Delta \xrightarrow{\varphi^\Delta} M^\Delta \to (\operatorname{Coker} \phi)^\Delta \to 0$$

is exact. Moreover, $\operatorname{Im}(\varphi)^\Delta \simeq \operatorname{Im}(\varphi^\Delta)$.

Proof. We denote the kernel $\ker \phi$ by K, the image $\operatorname{Im}(\varphi)$ by M', and the cokernel $\operatorname{Coker} \phi$ by Q.

We start with $\ker \varphi^\Delta$: We note that $(\ker \varphi^\Delta)(\mathfrak{X}) = K$, this induces a natural morphism $\alpha \colon K^\Delta \to \ker \varphi^\varphi$. We show that it is an isomorphism, it suffices to check that it induces an isomorphism on values over a basis of principal open subsets. Now recall that for any A-module L, we have an equality $L^\Delta(\operatorname{Spf} A_{\{f\}}) = \widehat{L_f}$ where the completion is taken with respect to the I-adic topology. Thus in order to check that α is an isomorphism it suffices to show that $\widehat{K_f}$ is naturally identified with $(\ker \varphi)(\operatorname{Spf} A_{\{f\}}) = \ker(\widehat{N_f} \to \widehat{M_f})$. Using the Artin-Rees Lemma 2.12.6 over the adhesive

[24] Topologically universally adhesive rings are by definition "t. u. rigid-Noetherian"

ring A_f, we conclude that the induced topologies on K_f and M'_f coincide with the I-adic ones. This implies that

$$\widehat{K_f} = \lim K_f/I^n K_f = \lim K_f/(I^n N_f \cap K_f) \text{ and } \widehat{M'_f} = \lim M'_f/I^n M'_f = \lim M'_f/(I^n M_f \cap M'_f)$$

This guarantees that we have two exact sequences:

$$0 \to \widehat{K_f} \to \widehat{M_f} \to \widehat{M'_f} \to 0,$$

$$0 \to \widehat{M'_f} \to \widehat{N_f}$$

In particular, we get that the natural map $\widehat{K_f} \to \ker(\widehat{M_f} \to \widehat{N_f})$ is an isomorphism. That shows that $K^\Delta \simeq \ker(\varphi^\Delta)$.

We prove the claim for $\operatorname{Im} \varphi^\Delta$: We note that since the category of $\mathcal{O}_\mathfrak{X}$-modules is abelian, we can identify $\operatorname{Im} \varphi^\Delta \simeq \operatorname{Coker}(K^\Delta \to N^\Delta)$. We observe that [FK18, Theorem I.7.1.1] and the established fact above that $\ker \varphi$ is adically quasi-coherent imply that the natural map $\mathcal{F}(\mathfrak{U})/K^\Delta(\mathfrak{U}) \to (\operatorname{Im} \varphi^\Delta)(\mathfrak{U})$ is an isomorphism for any affine open formal subscheme $\mathfrak{U}$. In particular, we have $(\operatorname{Im} \varphi^\Delta)(\mathfrak{X}) = M/K = M'$. Therefore, we have a natural map $(M')^\Delta \to \operatorname{Im} \varphi^\Delta$ and we show that it is an isomorphism. Again it suffices to show that this map is an isomorphism on values over a basis of principal open subsets. Then we use the identification $\mathcal{F}(\mathfrak{U})/K^\Delta(\mathfrak{U}) \simeq (\operatorname{Im} \varphi)(\mathfrak{U})$ and the short exact sequence

$$0 \to \widehat{K_f} \to \widehat{M_f} \to \widehat{M'_f} \to 0,$$

to finish the proof.

We show the claim for $\operatorname{Coker} \varphi^\Delta$: The argument is identical to the argument for $\operatorname{Im} \varphi$ once we proved that $\operatorname{Im} \varphi = \ker(\mathcal{G} \to \operatorname{Coker} \varphi)$ is adically quasi-coherent. □

Corollary 4.5.15. *Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in the Setup 4.5.1, let M an almost finitely generated A-module, and let N be any A-submodule of M. Then the following sequence*

$$0 \to N^\Delta \xrightarrow{\varphi^\Delta} M^\Delta \to (M/N)^\Delta \to 0$$

is exact.

Proof. We just apply Lemma 4.5.14 to the homomorphism $M \to M/N$ of almost finitely generated A-modules. □

Corollary 4.5.16. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of adically quasi-coherent, almost finite type $\mathcal{O}_\mathfrak{X}$-modules. Then $\ker \varphi$ is an adically quasi-coherent $\mathcal{O}_\mathfrak{X}$-module, $\operatorname{Coker} \varphi$ and $\operatorname{Im} \varphi$ are adically quasi-coherent $\mathcal{O}_\mathfrak{X}$-modules of almost finite type.*

Corollary 4.5.17. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1, and let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a morphism of almost almost finite type $\mathcal{O}_\mathfrak{X}^a$-modules. Then $\ker \varphi$ is an almost adically quasi-coherent $\mathcal{O}_\mathfrak{X}^a$-module, $\operatorname{Coker} \varphi$ and $\operatorname{Im} \varphi$ are $\mathcal{O}_\mathfrak{X}^a$-modules of almost finite type.*

Proof. We apply the exact functor $(-)_!$ to the map φ and reduce the claim to Corollary 4.5.16. □

Now we are ready to show that almost finite type and almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules share many good properties we would expect. The only subtle thing is that we do not know whether an adically quasi-coherent quotient of an adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-module is of almost finite type. The main extra complication here is that we need to be very careful with the adically quasi-coherent condition in the definition of almost finite type (resp. almost finitely presented) modules since that condition does not behave well in general.

Lemma 4.5.18. Let $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, then

- **(1)** If $\mathcal{F}$ is adically quasi-coherent of almost finite type, and $\mathcal{F}'$ is adically quasi-coherent then $\mathcal{F}''$ is adically quasi-coherent of almost finite type.
- **(2)** If $\mathcal{F}'$ and $\mathcal{F}''$ are adically quasi-coherent of almost finite type (resp. almost finitely presented) then so is $\mathcal{F}$.
- **(3)** If $\mathcal{F}$ is adically quasi-coherent of almost finite type and $\mathcal{F}''$ is adically quasi-coherent almost finitely presented then $\mathcal{F}'$ is adically quasi-coherent of almost finite type.
- **(4)** If $\mathcal{F}$ is adically quasi-coherent of almost finitely presented and $\mathcal{F}'$ is adically quasi-coherent of almost finite type then $\mathcal{F}''$ is adically quasi-coherent, almost finitely presented.

Proof. (**1**): Without loss of generality, we can assume that $\mathfrak{X} = \mathrm{Spf}\ A$ is an affine formal scheme. Then $\mathcal{F} \cong M^\Delta$ for some almost finitely generated A-module M, and $\mathcal{F}' \cong N^\Delta$ for some A-submodule $N \subset M$. Then Corollary 4.5.15 ensures that $\mathcal{F}'' \simeq (M/N)^\Delta$. In particular, it is adically quasi-coherent. Then the claim is an easy consequence of the vanishing theorem [FK18, Theorem I.7.1.1] and Lemma 2.5.15**(1)**.

(**2**): The difficult part is to show that $\mathcal{F}$ is adically quasi-coherent. In fact once we know that $\mathcal{F}$ is adically quasi-coherent, it is automatically of almost finite type (resp. almost finitely presented) by [FK18, Theorem I.7.1.1] and Lemma 2.5.15**(2)**.

In order to show that $\mathcal{F}$ is adically quasi-coherent, we may and do assume that $\mathfrak{X} = \mathrm{Spf}\ A$ is an affine formal R-scheme for some adhesive ring A. Then let us introduce A-modules $M' \coloneqq \mathcal{F}'(\mathfrak{X})$, $M \coloneqq \mathcal{F}(\mathfrak{X})$, and $M'' \coloneqq \mathcal{F}''(\mathfrak{X})$. We have the natural morphism $M^\Delta \to \mathcal{F}$ and we show that it is an isomorphism. The vanishing theorem [FK18, Theorem I.7.1.1] implies that we have a short exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

Thus M is almost finitely generated (resp. almost finitely presented) by Lemma 2.5.15**(2)**. Then Corollary 4.5.14 gives that we have a short exact sequence

$$0 \to M'^\Delta \to M^\Delta \to M''^\Delta \to 0$$

Using the vanishing theorem [FK18, Theorem I.7.1.1] once again we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M'^\Delta \arrow[r] \arrow[d] & M^\Delta \arrow[r] \arrow[d] & M''^\Delta \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact, and left and right vertical arrows are isomorphisms. That implies that the map $M^\Delta \to \mathcal{F}$ is an isomorphism.

(**3**): This easily follows from Lemma 2.5.15**(3)**, Lemma 4.5.16 and [FK18, Theorem I.7.1.1].

(**4**): This also easily follows from Lemma 2.5.15**(4)**, Lemma 4.5.16 and [FK18, Theorem I.7.1.1]. ☐

We also give the almost version of this Lemma:

Corollary 4.5.19. Let $0 \to \mathcal{F}'^a \xrightarrow{\varphi} \mathcal{F}^a \xrightarrow{\psi} \mathcal{F}''^a \to 0$ be an exact sequence of $\mathcal{O}_{\mathfrak{X}}^a$-modules, then

(1) If $\mathcal{F}^a$ is of almost finite type, and $\mathcal{F}'^a$ is almost adically quasi-coherent then $\mathcal{F}''^a$ is of almost finite type.

(2) If $\mathcal{F}'^a$ and $\mathcal{F}''^a$ are of almost finite type (resp. almost finitely presented) then so is $\mathcal{F}^a$.

(3) If $\mathcal{F}^a$ is of almost finite type and $\mathcal{F}''^a$ is almost finitely presented then $\mathcal{F}'^a$ is of almost finite type.

(4) If $\mathcal{F}^a$ is of almost finitely presented and $\mathcal{F}'^a$ is of almost finite type then $\mathcal{F}''^a$ is almost finitely presented.

Definition 4.5.20. We say that an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *almost coherent* if $\mathcal{F}^a$ is almost finite type and for any open set $\mathfrak{U}$ any finite type $\mathcal{O}_{\mathfrak{X}}^a$-submodule $\mathcal{G}^a \subset (\mathcal{F}|_{\mathfrak{U}})^a$ is an almost finitely presented $\mathcal{O}_{\mathfrak{U}}$-module.

We say that $\mathcal{F}$ is *(adically quasi-coherent) almost coherent* $\mathcal{O}_{\mathfrak{X}}$-module if $\mathcal{F}^a$ almost coherent (and $\mathcal{F}$ is adically quasi-coherent).

Remark 4.5.21. Lemma 4.5.10 ensures that any adically quasi-coherent almost coherent $\mathcal{O}_X$-module $\mathcal{F}$ is almost coherent.

Lemma 4.5.22. Let $\mathcal{F}^a$ be an $\mathcal{O}_{\mathfrak{X}}^a$-module on a topologically finitely presented formal R-scheme $\mathfrak{X}$. Then the following are equivalent:

(1) $\mathcal{F}^a$ is almost coherent.

(2) $\mathcal{F}^a$ is almost quasi-coherent and the $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U})$-module $\mathcal{F}^a(\mathfrak{U})$ is almost coherent for any open affine formal subscheme $\mathfrak{U} \subset \mathfrak{X}$.

(3) $\mathcal{F}^a$ is almost quasi-coherent and there is a covering of $\mathfrak{X}$ by open affine subschemes $(\mathfrak{U}_i)_{i\in I}$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ is almost coherent for each i.

In particular, an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is almost coherent if and only if it almost finitely presented.

Proof. The proof that these three notions are equivalent is identical to the proof of Lemma 4.5.22 modulo facts that we have already established in this chapter, especially Corollary 4.5.14.

As for the last claim, we recall that $\mathfrak{X}$ is topologically finitely presented over a topologically universally adhesive ring, so $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is coherent for any open affine $\mathfrak{U}$ [FK18, Prop. 0.8.5.23, Lemma I.1.7.4, Prop. I.2.3.3]. Then Lemma 2.6.13 and Lemma 2.6.15 prove the equivalence. ☐

Even though Lemma 4.5.22 says that the notion of almost coherence coincides with the notion of almost finite presentation, it shows that almost coherence is morally "the correct" definition. In what follows, we prefer to use the terminology of almost coherent sheaves as it is shorter and gives a better intuition from our point of view.

Lemma 4.5.23. **(1)** Any almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-submodule of an almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-module is almost coherent.

(2) Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism from an almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-module to an almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. Then $\ker\varphi$ is an almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-module.

(3) Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism of almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules. Then $\ker\varphi$ and $\operatorname{Coker}\varphi$ are almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

(4) Given a short exact sequence of $\mathcal{O}_{\mathfrak{X}}^a$-modules

$$0 \to \mathcal{F}'^a \to \mathcal{F}^a \to \mathcal{F}''^a \to 0,$$

if two out of three are almost coherent so is the third one.

Remark 4.5.24. There is also an evident version of this corollary for adically quasi-coherent almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Proof. The proof is identical to Corollary 4.1.12 once we have Lemma 4.5.16 and equivalence of almost coherent and almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules from Lemma 4.5.22. □

Corollary 4.5.25. Let $\mathfrak{X}$ be topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then the category $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,acoh}}$, $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{acoh}}$) of almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules (resp. adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules, resp. almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules) is a Weak Serre subcategory of $\mathbf{Mod}_{\mathfrak{X}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}}$, resp. $\mathbf{Mod}_{\mathfrak{X}}^a$).

4.6. **Formal Schemes. Basic Functors on Almost Coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.** This section is devoted to study how certain functors defined in Section 3.2 interact with the notions of almost (quasi-)coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules. The exposition follows Section 4.2 very closely.

We start with the affine situation, i.e. $\mathfrak{X} = \operatorname{Spf} A$. In this case, we note that the functor $(-)^\Delta\colon \mathbf{Mod}_A \to \mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc}}$ sends almost zero A-modules to almost zero $\mathcal{O}_{\mathfrak{X}}$-modules. Thus, it induces a functor

$$(-)^\Delta\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{\mathfrak{X}^a}.$$

Lemma 4.6.1. Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine formal R-scheme for R as in the Setup 4.5.1. Then the functor $(-)^\Delta\colon \mathbf{Mod}_A \to \mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc}}$ induces an equivalence $(-)^\Delta\colon \mathbf{Mod}_A^* \to \mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc},*}$ for any $* \in \{\mathrm{aft}, \mathrm{acoh}\}$. The quasi-inverse functor is given by $\Gamma(\mathfrak{X}, -)$.

Proof. We note that the functor $(-)^\Delta\colon \mathbf{Mod}_A \to \mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc}}$ induces an equivalence between the category of I-adically complete A-modules and adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules by [FK18, Theorem I.3.2.8]. Recall that all almost finite type modules are complete by Lemma 2.12.7. Thus it suffices to show that an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module is almost finitely generated (resp. almost coherent) if and only if so is $\Gamma(X, \mathcal{F})$. Now this follows from Lemma 4.5.13 and Lemma 4.5.22. □

Lemma 4.6.2. Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine formal R-scheme for R as in the Setup 4.5.1. Then the functor $(-)^\Delta\colon \mathbf{Mod}_A \to \mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc}}$ induces equivalences $(-)^\Delta\colon \mathbf{Mod}_{A^a}^* \to \mathbf{Mod}_{\mathfrak{X}^a}^*$ for any $* \in \{\mathrm{aft}, \mathrm{acoh}\}$. The quasi-inverse functor is given by $\Gamma(X, -)$.

Proof. The proof is analogous to Lemma 4.2.2 once Lemma 4.6.1 is verified. □

Now recall that for any R-scheme X, we can define the I-adic completion of X as a colimit $\operatorname{colim}(X_k, \mathcal{O}_{X_k})$ of the reductions $X_k := X \times_R \operatorname{Spec} R/I^{k+1}$ in the category of formal schemes. We refer to [FK18, §1.4(c)] for more details. This completion comes with a map of locally ringed spaces

$$c\colon \widehat{X} \to X\ .$$

One important example of a completion is $\widehat{\operatorname{Spec} A} = \operatorname{Spf} \widehat{A}$ for any R-algebra A[25]. We study the properties of the completion map in the case of a finitely presented R-scheme or an affine scheme $\operatorname{Spec} A$ for a topologically finitely presented R-algebra A.

[25]We note that $\widehat{A}$ is I-adically complete by [Sta21, Tag 05GG] since I is finitely generated.

Lemma 4.6.3. Let $X = \operatorname{Spec} A$ be an affine R-scheme for R as in the Setup 4.5.1. Suppose that A is either finitely presented or topologically finitely presented over R. Then the morphism $c\colon \widehat{X} \to X$ is flat and there is a functorial isomorphism $M^{\Delta} \cong c^*(\widetilde{M})$ for any almost finitely generated A-module M.

Proof. The flatness assertion is just [FK18, Proposition I.1.4.7 (2)]. The natural map

$$M \to \mathrm{H}^0(\mathfrak{X}, c^*(\widetilde{M}))$$

induces the map $M^{\Delta} \to c^*(\widetilde{M})$. In order to show that it is an isomorphism, it is enough to show that the map

$$\widehat{M_f} \to M_f \otimes_{A_f} \widehat{A_f}$$

is an isomorphism for any $f \in A$. This follows from Lemma 2.12.7, as each such A_f is I-adically adhesive. $\square$

Corollary 4.6.4. Let X be a locally finitely presented R-scheme for R as in the Setup 4.5.1. Then the morphism $c\colon \widehat{X} \to X$ is flat and c^* sends almost finite type $\mathcal{O}_X^a$-modules (resp. almost coherent $\mathcal{O}_X^a$-modules) to almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-modules (resp. almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules).

Similarly, c^* sends quasi-coherent almost finite type $\mathcal{O}_X$-modules (resp. quasi-coherent almost coherent $\mathcal{O}_X$-modules) to adically quasi-coherent almost finite type $\mathcal{O}_{\mathfrak{X}}$-modules (resp. adically quasi-coherent almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules)

Proof. The statement is local, so we can assume that $X = \operatorname{Spec} A$. Then the claim follows from Lemma 4.6.3. $\square$

Now we show that the pullback functor preserves almost finite type and almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.

Lemma 4.6.5. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally finitely presented formal R-scheme for R as in the Setup 4.5.1.

(1) Suppose that $\mathfrak{X} = \operatorname{Spf} B$, $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then $\mathfrak{f}^*(M^{\Delta})$ is functorially isomorphic to $(M \otimes_A B)^{\Delta}$ for any $M \in \mathbf{Mod}_A^{\mathrm{aft}}$.

(2) Suppose that $\mathfrak{X} = \operatorname{Spf} B$, $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then $\mathfrak{f}^*(M^{a,\Delta})$ is functorially isomorphic to $(M^a \otimes_{A^a} B^a)^{\Delta}$ for any $M^a \in \mathbf{Mod}_A^{a,\mathrm{aft}}$.

(3) The functor $\mathfrak{f}^*$ sends $\mathbf{Mod}_{\mathfrak{Y}}^{\mathrm{qc,aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{Y}}^{\mathrm{qc,acoh}}$) to $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,acoh}}$).

(4) The functor $\mathfrak{f}^*$ sends $\mathbf{Mod}_{\mathfrak{Y}^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{Y}^a}^{\mathrm{acoh}}$) to $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{acoh}}$).

Proof. We prove **(1)**, the proofs of other parts follow from it similarly to the proof Lemma 4.2.3.

We consider a commutative diagram

$$\begin{tikzcd} \operatorname{Spf} B \arrow[r, "c_B"] \arrow[d, "\mathfrak{f}"] & \operatorname{Spec} B \arrow[d, "f"] \\ \operatorname{Spf} A \arrow[r, "c_A"] & \operatorname{Spec} A \end{tikzcd}$$

where the map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the map induced by $\mathfrak{f}^{\#}\colon A \to B$. Then we have that $M^{\Delta} \simeq c_A^*\widetilde{M}$ by Lemma 4.6.3. Therefore,

$$\mathfrak{f}^*(M^{\Delta}) \simeq c_B^*(f^*\widetilde{M}) \simeq c_B^*(\widetilde{M \otimes_A B}) \simeq (M \otimes_A B)^{\Delta}$$

where the last isomorphism follows from Lemma 4.6.3 again. $\square$

The next thing we discuss is how the finiteness properties interact with tensor products.

Lemma 4.6.6. Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1.

(1) Suppose that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then $M^\Delta \otimes_{\mathcal{O}_\mathfrak{X}} N^\Delta$ is functorially isomorphic to $(M \otimes_A N)^\Delta$ for any $M, N \in \mathbf{Mod}_A^{\mathrm{aft}}$.

(2) Suppose that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then $M^{a,\Delta} \otimes_{\mathcal{O}_\mathfrak{X}^a} N^{a,\Delta}$ is functorially isomorphic to $(M^a \otimes_{A^a} N^a)^\Delta$ for any $M^a, N^a \in \mathbf{Mod}_{A^a}^{\mathrm{aft}}$.

(3) Let $\mathcal{F}, \mathcal{G}$ be two adically quasi-coherent almost finite type (resp. almost finitely presented) $\mathcal{O}_\mathfrak{X}$-modules. Then the $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{G}$ is adically quasi-coherent of almost finite type (resp. almost finitely presented).

(4) Let $\mathcal{F}^a, \mathcal{G}^a$ be two almost finite type (resp. almost coherent) $\mathcal{O}_\mathfrak{X}^a$-modules. Then the $\mathcal{O}_\mathfrak{X}^a$-module $\mathcal{F}^a \otimes_{\mathcal{O}_\mathfrak{X}^a} \mathcal{G}^a$ is of almost finite type (resp. almost coherent). The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.

Proof. Again, we only show **(1)** as the other parts follow from this similarly to the proof of Lemma 4.2.4 with the simplification that almost coherent and almost finitely presented modules coincide by our assumption on $\mathfrak{X}$ and R.

The proof of **(1)** is, in turn, similar to that of Lemma 4.6.5 **(1)**. We consider the completion morphism $c\colon \mathrm{Spf}\ A \to \mathrm{Spec}\, A$. Then we have a sequence of isomorphisms

$$M^\Delta \otimes_{\mathcal{O}_\mathfrak{X}} N^\Delta \simeq c^*(\widetilde{M}) \otimes_{\mathcal{O}_\mathfrak{X}} c^*(\widetilde{N}) \simeq c^*(\widetilde{M} \otimes_{\mathcal{O}_{\mathrm{Spec}\, A}} \widetilde{N}) \simeq c^*(\widetilde{M \otimes_A N}) \simeq (M \otimes_A N)^\Delta.$$

□

Finally, we deal with the functor $\underline{\mathcal{H}om}_{\mathcal{O}_X^a}(-,-)$. This is probably the most subtle functor considered in this section. We start with the following preparatory lemma:

Lemma 4.6.7. Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1.

(1) Suppose $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then the canonical map

$$\mathrm{Hom}_A(M,N)^\Delta \to \underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(M^\Delta, N^\Delta) \tag{4.4}$$

is an almost isomorphism for any almost coherent A-modules M and N.

(2) Suppose $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then there is a functorial isomorphism

$$\mathrm{alHom}_{A^a}(M^a, N^a)^\Delta \simeq \underline{al\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}^a}(M^{a,\Delta}, N^{a,\Delta}) \tag{4.5}$$

for any almost coherent A^a-module M^a and N^a. We also get a functorial almost isomorphism

$$\mathrm{Hom}_{A^a}(M^a, N^a)^\Delta \simeq^a \underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}^a}(M^{a,\Delta}, N^{a,\Delta}) \tag{4.6}$$

for any almost coherent A^a-module M^a and N^a.

(3) Suppose $\mathcal{F}$ and $\mathcal{G}$ are almost coherent $\mathcal{O}_\mathfrak{X}$-modules. Then $\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(\mathcal{F}, \mathcal{G})$ is an almost coherent $\mathcal{O}_\mathfrak{X}$-module.

(4) Suppose $\mathcal{F}^a$ and $\mathcal{G}^a$ are almost coherent $\mathcal{O}_\mathfrak{X}^a$-modules. Then

$$\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}^a}(\mathcal{F}^a, \mathcal{G}^a) \text{ (resp. } \underline{al\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}^a}(\mathcal{F}^a, \mathcal{G}^a))$$

is an almost coherent $\mathcal{O}_\mathfrak{X}$-module (resp. $\mathcal{O}_X^a$-module).

Proof. Again, the proof is absolutely analogous to Lemma 4.2.6 and Corollary 4.2.7 once **(1)** is proven. So we only give a proof of **(1)** here.

We note that both M and N are I-adically complete by Lemma 2.12.7. Now we use [FK18] to say that the natural map $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)$ is an isomorphism. This induces a morphism

$$\mathrm{Hom}_A(M,N)^\Delta \to \underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)\ .$$

In order to prove that it is an almost isomorphism, it suffices to show that the natural map

$$\mathrm{Hom}_A(M,N)\widehat{\otimes}_A A_{\{f\}} \to \mathrm{Hom}_{A_{\{f\}}}(M\widehat{\otimes}_A A_{\{f\}}, N\widehat{\otimes}_A A_{\{f\}})$$

is an almost isomorphism for any $f \in A$. Now we note that $\mathrm{Hom}_A(M,N)$ is almost coherent by Corollary 2.6.9. Thus, $\mathrm{Hom}_A(M,N)\otimes_A A_{\{f\}}$ is already complete, so the completed tensor product coincides with the usual one. Similarly, $M\widehat{\otimes}_A A_{\{f\}} \simeq M\otimes_A A_{\{f\}}$ and $N\widehat{\otimes}_A A_{\{f\}} \simeq N\otimes_A A_{\{f\}}$. Therefore, the question boils down to showing that the natural map

$$\mathrm{Hom}_A(M,N)\otimes_A A_{\{f\}} \to \mathrm{Hom}_{A_{\{f\}}}(M\otimes_A A_{\{f\}}, N\otimes_A A_{\{f\}})$$

is an almost isomorphism. This, in turn, follows from Lemma 2.9.11. ☐

4.7. **Formal Schemes. Approximation of Almost Coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.** The main goal of this section is to establish an analogue of Corollary 4.3.5 in the context of formal schemes. More precisely, we show that, for any "nice" formal scheme $\mathfrak{X}$, an almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ can be "approximated" by a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}_{\mathfrak{m}_0}$ up to $\mathfrak{m}_0 \subset \mathfrak{m}$ torsion. It turns out that this result is more subtle than its algebraic counterpart because, in general, we do not know if we can present an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module as a filtered colimit of finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules. Also colimits are much more subtle in the formal set-up due to the presence of topology. This seems unlikely that the method used in the proof Corollary 4.3.5 can be used in the formal set-up. Instead, we take another route and, instead, we first approximate $\mathcal{F}$ up to bounded torsion and then reduce to the algebraic case.

For the rest of the section, we fix a ring R as in the Set-up 4.5.1, and $\mathfrak{X}$ a topologically finitely presented formal R-scheme.

Definition 4.7.1. A map of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi\colon \mathcal{G} \to \mathcal{F}$ is an *FP-approximation* if $\mathcal{G}$ is a finitely presented $\mathcal{O}_{\mathfrak{X}}$-module, and $I^n(\mathrm{Ker}\,\phi)=0$, $I^n(\mathrm{Coker}\,\phi)=0$ for some $n>0$.

If $\mathfrak{m}_0 \subset \mathfrak{m}$ is a finitely generated sub-ideal of $\mathfrak{m}_0$, a map of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi\colon \mathcal{G} \to \mathcal{F}$ is an *FP-$\mathfrak{m}_0$-approximation* if it is an FP-approximation and $\mathfrak{m}_0(\mathrm{Coker}\,\phi)=0$.

Lemma 4.7.2. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme, and $\mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of almost finite type. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, $\mathcal{F}$ admits an FP-$\mathfrak{m}_0$-approximation.

Proof. Lemma 4.6.2 guarantees that $\mathcal{F} = M^\Delta$ for some almost finitely generated A-module M. Then, by definition, there is a submodule $N \subset M$ such that $\mathfrak{m}_0(M/N)$. By assumption, $U \coloneqq \mathrm{Spec}\,A \setminus \mathrm{V}(I)$ is noetherian, so $\widetilde{N}|_U$ is a finitely presented $\mathcal{O}_U$-module. Then [FK18, Lemma 0.8.1.6(2)] guarantees that there is a finitely presented A-module N' with a surjective map $N' \to N$ such that its kernel K is I^∞-torsion. In particular, $K \subset N'[I^\infty]$. But since A is I-adically complete and noetherian outside I, [FGK11, Theorem 5.1.2 and Definition 4.3.1] guarantees that $N'[I^\infty] = N'[I^n]$ for some $n \geq 0$. In particular, K is an I^n-torsion module.

Therefore, we have an exact sequence

$$0 \to K \to N' \to M \to Q \to 0,$$

where N' is finitely presented, M is almost finitely generated, $\mathfrak{m}_0 Q = 0$, and $I^n K = 0$ for some $n \geq 1$. Now Lemma 4.5.14 says that the following sequence is exact:

$$0 \to K^{\Delta} \to N'^{\Delta} \to M^{\Delta} \to Q^{\Delta} \to 0.$$

In particular, N'^{Δ} is a finitely presented $\mathcal{O}_{\mathfrak{X}}$-module, $\mathfrak{m}_0(Q^{\Delta}) = 0$, and $I^n(K^{\Delta})$. □

Lemma 4.7.3. [FK18, Exercise I.3.4] Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of finite type, and $\mathcal{G} \subset \mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodule. Then $\mathcal{G}$ is a filtered colimit $\mathcal{G} = \mathrm{colim}_{\lambda\in\Lambda}\, \mathcal{G}_\lambda$ of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodules of finite type such that, for all $\lambda \in \Lambda$, $\mathcal{G}/\mathcal{G}_\lambda$ is annihilated by I^n for some fixed $n > 0$.

Lemma 4.7.4. Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent, almost finitely generated $\mathcal{O}_{\mathfrak{X}}$-module, and $\phi_i \colon \mathcal{G}_i \to \mathcal{F}$ for $i = 1, 2$ two FP-$\mathfrak{m}_0$-approximations of $\mathcal{F}$ for some finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Then there is a commutative diagram

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, "q_1"'] \arrow[dr, "\phi_1"] & \\ \mathcal{H} \arrow[r, "\phi"] & \mathcal{F} \\ \mathcal{G}_2 \arrow[u, "q_2"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

where ϕ and q_i are FP-$\mathfrak{m}_0$-approximations for $i = 1, 2$.

Proof. By the assumption, there is an integer $c > 0$ such that $\ker(\phi_i)$ and $\mathrm{Coker}(\phi_i)$ are annihilated by I^c for $i = 0, 1$. Therefore, we may replace $\mathfrak{m}_0$ by $\mathfrak{m}_0 + I^c$ to assume that $\mathfrak{m}_0$ contains I^c.

Now we define $\mathcal{K}$ to be the kernel of the natural morphism $\mathcal{G}_1 \oplus \mathcal{G}_2 \to \mathcal{F}$. Note that it is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodule by Lemma 4.5.14. Therefore, Lemma 4.7.3 applies to the inclusion $\mathcal{K} \subset \mathcal{G}_1 \oplus \mathcal{G}_2$, so we can write $\mathcal{K} = \mathrm{colim}_{\lambda\in\Lambda}\, \mathcal{K}_\lambda$ as a filtered colimit of adically quasi-coherent, finite type $\mathcal{O}_{\mathfrak{X}}$-submodules of $\mathcal{G}_1 \oplus \mathcal{G}_2$ with $I^m(\mathcal{K}/\mathcal{K}_\lambda) = 0$ for some fixed $m > 0$ and every $\lambda \in \Lambda$. We define $\mathcal{H}_\lambda = (\mathcal{G}_1 \oplus \mathcal{G}_2)/\mathcal{K}_\lambda$, it comes with the natural morphisms

$$\phi_\lambda \colon \mathcal{H}_\lambda \to \mathcal{F},$$

$$q_{i,\lambda} \colon \mathcal{G}_i \to \mathcal{H}_\lambda$$

for $i = 1, 2$. We claim that these morphisms satisfy the claim of lemma for some $\lambda \in \Lambda$, i.e. ϕ_λ, and $q_{i,\lambda}$ are FP-$\mathfrak{m}_0$-approximations.

Since $\mathfrak{X}$ is topologically finitely presented (in particular, it is quasi-compact and quasi-separated), these claims can be checked locally. So we may and do assume that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then we use Lemma 4.6.2, [FK18, Theorem I.3.2.8, Proposition I.3.5.4] to reduce to the situation $\mathfrak{X} = \mathrm{Spf}\ A$, $\mathcal{F} = M^{\Delta}$, $\mathcal{G}_1 = N_1^{\Delta}$, $\mathcal{G}_2 = N_2^{\Delta}$ for some almost finitely generated A-module M, and finitely presented A-modules N_1, N_2 with maps of sheaves induced by homomorphisms $N_1 \to M$ and $N_2 \to M$. Then Lemma 4.5.14 guarantees that $\mathcal{K} = K^{\Delta}$ for $K = \ker(N_1 \oplus N_2 \to M)$, and $K = \mathrm{colim}_{\lambda\in\Lambda}\, K_\lambda$ for finitely generated A-submodules[26] K_λ with $I^m(K/K_\lambda) = 0$ for some fixed $m > 0$ and all $\lambda \in \Lambda$. So one can use Lemma 4.5.14 once again to conclude that it suffices (due to the assumption that $I^c \subset \mathfrak{m}_0$) to show that, for some $\lambda \in \Lambda$, the natural morphisms $(N_1 \oplus N_2)/K_\lambda \to M$, $N_i \to (N_1 \oplus N_2)/K_\lambda$ have kernels annihilated by some power of I, and cokernels annihilated by $\mathfrak{m}_0$.

[26]Here, $K_\lambda = \Gamma(\mathfrak{X}, \mathcal{K}_\lambda)$, so the equality follows from [Sta21, Tag 009F].

The kernels of $N_i \to (N_1 \oplus N_2)/K_\lambda$ (for $i = 1, 2$) embed into the respective kernels for the morphisms $N_1 \to M$, so they are automatically annihilated by some power of I for any $\lambda \in \Lambda$. Also, clearly, the morphism $(N_1 \oplus N_2)/K_\lambda \to M$ has kernel K/K_λ that is annihilated by I^m by the choice of K_λ.

Therefore, it suffices to show that we can choose $\lambda \in \Lambda$ such that $q_{i,\lambda}\colon N_i \to (N_1 \oplus N_2)/K_\lambda$ (for $i = 1, 2$) and $\phi_\lambda\colon (N_1 \oplus N_2)/K_\lambda \to M$ have cokernels annihilated by $\mathfrak{m}_0$. The latter case is automatic and actually holds for any $\lambda \in \Lambda$. So the only non-trivial thing we need to check is that $\mathfrak{m}_0(\operatorname{Coker} q_{i,\lambda}) = 0$ for some $\lambda \in \Lambda$.

Let $(m_1, \dots, m_d) \in \mathfrak{m}_0$ be a finite set of generators, and $\{y_{i,j}\}_{j \in J_i}$ a finite set of generators of N_i for $i = 1, 2$. Denote by $\overline{y_{i,j}}$ the image of $y_{i,j}$ in M. Define $x_{i,j,k} \in N_{2-i}$ to be a lift of $m_k\overline{y_{i,j}} \in M$ in N_{2-i} for $k = 1, \dots, d$, $i = 1, 2$ and $j \in J_i$. Note that elements $(m_k y_{1,j}, x_{1,j,k}) \in N_1 \oplus N_2$ and $(x_{2,j,k}, m_k y_{2,j}) \in N_1 \oplus N_2$ lie in K. So for some $\lambda \in \Lambda$, K_λ contains the elements $(m_k y_{1,j}, x_{1,j,k})$ and $(x_{2,j,k}, m_k y_{2,j})$. Then it is easy to see that the cokernels of $N_i \to (N_1 \oplus N_2)/K_\lambda$ are annihilated by $\mathfrak{m}_0$. This finishes the proof. □

Lemma 4.7.5. Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent, almost finitely type $\mathcal{O}_\mathfrak{X}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated.

Proof. Firstly, we note that Lemma 4.7.2 guarantees that the claim holds if $\mathfrak{X}$ is affine. Now choose a covering of $\mathfrak{X}$ by open affines $\mathfrak{X} = \bigcup_{i=1}^n \mathfrak{V}_i$, we know that claim on each $\mathfrak{V}_i$. So it suffices to show that, if $\mathfrak{X} = \mathfrak{U}_1 \cup \mathfrak{U}_2$ is union of two finitely presented open formal subschemes and $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated on both $\mathfrak{U}_1$ and $\mathfrak{U}_2$, then $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated on $\mathfrak{X}$.

Suppose that $\mathcal{G}_i \to \mathcal{F}|_{\mathfrak{U}_i}$ are FP-$\mathfrak{m}_0$-approximations on $\mathfrak{U}_i$ for $i = 1, 2$. Then the intersection $\mathfrak{U}_{1,2} := \mathfrak{U}_1 \cap \mathfrak{U}_2$ is again topologically finitely presented formal R-schene because $\mathfrak{X}$ is so (in particular, it is assumed to be quasi-compact and quasi-separated). Therefore, Lemma 4.7.4 guarantees that we can find another FP-$\mathfrak{m}_0$-approximation $\mathcal{H} \to \mathcal{F}|_{\mathfrak{U}_{1,2}}$ that is dominated by both $\mathcal{G}_i|_{\mathfrak{U}_{1,2}} \to \mathcal{F}|_{\mathfrak{U}_{1,2}}$ for $i = 1, 2$. Consider the $\mathcal{O}_{\mathfrak{U}_{1,2}}$-modules

$$\mathcal{K}_i := \ker(\mathcal{G}_i|_{\mathfrak{U}_{1,2}} \to \mathcal{H}) \text{ for } i = 1, 2.$$

Lemma 4.5.14 guarantees that both $\mathcal{K}_i$ are adically quasi-coherent $\mathcal{O}_\mathfrak{X}$-modules of finite type[27]. The fact that $\mathcal{G}_i|_{\mathfrak{U}_{1,2}} \to \mathcal{H}$ are FP-$\mathfrak{m}_0$-approximations ensures that both $\mathcal{K}_i$ are killed by some I^m for $m \geq 1$. In particular, we see that $\mathcal{K}_i \subset \mathcal{G}_i[I^m]|_{\mathfrak{U}_{1,2}}$, so they are naturally quasi-coherent sheaves on $\mathfrak{X}_{m-1} = \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R/I^m$. Therefore, one can use [Sta21, Tag 01PF] (applied to $\mathfrak{X}_{m-1}$) to extend $\mathcal{K}_i$ to

$$\widetilde{\mathcal{K}}_i \subset \mathcal{G}_i[I^m] \subset \mathcal{G}_i$$

where $\widetilde{\mathcal{K}}_i$ adically quasi-coherent $\mathcal{O}_\mathfrak{X}$-modules of finite type. Then $\mathcal{G}_i/\widetilde{\mathcal{K}}_i \to \mathcal{F}|_{\mathfrak{U}_i}$ are FP-$\mathfrak{m}_0$-approximations of $\mathcal{F}|_{\mathfrak{U}_i}$ that are isomorphic on the intersection. Therefore, they glue to a global FP-$\mathfrak{m}_0$-approximation $\mathcal{G} \to \mathcal{F}$. □

Theorem 4.7.6. Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_\mathfrak{X}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is an adically quasi-coherent, finitely generated (resp. finitely presented) $\mathcal{O}_\mathfrak{X}$-module $\mathcal{G}$ and a map $\phi\colon \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\operatorname{Coker} \phi) = 0$ and $\mathfrak{m}_0(\ker \phi) = 0$.

Proof. Without loss of generality, we can replace $\mathcal{F}$ by $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$, so we may and do assume that $\mathcal{F}$ is adically quasi-coherent.

[27] Since they are kernels of morphisms between coherent $\mathcal{O}_\mathfrak{X}$-modules

The case of almost adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ follows from Lemma 4.7.5. Indeed, there is an FP-$\mathfrak{m}_0$-approximation $\phi'\colon \mathcal{G}' \to \mathcal{F}$, so we define $\phi\colon \mathcal{G} \to \mathcal{F}$ to be the natural inclusion $\mathcal{G} := \mathrm{Im}(\phi') \to \mathcal{F}$. This gives the desired morphism as $\mathcal{G}$ is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of almost finite type by Corollary 4.5.16.

Now suppose $\mathcal{F}$ is an adically quasi-coherent, almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-module. Then we use Lemma 4.7.5 to find an FP-$\mathfrak{m}_0$-approximation $\phi'\colon \mathcal{G}' \to \mathcal{F}$. Now we note that any almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-module is almost coherent by Lemma 4.5.22. Therefore, $\ker \phi$ is again adically quasi-coherent, almost finitely presented. Therefore, we can find an FP-$\mathfrak{m}_0$-approximation $\phi''\colon \mathcal{G}'' \to \ker(\phi')$ by Lemma 4.7.5. Denote by $\phi'''\colon \mathcal{G}'' \to G'$ the composition of ϕ'' with the natural inclusion $\ker(\phi') \to \mathcal{G}'$. Now it is easy to check that $\phi\colon \mathrm{Coker}(\phi''') \to \mathcal{F}$ gives the desired "approximation". □

4.8. **Formal Schemes. Derived Category of Almost Coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.** We discuss the notion of the derived category of almost coherent sheaves on a formal scheme $\mathfrak{X}$. One major issue is that there the derived category of $\mathcal{O}_{\mathfrak{X}}$-modules with adically quasi-coherent cohomology sheaves is not well-defined, as adically quasi-coherent sheaves is not a Weak Serre subcategory of $\mathbf{Mod}_{\mathfrak{X}}$, so it is not even an abelian category. However, it would be useful for certain technical reasons to be able to work with that category.

In order to overcome this issue, we follow the strategy used in [Lur18] and define "$\mathbf{D}_{qc}(\mathfrak{X})$" completely on the derived level without really defining a good abelian notion of (adically) quasi-coherent sheaves. For the rest of the section, we fix a base ring R as in the Setup 4.5.1.

Definition 4.8.1. Let $\mathfrak{X}$ be a locally topologically finitely presented R-scheme. Then we define the *derived category of adically quasi-coherent sheave*s "$\mathbf{D}_{qc}(\mathfrak{X})$" as a full subcategory of $\mathbf{D}(\mathfrak{X})$ with elements $\mathcal{F}$ such that

- For every open affine $\mathfrak{U} \subset \mathfrak{X}$, $\mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F}) \in \mathbf{D}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}))$ is derived I-adically complete.
- For every inclusion $\mathfrak{U} \subset \mathfrak{V}$ of affine formal subschemes of $\mathfrak{X}$, the natural morphism
$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F})$$
is an isomorphism, where the completion is understood in the derived sense.

Remark 4.8.2. We refer to [Sta21, Tag 091N] and [Sta21, Tag 0995] for a self-contained discussion of the derived completion of modules and sheaves of modules respectively.

We want to give an interpretation of "$\mathbf{D}_{qc}(\mathfrak{X})$" in terms of A-modules for an affine formal scheme $\mathfrak{X} = \mathrm{Spf}\ A$. We recall that in the case of schemes, we have a natural equivalence $\mathbf{D}_{qc}(\mathrm{Spec}\, A) \simeq \mathbf{D}(A)$ and the map is induced by $\mathbf{R}\Gamma(\mathrm{Spec}\, A, -)$. In the case of formal schemes, it is not literally true. We need to impose certain completeness conditions.

Definition 4.8.3. Let A be a ring with a finitely generated ideal I. We define the $\mathbf{D}_{comp}(A, I) \subset \mathbf{D}(A)$ as a full triangulated subcategory consisting of I-adically derived complete objects.

Suppose now that $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine scheme, topologically finitely presented over R. We note that the natural functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(A)$ induces a functor
$$\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \text{“}\mathbf{D}_{qc}(\mathfrak{X})\text{”} \to \mathbf{D}_{comp}(A, I)\ .$$
The main claim is that this functor is an equivalence. This is the main content of [Lur18, Corollary 8.2.4.15]. We need to prove one technical result to ensure that our definitions are consistent with the definitions in Lurie's book.

Lemma 4.8.4. Let A be a topologically finitely presented R-algebra for R as in the Setup 4.5.1, let $f \in A$ be any element, and let $(x_1, \dots, x_d) = I$ be a choice of generators for the ideal of definition of R. Denote by $K(A_f; x_1^n, \dots, x_d^n)$ the Koszul complexes for the sequence $(x_1^n, \dots, x_d^n)$. Then the pro-systems $\{K(A_f; x_1^n, \dots, x_d^n)\}$ and $\{A_f/I^n\}$ are isomorphic in $\mathrm{Pro}(\mathbf{D}(A_f))$.

Proof. The proof is the same [Sta21, Tag 0921]. The only difference that one needs to use [FGK11, Theorem 4.2.2(2)(b)] in place of the usual the Artin-Rees lemma. $\square$

Lemma 4.8.5. Let A be a topologically finitely presented R-algebra for R as in the Setup 4.5.1, let $f \in A$ be any element. Then the completed localization $A_{\{f\}}$ coincides with the I-adic derived completion of A_f.

Proof. Choose some generators $I = (x_1, \dots, x_d)$. Then we know that the derived completion completion of A_f is given by $\mathbf{R}\lim_n K(A_f; x_1^n, \dots, x_d^n)$ where $K(A_f; x_1^n, \dots, x_d^n)$ is the Koszul complex for the sequence $(x_1^n, \dots, x_d^n)$. Lemma 4.8.4 implies that the pro-systems $\{K(A_f; x_1^n, \dots, x_d^n)\}$ and $\{A_f/I^n\}$ are naturally pro-isomorphic. Thus we have an isomorphism

$$\mathbf{R}\lim_n K(A_f; x_1^n, \dots, x_d^n) \cong \mathbf{R}\lim_n A_f/I^n \simeq A_{\{f\}} \ .$$

The last isomorphism uses the Mittag-Leffler criterion to ensure vanishing of $\lim^1$. $\square$

Theorem 4.8.6. [Lur18, Corollary 8.2.4.15] Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine, finitely presented formal scheme over R as in the Setup 4.5.1. Then the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon$ "$\mathbf{D}_{qc}(\mathfrak{X})$" $\to \mathbf{D}_{comp}(A, I)$ is an equivalence of categories.

Proof. The statement can be deduced from [Lur18, Corollary 8.2.4.15] by passing to the homotopy categories. We note that even though [Lur18, Corollary 8.2.4.15] uses ∞-categories, the cited proof can be rephrased in our situation without using any derived geometry. However, it would require quite a big digression, so instead we explain why our definitions are compatible with definitions in [Lur18].

Lemma 4.8.5 implies that the definition of Spf A in [Lur18] is compatible with the classical one. Now [Lur18, Proposition 8.2.4.18] ensures that our definition of "$\mathbf{D}_{qc}(\mathfrak{X})$" is equivalent to $h(\mathrm{Qcoh}(\mathfrak{X}))$ in the sense of [Lur18]. $\square$

Definition 4.8.7. We denote by

$$(-)^{L\Delta}\colon \mathbf{D}_{comp}(A, I) \to \text{“}\mathbf{D}_{qc}(\mathfrak{X})\text{”}$$

the pseudo-inverse to $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon$ "$\mathbf{D}_{qc}(\mathfrak{X})$" $\to \mathbf{D}_{comp}(A, I)$. We note that clearly

$$\mathbf{R}\Gamma(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta}) \simeq M \widehat{\otimes}_A A_{\{f\}}$$

for any $M \in \mathbf{D}_{comp}(A, I)$.

Remark 4.8.8. The functor $(-)^{L\Delta}$ *is not* compatible with the "abelian" functor $(-)^{\Delta}$ used the previous sections.

Our real goal is to show that there is an equivalence between $\mathbf{D}_{acoh}(A)$ and $\mathbf{D}_{qc,acoh}(\mathfrak{X})$. Theorem 4.8.6 will be a useful tool to prove this equivalence. We now give a precise definition of $\mathbf{D}_{qc,acoh}(\mathfrak{X})$.

Definition 4.8.9. We define $\mathbf{D}_{qc,acoh}(\mathfrak{X})$ (resp. $\mathbf{D}_{acoh}(\mathfrak{X})^a$) to be the full triangulated subcategory of $\mathbf{D}(\mathfrak{X})$ (resp. $\mathbf{D}(\mathfrak{X})^a$) consisting of complexes with adically quasi-coherent, almost coherent (resp. almost coherent) cohomology sheaves (resp. almost sheaves).

Remark 4.8.10. An argument similar to one in the proof of Lemma 4.4.5 shows that $\mathbf{D}_{acoh}(\mathfrak{X})^a$ is equivalent to the Verdier quotient $\mathbf{D}_{qc,acoh}(\mathfrak{X})/\mathbf{D}_{qc,\Sigma_{\mathfrak{X}}}(\mathfrak{X})$.

In order to show an equivalence $\mathbf{D}_{qc,acoh}(\mathfrak{X}) \simeq \mathbf{D}_{acoh}(A)$, our first goal is to show that $\mathbf{D}_{qc,acoh}$ lies inside "$\mathbf{D}_{qc}(\mathfrak{X})$". Even though it looks very plausible, it requires a proof that is not entirely trivial.

Lemma 4.8.11. Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{qc,acoh}(\mathfrak{X}) \to \mathbf{D}(A)$ is t-exact (with respect to the evident t-structures on both sides) and factors through $\mathbf{D}_{acoh}(A)$. More precisely, there is an isomorphism

$$\mathrm{H}^i\left(\mathbf{R}\Gamma\left(\mathfrak{X}, \mathcal{F}\right)\right) \simeq \mathrm{H}^0\left(\mathfrak{X}, \mathcal{H}^i\left(\mathcal{F}\right)\right) \in \mathbf{Mod}_A^{\mathrm{acoh}}$$

for any object $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X})$.

Proof. We note that the vanishing theorem [FK18, Theorem I.7.1.1] implies that we can use [Sta21, Tag 0D6U] with $N = 0$. Thus we see that the map $\mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F})) \to \mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \tau^{\geq i}\mathcal{F}))$ is an isomorphism for any integer i, and that $\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F}) \in \mathbf{D}_{acoh}(A)$ for any $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X})$. Combining it with the canonical isomorphism $\mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \tau^{\geq i}\mathcal{F})) \simeq \mathrm{H}^0(\mathfrak{X}, \mathcal{H}^i(\mathcal{F}))$ we get the desired result. $\square$

Lemma 4.8.12. Let $\mathfrak{X}$ be an locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then $\mathbf{D}_{qc,acoh}(\mathfrak{X})$ is naturally a full triangulated subcategory of "$\mathbf{D}_{qc}(\mathfrak{X})$".

Proof. Both $\mathbf{D}_{qc,acoh}(\mathfrak{X})$ and "$\mathbf{D}_{qc}(\mathfrak{X})$" are full triangulated subcategories of $\mathbf{D}(\mathfrak{X})$. Thus, it suffices to show that any $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X})$ lies in "$\mathbf{D}_{qc}(\mathfrak{X})$".

Lemma 4.8.11 and Corollary 2.12.8 imply that $\mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F}) \in \mathbf{D}_{comp}(A, I)$ for any open affine $\mathfrak{U} \subset \mathfrak{X}$. Now suppose $\mathfrak{U} \subset \mathfrak{V}$ is an inclusion of open affine formal subschemes in $\mathfrak{X}$. We consider the natural morphism

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F})\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F})$$

We note that $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is flat over $\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})$ by [FK18, Proposition I.4.8.1]. Thus, the complex

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \otimes^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})}$$

lies in $\mathbf{D}_{acoh}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}))$ by Lemma 2.8.1. Therefore, it also lies in $\mathbf{D}_{comp}(A, I)$ by Corollary 2.12.8. So we conclude that

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F})\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \simeq \mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \otimes^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})\ .$$

Using $\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})$-flatness of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, we conclude that the question boils down to show that

$$\mathrm{H}^i(\mathfrak{V}, \mathcal{F}) \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \mathrm{H}^i(\mathfrak{U}, \mathcal{F})$$

is an isomorphism for all i. Now Lemma 4.8.11 implies that this, in turn, reduces to showing that the natural map

$$\Gamma(\mathfrak{V}, \mathcal{H}^i(\mathcal{F})) \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \Gamma(\mathfrak{U}, \mathcal{H}^i(\mathcal{F}))$$

is an isomorphism. Without loss of generality, we may assume that $\mathfrak{X} = \mathfrak{V} = \operatorname{Spf} A$. Then $\mathcal{H}^i(\mathcal{F})$ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module, so it is isomorphic to M^Δ for some $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$ by Lemma 4.6.1. So the desired claim follows from [FK18, Lemma 3.6.4] and the observation that $M \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is already I-adically complete by Lemma 2.12.7. $\square$

Now we show that the $(-)^{L\Delta}$ functor sends $\mathbf{D}_{acoh}(A)$ to $\mathbf{D}_{qc,acoh}(\operatorname{Spf} A)$. This is also not entirely obvious as this derived version of $(-)^{L\Delta}$ a priori has nothing to do with the classical version of $(-)^\Delta$-functor defined on classically I-adically complete modules. The key is to show that these functors coincide on $\mathbf{Mod}_A^{\mathrm{acoh}}$.

Lemma 4.8.13. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then the functor $(-)^{L\Delta}\colon \mathbf{D}_{acoh}(A) \to$ "$\mathbf{D}_{qc}(\mathfrak{X})$" factors through $\mathbf{D}_{qc,acoh}(\mathfrak{X})$. Moreover, for any $M \in \mathbf{D}_{acoh}(A)$, there are functorial isomorphisms

$$\mathrm{H}^i(M)^\Delta \simeq \mathcal{H}^i(M^{L\Delta}).$$

Proof. We note that $\mathrm{H}^i(\mathfrak{X}, M^{L\Delta}) \simeq \mathrm{H}^i(M)$ by its very construction. Since $\mathcal{H}^i(M^{L\Delta})$ is canonically isomorphic to the sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{H}^i(\mathfrak{U}, M^{L\Delta}),$$

we get that there is a canonical map $\mathrm{H}^i(M) \to \Gamma(\mathfrak{X}, \mathcal{H}^i(M^\Delta))$. By the universal property of the classical $(-)^\Delta$ functor, we get a functorial morphism

$$\mathrm{H}^i(M)^\Delta \to \mathcal{H}^i(M^{L\Delta}).$$

Since $\mathrm{H}^i(M)$ is almost coherent, we only need to show that this map is an isomorphism for any i. This boils down (using almost coherence of $\mathrm{H}^i(M)$) to show that

$$\mathrm{H}^i(M) \otimes_A A_{\{f\}} \to \mathrm{H}^i(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta})\ .$$

for all $f \in A$. Now recall that $\mathbf{R}\Gamma(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta}) \simeq M\widehat{\otimes}^L_A A_{\{f\}}$ for any $f \in A$. Using that $M \in \mathbf{D}_{acoh}(A)$, $A_{\{f\}}$ is flat over A, and that almost coherent complexes are derived complete by Lemma 2.12.8, we conclude that the natural map

$$\mathrm{H}^i(M) \otimes_A A_{\{f\}} \to \mathrm{H}^i(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta})$$

is an isomorphism finishing the proof. $\square$

Corollary 4.8.14. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Suppose that $M \in \mathbf{D}(A)$ has almost zero cohomology modules. Then $\mathcal{H}^i(M^{L\Delta})$ is an almost zero, adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module for all integer i. In particular, $(-)^{L\Delta}$ induces a functor $(-)^{L\Delta}\colon \mathbf{D}_{acoh}(A)^a \to \mathbf{D}_{acoh}(\mathfrak{X})^a$.

Proof. This follows directly from the observation that any almost zero A-modules is almost coherent and the formula $\mathrm{H}^i(M)^\Delta \simeq \mathcal{H}^i(M^{L\Delta})$ established in Lemma 4.8.13. $\square$

Theorem 4.8.15. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{qc,acoh}(\mathfrak{X}) \to \mathbf{D}_{acoh}(A)$ is t-exact equivalence of triangulated categories with the pseudo-inverse $(-)^{L\Delta}$.

Proof. Lemma 4.8.11 implies that $\mathbf{R}\Gamma(\mathfrak{X}, -)$ induces the functor $\mathbf{D}_{qc,acoh}(\mathfrak{X}) \to \mathbf{D}_{acoh}(A)$ and that this functor is t-exact. Lemma 4.8.12 and Theorem 4.8.6 ensures that it is sufficient to show that $(-)^{L\Delta}$ sends $\mathbf{D}_{acoh}(A)$ to $\mathbf{D}_{qc,acoh}(\mathfrak{X})$, this follows from Lemma 4.8.13. $\square$

Now we can pass to the almost categories using Remark 4.8.10 to get the almost version of Theorem 4.8.15.

Corollary 4.8.16. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{acoh}(\mathfrak{X})^a \to \mathbf{D}_{acoh}(A)^a$ is a t-exact equivalence of triangulated categories with the pseudo-inverse $(-)^{L\Delta}$.

4.9. **Formal Schemes. Basic Functors on the Derived Categories of $\mathcal{O}_{\mathfrak{X}}^a$-modules.** We discuss the derived analogue of the results in Section 4.6. We show that the derived completion, derived tensor product, derived pullback, and derived almost Hom functors preserve complexes with almost coherent cohomology sheaves under certain conditions. For the rest of the section, we fix a ring R as in the Setup 4.5.1.

We start with the completion functor. We recall that we have defined the morphism of locally ringed spaces $c\colon \widehat{X} \to X$ for any R-scheme X. If X is locally finitely presented over R or $X = \operatorname{Spec} A$ for a topologically finitely presented R-algebra A, then c is a flat morphism as was shown in Lemma 4.6.3 and Corollary 4.6.4.

Lemma 4.9.1. Let $X = \operatorname{Spec} A$ be an affine R-scheme for R as in the Setup 4.5.1. Suppose that A is either finitely presented or topologically finitely presented over R. Suppose $M \in \mathbf{D}_{acoh}(A)$. Then $M^{L\Delta} \simeq \mathbf{L}c^*(\widetilde{M})$.

Proof. First of all, we show that $\mathbf{L}c^*(\widetilde{M}) \in \mathbf{D}_{qc,acoh}(\widehat{X})$. Indeed, the functor c^* is exact as c is flat. Thus, Lemma 4.6.3 guarantees that we have a sequence of isomorphisms

$$\mathcal{H}^i\left(\mathbf{L}c^*\left(\widetilde{M}\right)\right) \simeq c^*\left(\widetilde{\mathrm{H}^i(M)}\right) \simeq \left(\mathrm{H}^i(M)\right)^{\Delta}.$$

In particular, Theorem 4.8.6 ensures that the natural morphism

$$M \simeq \mathbf{R}\Gamma(X, \widetilde{M}) \to \mathbf{R}\Gamma(\widehat{X}, \mathbf{L}c^*(\widetilde{M}))$$

induces the morphism $M^{L\Delta} \to \mathbf{L}c^*(\widetilde{M})$. As c^* is exact, Lemma 4.8.13 implies that it is sufficient to show that the natural map

$$\mathrm{H}^i(M)^{\Delta} \to c^*(\widetilde{\mathrm{H}^i(M)})$$

is an isomorphism for all i. This follows from Lemma 4.6.3. $\square$

Corollary 4.9.2. Let X be a locally finitely presented R-scheme for R as in the Setup 4.5.1. Then $\mathbf{L}c^*$ induces functors $\mathbf{L}c^*\colon \mathbf{D}^*_{qc,acoh}(X) \to \mathbf{D}^*_{qc,acoh}(\widehat{X})$ (resp. $\mathbf{L}c^*\colon \mathbf{D}^*_{acoh}(X)^a \to \mathbf{D}^*_{acoh}(\widehat{X})^a$) for any $* \in \{\text{“ ”}, -, b, +\}$.

Proof. The claim is local, so it suffices to assume that $X = \operatorname{Spec} A$. Then it follows from exactness of c^* and Lemma 4.9.1. $\square$

Lemma 4.9.3. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally finitely presented formal R-scheme for R as in the Setup 4.5.1.

- **(1)** Suppose that $\mathfrak{X} = \operatorname{Spf}\ B$, $\mathfrak{Y} = \operatorname{Spf}\ A$ are affine formal R-schemes. Then there is a functorial isomorphism
$$\mathbf{L}\mathfrak{f}^*\left(M^{L\Delta}\right) \simeq (M \otimes_A B)^{L\Delta}$$
for any $M \in \mathbf{D}_{acoh}(A)$.
- **(2)** Suppose that $\mathfrak{X} = \operatorname{Spf}\ B$, $\mathfrak{Y} = \operatorname{Spf}\ A$ are affine formal R-schemes. Then there is a functorial isomorphism
$$\mathbf{L}\mathfrak{f}^*\left(M^{a,L\Delta}\right) \simeq (M^a \otimes_{A^a} B^a)^{L\Delta}$$
for any $M^a \in \mathbf{D}_{acoh}(A)$.
- **(3)** The functor $\mathbf{L}\mathfrak{f}^*$ carries $\mathbf{D}^-_{qc,acoh}(\mathfrak{Y})$ to $\mathbf{D}^-_{qc,acoh}(\mathfrak{X})$.
- **(4)** The functor $\mathbf{L}\mathfrak{f}^*$ carries $\mathbf{D}^-_{acoh}(\mathfrak{Y})^a$ to $\mathbf{D}^-_{acoh}(\mathfrak{X})^a$.

Proof. The proof is similar to Lemma 4.6.5. We use Lemma 4.9.1 and Lemma 4.8.13 to reduce to the analogous algebraic fact that was already proven in Lemma 4.2.3. □

Lemma 4.9.4. Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1.

(1) Suppose that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then there is a functorial isomorphism

$$M^{L\Delta} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} N^{L\Delta} \simeq (M \otimes^L_A N)^{L\Delta}$$

for any M, $N \in \mathbf{D}_{acoh}(A)$.

(2) Suppose that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then there is a functorial isomorphism

$$M^{a,L\Delta} \otimes^L_{\mathcal{O}^a_{\mathfrak{X}}} N^{a,L\Delta} \simeq (M^a \otimes^L_{A^a} N^a)^{L\Delta}$$

for any M^a, $N^a \in \mathbf{D}_{acoh}(A)^a$.

(3) Let $\mathcal{F}$, $\mathcal{G} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$. Then $\mathcal{F} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$.

(4) Let $\mathcal{F}^a$, $\mathcal{G}^a \in \mathbf{D}^-_{acoh}(\mathfrak{X})^a$. Then $\mathcal{F}^a \otimes^L_{\mathcal{O}^a_{\mathfrak{X}}} \mathcal{G}^a \in \mathbf{D}^-_{acoh}(\mathfrak{X})^a$.

Proof. Similarly to Lemma 4.9.3, we use Lemma 4.9.1 and Lemma 4.8.13 to reduce to the analogous algebraic fact that was already proven in Lemma 4.2.4. □

Now we discuss the $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(-,-)$ functor. Our strategy of showing that $\mathbf{R}\underline{al\mathcal{H}om}(-,-)$ preserves almost coherent complexes will be slightly different from the schematic case. The main technical problem is to define the map $\mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)^{L\Delta} \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_{\mathfrak{X}}}(M^{a,L\Delta}, N^{a,L\Delta})$ in the affine case.

The main issue is that we do not know if $(-)^{L\Delta}$ is a left adjoint to the functor of global section on the whole category $\mathbf{D}(\mathfrak{X})$; we only know that it becomes a pseudo-inverse to $\mathbf{R}\Gamma(\mathfrak{X}, -)$ after restriction to "$\mathbf{D}_{qc}(\mathfrak{X})$". However, the complex $\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$ itself does not usually lie inside "$\mathbf{D}_{qc}(\mathfrak{X})$". To overcome this issue, we will show that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$$

does lie in "$\mathbf{D}_{qc}(\mathfrak{X})$" for $M \in \mathbf{D}^-_{acoh}(A)$ and $N \in \mathbf{D}^+_{acoh}(A)$.

Since "$\mathbf{D}_{qc}(\mathfrak{X})$" was defined in a bit abstract way, it is probably the easiest way to show that $\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$ actually lies in $\mathbf{D}_{qc,acoh}(\mathfrak{X})$. That is sufficient by Lemma 4.8.12.

Lemma 4.9.5. Let $\mathfrak{X} = \mathrm{Spf}\ A$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Let $M, N \in \mathbf{Mod}^{\mathrm{acoh}}_A$ there are natural almost isomorphisms

$$\mathrm{Ext}^p_A(M,N)^\Delta \xrightarrow{\sim} \mathcal{E}xt^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)$$

for all integer p.

Proof. We recall that $\mathcal{E}xt^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)$ is canonically isomorphic to sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{U}}}(M^\Delta|_{\mathfrak{U}}, N^\Delta|_{\mathfrak{U}})\ .$$

In particular, there is a canonical map $\mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta) \to \Gamma(\mathfrak{X}, \mathcal{E}xt^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta))$. It induces the morphism

$$\mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)^\Delta \to \mathcal{E}xt^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)\ . \tag{4.7}$$

Now we note that the classical $(-)^{\Delta}$ functor and the derived version coincide on almost coherent modules by Lemma 4.8.13. Hence, the equivalence "$\mathbf{D}_{qc}(\mathfrak{X})$" $\simeq \mathbf{D}_{comp}(A, I)$ coming from Theorem 4.8.6 and Lemma 4.8.13 ensure that $\mathrm{Ext}^p_{\mathcal{O}_\mathfrak{X}}(M^\Delta, N^\Delta) \simeq \mathrm{Ext}^p_A(M, N)$. So the map (4.7) becomes the map

$$\mathrm{Ext}^p_A(M, N)^\Delta \to \mathcal{E}xt^p_{\mathcal{O}_\mathfrak{X}}(M^\Delta, N^\Delta)\ .$$

We note that $\mathrm{Ext}^p_A(M, N)$ is an almost coherent A-module by Proposition 2.6.19. Using that almost coherent modules are complete, we conclude that it suffices to show that

$$\mathrm{Ext}^p_A(M, N) \otimes_A A_{\{f\}} \to \mathrm{Ext}^p_{\mathrm{Spf}\ A_{\{f\}}}(M^\Delta|_{\mathrm{Spf}\ A_{\{f\}}}, N^\Delta|_{\mathrm{Spf}\ A_{\{f\}}})$$

is an almost isomorphism. Using Lemma 4.6.5 and the equivalence "$\mathbf{D}_{qc}(\mathfrak{X})$" $\simeq \mathbf{D}_{comp}(A, I)$ as above, we see that the map above becomes the canonical map

$$\mathrm{Ext}^p_A(M, N) \otimes_A A_{\{f\}} \to \mathrm{Ext}^p_{A_{\{f\}}}(M \otimes_A A_{\{f\}}, N \otimes_A A_{\{f\}})\ .$$

Finally, this map is an almost isomorphism by Proposition 2.9.12. □

Corollary 4.9.6. Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. Then

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$$

for $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$, and $\mathcal{G} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$.

Proof. The claim is local, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A$. Using the Ext-spectral sequence and Lemma 4.5.18 to reduce to the case $\mathcal{F}$and $\mathcal{F}$ in $\mathbf{Mod}_\mathfrak{X}^{\mathrm{qc,acoh}}$. Then Theorem 4.6.2 ensures that $\mathcal{F} = M^\Delta$ and $\mathcal{G} = N^\Delta$ for some M, $N \in \mathbf{Mod}_A^{\mathrm{acoh}}$. Then Lemma 4.9.5 guarantees that

$$\mathcal{H}^p\left(\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(\mathcal{F}, \mathcal{G})\right) \simeq^a \mathrm{Ext}^p_A(M, N)^\Delta.$$

In other words,

$$\widetilde{\mathfrak{m}} \otimes \mathcal{H}^p\left(\mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(\mathcal{F}, \mathcal{G})\right) \simeq \widetilde{\mathfrak{m}} \otimes \mathrm{Ext}^p_A(M, N)^\Delta\ .$$

Now $\mathrm{Ext}^p_A(M, N)^\Delta$ is an adically quasi-coherent, almost coherent $\mathcal{O}_\mathfrak{X}$-module by Proposition 2.6.19 and Lemma 4.6.1. So Lemma 4.5.10 guarantees that $\widetilde{\mathfrak{m}}\otimes\mathrm{Ext}^p_A(M, N)^\Delta$ is also adically quasi-coherent and almost coherent. Therefore, $\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$. □

Lemma 4.9.7. Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in the Setup 4.5.1.

(1) Suppose $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then there is a functorial isomorphism

$$\mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)^{L\Delta} \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_\mathfrak{X}}(M^{a,L\Delta}, N^{a,L\Delta})$$

for $M \in \mathbf{D}^-_{acoh}(A)^a$ and $N \in \mathbf{D}^+_{acoh}(A)^a$.

(2) Suppose $\mathcal{F}^a \in \mathbf{D}^+_{acoh}(\mathfrak{X})^a$ and $\mathcal{G}^a \in \mathbf{D}^-_{acoh}(\mathfrak{X})$ are almost coherent $\mathcal{O}^a_\mathfrak{X}$-modules. Then $\mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_\mathfrak{X}}(\mathcal{F}^a, \mathcal{G}^a) \in \mathbf{D}^+_{acoh}(\mathfrak{X})^a$.

Proof. We start with **(1)**. Proposition 3.5.8 implies the map

$$(\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_\mathfrak{X}}(M^\Delta, N^\Delta))^a \to \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}^a_\mathfrak{X}}(M^{a,\Delta}, N^{a,\Delta})$$

is an isomorphism in $\mathbf{D}(\mathfrak{X})^a$. Similarly, the map

$$(\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)^\Delta)^a \to \mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)^\Delta$$

is an isomorphism by Lemma 4.9.5. Thus it suffices to construct a functorial isomorphism

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M,N)^{L\Delta} \to \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta}) \ .$$

Now Lemma 4.8.13 and Corollary 4.9.6 guarantee that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta}) \in \mathbf{D}_{qc,acoh}(\mathfrak{X}).$$

Proposition 2.6.19, Lemma 4.6.1, and Lemma 4.5.10 also guarantee that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M,N)^{\Delta} \in \mathbf{D}_{qc,acoh}(\mathfrak{X}).$$

Thus, Theorem 4.8.6 ensures that in order to construct the desired isomorphism it suffices to do it after applying $\mathbf{R}\Gamma(\mathfrak{X}, -)$. Projection Formula (Theorem 3.3.6) and the definition of the $(-)^{L\Delta}$-functor provide us with functorial isomorphisms

$$\mathbf{R}\Gamma\left(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M,N)^{L\Delta}\right) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M,N)$$

$$\begin{aligned}\mathbf{R}\Gamma\left(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\right) &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\Gamma\left(\mathfrak{X}, \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\right) \\ &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(M^{L\Delta}, N^{L\Delta}) \\ &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A\left(M,N\right)\end{aligned}$$

where the last isomorphism uses equivalence from Theorem 4.8.6. Thus, we see

$$\mathbf{R}\Gamma\left(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M,N)^{L\Delta}\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\right).$$

As a consequence, we have a functorial isomorhism

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A\left(M,N\right)^{L\Delta} \xrightarrow{\sim} \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}}\left(M^{L\Delta}, N^{L\Delta}\right).$$

This induces the desired isomorphism

$$\mathbf{R}\mathrm{alHom}_{A^a}\left(M^a, N^a\right)^{L\Delta} \xrightarrow{\sim} \mathbf{R}\underline{al\mathcal{H}om}_{\mathcal{O}_{\mathfrak{X}}^a}\left(M^{a,L\Delta}, N^{a,L\Delta}\right).$$

(2) is an easy consequence of **(1)**, Proposition 2.6.19, and Corollary 4.8.14. □

5. Cohomological Properties of Almost Coherent Sheaves

5.1. **Almost Proper Mapping Theorem.** The main goal of this section is to prove the "Almost Proper Mapping Theorem" both in setup of both "nice" schemes and "nice" formal schemes. The theorem roughly says the derived pushforward of an almost coherent $\mathcal{O}_X$-module along a (topologically) finitely presented proper map is almost coherent.

The idea of the proof is rather easy: we "approximate" an almost finitely presented $\mathcal{O}_X$-module by finitely presented using Corollary 4.3.5 and then the usual Proper Mapping Theorem. However, there is a subtlety that the usual Proper Mapping Theorem is usually proven only for a (locally) noetherian base, and we are really interested in non-noetherian situation. So we use a more general version (in so-called "universally coherent" case) of the Proper Mapping Theorem from the book [FK18].

Definition 5.1.1. We say that a scheme Y is *universally coherent* if any scheme X that is locally of finite presentation over Y is coherent (i.e. the structure sheaf $\mathcal{O}_X$ is coherent).

Theorem 5.1.2 (Proper Mapping Theorem)**.** [FK18, Theorem I.8.1.3] Let Y be a universally coherent quasi-compact scheme, and $f: X \to Y$ a proper morphism of finite presentation. Then the functor $\mathbf{R}f_*$ sends $\mathbf{D}^*_{coh}(X)$ to $\mathbf{D}^*_{coh}(Y)$ for any $* \in \{\text{“ ”}, +, -, b\}$.

We want to generalize this theorem to the "almost world". So we pick a ring R and a fixed ideal $\mathfrak{m} \subset R$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. In this section, we always consider almost mathematics with respect to this ideal.

Theorem 5.1.3 (Almost Proper Mapping Theorem)**.** Let Y be a universally coherent quasi-compact R-scheme, and let $f : X \to Y$ be a proper, finitely presented morphism. Then

- The functor $\mathbf{R}f_*$ sends $\mathbf{D}^*_{qc,acoh}(X)$ to $\mathbf{D}^*_{qc,acoh}(Y)$ for any $* \in \{\text{“ ”}, +, -, b\}$.
- The functor $\mathbf{R}f_*$ sends $\mathbf{D}^*_{acoh}(X)^a$ to $\mathbf{D}^*_{acoh}(Y)^a$ for any $* \in \{\text{“ ”}, +, -, b\}$.
- The functor $\mathbf{R}f_*$ sends $\mathbf{D}^+_{acoh}(X)$ to $\mathbf{D}^+_{acoh}(Y)$.
- If Y has finite Krull dimension, then $\mathbf{R}f_*$ sends $\mathbf{D}^*_{acoh}(X)$ to $\mathbf{D}^*_{acoh}(Y)$ for any $* \in \{\text{“ ”}, +, -, b\}$.

Lemma 5.1.4. Let Y be a quasi-compact scheme of finite Krull dimension, and let $f : X \to Y$ be a finite type, quasi-separated morphism. Then X has finite Krull dimension, and f_* is of finite cohomological dimension on $\mathbf{Mod}_X$.

Proof. First of all, we show that X has finite Krull dimension. Indeed, the morphism $f\colon X \to Y$ is quasi-compact, therefore X is quasi-compact. Then it suffices to show that locally X has finite Krull dimension. So we can assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and the map is given by a finite type morphism $A \to B$. In that situation we have $\dim Y = \dim A$ and $\dim X = \dim B$. Thus, it is enough to show that the Krull dimension of a finite type A-algebra is finite. This readily reduces the question to the case of a polynomial algebra $\dim A[X_1, \dots, X_n]$. Now [AM69, Chapter 11, Exercise 6] implies that $\dim A[X_1, \dots, X_n] \le \dim A + 2n$.

Now we prove that f_* has finite cohomological dimension. We note that it suffices to show that there is an integer N such that for any open affine $U \subset Y$ the cohomology groups $\mathrm{H}^i(X_U, \mathcal{F})$ vanish for $i \ge N$ and any $\mathcal{O}_{X_U}$-module $\mathcal{F}$. We recall that f is quasi-separated, so X_U is quasi-compact, quasi-separated and $\dim X_U \le \dim X$ for any open $U \subset X$. Therefore, it is sufficient to show that on any spectral space X we have $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > \dim X$ and $\mathcal{F} \in \mathcal{A}b(X)$. This is proven in [Sch92, Corollary 4.6] (another reference is [Sta21, Tag 0A3G]). Thus we see that $N = \dim X$ does the job. ∎

Proof of Theorem 5.1.3. Step 0. Reduction to the case of bounded below derived categories: We note that f_* always has bounded cohomological dimension on $\mathbf{Mod}_X^{\mathrm{qc}}$. Indeed, for any $\mathcal{F} \in \mathbf{Mod}_X^{\mathrm{qc}}$ on a separated scheme X, we can compute $\mathrm{H}^i(X, \mathcal{F})$ by the *alternating* Čech complex for some finite covering of X by affines. Therefore, if X can be covered by N affines, the functor f_* restricted to $\mathbf{Mod}_X^{\mathrm{qc}}$ has cohomological dimension at most N.

Now we use [Sta21, Tag 0D6U] (alternatively, one can use [Lim19, Lemma 3.4]) to reduce the question of proving the claim for any $\mathcal{F} \in \mathbf{D}_{qc,acoh}(X)$ to the question of proving the claim for all its truncations $\tau^{\ge a}\mathcal{F}$. In particular, we can assume that $\mathcal{F} \in \mathbf{D}^+_{qc,acoh}(X)$. The case $\mathcal{F}^a \in \mathbf{D}^*_{acoh}(X)^a$ can be shown similarly. Actually, Proposition 3.5.23 and the observation that $\mathcal{F}^a_! \in \mathbf{D}_{qc,acoh}(X)$ imply that the results for $\mathbf{D}^*_{qc,acoh}(X)$ and $\mathbf{D}^*_{acoh}(X)^a$ are equivalent.

The same argument also works for $\mathbf{D}^+_{acoh}(X)$ provided that X, Y and f_* has finite cohomological dimension. Lemma 5.1.4 and [Sta21, Tag 0A3G] say that it holds whenever Y has finite Krull dimension.

Step 1. Reduction to the case quasi-coherent almost coherent sheaves: Using the Projection Formula (Lemma 3.3.6) (resp. Proposition 3.5.23), we see that in order to show $\mathbf{R}f_*$ sends $\mathbf{D}^+_{acoh}(X)$ to $\mathbf{D}^+_{acoh}(Y)$ (resp. $\mathbf{D}^+_{acoh}(X)^a$ to $\mathbf{D}^+_{acoh}(Y)^a$) it is sufficient to show the analogous result for

$\mathbf{D}^+_{qc,acoh}(X)$. Moreover, we can use the spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p f_* \mathcal{H}^q(\mathcal{F}) \Rightarrow \mathbf{R}^{p+q} f_*(\mathcal{F})$$

to reduce the claim to the fact that higher derived pushforwards of a quasi-coherent, almost coherent sheaf are quasi-coherent, almost coherent.

Step 2. The case of a quasi-coherent, almost coherent $\mathcal{O}_X$-module $\mathcal{F}$: We show that $\mathrm{R}^i f_*\mathcal{F}$ is a quasi-coherent, almost coherent $\mathcal{O}_Y$-module for any quasi-coherent, almost coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any i. First of all, we note that $\mathrm{R}^i f_*\mathcal{F}$ is quasi-coherent as higher pushforwards along quasi-compact, quasi-separated morphisms preserve quasi-coherence.

Now we show almost coherence of $\mathrm{R}^i f_*\mathcal{F}$. Note that it is sufficient to show that $\mathrm{R}^i f_*\mathcal{F}$ is almost finitely presented as Y is a coherent scheme (this follows from Lemma 4.1.15 and Lemma 4.1.16). We choose some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Then we use Corollary 4.3.5 to find a finitely presented $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism

$$\varphi : \mathcal{G} \to \mathcal{F}$$

such that $\ker(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by $\mathfrak{m}_1$. We define $\mathcal{O}_X$-modules

$$\mathcal{K} := \ker \varphi,\ \mathcal{M} := \mathrm{Im}\,\varphi \text{ and } \mathcal{Q} := \mathrm{Coker}\,\varphi,$$

so we have two short exact sequences

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{M} \to 0$$
$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{Q} \to 0$$

with sheaves $\mathcal{K}$ and $\mathcal{Q}$ killed by $\mathfrak{m}_1$. This easily shows that the natural homomorphisms

$$\mathrm{R}^i f_*(\varphi) : \mathrm{R}^i f_*\mathcal{G} \to \mathrm{R}^i f_*\mathcal{F}$$

have kernels and cokernels annihilated by $\mathfrak{m}_1^2$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ we see that $\mathfrak{m}_0(\ker \mathrm{R}^i f_*(\varphi)) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\,\mathrm{R}^i f_*(\varphi)) = 0$. Moreover, we know that $\mathrm{R}^i f_*\mathcal{G}$ is a finitely presented $\mathcal{O}_Y$-module by Theorem 5.1.2 ($\mathcal{G}$ is a coherent $\mathcal{O}_X$-module since X is a coherent scheme). Therefore we use Corollary 4.3.5 to conclude that $\mathrm{R}^i f_*\mathcal{F}$ is an almost finitely presented $\mathcal{O}_Y$-module for any $i \geq 0$. And this implies the almost coherence of $\mathrm{R}^i f_*\mathcal{F}$ as explained above. $\square$

Before we go to the formal version of this result, we need to establish a slightly more precise version of the usual Proper Mapping Theorem for formal schemes than the one in [FK18].

Theorem 5.1.5 (Proper Mapping Theorem)**.** Let R be as in Set-up 4.5.1, A a topologically finitely presented R-algebra, $\mathfrak{f}\colon \mathfrak{X} \to \mathrm{Spf}\ A$ a topologically finitely presented, proper morphism, and $\mathcal{F}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $\mathrm{H}^i(\mathfrak{X}, \mathcal{F})$ are coherent A-modules for all $i \geq 0$, and the natural morphism

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F})^\Delta \to \mathrm{R}^i \mathfrak{f}_*(\mathcal{F})$$

is an isomorphism for any $i \geq 0$.

Proof. Firstly, we use [FK18, Theorem I.11.1.2] to conclude that $\mathbf{R}\mathfrak{f}_*\mathcal{F} \in \mathbf{D}^+_{coh}(\mathrm{Spf}\ A)$. Therefore, Theorem 4.8.15 implies that $M := \mathbf{R}\Gamma(\mathrm{Spf}\ A, \mathbf{R}\mathfrak{f}_*\mathcal{F})$ lies in $\mathbf{D}^+_{acoh}(A)$, and

$$M^{L\Delta} \simeq \mathbf{R}\mathfrak{f}_*\mathcal{F}.$$

Moreover, Lemma 4.8.13 implies that the natural map

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F})^\Delta \simeq \mathrm{H}^i(M)^\Delta \to \mathrm{R}^i\mathfrak{f}_*\mathcal{F}$$

is an isomorphism. Finally, we conclude that

$$\mathrm{H}^i(\mathfrak{X},\mathcal{F}) \simeq \mathrm{H}^0\left(\mathfrak{X}, \mathrm{H}^i(\mathfrak{X},\mathcal{F})^\Delta\right) \simeq \mathrm{H}^0(\mathfrak{X}, \mathrm{R}^i\mathfrak{f}_*\mathcal{F})$$

must be coherent because $\mathrm{R}^i\mathfrak{f}_*\mathcal{F}$ is coherent. $\square$

Theorem 5.1.6 (Almost Proper Mapping Theorem)**.** Let $\mathfrak{Y}$ be a topologically finitely presented formal R-scheme for R as in the Setup 4.5.1. And let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a proper, topologically finitely presented morphism. Then

- The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^*_{qc,acoh}(\mathfrak{Y})$ for any $* \in \{\text{“ ”}, +, -, b\}$.
- The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{acoh}(\mathfrak{X})^a$ to $\mathbf{D}^*_{acoh}(\mathfrak{Y})^a$ for any $* \in \{\text{“ ”}, +, -, b\}$.
- The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^+_{acoh}(\mathfrak{X})$ to $\mathbf{D}^+_{acoh}(\mathfrak{Y})$.
- If $Y_0 := \mathfrak{Y} \times_{\mathrm{Spf}\ R} (\operatorname{Spec} R/\varpi)$ has finite Krull dimension, then $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{acoh}(\mathfrak{X})$ to $\mathbf{D}^*_{acoh}(\mathfrak{Y})$ for any $* \in \{\text{“ ”}, +, -, b\}$.

Moreover, if $\mathfrak{Y} = \mathrm{Spf}\ A$ is an affine scheme and $\mathcal{F}$ is an adically quasi-coherent, almost coherent $\mathcal{O}_\mathfrak{X}$-module, then $\mathrm{H}^n(\mathfrak{X},\mathcal{F})$ is almost coherent over A, and the natural map $\mathrm{H}^n(\mathfrak{X},\mathcal{F})^\Delta \to \mathrm{R}^n\mathfrak{f}_*\mathcal{F}$ is an isomorphism of $\mathcal{O}_\mathfrak{Y}$-modules for $n \geq 0$.

Lemma 5.1.7. Let $\mathfrak{Y}$ be a quasi-compact adic formal R-scheme, and let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a topologically finite type, quasi-separated morphism. Suppose that the reduction $Y_0 = \mathfrak{Y} \times_{\mathrm{Spf}\ R} (\operatorname{Spec} R/\varpi)$ (or equivalently the “special fiber” $\overline{\mathfrak{Y}} = \mathfrak{Y} \times_{\mathrm{Spf}\ R} \operatorname{Spec} R/\mathrm{Rad}(\varpi))$ is of finite Krull dimension. Then $\mathfrak{X}$ has finite Krull dimension, and $\mathfrak{f}_*$ is of finite cohomological dimension on $\mathbf{Mod}_\mathfrak{X}$.

Proof. The proof is identical to Lemma 5.1.4 once we notice that the underlying topological spaces of $\mathfrak{Y}, Y_0$ and $\overline{\mathfrak{Y}}$ are canonically identified. $\square$

Also, before going to the proof of Theorem 5.1.6 we need to establish one preliminary lemma.

Lemma 5.1.8. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y} = \mathrm{Spf}\ A$ be a morphism as in Theorem 5.1.6 with affine $\mathfrak{Y}$, and let $\mathcal{F} \in \mathbf{Mod}_\mathfrak{X}$ be an adically quasi-coherent, almost coherent sheaf. Then $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is an adically quasi-coherent, almost coherent $\mathcal{O}_\mathfrak{Y}$-module if

(1) the A-module $\mathrm{H}^q(\mathfrak{X},\mathcal{F})$ is almost coherent for any $q \geq 0$,

(2) for any $g \in A$, the canonical map

$$\mathrm{H}^q(\mathfrak{X},\mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_\mathfrak{U},\mathcal{F}),$$

where $\mathfrak{U} = \mathrm{Spf}\ A_{\{g\}} \to \mathfrak{Y} = \mathrm{Spf}\ A$, is an isomorphism for any $q \geq 0$.

Proof. Consider an A-module $M := \mathrm{H}^q(\mathfrak{X},\mathcal{F})$ that is almost coherent by hypothesis ((1)). So Lemma 2.12.7 guarantees that M is I-adically complete, and so M^Δ is an adically quasi-coherent, almost coherent $\mathcal{O}_\mathfrak{X}$-module. Now note that $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is the sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{H}^q(\mathfrak{X}_\mathfrak{U},\mathcal{F})$$

Thus there is a canonical map $M \to \mathrm{H}^0(\mathfrak{Y}, \mathrm{R}^q\mathfrak{f}_*\mathcal{F})$ that induces the morphism

$$M^\Delta \to \mathrm{R}^q\mathfrak{f}_*\mathcal{F}$$

The second hypothesis together with Lemma 2.8.1 and Lemma 2.12.7 ensures this map is an isomorphism on stalks (as the sheafification process preserves stalks). Therefore, $M^\Delta \to \mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is an isomorphism of $\mathcal{O}_\mathfrak{X}$-modules. In particular, $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is adically quasi-coherent and almost coherent. $\square$

Proof of Theorem 5.1.6. We use the same reduction as in the proof of Theorem 5.1.3 to reduce to the situation of an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. Moreover, the statement is local on $\mathfrak{Y}$, so we can assume that $\mathfrak{Y} = \operatorname{Spf} A$ is affine.

Now we show that both conditions in Lemma 5.1.8 are satisfied in our situation.

Step 1*:* $\mathrm{H}^q(\mathfrak{X}, \mathcal{F})$ *is almost coherent for every* $q \geq 0$*.* Fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

Theorem 4.7.6 guarantees that there is a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}_{\mathfrak{m}_1}$ and a morphism $\phi_{\mathfrak{m}_1} \colon \mathcal{G}_{\mathfrak{m}_1} \to \mathcal{F}$ such that its kernel and cokernel are annihilated by $\mathfrak{m}_1$. Then it is easy to see that the natural morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{G}_{\mathfrak{m}_1}) \to \mathrm{H}^q(\mathfrak{X}, \mathcal{F})$$

has kernel annihilated by $\mathfrak{m}_1^2$ and cokernel annihilated by $\mathfrak{m}_1$. In particular, both kernel and cokernel are annihilated by $\mathfrak{m}_0$. Since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal of $\mathfrak{m}$, it suffices to show that $\mathrm{H}^q(\mathfrak{X}, \mathcal{G}_{\mathfrak{m}_1})$ are coherent A-modules for any $q \geq 0$. This follows from Theorem 5.1.5.

Step 2*: canonical maps* $\mathrm{H}^q(\mathfrak{X}, \mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F})$ *are isomorphism for any* $g \in A$*,* $q \geq 0$*, and* $\mathfrak{U} = \operatorname{Spf} A_{\{g\}}$. Lemma 4.7.5 guarantees that $\mathcal{F}$ admits an FP-approximation $\phi \colon \mathcal{G} \to \mathcal{F}$. Using Lemma 4.5.14, we get short exact sequences of adically quasi-coherent sheaves

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{M} \to 0,$$

$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are annihilated by I^{n+1} for some $n \geq 0$. So $\mathcal{K}$ and $\mathcal{Q}$ can be identified with quasi-coherent sheaves on $\mathfrak{X}_n \coloneqq \mathfrak{X} \times_{\operatorname{Spf} A} \operatorname{Spec} A/I^{n+1}$. Therefore, the natural morphisms

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{K}) \otimes_A A_{\{g\}} \simeq \mathrm{H}^q(\mathfrak{X}_n, \mathcal{K}) \otimes_{A/I^{n+1}} (A/I^{n+1})_g \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U},n}, \mathcal{K}),$$

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{Q}) \otimes_A A_{\{g\}} \simeq \mathrm{H}^q(\mathfrak{X}_n, \mathcal{Q}) \otimes_{A/I^{n+1}} (A/I^{n+1})_g \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U},n}, \mathcal{Q})$$

are isomorphisms for $q \geq 0$. The morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{G}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{G}) \tag{5.1}$$

is an isomorphism by Theorem 5.1.5. In particular, the map (5.1) must be an isomorphism.

Finally, the five-lemma implies that the morphisms

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{M}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{M})$$

must be isomorphisms for all $q \geq 0$ because analogous maps for $\mathcal{K}$ and $\mathcal{G}$ are isomorphisms (and $A_{\{g\}}$ is flat over A). Applying the five-lemma again, we conclude that the morphisms

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F})$$

must be isomorphisms for all $q \geq 0$ because analogous maps for $\mathcal{M}$ and $\mathcal{G}$ are isomorphisms (and $A_{\{g\}}$ is flat over A). □

5.2. **Characterization of Quasi-Coherent, Almost Coherent Complexes.** The main goal of this Section is to show an almost analogue of [Sta21, Tag 0CSI]. This gives a useful characterization of objects in $\mathbf{D}^b_{qc,acoh}(X)$ on a separated, finitely presented R-scheme for a universally coherent R. This will be crucially used in our proof of the almost version of the Formal GAGA Theorem 5.3.2.

Our proof is very close to the proof of [Sta21, Tag 0CSI], but we need to make certain adjustments to make the arguments work in the almost coherent setting.

Theorem 5.2.1. Let R be an universally coherent ring with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is flat. Suppose that $\mathcal{F} \in \mathbf{D}_{qc}(\mathbf{P}^N_R)$ an element such that $\mathbf{R}\mathrm{Hom}_{\mathbf{P}^N}(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{acoh}(R)$ for $\mathcal{P} = \oplus_{i=0}^N \mathcal{O}(i)$. Then $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathbf{P}^N_R)$.

Proof. We follow the ideas of [Sta21, Tag 0CSG]. Denote the dg algebra $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{P})$ by S. A computation of cohomology groups of line bundles on $\mathbf{P}^N_R$ implies that S is a "discrete" non-commutative algebra that is finite and flat over R. [Sta21, Tag 0BQU][28] guarantees that the functor

$$- \otimes^{\mathbf{L}}_S \mathcal{P} \colon \mathbf{D}(S) \to \mathbf{D}_{qc}(\mathbf{P}^N)$$

is an equivalence of categories, and the map in the other direction is given by

$$\mathbf{R}\mathrm{Hom}(\mathcal{P}, -) \colon \mathbf{D}_{qc}(\mathbf{P}^N) \to \mathbf{D}(S)$$

So if we define $M := \mathbf{R}\mathrm{Hom}(\mathcal{P}, \mathcal{F}) \in \mathbf{D}(S)$, our assumptions imply that that the image of M in $\mathbf{D}(R)$ lands inside $\mathbf{D}^-_{acoh}(R)$. We need to show that this assumption guarantees that $\mathcal{F} \simeq M \otimes^{\mathbf{L}}_S \mathcal{P}$ lives in $\mathbf{D}^-_{qc,acoh}(\mathbf{P}^N)$. Moreover, using the convergence spectral sequence

$$\mathrm{E}_2^{p,q} = \mathcal{H}^p(\mathrm{H}^q(M) \otimes^{\mathbf{L}}_S \mathcal{P}) \Rightarrow \mathcal{H}^{p+q}(M \otimes^{\mathbf{L}}_S \mathcal{P})$$

shows that it is sufficient to assume that M is just an S-module. Then Lemma 2.8.4 implies that for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is a finitely presented right S-module N with a morphism $f \colon N \to M$ such that $\ker f$ and $\mathrm{Coker}\, f$ are annihilated by $\mathfrak{m}_0$. The universal coherence of R and [Sta21, Tag 0CSF] imply that $N \otimes^{\mathbf{L}}_S \mathcal{P} \in \mathbf{D}^-_{qc,coh}(\mathbf{P}^N)$. Now we note that the functor

$$- \otimes^{\mathbf{L}}_S \mathcal{P} \colon \mathbf{D}(S) \to \mathbf{D}_{qc}(\mathbf{P}^N)$$

is R-linear, so the standard argument shows that the cone of the morphism

$$f \otimes^{\mathbf{L}}_S \mathcal{P} \colon N \otimes^{\mathbf{L}}_S \mathcal{P} \to M \otimes^{\mathbf{L}}_S \mathcal{P}$$

has cohomology sheaves anihillated by $\mathfrak{m}_0 \mathcal{O}_X$. Finally, Lemma 2.5.7 says that $M \otimes^{\mathbf{L}}_S \mathcal{P}$ is in $\mathbf{D}^-_{qc,acoh}(\mathbf{P}^N)$. $\square$

Lemma 5.2.2. Let R be a universally coherent ring, and let X be a scheme separated and of finite presentation over R. Let $K \in \mathbf{D}_{qc}(X)$. If $\mathbf{R}\Gamma(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K)$ is in $\mathbf{D}^-_{acoh}(R)$ for every $E \in \mathbf{D}^-_{coh}(X)$, then $K \in \mathbf{D}^-_{qc,acoh}(X)$.

Proof. We follow the proof of [Sta21, Tag 0CSL]. The condition that $K \in \mathbf{D}^-_{qc,acoh}(X)$ is local on X as X is quasi-compact. Therefore, we can prove it locally around each point x. We use [Sta21, Tag 0CSJ] to find

- An open subset $U \subset X$ containing x.
- An open subset $V \subset \mathbf{P}^n_R$.
- A closed subset $Z \subset X \times_R \mathbf{P}^n_R$ with a point $z \in Z$ lying over x
- An object $E \in \mathbf{D}^-_{coh}(X \times_R \mathbf{P}^n_R)$.

with a lot of properties listed in the cited lemma. Even though the notations are pretty heavy, the only real properties of these object that we will use are that $x \in U$ and

$$\mathbf{R}q_*(\mathbf{L}p^* K \otimes^{\mathbf{L}} E)|_V = \mathbf{R}(U \to V)_*(K|_U)$$

[28] Note that they have slightly different notations for R and S

The last formula is proven in [Sta21, Tag 0CSK] and we refer to this lemma for a discussion of the morphism $U \to V$ that turns out to be a finitely presented closed immersion.

That being said, we note that the argument above shows that it is sufficient to show that $K|_U$ is almost coherent for each such U. Moreover, the formula $\mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)|_V = \mathbf{R}(U \to V)_*(K|_U)$, the fact that $U \to V$ is a finitely presented closed immersion and Lemma 2.8.4 imply that it is sufficient to show that $\mathbf{R}(U \to V)_*(K|_U) = \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)|_V$ lies in $\mathbf{D}^-_{qc,acoh}(V)$. In particular, it is enough to show that $\mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E) \in \mathbf{D}^-_{qc,acoh}(\mathbf{P}^n_R)$. The key is that we can check that condition using Theorem 5.2.1.

We define a sheaf $\mathcal{P} := \bigoplus_{i=0}^n \mathcal{O}_{\mathbf{P}^n}(i)$ and we compute

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\mathbf{P}^n}(\mathcal{P}, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)) &= \mathbf{R}\Gamma(\mathbf{P}^n, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{P}^\vee) \\
&= \mathbf{R}\Gamma(\mathbf{P}^n, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee)) \\
&= \mathbf{R}\Gamma(X \times_R \mathbf{P}^n_R, \mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee) \\
&= \mathbf{R}\Gamma(X, \mathbf{R}p_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee)) \\
&= \mathbf{R}\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}p_*(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee))
\end{aligned}$$

where the second and fifth equality come from the projection formula [Sta21, Tag 08EU]. Finally, we note that the Proper Mapping Theorem 5.1.2 implies that $\mathbf{R}p_*(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee) \in \mathbf{D}^-_{coh}(X)$, so the assumption says that

$$\mathbf{R}\mathrm{Hom}_{\mathbf{P}^n}(\mathcal{P}, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)) = \mathbf{R}\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}p_*(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee)) \in \mathbf{D}^-_{coh}(R)$$

Now Theorem 5.2.1 finishes the proof. □

Theorem 5.2.3. Let R be a universally coherent ring, and let X be a separated, finitely presented R-scheme. Let $\mathcal{F} \in \mathbf{D}^-_{qc}(X)$ be an object such that $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{acoh}(R)$ for any $\mathcal{P} \in \mathrm{Perf}(X)$, then $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(X)$. Analogously, if $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^b_{acoh}(R)$ for any $\mathcal{P} \in \mathrm{Perf}(X)$, then $\mathcal{F} \in \mathbf{D}^b_{qc,acoh}(X)$.

Proof. Once we have have Lemma 5.2.2 and the equality $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) = \mathbf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$, the first part of the Theorem is absolutely analogous to [Sta21, Tag 0CSH]. The second part now follows directly from [Sta21, Tag 09IS] and [BZNP17, Lemma 3.0.14]. □

5.3. **The GAGA Theorem.** The main goal of this section is to prove the formal GAGA Theorem for almost coherent sheaves. It roughly says that any adically quasi-coherent, almost coherent sheaf on a completion of a proper, finitely presented scheme admits an essentially unique algebraization, and the same holds for morphisms of those sheaves.

We start by recalling the statement of the classical formal GAGA Theorem. We start with a proper A-scheme for some complete adic noetherian ring A with the ideal of definition $\mathfrak{m}$. Then we consider the $\mathfrak{m}$-adic completion $\mathfrak{X}$ as a formal scheme over $\mathrm{Spf}\ A$. It comes with the natural morphism $c \colon \mathfrak{X} \to X$ of locally ringed spaces that induces a functor

$$c^* \colon \mathbf{Coh}_X \to \mathbf{Coh}_{\mathfrak{X}}$$

The GAGA Theorem says that it is an equivalence of categories. Let us say few words about the "classical" proof of this theorem. There are essentially three independent steps in the proof: the first one is to show that the morphism c is flat; the second one is to show that the functor c^* induces an isomorphism

$$c^* \colon \mathrm{H}^i(X, \mathrm{F}) \to \mathrm{H}^i(\mathfrak{X}, c^*\mathrm{F})$$

for any any $\mathrm{F} \in \mathbf{Coh}_X$ and any integer i. And the last one is to prove that any coherent sheaf $\mathcal{G} \in \mathbf{Coh}_{\mathbf{P}^N}$ admits a surjection of the form $\bigoplus_i \mathcal{O}(n_i)^{m_i} \to \mathcal{G}$. Though the first two steps generalizes to our Setup, there is no chance to have any analogue of the last statement. The reason is easy: existence of such a surjection would automatically imply that the sheaf $\mathcal{G}$ is of finite type, however almost coherent sheaves are usually not of finite type.

This issue suggests that we should take another approach to GAGA Theorems recently developed by J. Hall in his paper [Hal18]. The main advantage of this approach is that it firstly *constructs* a candidate for the algebraization, and only then he proves that this candidate works.

We start with the discussion of the GAGA functor. In what follows, we assume that R is a ring from the Setup 4.5.1. We pick a finitely presented R-scheme X, and we consider its I-adic completion $\mathfrak{X}$ that is a topologically finitely presented formal R-scheme. The formal scheme $\mathfrak{X}$ comes equipped with the canonical morphism of locally ringed spaces

$$c\colon (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to (X, \mathcal{O}_X)$$

that induces the functor

$$\mathbf{L}c^*\colon \mathbf{D}(X) \to \mathbf{D}(\mathfrak{X})$$

We now want to check that this functor "preserves" quasi-coherent, almost coherent objects. That is necessarily even to formulate the GAGA statement.

Lemma 5.3.1. Let R be a ring as in the Setup 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented A-scheme. Then the morphism c is flat, and the funtor $c^*\colon \mathbf{Mod}_X \to \mathbf{Mod}_{\mathfrak{X}}$ sends (quasi-coherent and) almost coherent sheaves to (adically quasi-coherent and) almost coherent sheaves. In particular, it induces functors

$$\mathbf{L}c^*\colon \mathbf{D}^*_{qc,acoh}(X) \to \mathbf{D}^*_{qc,acoh}(\mathfrak{X})$$

for any $* \in \{\text{“ ”}, +, -, b\}$.

Proof. The flatness assertion is just [FK18, Proposition I.1.4.7 (2)]. Flatness of c implies that it suffices to show that $c^*(\mathrm{G})$ is adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module for a quasi-coherent, almost coherent $\mathcal{O}_X$-module G. This claim is Zariski-local on X. Thus we can assume that $X = \operatorname{Spec} A$ is affine, so $\mathrm{G} \simeq \widetilde{M}$ for some almost finitely presented A-module M. This case is done in Lemma 4.6.3. $\square$

Theorem 5.3.2. Let R be a ring as in the Setup 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented, proper A-scheme. Then the functor

$$\mathbf{L}c^*\colon \mathbf{D}^*_{qc,acoh}(X) \to \mathbf{D}^*_{qc,acoh}(\mathfrak{X})$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

Corollary 5.3.3. Let R, A and X be as in Theorem 5.3.2. Then the functor

$$\mathbf{L}c^*\colon \mathbf{D}^*_{acoh}(X)^a \to \mathbf{D}^*_{acoh}(\mathfrak{X})^a$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

Corollary 5.3.4. Let R, A, and X be as in Theorem 5.3.2, and let $K \in \mathbf{D}_{qc,acoh}(X)$. Then the natural map

$$\beta_K\colon \mathbf{R}\Gamma(X, K) \to \mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}c^*K)$$

is an isomorphism. Moreover, the map β_K is an almost isomorphism for $K \in \mathbf{D}_{acoh}(X)$.

Proof. Note that the case of $K \in \mathbf{D}_{acoh}(X)$ follows from the case of $K \in \mathbf{D}_{qc,acoh}(X)$ due to Lemma 3.2.17 and Proposition 3.5.23. So it suffices to prove for $K \in \mathbf{D}_{qc,acoh}(X)$.

Now since we are allowed to replace K with $K[i]$ for any integer i, it suffices to show that

$$\mathrm{H}^0(\mathbf{R}\Gamma(X,K)) \simeq \mathrm{Hom}_X(\mathcal{O}_X, K) \to \mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}, \mathbf{L}c^*K) \simeq \mathrm{H}^0(\mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}c^*K)).$$

This follows from Theorem 5.3.2 and the observation that $\mathcal{O}_{\mathfrak{X}} \simeq \mathbf{L}c^*\mathcal{O}_X$. $\square$

We follow Jack Hall's proof of the GAGA Theorem very closely with according simplifications due to the flatness of the functor c^*. As he works entirely in the setting of the pseudo-coherent objects, and almost coherent sheaves may not be pseudo-coherent, we repeat some arguments in our setting.

Before going to the proof, we need to define the functor in the other direction. Recall that we always have a functor

$$\mathbf{R}c_*\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(X)$$

This functor is t-exact as $c\colon \mathfrak{X} \to X$ is topologically just a closed immersion. In particular, it preserves boundedness of complexes (in any direction). However, that functor usually does not preserve (almost) coherent objects as can be seen in the example of $\mathbf{R}c_*\mathcal{O}_{\mathfrak{X}} = c_*\mathcal{O}_{\mathfrak{X}}$. A way to fix it is to use a "so-called" quasi-coherator functor

$$\mathbf{R}Q_X\colon \mathbf{D}(X) \to \mathbf{D}_{qc}(X)$$

that is defined as the right adjoint to the inclusion $\iota\colon \mathbf{D}_{qc}(X) \to \mathbf{D}(X)$. It exists by [Sta21, Tag 0CR0]. So this allows us to define a functor

$$\mathbf{R}c_{qc}\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}_{qc}(X)$$

as the composition $\mathbf{R}c_{qc} := \mathbf{R}Q_X \circ \mathbf{R}c_*$.

Combining the adjunctions $(\mathbf{L}c^*, \mathbf{R}c_*)$ and $(\iota, \mathbf{R}Q_X)$, we conclude that we have a pair of the adjoint functors:

$$\mathbf{L}c^*\colon \mathbf{D}_{qc}(X) \rightleftarrows \mathbf{D}(\mathfrak{X}) :\mathbf{R}c_{qc}$$

That gives us the unit and counit morphisms

$$\eta\colon \mathrm{Id} \to \mathbf{R}c_{qc}\mathbf{L}c^* \text{ and } \varepsilon\colon \mathbf{L}c^*\mathbf{R}c_{qc} \to \mathrm{Id}$$

For future reference, we also note that the adjuntion and the monoidal property of the functor $\mathbf{L}c^*$ define a projection morphism

$$\pi_{\mathcal{G},\mathcal{F}}\colon \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathbf{R}c_{qc}\mathcal{F}) \to \mathbf{R}c_{qc}(\mathbf{L}c^*\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F})$$

for any $\mathcal{G} \in \mathbf{D}_{qc}(X)$ and any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$. Before discussing the actual proof of Theorem 5.3.2, we need to establish some formal properties of these functors. In particular, we need to verify that the unit and counit morphisms are isomorphisms in some easy special cases.

Lemma 5.3.5. Let R be a ring as in the Setup 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented A-scheme. Then there is an integer $N = N(X)$ such that $\mathbf{R}c_{qc}$ carries $\mathbf{D}^{\le n}_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^{\le n+N}_{qc}(X)$ (resp. $\mathbf{D}^{[a,n]}_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^{[a,n+N]}_{qc}(X)$) for any integer n. In particular, the natural map

$$\tau^{\ge a}\mathbf{R}c_{qc}\mathcal{F} \to \tau^{\ge a}(\mathbf{R}c_{qc}\tau^{\ge a-N}\mathcal{F})$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X})$ and any integer a.

Proof. We explain the proof that $\mathbf{R}c_{qc}$ carries $\mathbf{D}^{\leq n}_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^{\leq n+N}_{qc}(X)$; the case of $\mathbf{D}^{[a,n]}_{qc,acoh}(\mathfrak{X})$ is similar.

We start the proof by verifying the assumptions of [Sta21, Tag 0CSA] in our setting. Namely, we fix an object $\mathcal{F} \in \mathbf{D}^{\leq n}_{qc,acoh}(\mathfrak{X})$ and show that $\mathrm{H}^i(\mathbf{R}\Gamma(U, c_*\mathcal{F})) = 0$ for any open affine $U \subset X$ and any $i \geq n$. Indeed, we know that the functor $c_* \colon \mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}} \to \mathbf{Mod}_{\mathcal{O}_X}$ is exact as c is topologically just a closed immersion. Therefore, we see that

$$\mathrm{H}^i(\mathbf{R}\Gamma(U, c_*\mathcal{F})) = \mathrm{H}^i(\mathbf{R}\Gamma(\widehat{U}, \mathcal{F})) = \mathrm{H}^i(\widehat{U}, \mathcal{F}|_{\widehat{U}})$$

Lemma 4.8.11 implies that $\mathrm{H}^i(\widehat{U}, \mathcal{F}|_{\widehat{U}}) = 0$ for any $i \geq n$. Moreover, we know that $\mathbf{R}c_*\mathcal{F} \in \mathbf{D}^{\leq n}(X)$ as c_* is exact on $\mathbf{Mod}_{\mathfrak{X}}$ and $\mathcal{F} \in \mathbf{D}^{\leq n}(\mathfrak{X})$.

Now we apply [Sta21, Tag 0CSA] for $K = \mathbf{R}c_*\mathcal{F}$, $a = -\infty$ and $b = n$ to finish the proof of the first claim in the Lemma. One can check that the proof of that Lemma works well for $a = -\infty$.

The second claim of the lemma follows from the first claim and the distinguished triangle

$$\tau^{\leq a-N-1}\mathcal{F} \to \mathcal{F} \to \tau^{\geq a-N}\mathcal{F} \to \tau^{\leq a-N-1}\mathcal{F}[1]$$

Namely, we apply the exact functor $\mathbf{R}c_{qc}$ to this distinguished triangle to get that

$$\mathbf{R}c_{qc}\left(\tau^{\leq a-N-1}\mathcal{F}\right) \to \mathbf{R}c_{qc}\mathcal{F} \to \mathbf{R}c_{qc}\left(\tau^{\geq a-N}\mathcal{F}\right) \to \mathbf{R}c_{qc}\left(\tau^{\leq a-N-1}\mathcal{F}[1]\right)$$

is a distinguished triangle in $\mathbf{D}_{qc}(X)$ and that $\mathbf{R}c_{qc}(\tau^{\leq a-N-1}\mathcal{F}) \in \mathbf{D}^{\leq a-1}_{qc}(X)$. This implies that the map

$$\tau^{\geq a}\mathbf{R}c_{qc}\mathcal{F} \to \tau^{\geq a}\mathbf{R}c_{qc}\left(\tau^{\geq a-N}\mathcal{F}\right)$$

is an isomorphism. $\square$

Lemma 5.3.6. Let X be as in Theorem 5.3.2, $\mathcal{F} \in \mathbf{D}^{-}_{qc,acoh}(\mathfrak{X})$ and $\mathrm{G} \in \mathbf{D}^{-}_{qc}(X)$. Suppose that for each i there is n_i such that $I^{n_i}\mathcal{H}^i(\mathcal{F}) = 0$ and $I^{n_i}\mathcal{H}^i(\mathrm{G}) = 0$. Then the natural morphisms η_{G} and $\varepsilon\mathcal{F}$ are isomorphisms.

Proof. We prove the claim only for $\mathcal{F}$ as the other claim is similar.

Reduction to the case $\mathcal{F} \in \mathbf{D}^{b}_{qc,acoh}(\mathfrak{X})$: First of all, we note that it suffices to show that the natural maps

$$\tau^{\geq a}\mathcal{F} \to \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F}$$

is an isomorphism for any a. Moreover, we note that t-exactness of $\mathbf{L}c^*$ and Lemma 5.3.5 imply that there is an integer N such that the natural map $\tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F} \to \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{qc}\tau^{\geq a-N}\mathcal{F}$ is an isomorphism for any integer a. In particular, we have a commutative diagram

$$\begin{tikzcd}
\tau^{\geq a-N}\mathcal{F} \arrow[r] \arrow[d] & \mathbf{L}c^*\mathbf{R}c_{qc}(\tau^{\geq a-N}\mathcal{F}) \arrow[d] \\
\tau^{\geq a}\mathcal{F} \arrow[r] & \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F} \simeq \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{qc}\tau^{\geq a-N}\mathcal{F}
\end{tikzcd}$$

where the vertical maps induce isomorphisms in degree $\geq a$. Therefore, it suffices to prove the claim for $\tau^{\geq a-N}\mathcal{F}$. So we may and do assume that $\mathcal{F}$ is bounded.

Proof for a bounded $\mathcal{F}$: The case of a bounded $\mathcal{F}$ easily reduces to the case of an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module concentrated in degree 0. In that situation we have an adically quasi-coherent module $\mathcal{F}$ such that $I^{k+1}\mathcal{F} = 0$ for some k. That implies that $\mathcal{F} = i_{k,*}\mathcal{F}_k = \mathbf{R}i_{k,*}\mathcal{F}_k$ for the closed immersion $i_k\colon X_k \to \mathfrak{X}$. Now it is straightforward to see that the canonical map

$$\mathbf{R}i_{k,*}\mathcal{F}_k \to \mathbf{L}c^*\mathbf{R}c_{qc}(\mathbf{R}i_{k,*}\mathcal{F}_k)$$

is an isomorphism. The key is flatness of c and the observation that $\mathbf{R}c_*(\mathbf{R}i_{k,*}\mathcal{F}_k)$ is already quasi-coherent, so the quasi-coherator does nothing in this case. □

The other thing we need to check is that the map $\pi_{\mathcal{G},\mathcal{F}}$ is an isomorphism for $\mathcal{G} \in \mathrm{Perf}(X)$. As this statement is proven in [Hal18] without any pseudo-coherence assumption on $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$, we just cite it here.

Lemma 5.3.7. If $\mathrm{G} \in \mathbf{D}_{qc}(X)$ and $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$, then the natural projection morphism

$$\pi_{\mathrm{G},\mathcal{F}}\colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}c_{qc}\mathcal{F} \to \mathbf{R}c_{qc}(\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F})$$

is an isomorphism if G is perfect.

Proof. [Hal18, Lemma 4.3] □

Now we come to the key input ingredient. Even though $\mathbf{R}c_{qc}$ is quite abstract and difficult to compute in practice, it turns out that the Almost Proper Mapping Theorem allows us to check that this functor sends $\mathbf{D}^-_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^-_{qc,acoh}(X)$. That would give us a candidate for the algebraization.

Lemma 5.3.8. Let R be a ring as in the Setup 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented, proper A-scheme. Then $\mathbf{R}c_{qc}$ sends $\mathbf{D}^*_{qc,acoh}(\mathfrak{X})$ to $\mathbf{D}^*_{qc,acoh}(X)$ for $* \in \{-,\ b\}$.

Proof. We prove only the bounded above case as the other one follows from this using Lemma 5.3.5. We pick any $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$ and we use Theorem 5.2.3 to say that it is sufficient to show that $\mathbf{R}\mathrm{Hom}_X(\mathrm{P}, \mathbf{R}c_*\mathcal{F}) \in \mathbf{D}^-_{acoh}(R)$ for any perfect complex $\mathrm{P} \in \mathrm{Perf}(X)$. That turns out to be a formal consequence of the Almost Propper Mapping Theorem 5.1.6. Indeed, we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_X(\mathrm{P}, \mathbf{R}c_{qc}\mathcal{F}) &= \mathbf{R}\mathrm{Hom}_{\mathfrak{X}}(\mathbf{L}c^*\mathrm{P}, \mathcal{F})\\ &= \mathbf{R}\mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}, (\mathbf{L}c^*\mathrm{P})^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F})\\ &= \mathbf{R}\Gamma(\mathfrak{X}, (\mathbf{L}c^*\mathrm{P})^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}) \in \mathbf{D}^-_{acoh}(R),\end{aligned}$$

where the last formula comes from the fact that derived pullback and derived dual operations preserve perfect complexes, and for any $\mathcal{P} \in \mathrm{Perf}(\mathfrak{X})$ we have $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$. □

Finally, we are ready to give a proof of the GAGA Theorem.

Proof of Theorem 5.3.2. Claim 0: It suffices to show the theorem for $* = -$, i.e. for bounded above derived categories. Indeed, flatness of c^* implies that $\mathbf{L}c^*$ preserve boundedness (resp. boundedness above, resp. boundedness below), so it suffices to show that the natural morphisms

$$\eta_{\mathrm{G}}\colon \mathrm{G} \to \mathbf{R}c_{qc}\mathbf{L}c^*\mathrm{G}$$

$$\varepsilon_{\mathcal{F}}\colon \mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F} \to \mathcal{F}$$

are isomorphisms for any $\mathrm{G} \in \mathbf{D}_{qc,acoh}(X)$ and $\mathcal{F} \in \mathbf{D}_{qc,acoh}(\mathfrak{X})$.

We fix N as in Lemma 5.3.5. Then flatness of c^* and Lemma 5.3.5 guarantee that

$$\mathbf{R}c_{qc}\mathbf{L}c^*\tau^{\geq a}\mathrm{G} \in \mathbf{D}^{[a,\infty]}(X)$$

$$\mathbf{L}c^*\mathbf{R}c_{qc}\tau^{\geq a}\mathcal{F} \in \mathbf{D}^{[a,\infty]}(\mathfrak{X}).$$

Therefore, we see that η_{G} is an isomorphism on $\mathcal{H}^i$ for $i < a$ if and only if the same holds for $\eta_{\tau^{\le a-1}\mathrm{G}}$. Since a was arbitrary, we conclude that it suffices to show that η_{G} is an isomorphism for $\mathrm{G} \in \mathbf{D}^-_{qc,acoh}(X)$. Similar argument shows that it suffices to show that $\varepsilon_{\mathcal{F}}$ is an isomorphism for $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$. So it suffices to prove the theorem for $* = -$.

Before we formulate the next claim, we need to use the so-called "approximation by perfect complexes" [Sta21, Tag 08EL] to find some $\mathrm{P} \in \mathrm{Perf}(X)$ such that $\tau^{\ge 0}\mathrm{P} \simeq \mathcal{O}_X/I \simeq \mathcal{O}_{X_0}$ and whose support is equal to X_0. We note that it implies that all cohomology sheaves $\mathcal{H}^i(\mathrm{P})$ are killed by some power of I. We also denote its (derived) pullback by $\mathcal{P} := \mathbf{L}c^*\mathrm{P}$.

Claim 1: If $\mathrm{G} \in \mathbf{D}^-_{qc,acoh}(X)$ such that $\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \simeq 0$, then $\mathrm{G} \simeq 0$. Similarly, if $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$ such that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P} \simeq 0$, then $\mathcal{F} \simeq 0$.

We choose the maximal m (assuming that $\mathrm{G} \not\simeq 0$)such that $\mathcal{H}^m(\mathrm{G}) \neq 0$. Then we see that $\mathcal{H}^m(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}) \simeq \mathcal{H}^m(\mathrm{G}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} = \mathcal{H}^m(\mathrm{G})/I$. We have $(\mathcal{H}^m(\mathrm{G})/I)(U) = \mathcal{H}^m(\mathrm{G})(U)/I \simeq 0$ on any open affine U. So Nakayama's Lemma 2.5.19 implies that $\mathcal{H}^m(\mathrm{G})(U) \simeq 0$ for any such U. This contradicts the choice of m. The proof in the formal setup is the same once we notice that $\mathcal{H}^0(\mathcal{P}) = \mathcal{O}_{\mathfrak{X}}/I$.

Claim 2: The map $\eta_{\mathrm{G}} \colon \mathrm{G} \to \mathbf{R}c_{qc}\mathbf{L}c^*\mathrm{G}$ is an isomorphism for any $\mathrm{G} \in \mathbf{D}^-_{qc,acoh}(X)$.

Claim 1 implies that it is sufficient to show that the map

$$\varepsilon_{\mathrm{G}} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \to \mathbf{R}c_{qc}\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \tag{5.2}$$

is an isomorphism. Recall that the cohomology sheaves of P are killed by some power of ϖ. This property passes to $\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}$, so we can use Lemma 5.3.6 to get that the map

$$\varepsilon_{\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}} \colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \to \mathbf{R}c_{qc}\left(\mathbf{L}c^*\left(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} P\right)\right)$$

is an isomorphism. Now comes the key: we fit the morphism $\varepsilon_{\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}}$ into the following commutative triangle:

$$\begin{tikzcd}
\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \arrow[r, "\varepsilon_{\mathrm{G}} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}"] \arrow[d, "\varepsilon_{\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}}"'] & \mathbf{R}c_{qc}\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \arrow[d, "\pi_{\mathrm{P}, \mathbf{L}c^*\mathrm{G}}"] \\
\mathbf{R}c_{qc}(\mathbf{L}c^*(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)) \arrow[r] & \mathbf{R}c_{qc}(\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P})
\end{tikzcd}$$

where the bottom horizontal arrow is the isomorphism map induced by the monoidal structure on $\mathbf{L}c^*$. Moreover, we have already established that the left vertical arrow is an isomorphism, and right vertical arrow is an isomorphism due to Lemma 5.3.7. That shows that the top horizontal must be also an isomorphism.

Claim 3: The map $\varepsilon_{\mathcal{F}} \colon \mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F} \to \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathbf{D}^-_{qc,acoh}(\mathfrak{X})$.

We use Claim 1 again to say that it is sufficient to show that the map

$$\varepsilon_{\mathcal{F}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P} \colon \mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P}$$

is an isomorphism. But that map fits into the commutative diagram:

$$\begin{tikzcd}
\mathbf{L}c^*\mathbf{R}c_{qc}\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P} \arrow[rr, "\varepsilon_{\mathcal{F}}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P}"] \arrow[d] & & \mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P} \\
\mathbf{L}c^*(\mathbf{R}c_{qc}\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathrm{P}) \arrow[rr, "\mathbf{L}c^*(\pi_{\mathrm{P},\mathcal{F}})"] & & \mathbf{L}c^*\mathbf{R}c_{qc}(\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P}) \arrow[u, "\varepsilon_{\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P}}"']
\end{tikzcd}$$

where the left vertical morphism is the canonical isomorphism induced by the monoidal structure on $\mathbf{L}c^*$, the bottom morphism is an isomorphism by Lemma 5.3.7, and the right vertical morphism is an isomorphism by Lemma 5.3.6. This implies that the top horizontal morphism is an isomorphism and that finishes the proof. $\square$

5.4. **The Formal Function Theorem.** We derive the Formal Function Theorem for almost coherent sheaves from the Formal GAGA theorem in this Section. As an intermediate step, we compare the natural I-topology (see Definition 5.4.2) on cohomology groups of proper schemes to the I-adic topology. They turn out to coincide for proper schemes and almost coherent coefficient sheaves.

For the rest of the section, we fix a ring R as in the Setup 4.5.1 and a finitely presented or topologically finitely presented R-algebra A.

Remark 5.4.1. Both A and $\widehat{A}$ are also topologically universally adhesive by [FK18, Proposition 0.8.5.19], and they are (topologically universally) coherent by [FK18, Proposition 0.8.5.23].

For the next definition, we fix a finitely presented A-scheme X and an $\mathcal{O}_X$-module $\mathcal{F}$.

Definition 5.4.2. The *natural I-filtration* $\mathrm{F}^\bullet\mathrm{H}^i(X,\mathcal{F})$ is

$$\mathrm{F}^n\mathrm{H}^i(X,\mathcal{F}) := \mathrm{Im}\left(\mathrm{H}^i(X,I^n\mathcal{F}) \to \mathrm{H}^i(X,\mathcal{F})\right)$$

The *natural I-topology* on $\mathrm{H}^i(X,\mathcal{F})$ is the topology induced by the filtration $\mathrm{F}^\bullet\mathrm{H}^i(X,\mathcal{F})$.

Lemma 5.4.3. Let X be a finitely presented A-scheme, $\mathcal{F}$ an quasi-coherent almost finitely generated $\mathcal{O}_X$-module, and $\mathcal{G}\subset\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$. Then, for any n, there is m such that $I^m\mathcal{F}\cap\mathcal{G}\subset I^n\mathcal{G}$.

Proof. It suffices to assume that X is affine, in which case it follows from Lemma 2.12.6. $\square$

Lemma 5.4.4. Let X be a finitely presented A-scheme, $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent almost finitely generated $\mathcal{O}_X$-modules, and $\varphi\colon\mathcal{G}\to\mathcal{F}$ an $\mathcal{O}_X$-linear homomorphism such that $\ker(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by I^c for some integer c. Then, for every $i\geq 0$, the natural I-topology on $\mathrm{H}^i(X,\mathcal{F})$ coincides with the topology induced by the filtration

$$\mathrm{Fil}^n_{\mathcal{G}}\mathrm{H}^i(X,\mathcal{F}) = \mathrm{Im}(\mathrm{H}^i(X,I^n\mathcal{G})\to\mathrm{H}^i(X,\mathcal{F})).$$

Proof. Consider the short exact sequences

$$0\to\mathcal{K}\to\mathcal{G}\to\mathcal{H}\to 0,$$

$$0\to\mathcal{H}\to\mathcal{F}\to\mathcal{Q}\to 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are annihilated by I^c. The first short exact sequence induced the short exact sequence

$$0\to\mathcal{K}\cap I^m\mathcal{G}\to I^m\mathcal{G}\to I^m\mathcal{H}\to 0$$

for any $m \geq 0$. Lemma 5.4.3 implies that $\mathcal{K} \cap I^m\mathcal{G} \subset I^c\mathcal{K} = 0$ for large enough m. Therefore, the natural map $I^m\mathcal{G} \to I^m\mathcal{H}$ is an isomorphism for large enough m. Note that $\mathcal{H}$ is almost finitely generated and quasi-coherent, so we can replace $\mathcal{G}$ with $\mathcal{H}$ to assume that φ is injective.

Now clearly $\mathrm{Fil}^k_{\mathcal{G}}\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{F}^k\mathrm{H}^i(X, \mathcal{F})$ for every k. So it suffices to show that, for any k, there m such that $\mathrm{F}^m\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Fil}^k_{\mathcal{G}}\mathrm{H}^i(X, \mathcal{F})$. We consider the short exact sequence

$$0 \to \mathcal{G} \cap I^m\mathcal{F} \to I^m\mathcal{F} \to I^m\mathcal{Q} \to 0.$$

If $m \geq c$ we get that $\mathcal{G} \cap I^m\mathcal{F} = I^m\mathcal{F}$ because $I^c\mathcal{Q} \simeq 0$. Now we use Lemma 5.4.3 to conclude there is $m \geq c$ such that

$$I^m\mathcal{F} = \mathcal{G} \cap I^m\mathcal{F} \subset I^k\mathcal{G}$$

Therefore, $\mathrm{F}^m\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Fil}^k_{\mathcal{G}}\mathrm{H}^i(X, \mathcal{F})$. □

Lemma 5.4.5. Let X be a finitely presented A-scheme, $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent almost finitely generated $\mathcal{O}_X$-modules, and $\varphi\colon \mathcal{G} \to \mathcal{F}$ an $\mathcal{O}_X$-linear homomorphism such that $\ker(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by I^c for some integer c. Suppose that the natural I-topology on $\mathrm{H}^i(X, \mathcal{G})$ is the I-adic topology. Then the same holds for $\mathrm{H}^i(X, \mathcal{F})$.

Proof. Clearly, $I^n\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{F}^n\mathrm{H}^i(X, \mathcal{F})$. So it suffices to show that, for every n, there is an m such that $\mathrm{F}^m\mathrm{H}^i(X, \mathcal{F}) \subset I^n\mathrm{H}^i(X, \mathcal{F})$.

The assumption that the natural I-topology on $\mathrm{H}^i(X, \mathcal{G})$ coincides with the I-adic topology guarantees that $\mathrm{F}^k\mathrm{H}^i(X, \mathcal{G}) \subset I^n\mathrm{H}^i(X, \mathcal{G})$ for large enough k. Pick such k. Lemma 5.4.4 implies that

$$\mathrm{F}^m\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Im}(\mathrm{H}^i(X, I^k\mathcal{G}) \to \mathrm{H}^i(X, \mathcal{F}))$$

for large enough m. So we get, for such m, that

$$\mathrm{F}^m\mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Im}\left(\mathrm{H}^i(X, I^k\mathcal{G}) \to \mathrm{H}^i\left(X, \mathcal{F}\right)\right) \subset \mathrm{Im}\left(I^n\mathrm{H}^i\left(X, \mathcal{G}\right) \to \mathrm{H}^i\left(X, \mathcal{F}\right)\right) \subset I^n\mathrm{H}^i\left(X, \mathcal{F}\right)$$

for a large enough m. □

Theorem 5.4.6. Let X a proper, finitely presented A-scheme, and $\mathcal{F}$ a quasi-coherent, almost coherent $\mathcal{O}_X$-module. Then the natural I-topology on $\mathrm{H}^i(X, \mathcal{F})$ coincides with the I-adic topology for any $i \geq 0$.

Proof. Lemma 4.7.3 guarantees that there is a finitely presented $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism $\varpi\colon \mathcal{G} \to \mathcal{F}$ such that $I^c(\ker\varphi) = 0$ and $I^c(\mathrm{Coker}\,\varphi) = 0$. Lemma 5.4.5 then ensures that it suffices to prove the claim for $\mathcal{G}$. In this case, the claim follows [FK18, Proposition I.8.5.2 and Lemma 0.7.4.3] and Remark 5.4.1. □

We consider a proper, finitely presented A-scheme X, and an almost coherent $\mathcal{O}_X$-module $\mathcal{F}$. We denote the I-adic completion of X by $\mathfrak{X}$, so we have a commutative diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "c"] \arrow[d, "\widehat{f}"] & X \arrow[d, "f"] \\ \mathrm{Spf}\,(\widehat{A}) \arrow[r] & \mathrm{Spec}\,A \end{tikzcd}$$

Given this diagram we can consider four different cohomology groups:

$$\mathrm{H}^i(\mathfrak{X}, c^*\mathcal{F}),\ \widehat{\mathrm{H}^i(X, \mathcal{F})},\ \mathrm{H}^i(X, \mathcal{F}) \otimes_A \widehat{A},\ \text{and}\ \lim_n \mathrm{H}^i(X_n, \mathcal{F}_n).$$

All these groups have a natural structure of $\widehat{A}$-module, and it is straightforward to construct functorial in $\mathcal{F}$ homomorphisms

$$
\begin{tikzcd}
\mathrm{H}^i(X,\mathcal{F})\otimes_A \widehat{A} \arrow[r, "\alpha^i_{\mathcal{F}}"] \arrow[d, "\beta^i_{\mathcal{F}}"] & \widehat{\mathrm{H}^i(X,\mathcal{F})} \arrow[d, "\phi^i_{\mathcal{F}}"] \\
\mathrm{H}^i(\mathfrak{X}, c^*\mathcal{F}) \arrow[r, "\gamma^i_{\mathcal{F}}"] & \lim_n \mathrm{H}^i(X_n,\mathcal{F}_n)
\end{tikzcd}
\tag{5.3}
$$

We show that all these morphisms are (almost) isomorphisms.

Theorem 5.4.7. In the notation as above, all the maps $\alpha^i_{\mathcal{F}}$, $\beta^i_{\mathcal{F}}$, $\gamma^i_{\mathcal{F}}$, $\phi^i_{\mathcal{F}}$ are almost isomorphisms for any almost coherent $\mathcal{O}_X$-module $\mathcal{F}$. If $\mathcal{F}$ is quasi-coherent, almost coherent, then these maps are isomorphisms.

Proof. Step 0. Reduction to the case of a quasi-coherent, almost coherent sheaf $\mathcal{F}$: We observe that Lemma 3.3.2, Lemma 3.2.17 and the fact that limits of two almost isomorphic direct systems are almost the same allow us to replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}}\otimes\mathcal{F}$ to assume that $\mathcal{F}$ is quasi-coherent and almost coherent.

Step 1. $\alpha^i_{\mathcal{F}}$ is an isomorphism: This is just a consequence of Lemma 2.12.7 as we established in Theorem 5.1.3 that $\mathrm{H}^i(X,\mathcal{F})$ is an almost coherent A-module.

Step 2. $\beta^i_{\mathcal{F}}$ is an isomorphism: We note that the assumptions on A imply that the map $A\to\widehat{A}$ is flat by [FK18, Proposition 0.8.218]. Thus the flat base change for quasi-coherent cohomology groups implies that $\mathrm{H}^i(X,\mathcal{F})\otimes_A\widehat{A}\simeq\mathrm{H}^i(X_{\widehat{A}},\mathcal{F}_{\widehat{A}})$. Therefore, we may and do assume that A is ϖ-adically complete. Then the map $\mathrm{H}^i(X,\mathcal{F})\to\mathrm{H}^i(\mathfrak{X},c^*\mathcal{F})$ is an isomorphism by Theorem 5.3.2.

Step 3. $\alpha^i_{\mathcal{F}}$ is an injective: Theorem 5.4.6 and Corollary 5.3.4 imply that the I-adic topology of $\mathrm{H}^i(X,\mathcal{F})$ coincides with the natural I-topology. Therefore,

$$\widehat{\mathrm{H}^i(X,\mathcal{F})}\simeq\lim_n\frac{\mathrm{H}^i(X,\mathcal{F})}{\mathrm{Im}\left(\mathrm{H}^i(X,I^{n+1}\mathcal{F})\to\mathrm{H}^i(X,\mathcal{F})\right)}.$$

Clearly, we have an inclusion

$$\frac{\mathrm{H}^i(X,\mathcal{F})}{\mathrm{Im}\left(\mathrm{H}^i(X,I^{n+1}\mathcal{F})\to\mathrm{H}^i(X,\mathcal{F})\right)}\hookrightarrow\mathrm{H}^i(X_n,\mathcal{F}_n).$$

Therefore, we conclude that $\alpha^i_{\mathcal{F}}$ is injective by left exactness of the limit functor.

Step 4. $\gamma^i_{\mathcal{F}}$ is surjective: Recall that $\mathcal{F}\simeq\lim_k\mathcal{F}_k$ because $\mathcal{F}$ is adically quasi-coherent. Therefore, [FK18, Corollary 0.3.2.16] implies that it is sufficient to show that there is a basis of opens $\mathcal{B}$ such that, for every $\mathfrak{U}\in\mathcal{B}$,

$$\mathrm{H}^i(\mathfrak{U},\mathcal{F})=0 \text{ for } i\geq 1, \text{ and}$$
$$\mathrm{H}^0(\mathfrak{U},\mathcal{F}_{k+1})\to\mathrm{H}^0(\mathfrak{U},\mathcal{F}_k) \text{ is surjective for any } k\geq 0.$$

Vanishing of the higher cohomology groups of adically quasi-coherent sheaves on affine formal schemes (see [FK18, Theorem I.7.1.1]) implies that one can take $\mathcal{B}$ to be the basis consisting of open affine formal subschemes of $\mathfrak{X}$. Therefore, we get that $\gamma^i_{\mathcal{F}}$ is indeed surjective for any $i\geq 0$.

Step 5. $\alpha^i_{\mathcal{F}}$ and $\gamma^i_{\mathcal{F}}$ are isomorphisms: This follows formally from commutativity of Diagram 5.4 and the previous steps. ∎

5.5. **Almost Version of Grothendieck Duality.** For this section, we fix a universally coherent ring R with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. Since R is universally coherent, there is a good theory of $f^!$ functor for separated morphisms between finitely presented R-schemes[29].

Proposition 5.5.1. Let $f\colon X \to Y$ be a separated morphism of finitely presented R-schemes. Then $f^!$ sends $\mathbf{D}^+_{qc,acoh}(Y)$ to $\mathbf{D}^+_{qc,acoh}(X)$.

Proof. The only thing that we need to check here is that $f^!$ preserves almost coherence of cohomology sheaves. This statement is local, so we can assume that both X and Y are affine. Then we can choose a closed embedding $X \to \mathbf{A}^n_Y \to Y$. So, it suffices to prove the claim for a finitely presented closed immersion and for the morphism $\mathbf{A}^n_Y \to Y$.

In the case $f\colon X \to Y$ a finitely presented closed immersion, we know that

$$f^!(\mathcal{F}) \simeq \mathbf{R}\underline{\mathcal{H}om}_Y(f_*\mathcal{O}_X, \mathcal{F})$$

for any $\mathcal{F} \in \mathbf{D}^+_{qc}(Y)$. Since Y is a coherent scheme and f is finitely presented, we conclude that $f_*\mathcal{O}_X$ is an almost coherent $\mathcal{O}_Y$-module. Therefore, $f^!(\mathcal{F}) = \mathbf{R}\underline{\mathcal{H}om}_Y(f_*\mathcal{O}_X, \mathcal{F}) \in \mathbf{D}_{qc,acoh}(X)$ by Corollary 4.4.11.

Now we consider the case of a relative affine space $f\colon X = \mathbf{A}^n_Y \to Y$. In this case, the formula for $f^!$ is $f^!(\mathcal{F}) \simeq \mathbf{L}f^*\mathcal{F} \otimes^L_{\mathcal{O}_X} \Omega^n_{X/Y}[n]$. Then $\mathbf{L}f^*(\mathcal{F}) \in \mathbf{D}^+_{qc,acoh}(X)$ by Lemma 4.4.7**(4)**, and so $\mathbf{L}f^*\mathcal{F} \otimes^L_{\mathcal{O}_X} \Omega^n_{X/Y}[n] \in \mathbf{D}^+_{qc,acoh}(X)$ because $\Omega^n_{X/Y}$ is (non-canonically) isomorphic to $\mathcal{O}_X$. □

Now we use Proposition 5.5.1 to define the almost version of the upper shriek functor:

Definition 5.5.2. Let $f\colon X \to Y$ be a separated morphism of finitely presented R-schemes. We define $f^!_a\colon \mathbf{D}^+_{aqc}(Y)^a \to \mathbf{D}^+_{aqc}(X)^a$ as $f^!_a(\mathcal{F}) := (f^!(\mathcal{F}_!))^a$.

Remark 5.5.3. In what follows, we will usually denote the functor $f^!_a$ simply by $f^!$ as it will not cause any confusion.

Lemma 5.5.4. Let $f\colon X \to Y$ be a separated morphism of finitely presented R-schemes. Then $f^!$ carries $\mathbf{D}^+_{acoh}(Y)^a$ to $\mathbf{D}^+_{acoh}(X)^a$.

Proof. This follows from Proposition 5.5.1. □

Theorem 5.5.5. Let $f\colon X \to Y$ be as above. Suppose that f is proper. Then $f^!\colon \mathbf{D}^+_{aqc}(Y)^a \to \mathbf{D}^+_{aqc}(X)^a$ is right adjoint to the functor $\mathbf{R}f_*\colon \mathbf{D}^+_{aqc}(Y)^a \to \mathbf{D}^+_{aqc}(X)^a$.

We note that the theorem makes sense as $\mathbf{R}f_*$ carries $\mathbf{D}^+_{aqc}(X)^a$ into $\mathbf{D}^+_{aqc}(Y)$ by Lemma 4.4.9.

Proof. This follows from a sequence of canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(Y)^a}(\mathbf{R}f_*\mathcal{F}^a, \mathcal{G}^a) &\simeq \mathrm{Hom}_{\mathbf{D}(Y)}(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}, \mathcal{G}) && \text{Lemma 3.1.13} \\
&\simeq \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}), \mathcal{G}) && \text{Lemma 3.3.6} \\
&\simeq \mathrm{Hom}_{\mathbf{D}(X)}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, f^!(\mathcal{G})) && \text{Grothendieck Duality} \\
&\simeq \mathrm{Hom}_{\mathbf{D}(X)^a}(\mathcal{F}^a, f^!(\mathcal{G})^a) && \text{Lemma 3.1.13.}
\end{aligned}$$

□

[29] This theory does not seem to be addressed in the literature in this generality, however we all the arguments from [Sta21, Tag 0DWE] can be adapted to this level generality with little or no extra work.

Now suppose that $f\colon X \to Y$ be a proper morphism of finitely presented R-schemes, $\mathcal{F}^a \in \mathbf{D}^+_{aqc}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}^+_{aqc}(Y)^a$. Then we want to construct a canonical morphism

$$\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \to \mathbf{R}\underline{al\mathcal{H}om}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a).$$

Lemma 3.5.16 says that such a map is equivalent to a map

$$\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \otimes^L_{\mathcal{O}_X} \mathbf{R}f_*(\mathcal{F}^a) \to \mathcal{G}^a.$$

We construct the latter map as the composition

$$\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \otimes^L_{\mathcal{O}_X} \mathbf{R}f_*(\mathcal{F}^a) \to \mathbf{R}f_*\left(\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \otimes^L_{\mathcal{O}_X} \mathcal{F}^a\right) \to \mathbf{R}f_*f^!\mathcal{G}^a \to \mathcal{G}^a$$

where the first map is induced by the relative cup product ([Sta21, Tag 0B68]), the second map comes from Remark 3.5.15, and the last map is the counit of the $(\mathbf{R}f_*, f^!)$-adjunction.

Lemma 5.5.6. Let $f\colon X \to Y$ be a proper morphism of finitely presented R-schemes, $\mathcal{F}^a \in \mathbf{D}^-_{acoh}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}^+_{aqc}(Y)^a$. Then the map

$$\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \to \mathbf{R}\underline{al\mathcal{H}om}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a).$$

is an (almost) isomorphism in $\mathbf{D}^+_{aqc}(X)^a$.

Proof. We note that $\mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X(\mathcal{F}^a, f^!(\mathcal{G}^a))$ lies in $\mathbf{D}^+_{aqc}(Y)^a$ by Lemma 4.4.10 **(4)** and Lemma 4.4.9. Likewise, $\mathbf{R}\underline{al\mathcal{H}om}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a)$ lies in $\mathbf{D}^+_{aqc}(Y)^a$ by Theorem 5.1.3 and Lemma 4.4.10 **(4)**. Therefore, it suffices to show

$$\mathbf{R}\mathrm{Hom}_Y\left(\mathcal{H}^a, \mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X\left(\mathcal{F}^a, f^!\left(\mathcal{G}^a\right)\right)\right) \to \mathbf{R}\mathrm{Hom}_Y\left(\mathcal{H}^a, \mathbf{R}\underline{al\mathcal{H}om}_Y\left(\mathbf{R}f_*\left(\mathcal{F}^a\right), \mathcal{G}^a\right)\right)$$

is an isomorphism for any $\mathcal{H}^a \in \mathbf{D}^+_{aqc}(Y)^a$. This follows from the following sequence of isomorphisms:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_Y\left(\mathcal{H}^a, \mathbf{R}f_*\mathbf{R}\underline{al\mathcal{H}om}_X\left(\mathcal{F}^a, f^!\left(\mathcal{G}^a\right)\right)\right) &\simeq \mathbf{R}\mathrm{Hom}_X\left(\mathbf{L}f^*\mathcal{H}^a, \mathbf{R}\underline{al\mathcal{H}om}_X\left(\mathcal{F}^a, f^!\left(\mathcal{G}^a\right)\right)\right) \\
&\simeq \mathbf{R}\mathrm{Hom}_X\left(\mathbf{L}f^*\mathcal{H}^a \otimes^L_{\mathcal{O}_X} \mathcal{F}^a, f^!\left(\mathcal{G}^a\right)\right) \\
&\simeq \mathbf{R}\mathrm{Hom}_Y\left(\mathbf{R}f_*\left(\mathbf{L}f^*\mathcal{H}^a \otimes^L_{\mathcal{O}_X} \mathcal{F}^a\right), \mathcal{G}^a\right) \\
&\simeq \mathbf{R}\mathrm{Hom}_Y\left(\mathcal{H}^a \otimes \mathbf{R}f_*\left(\mathcal{F}^a\right), \mathcal{G}^a\right) \\
&\simeq \mathbf{R}\mathrm{Hom}_Y\left(\mathcal{H}^a, \mathbf{R}\underline{al\mathcal{H}om}_Y\left(\mathbf{R}f_*\left(\mathcal{F}^a\right), \mathcal{G}^a\right)\right).
\end{aligned}$$

The first isomorphism holds by Corollary 3.5.26. The second isomorphism holds by Corollary 3.5.16. The third isomorphism holds by Theorem 5.5.5. The fourth isomorphism holds by Proposition 4.4.12. The fifth equality holds by Corollary 3.5.16. □

Theorem 5.5.7. Let $f\colon X \to Y$ be as above. Suppose that f is smooth of pure dimension d. Then $f^!(-) \simeq \mathbf{L}f^*(-) \otimes^L_{\mathcal{O}_X} \Omega^d_{X/Y}[d]$

Proof. It follows from the corresponding statement in the classical Grothendieck Duality. □

We summarize all the results of this section in the following theorem:

Theorem 5.5.8. Let R be a universally coherent ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} \coloneqq \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$, and FPS_R be the category of finitely presented, separated R-schemes. Then there is a well-defined functor $(-)^!$ from FPS_R into the 2-category of categories such that

(1) $(X)^! = \mathbf{D}^+_{aqc}(X)^a$,

(2) for a smooth morphism $f\colon X \to Y$ of pure relative dimension d, $f^! \simeq \mathbf{L}f^*(-) \otimes^L_{\mathcal{O}_X^a} \Omega^d_{X/Y}[d]$.

(3) for a proper morphism $f\colon X \to Y$, $f^!$ is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{aqc}(X)^a \to \mathbf{D}^+_{aqc}(Y)^a$.

6. Almost Coherence of "p-adic Nearby Cycles"

6.1. **Introduction.** The main goal of this section is to study the "p-adic Nearby Cycles" sheaves $\mathbf{R}\nu_*\mathcal{O}^+_{X^\diamond}$ and $\mathbf{R}\nu_*\mathcal{O}^+_{X^\diamond}/p$ for a rigid-analytic variety X and versions with more general "coefficients" including $\mathcal{O}^+/p$ vector bundles in the v-topology, and sheaves of the form $\mathcal{O}^+_{X^\diamond}/p \otimes \mathcal{F}$ for a Zariski-constructible sheaf $\mathcal{F}$ (see Definition 6.1.7). These complexes turn to be very close to complexes of coherent sheaves that makes it possible to study étale cohomology groups of rigid-analytic varieties using (almost) coherent methods on the special fiber.

Before giving precise definitions, let us explain the main motivation to study these sheaves and their relation with étale cohomology of rigid-analytic varieties in the simplest case of the "nearby cycles" of the sheaf $\mathcal{O}^+_{X-\text{ét}}/p$. In [Sch13], P. Scholze proved ([Sch13, Theorem 5.1]) that the étale cohomology groups $\mathrm{H}^i(X, \mathbf{F}_p)$ are finite for any smooth, proper rigid-analytic variety X over an algebraically closed p-adic non-archimedean field C. There are two important ingredients: the almost primitive comparison theorem that says that $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ are *almost* isomorphic to $\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p$, and the almost finiteness of $\mathrm{H}^i(X, \mathcal{O}^+_{X+\text{ét}}/p)$.

The proof of the almost finiteness result in [Sch13] uses properness of the space X in a very elaborate way; namely, he constructs some "good covering" of X by affinoids and then shows that there is enough cancelation in the Čech-to-Derived spectral sequence associated with that covering. We note that the second page of this spectral sequence has all terms being not almost finitely generated, but mysteriously there is enough cancellations in this spectral sequence so that the terms on the ∞-page turn out to be almost finitely generated. We refer to [Sch13, §5] for the details of this proof.

Our main goal is to give a more geometric way to prove that almost finiteness result. Instead of constructing some explicit "nice" covering of X, we separate the problem into two different problems. We choose an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ of X and consider the associated morphism of ringed topoi

$$t\colon (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}})$$

that induces the morphism

$$t\colon (X_{\text{ét}}, \mathcal{O}^+_{X+\text{ét}}/p) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}}/p) = (\mathfrak{X}_0, \mathcal{O}_{X_0})$$

where $\mathfrak{X}_0 \coloneqq \mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}_C} \mathrm{Spec}\, \mathcal{O}_C/p$ is the mod-p fiber of $\mathfrak{X}$. Then one can write

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

so one can separately study the "nearby cycles" complex $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$ and its derived global sections on $\mathfrak{X}_0$.

The key is that now $\mathfrak{X}$ is proper over Spf $\mathcal{O}_K$ by [L9̈0, Lemma 2.6][30] (or [Tem00, Corollary 4.4 and 4.5]). Thus the Almost Proper Mapping Theorem 5.1.3 tells us that, for the purpose of proving almost finiteness of $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$, it is sufficient only to show that $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p \in \mathbf{D}^+_{acoh}(X)$ has almost coherent cohomology sheaves.

The main advantage now is that we can study the "nearby cycles" $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$ *locally* on the formal model $\mathfrak{X}$. So this holds for any admissible formal model and not only for the proper ones. Moreover, the only place where we use properness of X in our proof is to get properness of the formal model $\mathfrak{X}$ to be able to use the Almost Proper Mapping Theorem 5.1.3. This allows us to

[30] Strictly speaking, his proof is written under the assumption that $\mathcal{O}_K$ is discretely valued. However, it can be easily generalized to the of a general rank-1 complete valuation ring $\mathcal{O}_K$.

avoid all elaborate spectral sequence arguments at the same time making the essential part of the proof local on $\mathfrak{X}$.

Now we discuss how we get almost coherentness of $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$. We will actually prove a much stronger almost coherentness statement that holds for all $\mathcal{O}^+/p$-vector bundle in the v-topology. However, we find it instructive to discuss the simplest case first.

The main idea of the proof is similar to the idea behind the proof [Sch13, Lemma 5.6]: we reduce the general case to the case of an affine $\mathfrak{X}$ with "nice" coordinates, where everything can be reduced to almost coherentness of certain continuous group cohomology via perfectoid techniques. In order to make it work, we have to pass to a finer topology that allows towers of finite étale morphisms. There are different possible choices, we find the formalims of v-topology on the associated diamond $X^\diamondsuit$ of X (in the sense of [Sch17]) to be the most convenient for our purposes (see Appendix C.

The case of a general $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle (see Definition 6.1.1) will cause us more trouble; we will use the structure results from Section C.4 to handle a general $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. The main crucial results is that the category of étale $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles is actually equivalent to the category of $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundles and that, locally, any $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle can be trivialized by some very particular étale covering (see Corollary C.4.10 for both results).

That being said, we can move to the formulation of the main theorem of this section. We refer to Appendix C for the definition of the quasi-proétale and v-topologies on $X^\diamondsuit$ for a rigid-analytic variety over a non-archimedean field K. These sites come with their "integral" structure sheaves $\mathcal{O}^+_{X^\diamondsuit}$, $\mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}$, and $\mathcal{O}^+_{X_{\text{ét}}}$ (see Definition C.3.1) and a diagram of morphisms of ringed sites (see Diagram C.1 and C.2):

$$\begin{tikzcd}\left(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}\right) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"] & \left(X^\diamondsuit_{\text{qproét}}, \mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}\right) \arrow[r, "\mu"] & \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}\right) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}})\end{tikzcd} \tag{6.1}$$

and the mod-p version

$$\begin{tikzcd}\left(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}/p\right) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"] & \left(X^\diamondsuit_{\text{qproét}}, \mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}/p\right) \arrow[r, "\mu"] & \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}_0})\end{tikzcd} \tag{6.2}$$

If there is any ambiguity in the meaning of ν, we then denote it by $\nu_{\mathfrak{X}}$ to explicitly specify the formal model for these functors.

Recall that for a perfectoid field K, the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal of almost mathematics with flat $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$ by by Lemma B.6. For the rest of this section, we fix a p-adic perfectoid field K, and always do almost mathematics with respect to the ideal $\mathfrak{m}$.

We are ready to formulate our first main result. We thank B. Heuer for suggesting this formulation.

Definition 6.1.1. An $\mathcal{O}^+_{X^\diamondsuit}/p$-module $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamondsuit}/p$-*vector bundle* if, v-locally on $X^\diamondsuit$, it is isomorphic to $(\mathcal{O}^+_{X^\diamondsuit}/p)^r$ for some integer r.

An $\mathcal{O}^+_{X^\diamond}/p$-module $\mathcal{E}$ is a *very small* $\mathcal{O}^+_{X^\diamond}/p$*-vector bundle* if there is a finite étale surjective morphism $V \to U$ such that $\mathcal{E}|_{V_v^\diamond} \simeq (\mathcal{O}^+_{V^\diamond}/p)^r$ for some integer r.

An $\mathcal{O}^+_{X^\diamond}/p$-module $\mathcal{E}$ is a *small* $\mathcal{O}^+_{X^\diamond}/p$*-vector bundle* if, for each point $x \in X$, there is an open affinoid $x \in U \subset X$ such that $\mathcal{E}|_{U^\diamond}$ is very small.

Theorem 6.1.2. Let $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then

(1) the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X}_0)^a$;

(2) for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$ with the adic generic fiber X, the natural map

$$\widetilde{\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$;

(4) if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small, then

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a;$$

(5) if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small.

In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$\left(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_{i,0})^a.$$

for each $i \in I$.

Remark 6.1.3. We refer to Definition 4.4.1 and Definition 4.4.2 for the precise definition of all derived categories appearing in Theorem 6.1.2. In order to avoid any confusion, we explicitly mention that the expression $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a$ means that it is concentrated in degrees $[0, d]$ in the derived category of almost sheaves. In particular, this is equivalent that cohomology sheaves of the complex $\mathbf{R}\nu_*\mathcal{E}$ are almost zero in degrees larger than d.

Remark 6.1.4. We note that Theorem 6.1.2 (1) implies that the nearby cycles $\mathbf{R}\nu_*\mathcal{E}$ is quasi-coherent on the nose (as opposed to being almost quasi-coherent). This is quite unexpected to the author since all previous results on the cohomology groups of $\mathcal{O}^+/p$ were only available in the almost category.

Remark 6.1.5. If $K = C$ is algebraically closed, the proof gives a non-almost version of cohomological bound. Namely, we see that

$$\mathbf{R}\nu_*\left(\mathcal{E}\right) \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X}_0).$$

However, we do not know if, under the assumption of Part (4), $\mathbf{R}\nu_*\left(\mathcal{E}\right)$ is concentrated in degrees $[0, d]$ on the nose.

Remark 6.1.6. We do not know if an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ in the formulation of Theorem 6.1.2 is really necessary or just an artefact of the proof. More importantly, we do not know if, for every $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$, there is an admissible formal model $\mathfrak{X}$ such that the "nearby cycles" sheaf $\mathbf{R}\nu_{\mathfrak{X},*}\mathcal{E}$ lies in $\mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a$.

One can prove a slightly more precise version in case $\mathcal{E}$ comes as a tensor product of an $\mathbf{F}_p$-local system and $\mathcal{O}^+_{X^\diamond}/p$. More generally, one can slightly generalize the result to the class of Zariski-constructible sheaves.

Definition 6.1.7. [Han20] An étale sheaf $\mathcal{F}$ of $\mathbf{F}_p$-modules is a *local system* if it is a locally constant sheaf with finite stalks.

An étale sheaf $\mathcal{F}$ of $\mathbf{F}_p$-modules is *Zariski-constructible* if there is a locally finite stratification $X = \bigsqcup_{i\in I} Z_i$ into Zariski locally closed subspaces Z_i such that $\mathcal{F}|_{Z_i}$ is a local system.

The category $\mathbf{D}_{zc}(X; \mathbf{F}_p)$ is a full subcategory of $\mathbf{D}(X_{\text{ét}}; \mathbf{F}_p)$ consisting of objects with Zariski-constructible cohomology sheaves.

Remark 6.1.8. Any Zariski-constructible sheaf $\mathcal{F}$ is overconvergent, i.e., for any morphism $\overline{\eta} \to \overline{s}$ of geometric points in $X_{\text{ét}}$, the specialization map $\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}}$ is an isomorphism.

Note that any sheaf of $\mathbf{F}_p$-modules on $X_{\text{ét}}$ can be treated as a sheaf on any of the sites $X^\diamondsuit_v$, $X^\diamondsuit_{\text{qproét}}$, or $X_{\text{proét}}$ via the pullback functors along the morphisms in Diagram 6.1. In what follows, we abuse notation and implicitly treat a sheaf $\mathcal{F}$ as a sheaf on any of those sites. We also denote the tensor product $\mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_X/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_X/p$ in what follows.

Now we discuss an integral version of Theorem 6.1.2.

Theorem 6.1.9. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$. Then

(1) there is an isomorphism $\mathbf{R}t_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\nu_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)$;

(2) the nearby cycles $\mathbf{R}\nu_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$, and $\mathbf{R}\nu_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{acoh}(\mathfrak{X}_0)^a$;

(3) for an affine admissible $\mathfrak{X} = \operatorname{Spf} A$, the natural map

$$\mathrm{H}^i\left(X^\diamondsuit_v, \widetilde{\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p}\right) \to \mathrm{R}^i\nu_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)$$

is an isomorphism for every $i \geq 0$;

(4) the formation of $\mathrm{R}^i\nu_* \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}_0^* \left(\mathrm{R}^i\nu_{\mathfrak{X},*} \left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\right)\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*} \left(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\right)$$

is an isomorphism for any $i \geq 0$;

Definition 6.1.10. An $\mathcal{O}^+_{X^\diamond}$-module $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamond}$*-vector bundle* if, v-locally on $X^\diamondsuit$, it is isomorphic to $(\mathcal{O}^+_{X^\diamond})^r$ for some integer r.

An $\mathcal{O}^+_{X^\diamond}$-vector bundle $\mathcal{E}$ is a *very small* $\mathcal{O}^+_{X^\diamond}$*-vector bundle* if $\mathcal{E}/p\mathcal{E}$ is a very small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle (see Definition 6.1.1).

An $\mathcal{O}^+_{X^\diamond}$-vector bundle $\mathcal{E}$ is a *small* $\mathcal{O}^+_{X^\diamond}$*-vector bundle* if $\mathcal{E}/p\mathcal{E}$ is a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle (see Definition 6.1.1).

Theorem 6.1.11. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then

(1) the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X})^a$;

(2) for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$ with the adic generic fiber X, the natural map

$$\mathrm{H}^i\left(X^\diamondsuit_v, \mathcal{E}\right)^\Delta \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamondsuit}\right)$$

is an isomorphism for any $i \geq 0$;

(4) if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamondsuit}$ is very small, then

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a;$$

(5) if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamondsuit}$ is very small.

In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_i)^a.$$

for each $i \in I$.

Remark 6.1.12. We refer to Definition 4.8.9 for the precise definition of all derived categories appearing in Theorem 6.1.11.

Remark 6.1.13. One can also prove a version of Theorem 6.1.11 for Zariski-constructible $\mathbf{Z}_p$-sheaves in the sense of [BH21, Definition 3.32]. However, we prefer not to do this here as it does not require new ideas but makes the exposition a heavier in terms of terminology.

For the version of Theorem 6.1.11 with the pro-étale site $X_{\text{proét}}$ as defined in [Sch13] and [Sch16], see Theorem 6.13.6

The rest of the paper is devoted to proving Theorem 6.1.9, Theorem 6.1.2, and Theorem 6.1.11 and discussing their applications. We have decided to work entirely in the v-site of $X^\diamondsuit$ because it is quite flexible for different types of arguments (e.g. proper descent, torsors under pro-finite groups, etc.). However, most of the arguments can be carried over in a more classical pro-étale site defined in [Sch13]. However, it seems difficult to show that $\mathrm{R}^i\nu_*\mathcal{E}$ are quasi-coherent (as opposed to almost quasi-coherent) using that version of the pro-étale site (however the quasi-proétale site is sufficient for these purposes), and it is also crucial to argue on the level of diamonds for the purpose of getting a cohomological bound on $\mathrm{R}^i\nu_*\mathcal{E}$ in the small case.

6.2. **Digression: Geometric Points.** In this section, we discuss some preliminary results that will be both used in the proof of Theorem 6.1.9 and in deriving applications out of it.

We start the section by recalling some definitions.

Definition 6.2.1. [Tem21, 2.1.4] An extension of non-archimedean fields[31] $K \subset L$ is *topologically algebraic* if the algebraic closure of K in L is dense in L. Equivalently, $K \subset L$ is topologically algebraic if L is a non-archimedean subfield of $\widehat{\overline{K}}$ completed algebraic closure of K.

[31] Recall that non-archimedean fields are complete by our convention.

Lemma 6.2.2. **(1)** Let $K \subset L$ and $L \subset M$ be two topologically algebraic extensions of non-archimedean fields. Then $K \subset M$ is also topologically algebraic.

(2) Let

$$\begin{array}{ccc} N & \longleftarrow & L \\ \uparrow & & \uparrow \\ M & \longleftarrow & K \end{array}$$

be a commutative diagram of non-archimedean fields such that LM is dense in N and $K \subset L$ is topologically algebraic. Then $M \subset N$ is also a topologically algebraic extension.

Proof. **(1)** : We know that $L \subset \widehat{\overline{K}}$ and $M \subset \widehat{\overline{L}}$ since both extensions are topologically algebraic. Therefore, $M \subset \widehat{\overline{L}} \subset \widehat{\overline{K}}$ proving that $K \subset M$ is topologically algebraic.

(2) : First, we note that

$$LM \subset \widehat{\overline{K}}M \subset \widehat{\overline{K}M} \subset \widehat{\overline{M}}.$$

Secondly, we note that since $LM \subset N$ is dense, the inclusion $LM \subset \widehat{\overline{M}}$ uniquely extends to an inclusion $N \subset \widehat{\overline{M}}$ proving that $M \subset N$ is topologically algebraic. □

Definition 6.2.3. A *geometric point* above point $x \in X$ of an analytic adic space X is a morphism $x\colon \operatorname{Spa}(C(x), C(x)^+) \to X$ such that $C(x)$ is an algebraically closed non-archimedean field, and the corresponding extension of completed residue fields $\widehat{k(x)} \subset C(x)$ is a topologically algebraic extension.

Remark 6.2.4. If $\operatorname{Spa}(C(x), C(x)^+) \to X$ is a geometric point, then $C(x)$ can be identified with the completed algebraic closure of $\widehat{k(x)}$ (or, equivalently, of $k(x)$) and $C(x)^+$ with a valuation ring extending $\widehat{k(x)}^+$ (or, equivalently, $k(x)^+$). Therefore, Definition 6.2.3 is more restrictive than [Hub96, Definition 2.5.1], but coincides with the subclass of geometric points constructed in [Hub96, (2.5.2)].

Lemma 6.2.5. Let K be a non-archimedean field with an open and bounded valuation sub-ring $K^+ \subset K$ and a pseudo-uniformizer ϖ. Let $f\colon X \to Y$ be a morphism of locally of finite type (K, K^+)-adic spaces, and $\overline{y}\colon \operatorname{Spa}(C(y), C(y)^+) \to Y$ be a geometric point above $y \in Y$. Then the natural morphism

$$a\colon i^{-1}(\mathcal{O}^+_{X_{\text{ét}}}/\varpi) \to \mathcal{O}^+_{X_{\overline{y},\text{ét}}}/\varpi$$

is an isomorphism where $i\colon X_{\overline{y}} \to X$ is the "projection" of the geometric fiber $X_{\overline{y}} \coloneqq X \times_Y \operatorname{Spa}(C(y), C(y)^+)$ back to X.

Proof. [Hub96, Proposition 2.5.5] ensures that it suffices to show that a is an isomorphism on stalks above geometric points of $X_{\overline{y}}$. Now note that Lemma 6.2.2 implies that any geometric point $\overline{x}\colon \operatorname{Spa}(C(x), C(x)^+) \to X_{\overline{y}}$ defines a geometric point $\overline{x}'\colon \operatorname{Spa}(C(x), C(x)^+) \to X$ of X by taking the compostition of $\overline{x}$ with i. So it is enough to show that the natural map

$$(\mathcal{O}^+_{X_{\text{ét}}}/\varpi)_{\overline{x}'} \simeq \left(i^{-1}(\mathcal{O}^+_{X_{\text{ét}}}/\varpi)\right)_{\overline{x}} \to (\mathcal{O}^+_{X_{\overline{y},\text{ét}}}/\varpi)_{\overline{x}} \tag{6.3}$$

is an isomorphism. But [Hub96, Proposition 2.6.1] naturally identifies both sides of (6.3) with $C(x)^+/\varpi C(x)^+$ finishing the proof. □

Remark 6.2.6. Lemma 6.2.5 is very specific to the adic geometry (and probably quite counter-intuitive from algebraic point of view). Its scheme-theoretic version with $\mathcal{O}^+/\varpi$ replaced with $\mathcal{O}$ is very false. The main feature of analytic adic geometry (implicitly) used in the proof is that the morphism $\mathcal{O}^+_{X,x} \to k(x)^+$ becomes an isomorphism after ϖ-adic completion.

Lemma 6.2.7. Let (C, C^+) be a Huber pair of an algebraically closed non-archimedean field C, an open and bounded valuation sub-ring $C^+ \subset C$ and a pseudo-uniformizer $\varpi \in C^+$. Suppose that $(C, C^+) \to (D, D^+)$ is a finite morphism of complete Huber pairs with a local ring D. Then the natural morphism

$$C^+/\varpi C^+ \to D^+/\varpi D^+$$

is an isomorphism.

Proof. Firstly, we show that $C^+/\varpi C^+ \to D^+/\varpi D^+$ is injective. Suppose, let $\bar{c} \in C^+/\varpi C^+$ be an element in the kernel, lift it to $c \in C^+$. The assumption on c implies that $c = \varpi d$ for some $d \in D^+$. But then $d = c/\varpi \in C \cap D^+ = C^+$. Therefore, $\bar{c} = 0$ in $C^+/\varpi C^+$.

Now we check surjectivity. Since D is a local ring that is finite over an algebraically closed field C, we conclude that D is an Artin local ring and $D/\mathrm{nil}(D) \simeq C$. Therefore, for every $d \in D^+$, we can find $c \in C$ and $d' \in \mathrm{nil}(D)$ such that $d = c + d'$. Since $\mathrm{nil}(D) \subset D^{\circ\circ} \subset D^+$, we conclude that $c = d - d' \in D^+ \cap C = C^+$. Now note taht d'/ϖ is still a nilpotent element of D, thus $d'/\varpi \in \mathrm{nil}(D) \subset D^+$. Thus,

$$d = c + \varpi(d'/\varpi)$$

proving that $C^+/\varpi C^+ \to D^+/\varpi D^+$ is surjective. □

Corollary 6.2.8. Let (K, K^+) be a Huber pair of a p-adic non-archimedean field K and an open and bounded valuation sub-ring $K^+ \subset K$. Let $f\colon X \to Y$ be a finite morphism of locally finite type (K, K^+)-adic spaces. Then the natural morphism

$$c\colon f_*\left(\underline{\mathbf{F}}_p\right) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \to f_*(\mathcal{O}^+_{X_{\text{ét}}}/p)$$

is an isomorphim on $Y_{\text{ét}}$.

Proof. We use [Hub96, Proposition 2.5.5] to ensure that it suffices to show that c is an isomorphism on stalks at geometric points. Thus [Hub96, Proposition 2.6.1] and Lemma 6.2.5 imply that it suffices to assume that $\bar{y} = Y = \mathrm{Spa}\,(C, C^+)$ with an algebraically closed p-adic non-archimedean field C. In this case, $X = \mathrm{Spa}\,(D, D^+)$ for some finite morphism of Huber pairs $(C, C^+) \to (D, D^+)$. In particular, D is a finite C-algebra, so it is a finite direct product of local artinian C-algebra. By passing to a direct factor of D (or, geometrically, to a connected component of $\mathrm{Spa}\,(D, D^+)$), we can assume that D is local. In particular, D does not have any idempotents, and so $\mathrm{Spa}\,(D, D^+)$ is connected. Thus [Hub96, Proposition 2.6.1] ensures that

$$\left(f_*\underline{\mathbf{F}}_p \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right)_{\bar{y}} \simeq \mathrm{H}^0(X, \mathbf{F}_p) \otimes C^+/pC^+ \simeq C^+/pC^+,$$

where $\mathrm{H}^0(X, \mathbf{F}_p) \simeq \mathbf{F}_p$ by the connectivity assumption on $\mathrm{Spa}\,(D, D^+)$.

Now we observe that $\mathrm{Spa}\,(D, D^+)_{\mathrm{red}} \simeq \mathrm{Spa}\,(C, C^+)$, so all étale sheaves on $\mathrm{Spa}\,(D, D^+)$ do not have higher cohomology groups. Thus, we have

$$\left(f_*(\mathcal{O}^+_{X_{\text{ét}}}/p)\right)_{\bar{y}} \simeq \mathrm{H}^0(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq D^+/pD^+.$$

In particular, the question boils down to showing that the natural map

$$C^+/pC^+ \to D^+/pD^+$$

is an isomorphism. This was already done in Lemma 6.2.7. □

Corollary 6.2.9. Let K be a p-adic non-archimedean field, $f\colon X \to Y$ a finite morphism of rigid-analytic varieties over K, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism

$$c\colon f_*(\mathcal{F}) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \to f_*\left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

is an isomorphim on $Y_{\text{ét}}$.

Proof. We recall that [BH21, Proposition 3.6] says that $\mathbf{D}^b_{zc}(X; \mathbf{F}_p)$ is a thick triangulated subcategory of $\mathbf{D}(X_{\text{ét}}; \mathbf{F}_p)$ generated by objects of the form $g_*\left(\underline{\mathbf{F}}_p\right)$ for finite morphisms $g\colon X' \to X$. Since both claims in the question satisfy the 2-out-of-3 property and are preserved by passing to direct summands, it suffices to prove the claim only for $\mathcal{F} = g_*\left(\underline{\mathbf{F}}_p\right)$. In this situation, the claim follows from Corollary 6.2.8 by a sequence of isomorphisms

$$\begin{aligned} f_*\left(g_*\left(\underline{\mathbf{F}}_p\right)\right) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p &\simeq (f\circ g)_*\left(\underline{\mathbf{F}}_p\right) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \\ &\simeq (f\circ g)_*\left(\mathcal{O}^+_{X'_{\text{ét}}}/p\right) \\ &\simeq f_*\left(g_*\mathcal{O}^+_{X'_{\text{ét}}}/p\right) \\ &\simeq f_*\left(g_*\underline{\mathbf{F}}_p \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right). \end{aligned}$$

□

6.3. **Applications.** The main goal of this section is to discuss some applications of Theorem 6.1.9.

For the rest of the section, we fix a p-adic algebraically closed field C with its rank-1 valuation ring $\mathcal{O}_C$, maximal ideal $\mathfrak{m} \subset \mathcal{O}_C$, and a good pseudo-uniformizer $\varpi \in \mathcal{O}_C$ (see Definition B.1.6). We always do almost mathematics with respect to the ideal $\mathfrak{m}$ in this section. If we need to consider a more general non-archimedean field, we denote it by K.

One non-trivial consequence of Theorem 6.1.11 is that v cohomology groups of $\mathcal{O}^+_{X^\diamond}$-vector bundles have bounded p-torsion.

Lemma 6.3.1. Let K be a p-adic perfectoid field, $\mathfrak{X} = \operatorname{Spf} A_0$ an affine admissible formal $\mathcal{O}_K$-scheme the adic generic fiber X, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then the cohomology groups $\mathrm{H}^i(X^\diamond_v, \mathcal{E})$ are almost finitely presented over A_0. In particular, they are p-adically complete and have bounded torsion p^∞-torsion.

Proof. This is a straightforward consequence of Theorem 6.1.11, Lemma 2.12.5 and Lemma 2.12.7. □

Remark 6.3.2. Lemma 6.3.1 implies that v cohomology groups of $\mathcal{O}^+_{X^\diamond}$ behave pretty differently from analytic cohomology groups of $\mathcal{O}^+_X$. Indeed, see [Bha, Remark 9.3.4] (that can be easily adapted to the p-adic situation) for an example of an affinoid with unbounded p-torsion in $\mathrm{H}^1_{\text{an}}(X, \mathcal{O}^+_X)$.

Theorem 6.3.3. Let K be a p-adic perfectoid field, X a proper rigid-analytic K-variety of dimension d, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle (resp. $\mathcal{O}^+_{X^\diamond}/p$-vector bundle). Then

$$\mathbf{R}\Gamma(X^\diamond_v, \mathcal{E}) \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_K)^a.$$

Proof. We firstly deal with the case of an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$. We choose choosing an admissible formal model $\mathfrak{X}$ of the rigid-analytic variety X. It is necessarily proper by [L9̈0, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]). Now Theorem 6.1.2 (and [Sch13, Corollary 3.17(i)]) implies that

$$\mathbf{R}\nu_*(\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X}_0)^a.$$

Recall that the underlying topological spaces of the mod-p fiber $\mathfrak{X}_0$ and the special fiber $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}\, \mathcal{O}_C} \mathrm{Spec}\, \mathcal{O}_C/\mathfrak{m}$ are the same. Thus [FK18, Corollary II.10.1.11] implies that $\mathfrak{X}_0$ has Krull dimension d. Therefore, Theorem 5.1.3, [Sta21, Tag 0A3G] and Lemma 3.3.6 imply that

$$\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E})^a \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\left(\mathcal{E}\right)\right)^a \in \mathbf{D}^{[0,3d]}_{acoh}(\mathcal{O}_K/p)^a.$$

Now we need to get a better cohomological estimate. Lemma 6.7.4 implies (by choosing an affinoid cover of X) that

$$\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E}) \otimes \mathcal{O}_K/p \to \mathbf{R}\Gamma(X_{C,v}^{\diamondsuit}, \mathcal{E})$$

is an isomorphism, where C is a completed algebraic closure of K. Then Lemma 2.10.5 and faithful flatness of $\mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_C/p\mathcal{O}_C$ implies that it suffices to prove the claim under the additional assumption that $K = C$ is algebraically closed. Then we consider

$$\mathcal{E}' :- \mathbf{R}\mu_*\mathbf{R}\lambda * \mathcal{E},$$

this is an $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle (concentrated in degree 0) by Theorem C.4.8 and Theorem C.4.5. So it suffices to show that

$$\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E}) \simeq \mathbf{R}\Gamma(X_{\text{ét}}, \mathcal{E}')$$

is concentrated in degrees $[0, 2d]$. This follows from [Hub96, Corollary 2.8.3] and finishes the proof for $\mathcal{O}^+_{X^\diamond}/p$-modules.

Now if $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamond}$-vector, we see that

$$[\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E})/p] \simeq \mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E}/p) \in \mathbf{D}^{[0,3d]}_{acoh}(\mathcal{O}_C/p)^a,$$

and

$$[\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E})/p] \simeq \mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{E}/p) \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_C/p)^a$$

if $\mathcal{E}$ is small. So we conclude the claim by Corollary 2.13.3, and Lemma C.3.5 **(3)**. ☐

Another application of the results in Section 6 is the finiteness properties of Zariski-constructible sheaves. We show that cohomology groups of a Zariski-constructible sheaf on a proper space are finite, and that the "p-adic nearby cycles" commute with proper pushforward establishing a similar behaviour to the algebraic nearby cycles.

We start with the finiteness properties:

Lemma 6.3.4. Let X be a proper rigid-analytic variety over C of dimension d, and $\mathcal{F}$ a Zariski-constructible sheaf of $\mathbf{F}_p$-modules on $X_{\text{ét}}$. Then $\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_C/p)^a$.

Proof. The proof is analogous to the proof of Theorem 6.3.3 using Theorem 6.1.9 in place of Theorem 6.1.2. ☐

Lemma 6.3.5. Let X be a proper rigid-analytic variety over C of dimension d. Then

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{F}_p)$$

and the natural morphism

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p \to \mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p)$$

is an almost isomorphism.

Proof. *Step* 1. $\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}^\flat_C)^a$. We consider the tilted integral structure sheaf $\mathcal{O}^{\flat,+}_{X^\diamondsuit}$ (see Definition C.3.4). Lemma C.3.5 **(4)** ensures that $\mathcal{O}^{\flat,+}_{X^\diamondsuit}$ is derived $\varpi^\flat$-adically complete and Lemma C.3.5 **(5)** implies that

$$[\mathcal{O}^{\flat,+}_{X^\diamondsuit}/\varpi^\flat] \simeq [\mathcal{O}^+_{X^\diamondsuit}/p] \simeq \mathcal{O}^+_{X^\diamondsuit}/p.$$

Therefore, [Sta21, Tag 0BLX] guarantees that $\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit}\right) \in \mathbf{D}\left(\mathcal{O}^\flat_C\right)$ is derived $\varpi^\flat$-adically complete. Moreover, Lemma 6.3.4 implies

$$\left[\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit}\right)^a/\varpi^\flat\right] \simeq \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}/p)^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_C/p)^a.$$

Thus Corollary 2.13.3 applied to $R = C^+ = \mathcal{O}^\flat_C$ implies that $\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit}\right)^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}^\flat_C)^a$.

Step 2. *$\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{F}_p)$ and the natural morphism $\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes C^\flat \to \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})$ is an isomorphism.* After inverting $\varpi^\flat$, Step 1 implies that

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit}) \in \mathbf{D}^{[0,2d]}_{coh}(C^\flat).$$

Now $\mathcal{O}^\flat_{X^\diamondsuit}$ is a sheaf of $\mathbf{F}_p$-algebras, so there is a natural Frobenius morphism

$$F\colon \mathcal{O}^{\flat,+}_{X^\diamondsuit} \xrightarrow{f\mapsto f^p} \mathcal{O}^{\flat,+}_{X^\diamondsuit}.$$

that can be easily seen to be an isomorphism by Lemma C.3.5 **(2)** (and Remark B.1.3). Now we use the Artin-Shreier short exact sequence

$$0 \to \underline{\mathbf{F}}_p \to \mathcal{O}^\flat_{X^\diamondsuit} \xrightarrow{F-\mathrm{Id}} \mathcal{O}^\flat_{X^\diamondsuit} \to 0$$

on the v-site $X^\diamondsuit_v$ to get the associated long exact sequence[32]

$$\dots \to \mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit}) \xrightarrow{\mathrm{H}^i(F)-\mathrm{Id}} \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit}) \to \mathrm{H}^{i+1}(X, \mathbf{F}_p) \to \dots$$

We already know that each group $\mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})$ is a finitely generated $C^\flat$-vector space, each $\mathrm{H}^i(F)$ is a frobenius-linear automorphism, and $C^\flat$ is an algebraically closed field of characteristic p (see [Sch12, Theorem 3.7]). Thus (the proof of) [Sta21, Tag 0A3L] ensures that $\mathrm{H}^i(F) - \mathrm{Id}$ is surjective for each $i \geq 0$ (so $\mathrm{H}^i(X, \mathbf{F}_p) \simeq \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})^{F=1}$) and the natural morphism

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes C^\flat \to \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})$$

is an isomoprhism. In particular, $\dim_{\mathbf{F}_p} \mathrm{H}^i(X, \mathbf{F}_p) = \dim_{C^\flat} \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})$, $\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{F}_p)$, and the natural morphism

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes C^\flat \to \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^\flat_{X^\diamondsuit})$$

is an isomorphism.

Step 3. *The natural morphism $\mathbf{R}\Gamma(X, \mathbf{F}_p)\otimes\mathcal{O}_C/p \to \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}/p)$ is an almost isomorphism.* It suffices to show that

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes \mathcal{O}^\flat_C \to \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit})$$

is an almost isomorphism. The version with $\mathcal{O}^+_{X^\diamondsuit}/p$ then would follow by taking the derived mod-$\varpi^\flat$ reduction. Therefore, it is enough to show that

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}^\flat_C \to \mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^{\flat,+}_{X^\diamondsuit})$$

[32] We implicitly use that $\mathrm{H}^i(X, \mathbf{F}_p) \simeq \mathrm{H}^i(X^\diamondsuit_v, \mathbf{F}_p)$ by [Sch17, Proposition 14.7, 14.8, and Lemma 15.6].

is an almost isomorphism for each $i \geq 0$. Consider a commutative diagram

$$\begin{tikzcd} \mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C^\flat \arrow[r, "\alpha"] \arrow[d, "i"] & \mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+}) \arrow[d] \\ \mathrm{H}^i(X, \mathbf{F}_p) \otimes C^\flat \arrow[r, "\beta"] & \mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat}). \end{tikzcd}$$

By Step 2, we know that β is an isomorphism. Since i is injective, we conclude that α is injective as well. So it only suffices to show that α is almost surjective.

Since β is an isomorphism and $\mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$ is almost coherent by Step 1, it is easy to see that there is an integer n such that $(\varpi^\flat)^n x \in \mathrm{Im}(\alpha)$ for any $x \in \mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$. Now we note that both the source and the target of α have a natural Frobenius action and α respects those actions: the action on the source comes from the Frobenius action on $\mathcal{O}_C^\flat$ and the Frobenius action on the target comes from the Frobenius action on $\mathcal{O}_{X^\diamondsuit}^{\flat,+}$. The action on $\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C^\flat$ is an isomorphism because $\mathcal{O}_C^\flat$ is perfect, and the action on $\mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$ is an isomorphism because Frobenius is already an isomorphism on the sheaf $\mathcal{O}_{X^\diamondsuit}^{\flat,+}$ by Lemma C.3.5 **(2)** (and Remark B.1.3). Therefore, it makes sense to consider the inverse Frobenius action F^{-1} on both modules and α commutes with this action.

Now we pick any element $x \in \mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$. Since F is an isomorphism on $\mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$, there is $x' \in \mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}_{X^\diamondsuit}^{\flat,+})$ such that $F^m(x') = x$. By the discussion above, there is $y' \in \mathrm{H}^i(X_v^\diamondsuit, \mathbf{F}_p) \otimes \mathcal{O}_C^\flat$ such that $\alpha(y') = (\varpi^\flat)^N x'$. Therefore,

$$\left(\varpi^\flat\right)^{N/p^m} x = F^{-m}\left(\left(\varpi^\flat\right)^N x'\right) = F^{-m}\left(\alpha\left(y'\right)\right) = \alpha\left(F^{-m}\left(y'\right)\right).$$

Thus $(\varpi^\flat)^{N/p^m} x = \alpha(y)$ where $y = F^{-m}(y') \in \mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C^\flat$. Since N/p^m can be made arbitrary small by increasing m, we conclude that α is almost surjective. □

Lemma 6.3.6. Let X be a proper rigid-analytic variety over C of dimension d, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then

$$\mathbf{R}\Gamma(X, \mathcal{F}) \in \mathbf{D}_{coh}(\mathbf{F}_p).$$

Proof. We recall that [BH21, Proposition 3.6] says that $\mathbf{D}^b_{zc}(X, \mathbf{F}_p)$ is a thick triangulated subcategory of $\mathbf{D}(X_{\text{ét}}; \mathbf{F}_p)$ generated by objects of the form $f_*(\underline{\mathbf{F}}_p)$ for finite morphisms $f\colon X' \to X$. Since both claims in the question satisfy the 2-out-of-3 property and are preserved by passing to direct summands, it suffices to prove the claim only for $\mathcal{F} = f_*(\underline{\mathbf{F}}_p)$. Then the claim follows from Lemma 6.3.5 since

$$\mathbf{R}\Gamma\left(X, f_*\left(\underline{\mathbf{F}}_p\right)\right) \simeq \mathbf{R}\Gamma(X', \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{F}_p).$$

□

Lemma 6.3.7. Let K be a p-adic perfectoid field K, $f\colon X \to Y$ a proper morphism of rigid-analytic varieties over K, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism

$$\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \to \mathbf{R}f_*(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p)$$

is an almost isomorphism.

Proof. The claim is local on Y, so we can assume that Y is affinoid. Then a similar trick as in the proof of Corollary 6.3.6 allows us to reduce to the case $\mathcal{F} = g_*\left(\underline{\mathbf{F}}_p\right)$ for a finite map $g\colon X' \to X$. So Corollary 6.2.8 implies that it suffices to prove the claim for the morphism $f \circ g\colon X' \to Y$ and $\mathcal{F} = \underline{\mathbf{F}}_p$.

Now [Hub96, Proposition 2.5.5] guarantees that it suffices to show the claim on stalks at geometric points. Therefore, by Lemma 6.2.5 we reduce the question to showing that, for any proper adic space X over a geometric point $\mathrm{Spa}\,(C, C^+)$, the natural morphism

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes C^+/p \to \mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p).$$

is an almost isomorphism. Denote by $X^\circ \coloneqq X \times_{\mathrm{Spa}\,(C,C^+)} \mathrm{Spa}\,(C, C^\circ)$. We have an isomorphism $\mathbf{R}\Gamma(X, \mathbf{F}_p) \simeq \mathbf{R}\Gamma(X^\circ, \mathbf{F}_p)$ by [Hub96, Proposition 8.2.3(ii)], an almost isomorphism

$$C^+/pC^+ \simeq^a \mathcal{O}_C/p\mathcal{O}_C$$

by Lemma 2.11.1, and an almost isomorphism $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq^a \mathbf{R}\Gamma(X^\circ, \mathcal{O}^+_{X^\circ_{\text{ét}}}/p)$ by Corollary C.3.12 and Corollary C.3.15. Thus we may replace (C, C^+) with $(C, \mathcal{O}_C)$ and X with X° to assume that $\mathrm{Spa}\,(C, \mathcal{O}_C)$ is a geometric point of rank-1. In this case, the claim was already proven in Lemma 6.3.6. $\square$

Corollary 6.3.8. Let X be a proper rigid-analytic variety over C of dimension d, and $\mathcal{F}$ a Zariski-constructible étale $\mathbf{F}_p$-sheaf. Then

$$\mathbf{R}\Gamma(X, \mathcal{F}) \in \mathbf{D}^{[0,2d]}_{coh}(\mathbf{F}_p).$$

Proof. Lemma 6.3.6 already implies that $\mathbf{R}\Gamma(X, \mathcal{F}) \in \mathbf{D}_{coh}(\mathbf{F}_p)$, so we only need to show that this complex is concentrated in degrees $[0, 2d]$. Now Lemma 6.3.7 (applied to $Y = \mathrm{Spa}\,(C, \mathcal{O}_C)$) and Lemma 6.3.4 ensure that

$$(\mathbf{R}\Gamma\,(X, \mathcal{F}) \otimes \mathcal{O}_C/p)^a \simeq \mathbf{R}\Gamma\left(X, \mathcal{F} \otimes \mathcal{O}^+_X/p\right)^a \simeq \mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{F} \otimes \mathcal{O}^+_{X^\diamondsuit}/p\right)^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathcal{O}_C/p)^a$$

implying that $\mathbf{R}\Gamma(X, \mathcal{F})$ must be concentrated in degrees $[0, 2d]$. $\square$

Now we show that p-adic nearby cycles commute with proper morphisms.

Corollary 6.3.9. Let K be a p-adic perfectoid field K, $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ a proper morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamondsuit}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamondsuit}/p\right)\right)$$

is an almost isomorphism.

Proof. Firstly, note that $\mathbf{R}f_*\mathcal{F}$ has overconvergent cohomology sheaves by [Hub96, Proposition 8.2.3(ii)] and Remark 6.1.8. Therefore, Lemma C.5.10 implies that

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamondsuit}/p\right) \simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right),$$

where $t_{\mathfrak{Y}}\colon \left(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\right) \to (\mathfrak{Y}_0, \mathcal{O}_{\mathfrak{Y}_0})$ is the natural morphism of ringed sites. Similarly, we have an isomorphism

$$\mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamondsuit}/p\right)\right) \simeq \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right).$$

Therefore, it suffices to show that the natural morphism

$$\mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right)$$

is an isomorphism.

For this, we use a commutative diagram of ringed sites

$$\begin{tikzcd} \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \arrow[r, "t_{\mathfrak{X}}"] \arrow[d, "f"] & (\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}) \arrow[d, "\mathfrak{f}_0"] \\ \left(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\right) \arrow[r, "t_{\mathfrak{Y}}"] & (\mathfrak{Y}_0, \mathcal{O}_{\mathfrak{Y}_0}) \end{tikzcd}$$

implies that

$$\mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{X},*}\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)\right) \simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)\right).$$

Therefore, the morphism

$$\mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{Y},*}\left(\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\right)\right) \simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)\right)$$

is an almost isomorphism by Lemma 6.3.7 and Proposition 3.5.23. □

6.4. **Perfectoid Covers of Affinoids.** The main goal of this section is to show almost vanishing of higher v-cohomology groups of a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle on an affinoid perfectoid. Later we will apply it to certain pro-étale coverings of $\mathrm{Spa}\,(A, A^+)$ to reduce the computation of v-cohomology groups to the computation of Čech cohomology groups.

Set-up 6.4.1. We fix

- **(1)** a p-adic perfectoid field K with its rank-1 open and bounded valuation ring $\mathcal{O}_K$ and a good pseudo-uniformizer[33] $\varpi \in \mathcal{O}_K$ as in (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$),
- **(2)** an affine admissible formal scheme $\mathfrak{X} = \mathrm{Spf}\ A_0$ with an adic generic fiber $X = \mathrm{Spa}\,(A, A^+)$, and an affinoid perfectoid pair (A_∞, A^+_∞) (see Definition B.1.1) with a morphism $(A, A^+) \to (A_\infty, A^+_\infty)$ such that $\mathrm{Spd}\,(A_\infty, A^+_\infty) \to \mathrm{Spd}\,(A, A^+)$ is a v-covering (see Definition C.1.1 and Definition C.1.5);
- **(3)** a very small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ (see Definition 6.1.1).

Definition 6.4.2. We say that a p-torsionfree (equivalently, ϖ-torsionfree) $\mathcal{O}_K$-algebra R is *integrally perfectoid* if the Frobenius homomorphism $R/\varpi R \xrightarrow{x\mapsto x^p} R/\varpi^p R = R/pR$ is an isomorphism.

Remark 6.4.3. This definition coincides with [BMS18, Definition 3.5] for p-torsionfree $\mathcal{O}_K$-algebras by [BMS18, Lemma 3.10]. In particular, A^+_∞ is an integral perfectoid $\mathcal{O}_K$-algebra by [BMS18, Lemma 3.20].

Lemma 6.4.4. Under the assumption of Set-up 6.4.1, let $\mathfrak{f}\colon \mathrm{Spf}\ B_0 \to \mathrm{Spf}\ A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes. Then $B^+_\infty := B_0\widehat{\otimes}_{A_0}A^+_\infty$ is p-torsionfree integrally perfectoid $\mathcal{O}_K$-algebra.

Proof. Firstly, we note that $A_0 \to B_0$ is a flat morphism by [FK18, Proposition I.4.8.1], so $B_0 \otimes_{A_0} A^+_\infty$ is ϖ-torsion free. Since the ϖ-adic completion of a ϖ-torsionfree algebra is ϖ-torsionfree, we conclude that $B^+_\infty = B_0\widehat{\otimes}_{A_0}A^+_\infty$ is ϖ-torsionfree. We see that the only thing we are left to show is that the Frobenius morphism

$$B^+_\infty/\varpi B^+_\infty \to B^+_\infty/\varpi^p B^+_\infty$$

[33] See Definition B.1.6

is an isomorphism. We consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} B_\infty^+/\varpi \arrow[dr, "F"] \arrow[drr, "\Phi_B^*"] \arrow[ddr, "\mathfrak{f}_\infty/\varpi"'] & & \\
& \operatorname{Spec}\left(B_\infty^+/\varpi^p \otimes_{A_\infty^+/\varpi^p} A_\infty^+/\varpi\right) \arrow[r, "\Phi_A^*\times B_0"'] \arrow[d] & \operatorname{Spec} B_\infty^+/\varpi^p \arrow[d, "\mathfrak{f}_\infty/\varpi^p"] \\
& \operatorname{Spec} A_\infty^+/\varpi \arrow[r, "\Phi_A^*"] & \operatorname{Spec} A_\infty^+/\varpi^p.
\end{tikzcd}$$

We need to show that Φ_B^* is an isomorphism. We know that $\mathfrak{f}_\infty/\varpi^p$ and $\mathfrak{f}_\infty/\varpi$ are étale morphisms since $\mathfrak{f}$ is, and the Frobenious Φ_A^* an isomorphism by Remark 6.4.3. Therefore, the morphism

$$\operatorname{Spec}\left(B_\infty^+/\varpi^p \otimes_{A_\infty^+/\varpi^p} A_\infty^+/\varpi\right) \to \operatorname{Spec} A_\infty^+/\varpi$$

is étale as a base change of étale $\mathfrak{f}_\infty/\varpi^p$, and the morphism

$$\operatorname{Spec}\left(B_\infty^+/\varpi^p \otimes_{A_\infty^+/\varpi^p} A_\infty^+/\varpi\right) \to \operatorname{Spec} B_\infty^+/\varpi^p$$

is an isomorphism as a pullback of an isomorphism. Thus, we conclude that F is an étale morphism as a morphism between étale A_∞^+/ϖ-schemes. Therefore, Φ_B^* is also an étale morphism as a composition of an étale morphism and an isomorphism. However, Φ_B^* is a bijective radiciel morphism since it is an absolute Frobenius. Thus we conclude that it must be an isomorphism as any étale, bijective radiciel morphism is an isomorphism by [Gro63, Théorème 5.1]. □

Corollary 6.4.5. Under the assumption of Set-up 6.4.1, let $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes. Then

$$(B_\infty, B_\infty^+) \coloneqq \left(\left(B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)[1/p], B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)$$

is a perfectoid pair.

Proof. Lemma 6.4.4 states that $B_\infty^+ = B_0 \widehat{\otimes}_{A_0} A_\infty^+$ is a p-torsionfree integral perfectoid. Now $B_0 \otimes_{A_0} A_\infty^+$ is integrally closed in $B_0 \otimes_{A_0} A_\infty^+[1/p]$ because A^+ is integrally closed in A and B_0 is étale over A_0. Therefore, [Bha, Lemma 5.1.2] ensures that the same holds after completion, i.e. B_∞^+ is integrally closed in B_∞. Thus [BMS18, Lemma 3.20] guarantees that (B_∞, B_∞^+) is a perfectoid pair. □

Lemma 6.4.6. Under the assumption of Set-up 6.4.1, let $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes with adic generic fiber $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$. Then the natural morphism

$$\left(\left(B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)[1/p], B_0 \widehat{\otimes}_{A_0} A_\infty^+\right) \to \left(B \widehat{\otimes}_A A_\infty, (B \widehat{\otimes}_A A_\infty)^+\right)$$

is an isomorphism of Huber-Tate pairs.

Proof. By [Hub93b, Lemma 1.6], $B \widehat{\otimes}_A A_\infty \simeq \left(B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)[1/p]$. Now $(B \widehat{\otimes}_A A_\infty)^+$ is defined to be the integral closure of the image of the map

$$B^+ \widehat{\otimes}_{A^+} A_\infty^+ \to B \widehat{\otimes}_A A_\infty.$$

By [Hub93b, Lemma 1.6], we also have

$$B^+ \widehat{\otimes}_{A^+} A_\infty^+ \simeq \left(B^+ \otimes_{A^+} A_\infty^+\right) \otimes_{B_0 \otimes_{A_0} A_\infty^+} \left(B_0 \widehat{\otimes}_{A_0} A_\infty^+\right).$$

Since $B^+ \otimes_{A^+} A^+_\infty$ is integral over $B_0 \otimes_{A_0} A^+_\infty$, we conclude that $B^+\widehat{\otimes}_{A^+}A^+_\infty$ is integral over $B_0\widehat{\otimes}_{A_0}A^+_\infty$. In particular, we see that $(B\widehat{\otimes}_A A_\infty)^+$ is integral over $B_0\widehat{\otimes}_{A_0}A^+_\infty$. However, Corollary 6.4.5 implies that $B_0\widehat{\otimes}_{A_0}A^+_\infty$ is a sub-algebra of $B\widehat{\otimes}_A A_\infty$ that is integrally closed in $B\widehat{\otimes}_A A_\infty$. Thus we must have an equality

$$B_0\widehat{\otimes}_{A_0}A^+_\infty \simeq (B\widehat{\otimes}_A A_\infty)^+.$$

□

Remark 6.4.7. It will be crucial for our arguments later that $(B\widehat{\otimes}_A A_\infty)^+$ is *equal* to $B_0\widehat{\otimes}_{A_0}A^+_\infty$ and not simply to its integral closure. Taking an integral closure of these "big" non-noetherian rings may ruin many finiteness properties.

Lemma 6.4.8. Under the assumption of Set-up 6.4.1, let $M_\mathcal{E}$ be an A^+_∞/pA^+_∞-module

$$M_\mathcal{E} \coloneqq \mathrm{H}^0\left(\mathrm{Spd}\,(A_\infty, A^+_\infty)_v, \mathcal{E}\right).$$

Then $M_\mathcal{E}$ is an almost faithfully flat, almost finitely presented A^+_∞/pA^+_∞-module, and for every morphism $\mathrm{Spa}\,(D, D^+) \to \mathrm{Spa}\,(A_\infty, A^+_\infty)$ of affinoid perfectoids, the natural morphism

$$M_\mathcal{E} \otimes_{A^+_\infty/p} D^+/p \to \mathrm{H}^0\left(\mathrm{Spd}\,(D, D^+)_v, \mathcal{E}\right)$$

is an almost isomorphism[34]. Moreover,

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A^+_\infty)_v, \mathcal{E}\right) \simeq^a 0$$

for $i > 0$.

Proof. Step 1. $\mathrm{H}^0\left(\mathrm{Spd}(A_\infty, A^+_\infty)_v, \mathcal{E}\right)$ *is almost flat and almost finitely presented:* The very smallness assumption implies that there is a finite étale surjection $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A_\infty, A^+_\infty)$ such that $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)} \simeq (\mathcal{O}^+_{X^\diamond}/p)^r$ for some integer $r \geq 0$. The adic space $\mathrm{Spa}\,(B, B^+)$ is affinoid perfectoid by [Sch13, Theorem 7.9].

The natural morphism $A^+ \to B^+$ is almost finitely presented and almost faithfully flat by [Sch13, Theorem 7.9] (see also [Bha, Theorem 10.0.9] for the almost *faithfully* flat part). Since $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)}$ is trivial, Lemma C.3.5 **(1)** implies that

$$\mathrm{H}^0\left(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\right) \simeq^a (B^+/pB^+)^r.$$

In particular, it is almost flat and almost finitely presented. We now want to descent these properties to $\mathrm{H}^0\left(\mathrm{Spd}\,(A_\infty, A^+_\infty)_v, \mathcal{E}\right)$. For this, we use Proposition C.1.6 to recall that diamondification commutes with fiber products, and so

$$\begin{aligned}\mathrm{Spd}\,(B, B^+) \times_{\mathrm{Spd}\,(A_\infty, A^+_\infty)} \mathrm{Spd}\,(B, B^+) &\simeq \left(\mathrm{Spa}\,(B, B^+) \times_{\mathrm{Spa}\,(A_\infty, A^+_\infty)} \mathrm{Spa}\,(B, B^+)\right)^\diamond \\ &\simeq \mathrm{Spd}\,\left(B\widehat{\otimes}_{A_\infty}B, (B\widehat{\otimes}_{A_\infty}B)^+\right).\end{aligned}$$

By the proof of [Sch12, Proposition 6.18] (and Lemma B.1.7), we see that $B^+\widehat{\otimes}_{A^+_\infty}B^+ \to (B\widehat{\otimes}_{A_\infty}B)^+$ is an almost isomorphism (while, a priori, the latter group is the integral closure of the former one inside $B\widehat{\otimes}_{A_\infty}B$). In particular,

$$B^+/p \otimes_{A^+_\infty/p} B^+/p \simeq^a (B\widehat{\otimes}_{A_\infty}B)^+/p(B\widehat{\otimes}_{A_\infty}B)^+.$$

Thus

$$\mathrm{H}^0\left(\mathrm{Spd}\,\left(B\widehat{\otimes}_{A_\infty}B, (B\widehat{\otimes}_{A_\infty}B)^+\right)_v, \mathcal{E}\right) \simeq^a \left((B^+/p)^{\otimes^2_{A^+_\infty/p}}\right)^r$$

[34] We note that $\mathcal{E}$ is a sheaf on a (big) v-site of $\mathrm{Spd}\,(A, A^+)$, so it makes sense to evaluate $\mathcal{E}$ on $\mathrm{Spd}\,(D, D^+)$.

and the natural morphism

$$\mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right)\otimes_{B^+/p}(B^+/p)^{\otimes^2_{A^+_\infty/p}}\to \mathrm{H}^0\left(\mathrm{Spd}\,\left(B\widehat{\otimes}_{A_\infty}B,(B\widehat{\otimes}_{A_\infty}B)^+\right)_v,\mathcal{E}\right)$$

is an almost isomorphism. We use the sheaf condition and the previous discussion to get an almost exact sequence

$$0\to \mathrm{H}^0(\mathrm{Spd}\,\left(A_\infty,A^+_\infty\right)_v,\mathcal{E})\to \mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right)\to \mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right)\otimes_{B^+/p}\left((B^+/p)^{\otimes 2}\right).$$

Theorem 2.10.3 applied to the almost faithfully flat morphism $A^+_\infty/pA^+_\infty\to B^+/pB^+$ implies that the natural morphism

$$\mathrm{H}^0\left(\mathrm{Spd}\,\left(A_\infty,A^+_\infty\right)_v,\mathcal{E}\right)\otimes_{A^+_\infty/p}B^+/p\to \mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right) \tag{6.4}$$

is an almost isomorphism. By the computation above, we know that $\mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right)$ is almost faithfully flat and almost finitely presented over B^+/pB^+. Thus, the faithfully flat descent for flatness and almost finitely presented modules (see Lemma 2.10.5 and Lemma 2.10.7) implies that $\mathrm{H}^0\left(\mathrm{Spd}\,\left(A_\infty,A^+_\infty\right)_v,\mathcal{E}\right)$ is almost faithfully flat and almost finitely presented over A^+_∞/pA^+_∞.

Step 2. $\mathrm{H}^0\left(\mathrm{Spd}(A_\infty,A^+_\infty)_v,\mathcal{E}\right)$ *almost commutes with base change:* By the proof of [Sch12, Proposition 6.18] (and Lemma B.1.7), we know that $\mathrm{Spa}\,(B,B^+)\times_{\mathrm{Spa}\,(A_\infty,A^+_\infty)}\mathrm{Spa}\,(D,D^+)$ exists as an adic space and is represented by $\mathrm{Spa}\,(R,R^+)$ for a perfectoid pair (R,R^+) such that

$$B^+/p\otimes_{A^+_\infty/p}D^+/p\to R^+/p \tag{6.5}$$

is an almost isomorphism. Thus the proof of Step 1 and (6.5) imply that

$$\mathrm{H}^0\left(\mathrm{Spd}\,(D,D^+)_v,\mathcal{E}\right)\otimes_{A^+_\infty/p}B^+/p\to \mathrm{H}^0\left(\mathrm{Spd}\,(R,R^+)_v,\mathcal{E}\right)$$

is an almost isomorphism. Now we wish to show the natural morphism

$$\mathrm{H}^0\left(\mathrm{Spd}\,(A_\infty,A^+_\infty)_v,\mathcal{E}\right)\otimes_{A^+_\infty/pA^+_\infty}D^+/pD^+\to \mathrm{H}^0\left(\mathrm{Spd}\,(D,D^+)_v,\mathcal{E}\right)$$

is an almost isomorphism. By the faithfully flat descent, it suffices to check after tensoring against B^+/pB^+ over A^+_∞/pA^+_∞. Therefore, we use (6.4) and (6.5) to see that it suffices to show that

$$\mathrm{H}^0\left(\mathrm{Spd}\,(B,B^+)_v,\mathcal{E}\right)\otimes_{B^+/p}R^+/p\to \mathrm{H}^0\left(\mathrm{Spd}\,(R,R^+)_v,\mathcal{E}\right)$$

is an almost isomorphism. Now Lemma C.3.5 **(1)** almost identifies (in the technical sense) this morphism with the identity moprhism

$$(R^+/pR^+)^r\to (R^+/pR^+)^r$$

since $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)}$ is a trivial $\mathcal{O}^+/p$-vector bundle of rank r.

Step 3. $\mathrm{H}^i\left(\mathrm{Spd}(A_\infty,A^+_\infty)_v,\mathcal{E}\right)$ *is almost zero for* $i>0$*:* As above, we use that

$$\mathrm{Spa}\,(B,B^+)\to \mathrm{Spa}\,(A_\infty,A^+_\infty)$$

is a finite étale morphism of affinoid perfectoids to conclude that all fiber products

$$\mathrm{Spa}\,(B,B^+)^{j/\mathrm{Spd}\,(A_\infty,A^+_\infty)}$$

are represented by affinoid perfectoids $\mathrm{Spa}\,(B_j,B^+_j)$ and the natural morphisms

$$(B^+/pB^+)^{\otimes^j_{A^+_\infty/pA^+_\infty}}\to B^+_j/pB^+_j$$

are almost isomorphisms. Since each restriction $\mathcal{E}|_{\mathrm{Spd}\,(B_j,B^+_j)}$ is trivial, Lemma C.3.5 **(1)** ensures that higher cohomology of $\mathcal{E}$ on $\mathrm{Spd}\,(B_j,B^+_j)$ almost vanish. Thus $\mathbf{R}\Gamma\left(\mathrm{Spd}\,(A_\infty,A^+_\infty)_v,\mathcal{E}\right)$ is almost isomorphism to the Čech complex associated to the covering $\mathrm{Spd}\,(B, B^+) \to \mathrm{Spd}\,(A_\infty, A_\infty^+)$. Step 2 implies that this complex is almost isomorphic to the standard Amitsur complex

$$0 \to M_{\mathcal{E}} \to M_{\mathcal{E}} \otimes_{A_\infty^+/p} B^+/p \to M_{\mathcal{E}} \otimes_{A_\infty^+/p} B^+/p \otimes_{A_\infty^+/p} B^+/p \to \dots$$

Almost exactness of this complex follows from Lemma 2.10.4. □

6.5. **Strictly Totally Disconnected Covers of Affinoids.** The main goal of this section is to get rid of the almost mathematics in Lemma 6.4.8 under some stronger assumptions on A_∞ (and on $\mathcal{E}$).

Set-up 6.5.1. We fix

(1) a p-adic perfectoid field K with its rank-1 open and bounded valuation ring $\mathcal{O}_K$ and a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$),

(2) an affine admissible formal scheme $\mathfrak{X} = \mathrm{Spf}\ A_0$ with an adic generic fiber $X = \mathrm{Spa}\,(A, A^+)$;

(3) a strictly totally disconnected affinoid perfectoid $\mathrm{Spa}\,(A_\infty, A_\infty^+)$ (see Definition C.2.1) with a morphism

$$\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$$

such that $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a v-covering and all fiber products

$$\mathrm{Spa}\,(A_\infty, A_\infty^+)^{j/\,\mathrm{Spa}\,(A,A^+)}$$

are strictly totally disconnected affinoid perfectoid spaces;

(4) an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ such that $\mathcal{E}|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)} \simeq (\mathcal{O}^+_{\mathrm{Spd}\,(A_\infty, A_\infty^+)}/p)^r$ for some integer r.

Corollary 6.5.2. Under the assumption of Set-up 6.5.1, let $\mathfrak{f}\colon \mathrm{Spf}\ B_0 \to \mathrm{Spf}\ A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes. Then

$$(B_\infty, B_\infty^+) \coloneqq \left(\left(B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)[1/p], B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)$$

is a perfectoid pair and $\mathrm{Spa}\,(B_\infty, B_\infty^+)$ is a strictly totally disconnected (affinoid) perfectoid.

Proof. Corollary 6.4.5 already implies that $\mathrm{Spa}\,(B_\infty, B_\infty^+)$ is an affinoid perfectoid. Moreover, Lemma 6.4.6 implies that

$$\mathrm{Spa}\,(B_\infty, B_\infty^+) \simeq \mathrm{Spa}\,(B, B^+) \times_{\mathrm{Spa}\,(A,A^+)} \mathrm{Spa}\,(A_\infty, A_\infty^+),$$

where $\mathrm{Spa}\,(B, B^+)$ is the generic fiber of $\mathrm{Spf}\ B_0$. So $\mathrm{Spa}\,(B_\infty, B_\infty^+) \to \mathrm{Spa}\,(A_\infty, A_\infty^+)$ is an étale morphism, and so the claim follows from Lemma C.2.6. □

Lemma 6.5.3. Under the assumption of Set-up 6.5.1, let $M_{\mathcal{E}}$ be an A_∞^+/pA_∞^+-module

$$M_{\mathcal{E}} \coloneqq \mathrm{H}^0\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\right).$$

Then $M_{\mathcal{E}}$ is (non-canonically) isomorphic to $(A_\infty^+/pA_\infty^+)^r$, and for every morphism $\mathrm{Spa}\,(D, D^+) \to \mathrm{Spa}\,(A_\infty, A_\infty^+)$ of strictly totally disconnected affinoid perfectoids, the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} D^+/p \to \mathrm{H}^0\left(\mathrm{Spd}\,(D, D^+)_v, \mathcal{E}\right)$$

is an isomorphism. Moreover,

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \simeq 0$$

for $i, j \geq 1$.

Proof. Once we fixed an isomorphism

$$\mathcal{E}|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)} \simeq \left(\mathcal{O}^+_{X^\diamond}/p\right)^r|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)},$$

the isomorphism $M_{\mathcal{E}} \simeq (A_\infty^+/A_\infty^+)^r$ follows from Corollary C.3.13. An isomorphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} D^+/p \simeq^a \mathrm{H}^0(\mathrm{Spd}\,(D, D^+)_v, \mathcal{E})$$

is then clear. And vanishing

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \simeq 0$$

for $i, j \geq 1$ also follows from Corollary C.3.13. because we assume that all fiber products

$$\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$$

are representable by strictly totally disconnected (affinoid) perfectoid spaces. ∎

Corollary 6.5.4. Under the assumption of Set-up 6.5.1, let $\mathfrak{f}\colon \mathrm{Spf}\, B_0 \to \mathrm{Spf}\, A_0$ is an étale morphism with (B_∞, B_∞^+) a perfectoid pair as in Corollary 6.4.5. Then the natural morphism

$$\Gamma\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \otimes_{A_0/pA_0} B_0/pB_0 \to \Gamma\left(\mathrm{Spd}\,(B_\infty, B_\infty^+)_v^{j/\mathrm{Spd}\,(B,B^+)}, \mathcal{E}\right).$$

is an isomorphism for $j \geq 1$.

Proof. By definition, all fiber products $\mathrm{Spa}\,(A_\infty, A_\infty^+)^{j/\,\mathrm{Spa}\,(A,A^+)}$ satisfy the assumption of Set-up 6.5.1, so Proposition C.1.6 **(6)** ensures that it suffices to show the claim for $j = 1$. In this case, the result follows from Lemma 6.5.3 and Corollary 6.5.2. ∎

Corollary 6.5.5. Under the assumption of Set-up 6.5.1, let $\mathfrak{f}\colon \mathrm{Spf}\, B_0 \to \mathrm{Spf}\, A_0$ is an étale morphism with (B_∞, B_∞^+) a perfectoid pair as in Corollary 6.4.5. Then the natural morphism

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E}\right) \otimes_{A_0/pA_0} B_0/pB_0 \to \mathrm{H}^i\left(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\right).$$

is an isomorphism for $i \geq 0$.

Proof. Again, by definition, all fiber products $\mathrm{Spa}\,(A_\infty, A_\infty^+)^{j/\,\mathrm{Spa}\,(A,A^+)}$ satisfy the assumption of Set-up 6.5.1, so Lemma 6.5.3 implies that

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \simeq 0$$

for $i, j \geq 1$. Therefore, cohomology groups $\mathrm{H}^i(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E})$ can be computed via cohomology of the Čech complex associated to the covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$. By Corollary 6.5.2, the same applies to $\mathrm{Spa}\,(B, B^+)$ and the Čech complex associated to the covering $\mathrm{Spd}\,(B_\infty, B_\infty^+) \to \mathrm{Spd}\,(B, B^+)$. Therefore, the claim follows from Corollary 6.5.4. ∎

Corollary 6.5.6. Under the assumption of Set-up 6.4.1, let $K \subset C$ be a completed algebraic closure of K, and $\mathrm{Spa}\,(A_C, A_C^+) = \mathrm{Spa}\,(A, A^+) \times_{\mathrm{Spa}\,(K, \mathcal{O}_K)} \mathrm{Spa}\,(C, \mathcal{O}_C)$. Then the natural morphism

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E}\right) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i\left(\mathrm{Spd}\,(A_C, A_C^+)_v, \mathcal{E}\right).$$

is an almost isomorphism.

Proof. The proof is similar to that of Corollary 6.5.4 and Corollary 6.5.5. The only change that we need to make is that the fiber product

$$\mathrm{Spa}\,(A_\infty, A_\infty^+) \times_{\mathrm{Spa}\,(K, \mathcal{O}_K)} \mathrm{Spa}\,(L, \mathcal{O}_L)$$

is a strictly totally disconnected affinoid perfectoid with the $+$-ring *almost* isomorphic to $A_\infty^+ \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_L$. The strictly totally disconnected claim follows from Lemma C.2.6 and almost computation of the $+$-ring follows from the proof of [Sch13, Proposition 6.18]. ∎

6.6. **Perfectoid Torsors.** We apply the results of Section 6.4 to certain pro-étale covers of $\mathrm{Spa}\,(A, A^+)$ to see that computation of v-cohomology sheaves can be often reduced to the computation of certain continuous cohomology groups. To make this precise, we need to define the notion of a G-torsor under a pro-finite group G.

Definition 6.6.1. A *v-sheaf $\underline{G}$ associated to a pro-finite group G* is a v-sheaf $\underline{G}\colon \mathrm{Perf}^{\mathrm{op}} \to \mathrm{Sets}$ such that $\underline{G}(S) = \mathrm{Hom}_{\mathrm{cont}}(|S|, G)$.

A morphism of v-sheaves $X \to Y$ is a *$\underline{G}$-torsor* if it is a v-surjection and there is an action $a\colon \underline{G} \times X \to X$ over Y such that the morphism $a \times_Y p_2\colon \underline{G} \times X \to X \times_Y X$ is an isomorphism, where $p_2\colon \underline{G} \times X \to X$ is the canonical projection.

Remark 6.6.2. If a pro-finite group G is a cofiltered limit of finite groups $G \simeq \lim_I G_i$, then $\underline{G} \simeq \lim_I \underline{G_i}$.

Now we can formulate the precise set-up we are going to work in.

Set-up 6.6.3. We fix

(1) a p-adic perfectoid field K with its rank-1 open and bounded valuation ring $\mathcal{O}_K$ and a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$);

(2) an affinoid rigid-analytic space $\mathrm{Spa}\,(A, A^+)$ over K with an admissible formal $\mathcal{O}_K$-model $\mathrm{Spa}\,(A_0)$, and a morphism $(A, A^+) \to (A_\infty, A_\infty^+)$ such (A_∞, A_∞^+) is a perfectoid pair and $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor under a pro-finite group Δ_∞;

(3) a very small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$.

We start the section by studying the structure of the fiber products $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ for $j \geq 1$. For a general v-cover, we cannot say much about these fiber products. But the situation is much better in the case of $\underline{G}$-torsors.

Lemma 6.6.4. Under the assumption of Set-up 6.6.3, the fiber product $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ is represented by an affinoid perfectoid[35] $\mathrm{Spa}\,(T_j, T_j^+)$ for every $j \geq 0$. Moreover, for every $j \geq 0$,

$$\left(T_j, T_j^+\right) \simeq \left(\mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^\flat), \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^{\flat,+})\right)$$

and $T_j^{\sharp,+}/pT_j^{\sharp,+} \simeq T_j^+/\varpi^\flat T_j^+ \simeq \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^+/pA_\infty^+)$.

Proof. We first show that $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ are representable by affinoid perfectoids. Since $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor, we get

$$\begin{aligned}\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)} &\simeq \mathrm{Spd}\,(A, A^+) \times \underline{\Delta}_\infty^{j-1} \\ &\simeq \lim_I \left(\mathrm{Spa}\,(A_\infty^\flat, A_\infty^{\flat,+}) \times \underline{\Delta}_i^{j-1}\right) \\ &\simeq \lim_I \left(\mathrm{Spa}\,\left(\mathrm{Map}(\Delta_i^{j-1}, A_\infty^\flat), \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})\right)\right)\end{aligned}$$

is a cofiltered limit of affinoid perfectoid spaces, so it is an affinoid perfectoid space $\mathrm{Spa}\,(T_j, T_j^+)$ by [Sch17, Proposition 6.5]. Moreover, *loc. cit.* implies that T_j^+ is equal to the $\varpi^\flat$-adic completion

[35]Recall that $\mathrm{Spd}\,(A_\infty, A_\infty^+)$ is itself represented by an affinoid perfectoid $\mathrm{Spa}\,(A_\infty^\flat, A_\infty^{\flat,+})$.

of the filtered colimit $\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})$ and $T_j = T_j^+[\frac{1}{\varpi^\flat}]$. In particular, we already see that

$$\begin{aligned} T_j^{\sharp,+}/pT_j^{\sharp,+} \simeq T_j^+/(\varpi)^\flat T_j^+ &\simeq \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})\right)/(\varpi)^\flat \\ &\simeq \mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+}/(\varpi)^\flat A_\infty^{\flat,+}) \\ &\simeq \mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{+}/\varpi A_\infty^{+}) \\ &\simeq \mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{+}/p A_\infty^{+}) \\ &\simeq \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^{+}/p A_\infty^{+}). \end{aligned}$$

Now we compute T_j^+ and T_j. We start with T_j^+:

$$\begin{aligned} T_j^+ &\simeq \lim_n \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})/(\varpi^\flat)^n\right) \\ &\simeq \lim_n \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+})\right) \\ &\simeq \lim_n \mathrm{Map}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+}\right) \\ &\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, \lim_n A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+}\right) \\ &\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}\right). \end{aligned}$$

Since Δ_∞ is compact and $A_\infty^\flat \simeq A_\infty^{\flat,+}[\frac{1}{\varpi^\flat}]$, we also have

$$\begin{aligned} T_j &\simeq T_j^+[1/\varpi^\flat] \\ &\simeq \mathrm{colim}_{\times \varpi^\flat, n}\, \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}\right) \\ &\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, \mathrm{colim}_{\times \varpi^\flat} A_\infty^{\flat,+}\right) \\ &\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat}\right) \end{aligned}$$

finishing the proof. □

Note that since $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor, there is a canonical continuous A^+-linear action of Δ_∞ on A_∞^+. Now we want to relate v-cohomology groups of $\mathcal{E}$ to the continuous group cohomology of Δ_∞. This is done in the following lemmas:

Lemma 6.6.5. Under the assumption of Set-up 6.6.3, we define $M_\mathcal{E}$ to be an A_∞^+/pA_∞^+-module $\mathrm{H}^0\,(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E})$. Then $M_\mathcal{E}$ is almost faithfully flat, almost finitely presented A_∞^+/pA_∞^+-module, and

$$\mathrm{H}^0(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}) \simeq^a \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, M_\mathcal{E}) \simeq^a \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, (M_\mathcal{E}^a)_!),$$

$$\mathrm{H}^i(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}) \simeq^a 0$$

for every $i, j \geq 1$.

Proof. Lemma 6.6.4 implies that all fiber products $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ satisfy the assumptions of Lemma 6.4.8. Thus Lemma 6.4.8 and the computation of fiber products in Lemma 6.6.4 imply that

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \simeq^a 0$$

for every $i, j \geq 1$, and the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/pA_\infty^+} \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/p\right) \to \mathrm{H}^0\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right)$$

is an almost isomorphism for every $j \geq 1$. Thus it suffices to show that the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/p\right) \to \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, M_{\mathcal{E}})$$

is an isomorphism. This can be done by writing $\Delta_\infty = \lim_I \Delta_i$ and reducing to the case of a finite group similarly to the proof of Lemma 6.6.4. The almost isomorphism

$$\mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, M_{\mathcal{E}}) \simeq^a \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, (M_{\mathcal{E}}^a)_!)$$

is achieved similarly using that $(-)_!$ commutes with colimits being a left adjoint functor. □

Lemma 6.6.6. Under the assumption of Set-up 6.4.1, we define $M_{\mathcal{E}}$ to be an A_∞^+/pA_∞^+-module $\mathrm{H}^0\,(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E})$. Then there is a canonical continuous action of Δ_∞ on $(M_{\mathcal{E}}^a)_!$ compatible with the action of Δ_∞ on A_∞^+/pA_∞^+, i.e. $g(am) = g(a)g(m)$ for any $a \in A_\infty^+/pA_\infty^+$ and $m \in M_{\mathcal{E}}$.

Proof. Lemma 6.6.4 ensures that the fiber product $\mathrm{Spd}\,(A_\infty, A_\infty^+) \times_{\mathrm{Spd}\,(A,A^+)} \mathrm{Spd}\,(A_\infty, A_\infty^+)$ is represented by an affinoid perfectoid $\mathrm{Spa}\,(T_2, T_2^+)$. Therefore, we can uniquely write it as $\mathrm{Spd}\,(S, S^+)$ for an untilt of (T_2, T_2^+) corresponding to the morphism $\mathrm{Spa}\,(T_2, T_2^+) \to \mathrm{Spd}\,(A, A^+) \to \mathrm{Spd}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Lemma 6.4.8 implies that the descent data for the sheaf $\mathcal{E}$ provides us with an $(S^+/pS^+)^a$-isomorphism

$$\left(S^+/p\right)^a \otimes_{\left(A_\infty^+/p\right)^a} (M_{\mathcal{E}})^a \to (M_{\mathcal{E}})^a \otimes_{\left(A_\infty^+/p\right)^a} \left(S^+/p\right)^a$$

satisfying the cocycle condition. By Corollary 2.2.4 **(2)**, this defines an $\left(A_\infty^+/pA_\infty^+\right)^a$-linear morphism

$$(M_{\mathcal{E}})^a \to (M_{\mathcal{E}})^a \otimes_{\left(A_\infty^+/p\right)^a} \left(S^+/p\right)^a.$$

By Lemma 6.4.8 and Lemma 6.6.5, this is equivalent to an $\left(A_\infty^+/pA_\infty^+\right)^a$-linear morphism

$$(M_{\mathcal{E}})^a \to \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty, (M_{\mathcal{E}}^a)_!\right)^a.$$

By Lemma 2.1.9 **(3)**, this is the same as an (A_∞^+/pA_∞^+)-linear morphism

$$\varphi\colon\ (M_{\mathcal{E}}^a)_! \to \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty, (M_{\mathcal{E}}^a)_!\right).$$

This defines a morphism

$$\gamma\colon \Delta_\infty \to \mathrm{Hom}_{A_\infty^+/p}\left((M_{\mathcal{E}})_!\,, (M_{\mathcal{E}})_!\right)$$

by the rule

$$\gamma(g)(m) = (\phi(m))(g).$$

One checks that the cocycle condition translates into the statement that γ is a group homomorphism, i.e. it defines an action of Δ_∞. Likewise, one checks that A_∞^+/pA_∞^+-linearity of ϕ translates in to the fact that this action is compatible with the action on A_∞^+/pA_∞^+. And continuity of ϕ translates into the fact that γ defines a continuous action, i.e. the natural morphism

$$\mathrm{colim}_{U_i \triangleleft \Delta_\infty, \mathrm{open}} (M_{\mathcal{E}}^a)_!^{U_i} \to (M_{\mathcal{E}}^a)_!^{\Delta_\infty}$$

is an isomorphism. □

Corollary 6.6.7. Under the assumption of Set-up 6.4.1, we define $M_{\mathcal{E}}$ to be an A_∞^+/pA_∞^+-module $\mathrm{H}^0\,(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E})$. Then

$$\mathrm{H}^i(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E}) \simeq^a \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, (M_{\mathcal{E}}^a)_!).$$

Proof. Lemma 6.6.5 implies that

$$\mathrm{H}^i\left(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\right) \simeq^a 0$$

for $i, j \geq 1$. Therefore, the cohomology groups $\mathrm{H}^i(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E})$ can be almost computed via cohomology of the Čech complex associated to the covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$. Moreover, Lemma 6.6.5 also implies that the terms of this complex can be almost identified with the bar complex computing continuous cohomology of the pro-finite group Δ_∞ with coefficients in the discrete module $(M_{\mathcal{E}}^a)_!$. We leave it to the reader to verify that the differentials in the Čech complex coincide with the differentials in the bar complex computing continuous cohomology. □

For the future reference, we also discuss the following base change result:

Lemma 6.6.8. Let G be a pro-finite group, and let M be a discrete R-module that has a continuous R-linear action of G. Suppose that $R \to A$ is a flat homomorphisms of rings. Then the canonical morphism $\mathrm{H}^i_{\mathrm{cont}}(G, M) \otimes_R A \to \mathrm{H}^i_{\mathrm{cont}}(G, M \otimes_R A)$ is an isomorphism for $i \geq 0$.

Proof. This is a combination of two facts: filtered colimits commute with tensor product, cohomology of finite groups commute with flat base change (in particular, invariants commute with base change). Indeed, Lemma follows from a sequence of isomorphisms

$$\begin{aligned}
\mathrm{H}^i_{\mathrm{cont}}(G, M) \otimes_R A &\cong (\mathrm{colim}_{H \triangleleft G, \mathrm{open}} \mathrm{H}^i(G/H, M^H)) \otimes_R A \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}} (\mathrm{H}^i(G/H, M^H) \otimes_R A) \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}} \mathrm{H}^i(G/H, M^H \otimes_R A) \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}} \mathrm{H}^i(G/H, (M \otimes_R A)^H) \\
&\simeq \mathrm{H}^i_{\mathrm{cont}}(G, M \otimes_R A)
\end{aligned}$$

□

6.7. **Nearby Cycles are Quasi-Coherent.** We start the proof Theorem 6.1.9 and Theorem 6.1.2 in this Section. Namely, we show that the complex $\mathbf{R}\nu_*(\mathcal{E})$ are quasi-coherent and commutes with étale base change for a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$. The main idea is to apply the results of Section 6.4 to a particular perfectoid covering of X.

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (see Definition B.1.6). We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 6.7.1. Let $X = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then there is a strictly totally disconnected affinoid perfectoid $\mathrm{Spa}\,(A_\infty, A_\infty^+)$ with a morphism $\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$ such that

1. the morphism $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a v-covering;
2. all fiber products $\mathrm{Spa}\,(A_\infty, A_\infty^+)^{j/\,\mathrm{Spa}\,(A,A^+)}$ are strictly totally disconnected affinoid perfectoids;
3. $\mathcal{E}|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)} \simeq \left(\mathcal{O}^+_{X^\diamond}/p\right)^r|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)}$ for some integer r.

Proof. Lemma C.2.10 ensures that there is a morphism $\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$ satisfying the first two properties. Now Lemma C.4.4 ensures that $\mathcal{E}|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)} \simeq \left(\mathcal{O}^+_{X^\diamond}/p\right)^r|_{\mathrm{Spd}\,(A_\infty, A_\infty^+)}$. □

Lemma 6.7.2. Let $\mathfrak{X} = \mathrm{Spf}\ A_0$ an admissible affine formal $\mathcal{O}_K$-scheme with an affinoid generic fiber $X = \mathrm{Spa}\,(A, A^+)$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. Then $\mathrm{R}^i\nu_*(\mathcal{E})$ is quasi-coherent for $i \geq 0$. More precisely, the natural morphism

$$\widetilde{\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E})} \to \mathrm{R}^i\nu_*\,(\mathcal{E})$$

is an isomorphism for any $i \geq 0$.

Proof. The universal property of the tilde-construction implies that we do have a natural morphism

$$c\colon \widetilde{\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E})} \to \mathrm{R}^i\nu_*(\mathcal{E}).$$

Recall that $\mathrm{R}^i\nu_*(\mathcal{E})$ is the sheafification of a presheaf defined by

$$\mathfrak{U} \mapsto \mathrm{H}^i(\mathfrak{U}^\diamondsuit_{K,v}, \mathcal{E}).$$

Thus, in order to show that c is an isomorphism, it suffices to show that the natural morphism

$$\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E}) \otimes_{A_0/pA_0} (A_0/pA_0)_f \to \mathrm{H}^i(\mathfrak{U}^\diamondsuit_{K,v}, \mathcal{E})$$

is an isomorphism for any open formal subscheme $\mathrm{Spf}\,(A_0)_{\{f\}} \subset \mathrm{Spf}\ A_0$. We choose a covering $\mathrm{Spa}\,(A_\infty, A_\infty) \to \mathrm{Spa}\,(A, A^+)$ from Lemma 6.7.1. Then the result follows from Corollary 6.5.5 since $(A, A^+) \to (A_\infty, A^+_\infty)$ and $\mathcal{E}$ fit into Set-up 6.5.1. □

Theorem 6.7.3. Let $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathfrak{X}_K$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. Then $\mathrm{R}^i\nu_*(\mathcal{E})$ is quasi-coherent for $i \geq 0$. Furthermore, if $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ an étale morphism with generic fiber $f\colon Y \to X$, then the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\mathcal{E}\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamondsuit_v}\right)$$

is an isomorphism for any $i \geq 0$.

Proof. Both claims are local on $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A_0$ and $\mathfrak{Y} = \mathrm{Spf}\ B_0$ are affine. Then quasi-coherence of $\mathrm{R}^i\nu_*(\mathcal{E})$ follows from Lemma 6.7.2. In order to show that

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\mathcal{E}\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamondsuit_v}\right),$$

it suffices to show that the natural morphism

$$\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E}) \otimes_{A_0/pA_0} B_0/pB_0 \to \mathrm{H}^i(Y^\diamondsuit_v, \mathcal{E})$$

is an isomorphism. This follows from Corollary 6.5.5 using the covering $\mathrm{Spa}\,(A_\infty, A^+_\infty) \to \mathrm{Spa}\,(A, A^+)$ from Lemma 6.7.1. □

For the future reference, we also prove the following result:

Lemma 6.7.4. Let $X = \mathrm{Spa}\,(A, A^+)$ be an affinoid rigid-analytic space over K, $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle, and $K \subset C$ a completed algebraic closure of K. Then

$$\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E}) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i(X^\diamondsuit_{C,v}, \mathcal{E})$$

is an almost isomorphism.

Proof. This follows directly from Corollary 6.5.6 using the covering $\mathrm{Spa}\,(A_\infty, A^+_\infty) \to \mathrm{Spa}\,(A, A^+)$ from Lemma 6.7.1. □

6.8. **Nearby Cycles are Almost Coherent for Smooth X and small $\mathcal{E}$.** The main goal of this section is to show that the complex $\mathbf{R}\nu_*(\mathcal{E})$ has almost coherent cohomology sheaves for an admissible formal $\mathcal{O}_K$-scheme with *smooth* generic fiber. The main idea is to apply the results of Section 6.6 to a particular "small" perfectoid torsor cover of X, where one has a good control over the structure group Δ_∞.

For the rest of the section we fix a p-adic perfectoid field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

We first discuss the overall strategy of the proof. We proceed in four steps: firstly, we show the result for the formal $\widehat{\mathbf{G}}_m^n$ and $\mathcal{E} = \mathcal{O}^+_{X^\diamond}/p$, then we deduce the result for affine formal schemes such that the adic generic fiber admits a map to a torus $\mathbf{T}^n_C$ that is a composition of finite étale maps and rational embeddings. After that we show the result in general by choosing a "good" covering of $\mathfrak{X}$ possibly after an admissible blow-up of $\mathfrak{X}$ to finish the proof for $\mathcal{E} = \mathcal{O}^+_{X^\diamond}/p$. We reduce the general case to the case $\mathcal{E} = \mathcal{O}^+_{X^\diamond}/p$ via Corollary C.4.10.

The main ingredient for the third step is Achinger's result ([Ach17, Proposition 6.6.1]) that any étale morphism $g\colon \operatorname{Spa}(A, A^+) \to \mathbf{D}^n_K$ can be replaced with a finite étale morphism

$$g'\colon \operatorname{Spa}(A, A^+) \to \mathbf{D}^n_K.$$

The proof of this result in [Ach17] is given only for rigid-analytic varieties over the fraction field of a discrete valuation ring, but we need to apply it in the perfectoid situation that is never discretely valued. So Appendix D provides the reader with a detailed proof of this result without any discreteness assumptions.

Now we begin to realize the strategy sketched above. We consider $\mathfrak{X} = \operatorname{Spf} \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$, and set $R^+ \coloneqq \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$ and $R^+_m \coloneqq \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle$. We note that the map $\operatorname{Spf} R^+_m \to \operatorname{Spf} R^+$ defines a $\mu^n_{p^m}$-torsor, thus $\mu^n_{p^m}$ continuously acts on R^+_m by R^+-linear automorphisms.

Now we consider an R-algebra

$$R^+_\infty = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle = \left(\operatorname{colim}_n R^+_m\right)^\wedge$$

where $^\wedge$stands for the p-adic completion. It comes with a continuous R-linear action of the group $\Delta_\infty \coloneqq \mathbf{Z}_p(1)^n = T_p(\mu_{p^\infty})$ on R^+_∞. We trivialize $\mathbf{Z}_p(1)$ by choosing some compatible system of p^i-th roots of unity $(\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots)$. In order to describe the action of Δ_∞ on R^+_∞ we need the following definition:

Definition 6.8.1. For any $a \in \mathbf{Z}[1/p]$, we define ζ^a as $\zeta_{p^l}^{ap^l}$ whenever $ap^l \in \mathbf{Z}$. It is clear to see that this definition does not depend on a choice of l.

Essentially by definition, the k-th basis vector $\gamma_k \in \Delta_\infty \simeq \mathbf{Z}_p^n$ acts on R^+_∞ as

$$\gamma_k(T_1^{a_1} \dots T_n^{a_n}) = \zeta^{a_k} T_1^{a_1} \dots T_n^{a_n}.$$

Lemma 6.8.2. [Sch13, Lemma 5.5] Let R^+, R^+_∞ and Δ_∞ be as above. Then the cohomology groups $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty)$ are almost coherent R^+/pR^+_∞-modules. And the natural map

$$\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p) \otimes_{R^+/p} A^+/p \to \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p \otimes_{R^+/p} A^+/p)$$

is an isomorphism for a p-torsionfree R^+-algebra A^+ and $i \geq 0$.

Proof. We note that R^+/pR^+ is an almost noetherian ring by Theorem 2.11.4. Thus Theorem 2.7.8 implies that $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty)$ is almost coherent if it is almost finitely generated.

Now [BMS18, Lemma 7.3] says that $\mathbf{R}\Gamma_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty)$ is computed by means of the Koszul complex $K\left(R^+_\infty/pR^+_\infty; \gamma_1 - 1, \dots, \gamma_n - 1\right)$. Then, similarly to [Bha18, Lemma 4.6], we can write

$$K\left(R^+_\infty/p; \gamma_1 - 1, \dots, \gamma_n - 1\right) = K\left(R^+/p; 0, 0, \dots, 0\right) \oplus \bigoplus_{(a_1,\dots,a_n)\in(\mathbf{Z}[1/p]\cap(0,1))^n} K\left(R^+/p; \zeta^{a_1} - 1, \dots, \zeta^{a_n} - 1\right)$$

We observe that

$$\mathrm{H}^i\left(K\left(R^+/pR^+; 0, 0, \dots, 0\right)\right) = \wedge^i\left(R^+/pR^+\right)$$

is a free finitely presented R^+/pR^+-module. For each $(a_1, \dots, a_n) \in (\mathbf{Z}[1/p] \cap (0, 1))^n$, we can assume that a_1 has the minimal p-adic valuation for the purpose of proving that

$$K(R^+_\infty/pR^+_\infty; \gamma_1 - 1, \dots, \gamma_n - 1)$$

has almost finitely finitely generated cohomology groups. Then [BMS18, Lemma 7.10] implies that $\mathrm{H}^i\left(K\left(R^+/pR^+; \zeta^{a_1} - 1, \dots, \zeta^{a_n} - 1\right)\right)$ is finitely presented over R^+/pR^+ and $\zeta^{a_1} - 1$-torsion module. Note that

$$v_p(\zeta^{a_1} - 1) = v_p(\zeta_{p^l} - 1) = \frac{v(p)}{p^l - p^{l-1}} \to 0$$

where $a_1 = b/p^l$ with $\gcd(b, p) = 1$. Moreover, for any $h \in \mathbf{Z}$, there are only finitely many indexes $(a_1, \dots, a_n) \in (\mathbf{Z}[1/p] \cap (0, 1))^n$ with $v_p(a_j) \geq h$. This implies that

$$\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty) = \mathrm{H}^i\left(K\left(R^+_\infty/p; \gamma_1 - 1, \dots, \gamma_n - 1\right)\right)$$

is a finitely presented R^+/pR^+-module up to any ϖ^{1/p^n}-torsion. In particular, this module is almost finitely presented.

Now we show that $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty)$ commutes with base change for any $\mathcal{O}_K$-flat algebra A^+. In order to show this, we observe the $(R^+/pR^+)[\Delta_\infty]$-module R^+_∞/pR^+_∞ comes as a tensor product $M \otimes_{\mathcal{O}_K/p} R^+/p$ for the $(\mathcal{O}_K/p\mathcal{O}_K)[\Delta_\infty]$-module

$$M \coloneqq \bigoplus_{(a_1,\dots,a_n)\in(\mathbf{Z}[1/p]\cap[0,1))^n} (\mathcal{O}_K/p\mathcal{O}_K) T_1^{a_1} \dots T_n^{a_n}$$

where the basis element γ_k acts by

$$\gamma_k(T_1^{a_1} \dots T_n^{a_n}) = \zeta^{a_k} T_1^{a_1} \dots T_n^{a_n}.$$

Therefore, the desired claim follows from a sequence of isomorphisms

$$\begin{aligned}
\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p) \otimes_{R^+/p} A^+/p &\simeq \left(\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, M) \otimes_{\mathcal{O}_K/p} R^+/p\right) \otimes_{R^+/p} A^+/p \\
&\simeq \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, M) \otimes_{\mathcal{O}_K/p} A^+/p \\
&\simeq \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, M \otimes_{\mathcal{O}_K/p} A^+/p) \\
&\simeq \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p \otimes_{R^+/p} A^+/p),
\end{aligned}$$

where the third isomorphism uses Lemma 6.6.8. □

Lemma 6.8.2 combined with Corollary 6.6.7 essentially settle the first step of our strategy. Now we move to the second step. We start with the following preliminary result:

Lemma 6.8.3. Let A_0 be a topologically finitely presented $\mathcal{O}_K$-algebra, and P a topologically free A_0-module, i.e. $P = \widehat{\bigoplus}_I A_0$ for some set I. Then M is A_0-flat.

Proof. We start the proof by noting that [Sta21, Tag 00M5] guarantees that it suffices to show that $\mathrm{Tor}_1^{A_0}(P, M) = 0$ for any finitely presented A_0-module M. We choose a presentation

$$0 \to Q \to A_0^n \to M \to 0$$

and observe that Q is finitely presented because A_0 is coherent. So vanishing of Tor_1 is equivalent to showing that

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$$

is injective.

Now note that $Q[p^\infty]$, $A_0^n[p^\infty]$, and $M[p^\infty]$ are bounded by [Bos14, Lemma 7.3/7], so the same holds for $\bigoplus_I Q$, $\bigoplus_I A_0^n$, and $\bigoplus_I M$. Therefore, the usual p-adic completions of $\bigoplus_I Q$, $\bigoplus_I A_0^n$ and $\bigoplus_I M$ coincide with their derived p-adic completions. Since derived p-adic completion is exact (in the sense of triangulated categories) and coincides with the usual one on these modules, we get that the sequence

$$0 \to \widehat{\bigoplus_I} Q \to \widehat{\bigoplus_I} A_0^n \to \widehat{\bigoplus_I} M \to 0$$

is exact.

Now we want to show that this short exact sequence is the same as the sequence

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n \to P \otimes_{A_0} M \to 0.$$

As a consequence, this will prove that $P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$ is injective.

For each A_0-module N, there is a canonical map

$$P \otimes_{A_0} N \to \widehat{\bigoplus_I} N.$$

So we have a morphism of sequences:

$$\begin{tikzcd}
 & P \otimes_{A_0} Q \arrow[r] \arrow[d] & P \otimes_{A_0} A_0^n \arrow[r] \arrow[d] & P \otimes_{A_0} M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\bigoplus}_I Q \arrow[r] & \widehat{\bigoplus}_I A_0^n \arrow[r] & \widehat{\bigoplus}_I M \arrow[r] & 0.
\end{tikzcd}$$

The map $A_0^n \otimes_{A_0} P \to \widehat{\bigoplus}_I A_0^n$ is an isomorphism because $A_0^n \otimes_{A_0} P = P^n$ is already p-adically complete. This implies that the arrow

$$M \otimes_{A_0} P \to \widehat{\bigoplus_I} M$$

is surjective. But then

$$P \otimes_{A_0} Q \to \widehat{\bigoplus_I} Q$$

is surjective since M was an arbitrary finitely presented A-module. Now a diagram chase implies that

$$M \otimes_{A_0} P \to \widehat{\bigoplus_I} M$$

is also injective. And, therefore, it is an isomorphism. So

$$P \otimes_{A_0} Q \to \widehat{\bigoplus_I} Q$$

is also an isomorphism. Therefore, these two sequences are the same. In particular,

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$$

is injective. $\square$

To establish the second part of our strategy, we will also need a slightly refined version of [Sch13, Lemma 4.5] specific to the situation of an étale morphism to a torus.

We recall that we have defined

$$R^+ \coloneqq \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle,$$

$$R^+_m \coloneqq \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle, \text{ and}$$

$$R^+_\infty = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle = \big(\mathrm{colim}_n\ R^+_m\big)^\wedge$$

and a group $\Delta_\infty \simeq \mathbf{Z}_p^m$ continuously acts on R^+_∞. We also define R (resp. R_m, R_∞) as $R^+[1/p]$ (resp. $R^+_m[1/p]$, $R^+_\infty[1/p]$). For an étale morphism $\mathrm{Spa}\,(A, A^+) \to \mathrm{Spa}\,(R, R^+) = \mathbf{T}^n$ we define a Huber pair

$$\big(A_m, A_m^+\big) \coloneqq \big(R_m \otimes_R A, (R_m \otimes_R A)^+\big) = \big(R_m \widehat{\otimes}_R A, (R_m \widehat{\otimes}_R A)^+\big)\,,$$

where $(R_m \otimes_R A)^+$ is the *integral closure* of the image of $R^+_m \otimes_{R^+} A^+$ in $R_m \otimes_R A$. Similarly, we define

$$A^+_\infty \coloneqq \big(\mathrm{colim}_n\ A^+_m\big)^\wedge$$

and $A_\infty \coloneqq A^+_\infty[1/p]$.

Lemma 6.8.4. [Sch13, Lemma 4.5] Let $\mathrm{Spa}\,(A, A^+) \to \mathrm{Spa}\,(R, R^+) = \mathbf{T}^n$ be a morphism that is a composition of a finite étale maps and rational embeddings. Then (A_∞, A^+_∞) is an affinoid perfectoid pair, $\mathrm{Spd}\,(A_\infty, A^+_\infty) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor, and, for any $n \in \mathbf{Z}$, there exists m such that the morphism

$$A^+_m \widehat{\otimes}_{R^+_m} R^+_\infty \to A^+_\infty$$

is injective with cokernel annihilated by ϖ^{1/p^n}.

Proof. We note that [Sch13, Lemma 4.5] proves that (A_∞, A^+_∞) is an affinoid perfectoid (denoted by (S_∞, S^+_∞) there). By construction (and Proposition C.1.6 **(6)**), $\mathrm{Spd}\,(A_m, A^+_m) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{(\mathbf{Z}/p^m\mathbf{Z})}^n$-torsor. So $\mathrm{Spd}\,(A_\infty, A^+_\infty) \simeq \lim_m \mathrm{Spd}\,(A_m, A^+_m)$ (see Proposition C.1.6 **(5)**) is a $\underline{\Delta}_\infty \simeq \lim_m \underline{(\mathbf{Z}/p^n\mathbf{Z})}^n$-torsor. Therefore, we are only left to show that, for any $n \in \mathbf{Z}$, there exists m such that the morphism

$$A^+_m \widehat{\otimes}_{R^+_m} R^+_\infty \to A^+_\infty$$

is injective with cokernel annihilated by ϖ^{1/p^n}.

We denote by $\widetilde{A}_m$ the p-adic completion of p-torsionfree quotient of $A^+_m \otimes_{R^+_m} R^+_\infty$ ($\widetilde{A}_m$ is denoted by A_m in [Sch13, Lemma 4.5]). Then [Sch13, Lemma 4.5] shows that, for any $n \in \mathbf{Z}$, there exists m such that the map $\widetilde{A}_m \to A^+_\infty$ has cokernel annihilated by ϖ^{1/p^n}. Moreover, the map becomes an isomorphism after inverting p. We observe that this implies that $\widetilde{A}_m \to A^+_\infty$ is injective as the kernel should be p^∞-torsion, but the p-adic completion of a p-torsionfree ring is p-torsionfree. Thus the only thing we need to show is that $A^+_m \otimes_{R^+_m} R^+_\infty$ is already p-torsionfree for any m. We note that R^+_∞ is topologically free as an R^+_m-module because

$$\begin{aligned} R^+_\infty = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle &= \widehat{\bigoplus}_{(b_1, \dots, b_n) \in \mathbf{Z}^n \setminus m\mathbf{Z}^n} \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle T_1^{1/p^{b_1}} \dots T_n^{1/p^{b_n}} \\ &= \widehat{\bigoplus}_{(b_1, \dots, b_n) \in \mathbf{Z}^n \setminus m\mathbf{Z}^n} R^+_m \cdot T_1^{1/p^{b_1}} \dots T_n^{1/p^{b_n}}. \end{aligned}$$

Thus, R^+_∞ is R^+_m-flat for any m by Lemma 6.8.3. Therefore, $A^+_m \otimes_{R^+_m} R^+_\infty$ is flat over A^+_m, so it is, in particular, $\mathcal{O}_K$-flat. As a consequence, it does not have any non-zero p-torsion. This finishes the proof. $\square$

Lemma 6.8.5. Let $\mathfrak{X} = \mathrm{Spf}\, A_0$ be an affine admissible formal $\mathcal{O}_K$-scheme with generic fiber $X = \mathrm{Spa}\,(A, A^+)$ that admits a map $f\colon X \to \mathbf{T}^n = \mathrm{Spa}\,(R, R^+)$ that factors as a composition of finite étale morphisms and rational embeddings. Then the cohomology groups

$$\mathrm{H}^i(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p)$$

are almost coherent A_0/pA_0-modules for $i \geq 0$.

Proof. We denote the completed algebraic closure of K by C. Then we note that Lemma 6.7.4 implies that

$$\mathrm{H}^i(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i(X_{C,v}^{\diamondsuit}, \mathcal{O}^+_{X_C^\diamond}/p)$$

is an almost isomorphism for all $i \geq 0$. Therefore, faithful flatness of the morphism $\mathcal{O}_K/p \to \mathcal{O}_C/p$ and Lemma 2.10.5 imply that it suffices to prove the claim under the additional assumption that $K = C$ is algebraically closed.

Theorem 2.11.4 ensures that A_0 is an almost noetherian ring, thus it suffices to show that $\mathrm{H}^i(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p)$ are almost finitely generated A_0/pA_0-modules.

Now the generic fiber X is smooth over C, so [BGR84, Corollary 6.4.1/5] implies that $A^+ = A^\circ$ is a flat, topologically finitely type $\mathcal{O}_C$-algebra that is finite over A_0. Thus Lemma 2.8.3 ensures that it suffices to show that $\mathrm{H}^i(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p)$ is almost finitely generated A^+/pA^+-modules for $i \geq 0$. We note that A^+ is almost noetherian as a topologically finitely generated $\mathcal{O}_C$-algebra, so almost coherent and almost finitely generated A^+-modules coincide.

We consider a $\underline{\Delta}_\infty$-torsor $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ that is constructed in Lemma 6.8.4. Thus Corollary 6.6.7 ensures that

$$\mathbf{R}\Gamma(X_v^{\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p) \simeq^a \mathbf{R}\Gamma_{\mathrm{cont}}(\Delta_\infty, A^+_\infty/pA^+_\infty).$$

So we reduce the problem to showing that the complex $\mathbf{R}\Gamma_{\mathrm{cont}}(\Delta_\infty, A^+_\infty/pA^+_\infty)$ has almost finitely generated cohomology modules.

Now we pick any $\varepsilon \in \mathbf{Q}_{>0}$ and use Lemma 6.8.4 to find m such that the map

$$A_m^+ \widehat{\otimes}_{R_m^+} R_\infty^+ \to A_\infty^+$$

is injective with cokernel killed by p^ε. Thus we conclude that the map

$$A_m^+/p \otimes_{R_m^+/p} R_\infty^+/p \to A_\infty^+/p$$

has kernel and cokernel annihilated by p^ε. Then it is clear that the induced map

$$\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, A_m^+/p \otimes_{R_m^+/p} R_\infty^+/p) \to \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, A_\infty^+/p)$$

has kernel and cokernel annihilated by $p^{2\varepsilon}$ for any $i \geq 0$. Therefore, Lemma 2.5.7 implies that it is sufficient to show $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, A_m^+/p \otimes_{R_m^+/p} R_\infty^+/p)$ is almost finitely generated over A^+/pA^+ for any $m \geq 0$ and any $i \geq 0$.

The trick now is to consider the subgroup $p^m\Delta_\infty$ that acts trivially on A_m^+/pA_m^+ to pull it out of cohomology group by Lemma 6.8.2. More precisely, we consider the Hochschild–Serre spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i\left(\Delta_\infty/p^m\Delta_\infty, \mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, A_m^+/p \otimes_{R_m^+/p} R_\infty^+/p)\right) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{cont}}(\Delta_\infty, A_m^+/p \otimes_{R_m^+/p} R_\infty^+/p)$$

We recall that group cohomology of any finite group G can be computed by an explicit bar complex. Namely, for a G-module M, the complex looks like

$$\mathrm{C}^0(G, M) \xrightarrow{\mathrm{d}^0} \mathrm{C}^1(G, M) \xrightarrow{\mathrm{d}^1} \dots$$

where

$$\mathrm{C}^i(G, M) = \{f : G^i \to M\} \simeq M^{\oplus i \cdot \# G}$$

and

$$\begin{aligned} d^i(f)(g_0, g_1, \dots, g_i) &= g_0 \cdot f(g_1, \dots, g_i) + \\ &\sum_{j=1}^{i} (-1)^j f(g_0, \dots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \dots, g_i) + (-1)^{i+1} f(g_0, \dots, g_{i-1}). \end{aligned}$$

In case M is an A^+/pA^+-module and G acts A^+/pA^+-linearly on M, all the terms $\mathrm{C}^i(G, M)$ have a natural structure of an A^+/pA^+-module, and the differentials are A^+/pA^+-linear. Moreover, the terms $\mathrm{C}^i(G, M)$ are finite direct sums of M as an A^+/pA^+-module. In particular, they are almost coherent, if so is M. Thus Lemma 2.6.8 guarantees that all cohomology groups $\mathrm{H}^i(G, M)$ are almost coherent over A^+/pA^+ if M is almost coherent (equivalently, almost finitely generated) over A^+/pA^+.

We now apply this observation (together with Lemma 2.6.8) for

$$G = \Delta_\infty / p^m \Delta_\infty \text{ and } M = \mathrm{H}^j_{\mathrm{cont}}(p^m \Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p)$$

to conclude that it suffices to show $\mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p)$ is almost coherent (equivalently, almost finitely generated) over A^+/pA^+ for any $j \geq 0$, $m \geq 0$. We note that A^+_m is finite over A^+ by [BGR84, Corollary 6.4.1/5]. Thus Lemma 2.8.3 implies that it is enough to show that $\mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p)$ is almost finitely generated over A^+_m/pA^+_m for $i \geq 0$ and $m \geq 0$. Now we can use Lemma 6.8.2 to write

$$\mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p) \simeq \mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, R^+_\infty/p) \otimes_{R^+_m/p} A^+_m/p$$

Moreover, Lemma 6.8.2 guarantees that $\mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, R^+_\infty/p)$ is almost finitely generated over R^+_m/pR^+_m. Thus $\mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, R^+_\infty/p) \otimes_{R^+_m/p} A^+_m/p$ is almost finitely generated over A^+_m/pA^+_m by Lemma 2.8.1. □

Corollary 6.8.6. Let $\mathfrak{X} = \mathrm{Spf}\ A_0$ and $X = \mathrm{Spa}\,(A, A^+)$ be as in Lemma 6.8.5, and let $\mathcal{E}$ be a very small $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. Then $\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E})$ are almost coherent over A_0/pA_0.

Proof. Similarly to the proof of Lemma 6.8.5, we can assume that $K = C$ is algebraically closed and $A_0 = A^\circ = A^+$ is almost noetherian.

By assumption, we can find a finite étale surjection $Y \to X$ that splits $\mathcal{E}$. Since X is noetherian, we can dominate it by a Galois cover to assume that $Y \to X$ is a G-torsor for a finite group G such that $\mathcal{E}|_{Y^\diamondsuit_v} \simeq (\mathcal{O}^+_{Y^\diamondsuit}/p)^r$ for some r. Then we we have the Hochschild–Serre spectral sequence

$$\mathrm{E}^{i,j}_2 = \mathrm{H}^i\left(G, \mathrm{H}^j\left(Y^\diamondsuit_v, (\mathcal{O}^+_{Y^\diamondsuit}/p)^r\right)\right) \Rightarrow \mathrm{H}^{i+j}(X^\diamondsuit_v, \mathcal{E})$$

Similarly to the proof of Lemma 6.8.5, the argument with the explicit bar complex computing $\mathrm{H}^i(G, -)$ implies that it is sufficient to show that $\mathrm{H}^j\left(Y^\diamondsuit_v, \left(\mathcal{O}^+_{Y^\diamondsuit}/p\right)^r\right)$ is almost coherent over $\mathcal{O}^+_{Y^\diamondsuit}(Y^\diamondsuit)/p$ for $j \geq 0$. But this is done in Lemma 6.8.5. □

Lemma 6.8.7. Let K be a p-adic perfectoid field, and $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathfrak{X}_K$. Let $\mathcal{E}$ be a small $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle on $X^\diamondsuit_v$. Then there is a collection of

(1) an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$,

(2) a finite open affine cover $\mathfrak{X}' = \bigcup_{i \in I} \mathfrak{U}_i$, such that, for every $i \in I$, the restriction $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamondsuit_v}$ is very small.

Proof. By the smallness assumption, there is a finite open cover $X = \bigcup_{i\in I} U_i$ such that $\mathcal{E}|_{(U_{i,K})^\diamondsuit_v}$ can be trivialized by a finite étale surjection. Therefore, [Bos14, Lemma 8.4/5] implies that there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ with a covering $\mathfrak{X}' = \bigcup_{i\in I} \mathfrak{U}_i$ such that $\mathfrak{U}_{i,K} = U_i$. We can refine $\mathfrak{U}$ to assume that each $\mathfrak{U}_i = \mathrm{Spf}\ A_{i,0}$ is affine. $\square$

Theorem 6.8.8. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with smooth adic generic fiber X and mod-p fiber $\mathfrak{X}_0$. Then

$$\mathbf{R}\nu_*(\mathcal{E})^a \in \mathbf{D}^+_{acoh}(\mathfrak{X}_0)^a$$

for any small $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle $\mathcal{E}$.

Proof. Firstly, we note that the claim is clearly Zariski-local on $\mathfrak{X}$ and descends through rig-isomorphisms by the Almost Proper Mapping Theorem 5.1.3. Thus Lemma 6.8.7 implies that it suffices to prove the theorem for $\mathfrak{X} = \mathrm{Spf}\ A_0$ an affine formal $\mathcal{O}_K$-scheme and a very small $\mathcal{E}$.

Now we note that $\mathfrak{X}$ is rig-smooth in terminology of [BLR95, §3]. Thus, [BLR95, Proposition 3.7] states that there is an admissible blow-up $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ and a covering of $\mathfrak{X}'$ by open affine formal subschemes $\mathfrak{U}'_i$ with rig-étale morphisms $\mathfrak{f}'_i\colon \mathfrak{U}'_i \to \widehat{\mathbf{A}}^{n_i}_{\mathcal{O}_K}$, i.e. the adic generic fibers $\mathfrak{f}'_{i,K}\colon \mathfrak{U}'_{i,K} \to \mathbf{D}^{n_i}_K$ are étale. We apply the Almost Proper Mapping Theorem 5.1.3 again to conclude that it suffices to show the theorem for $\mathfrak{X}'$. Moreover, since the claim is Zariski-local on $\mathfrak{X}$, we can even pass to each $\mathfrak{U}'_i$ separately. So we reduce to the case $\mathfrak{X} = \mathrm{Spf}\ A_0$ is affine, admits a rig-étale morphism $\mathfrak{f}'\colon : \mathfrak{X} \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$, and $\mathcal{E}$ is very small.

We wish to reduce the question to the situation of Corollary 6.8.6, though we are still not quite there. The key trick now is to use Theorem D.4 to find a finite rig-étale morphism $\mathfrak{f}\colon \mathfrak{X} \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$. In particular, the generic fiber $\mathfrak{f}_K\colon X \to \mathbf{D}^d_K$ is a finite étale morphism. So the only thing we are left to do is to embedd $\mathbf{D}^d_K$ into $\mathbf{T}^d_K$ as a rational subset. This is done by observing that

$$\mathbf{D}^d_K \simeq \mathbf{T}^d_K\left(\frac{T_1-1}{p}, \dots, \frac{T_d-1}{p}\right) \subset \mathbf{T}^d_K.$$

In particular, X admits an étale morphism to a torus that is a composition of a finite étale morphism and a rational embedding. Therefore, Corollary 6.8.6 implies that

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E})^a \in \mathbf{D}^+_{acoh}(A_0/pA_0)^a.$$

Finally, we note that Lemma 6.7.3 ensures that $\widetilde{\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E})} \simeq^a \mathbf{R}\nu_*(\mathcal{E})$, so

$$\mathbf{R}\nu_*(\mathcal{E})^a \in \mathbf{D}^+_{acoh}(\mathfrak{X}_0)^a$$

by Theorem 4.4.6. $\square$

6.9. **Nearby Cycles are Almost Coherent for General X and $\mathcal{E}$.** The main goal of this section is to generalize Theorem 6.8.8 to the case of a general generic fiber X and any $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle $\mathcal{E}$. The idea is to reduce the general case to the smooth case by means of Lemma 5.4.4, resolution of singularities, and proper hyperdescent.

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (see Definition B.1.6). We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 6.9.1. Let Spf A_0 be an admissible affine formal $\mathcal{O}_K$-scheme with adic generic fiber $\mathrm{Spa}\,(A, A^+)$. Let $f\colon X \to \mathrm{Spa}\,(A, A^+)$ be a proper morphism with smooth X, and $\mathcal{E}$ is an $\mathcal{O}^+_{\mathrm{Spd}\,(A,A^+)}/p$-vector bundle. Then $\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E})$ is an almost coherent A_0/pA_0-module for any $i \geq 0$.

Proof. Step 1*: $\mathcal{E}$ is small.* By the theory of formal models (see [BL93, Assertion (c) on p.307]), we can choose an admissible formal $\mathcal{O}_K$-model $\mathfrak{X}$ of X with a morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathrm{Spa}\, A_0$ such that $\mathfrak{f}_K = f$. The map f is proper by [L9̈0, Lemma 2.6] (or [Tem00, Corollary 4.4 and 4.5]). Now we can compute

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \simeq \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_* (\mathcal{E}))$$

Theorem 6.8.8 reads that $\mathbf{R}\nu_*(\mathcal{E}) \in \mathbf{D}^+_{acoh}(\mathfrak{X}_0)$ as X is smooth. Thus Theorem 5.1.3 implies that

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \simeq \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}\nu_*(\mathcal{E})) \in \mathbf{D}^+_{acoh}(A_0/pA_0).$$

Step 2*: General* $\mathcal{E}$. Lemma 5.4.4 implies that there is a finite étale morphism $g\colon \mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A, A^+)$ such that $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)}$ is trivial. Without loss of generality, we can assume that g is a G-torsor for some finite group G. Then the base change morphism $Y = X_B \to X$ is also a G-torsor. Then we we have the Hochschild–Serre spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i\left(G, \mathrm{H}^j\left(Y^\diamondsuit_v, \mathcal{E}\right)\right) \Rightarrow \mathrm{H}^{i+j}(X^\diamondsuit_v, \mathcal{E})$$

Similarly to the proof of Lemma 6.8.5, the argument with the explicit bar complex computing $\mathrm{H}^i(G, -)$ implies that it is sufficient to show that $\mathrm{H}^j\left(Y^\diamondsuit_v, \mathcal{E}\right)$ is almost coherent over A_0/pA_0 for $j \geq 0$. But this follows from Step 1 because Y is smooth and proper over $\mathrm{Spa}\,(A, A^+)$. $\square$

Now we recall the notion of a hypercovering that will be crucial for our proof. We refer to [Sta21, Tag 01FX] and [Con] for more detail.

Definition 6.9.2. Let $\mathcal{C}$ be a category admitting finite limits. Let $\mathbf{P}$ be a class of morphisms in $\mathcal{C}$ which is stable under base change, preserved under composition (hence under products), and contains all isomorphisms. A simplicial object $X_\bullet$ in $\mathcal{C}$ is said to be a $\mathbf{P}$-*hypercovering* if, for all $n \geq 0$, the natural adjunction map[36]

$$X_\bullet \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet))$$

induces a map $X_{n+1} \to (\mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet)))_{n+1}$ in degree $n+1$ which is in $\mathbf{P}$. If $X_\bullet$ is an augmented simplicial complex, we make a similar definition but also require the case $n = -1$ (and we then say $X_\bullet$ is a $\mathbf{P}$-hypercovering of X_{-1}).

Lemma 6.9.3. Let X be a quasi-compact, quasi-separated rigid-analytic variety over K. Then there is a proper hypercovering $a\colon X_\bullet \to X$ such that all X_i are smooth over K.

Proof. First of all, we note that quasi-compact rigid-analytic varieties over $\mathrm{Spa}\,(K, \mathcal{O}_K)$ admit resolution of singularities by [Tem12, Theorem 5.2.2]. Thus, the proof of [Con, Theorem 4.16] (or [Sta21, Tag 0DAX]) carries over to show that there is a proper hypercovering $a\colon X_\bullet \to X$ such that all X_i are smooth over $\mathrm{Spa}\,(K, \mathcal{O}_K)$. $\square$

Lemma 6.9.4. Let $a\colon X_\bullet \to X$ be a proper hypercovering of a rigid-analytic variety X. Then $a^\diamondsuit\colon X^\diamondsuit_\bullet \to X^\diamondsuit$ is a v-hypercovering of $X^\diamondsuit$.

[36] See [Con, §3] (or [Sta21, Tag 0AMA]) for the definition of the coskeleton functor.

Proof. The functor $(-)^\diamondsuit$ commutes with fiber products by Proposition C.1.6 **(6)**. So

$$((\mathrm{cosk}_n(\mathrm{sk}_n X_\bullet))_{n+1})^\diamondsuit \simeq (\mathrm{cosk}_n(\mathrm{sk}_n X_\bullet^\diamondsuit))_{n+1}.$$

Therefore, the only thing we need to show that $(-)^\diamondsuit$ sends proper coverings to v-coverings. This follows from Lemma C.1.13 and Example C.1.11. □

Theorem 6.9.5. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X and mod-p fiber $\mathfrak{X}_0 := \mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_K/p$. Then

$$\mathbf{R}\nu_*(\mathcal{E}) \in \mathbf{D}^+_{acoh}(\mathfrak{X}_0)$$

for any $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle $\mathcal{E}$.

Proof. The claim is Zariski-local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A_0$ is affine. Thus Lemma 6.7.3 and Theorem 4.4.6 ensure that it suffices to show that

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \in \mathbf{D}^+_{acoh}(A_0/pA_0).$$

Lemma 6.9.3 shows that there is a proper hypercovering $a\colon X_\bullet \to X$ with smooth X_i, and Lemma 6.9.4 implies that $a\colon X^\diamondsuit_\bullet \to X^\diamondsuit$ is then a v-hypercovering.

The proof of [Sta21, Tag 01GY] implies that there is a spectral sequence

$$\mathrm{E}_1^{i,j} = \mathrm{H}^j\left(X^\diamondsuit_{i,v}, \mathcal{E}\right) \Rightarrow \mathrm{H}^{i+j}(X^\diamondsuit_v, \mathcal{E}).$$

Lemma 6.9.1 guarantees that $\mathrm{H}^j(X^\diamondsuit_{i,v}, \mathcal{E})$ is almost coherent over A_0/pA_0 for every $i, j \geq 0$. Therefore, Lemma 2.6.8 guarantees that $\mathrm{H}^{i+j}(X^\diamondsuit_v, \mathcal{E})$ is almost coherent A_0/pA_0 for every $i+j \geq 0$. □

6.10. **Cohomological Bound on Nearby Cycles.** The main goal of this section is to show that $\mathbf{R}\nu_*(\mathcal{E})$ is almost concentrated in degrees $[0, d]$ for a very small vector bundle $\mathcal{E}$. This claim turns out to be pretty hard. In order to achieve this result we have to use a recent notion of perfectoidization developed in [BS22] that give a stronger version of the almost purity theorem in the world of diamonds. Our approach is very much motivated by [Guo19, Proposition 7.5.2].

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

One may notice that all previous sections did not really use much that we work on the v-site $X^\diamondsuit_v$ of a diamond associated to a rigid-analytic variety X rather than its pro-étale site $X_{\text{proét}}$. Most arguments can be carried over in the pro-étale site. However, it is crucial to work on the level of diamonds in this section. The main observation is that the functor

$$(-)^\diamondsuit\colon \{\text{(Pre-)Adic Analytic Spaces}\} \to \{\text{Diamonds}\}$$

is not fully faithful, so it is a priori possible that a non-perfectoid (pre)-adic space becomes representable by an affinoid perfectoid after diamondification. An explicit construction of such examples is the crux of our argument in this section. In order to construct them, we need the following theorem of B. Bhatt and P. Scholze:

Theorem 6.10.1. [BS22, Theorem 10.11] Let R be an integral perfectoid ring[37]. Let $R \to S$ be the p-adic completion of an integral map. Then there exists an integral perfectoid ring S_{perfd} together with a map of R-algebras $S \to S_{\mathrm{perfd}}$, such that it is initial among all of the R-algebra maps $S \to S'$ for S' being integral perfectoid.

[37]We use [BMS18, Definition 3.5] as the definition for integral perfectoid rings here. This definition coincides with Definition 6.4.2 in the p-torsionfree case, but it is less restrictive in general.

Now we show how we can use this result to get cohomological bound on $\mathbf{R}\nu'_*(\mathcal{E})$. We recall that a torus

$$\mathbf{T}^d = \mathrm{Spa}\left(K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle\right) = \mathrm{Spa}\,(R, R^+)$$

admits a map

$$\mathbf{T}^d_\infty = \mathrm{Spa}\left(K\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle\right) \to \mathbf{T}^d$$

such that $\mathbf{T}^d_\infty$ is an affinoid perfectoid, and the map becomes a $\underline{\Delta}_\infty = \underline{\mathbf{Z}_p(1)}^d$-torsor after applying the diamondification functor.

Now we can embed a d-dimensional disk $\mathbf{D}^d$ as a rational subdomain

$$\mathbf{D}^d = \mathbf{T}^d\left(\frac{T_1 - 1}{p}, \dots, \frac{T_n - 1}{p}\right) \subset \mathbf{T}^d,$$

so the fiber product

$$\mathbf{D}^d_\infty = \mathbf{D}^d \times_{\mathbf{T}^d} \mathbf{T}^d_\infty \to \mathbf{D}^d$$

is again an affinoid perfectoid covering of $\mathbf{D}^d$ by Lemma 6.8.4.

If $X = \mathrm{Spa}\,(A, A^+) \to \mathbf{D}^d$ is an arbitrary finite morphism, then the fiber product $X \times_{\mathbf{D}^d} \mathbf{D}^d_\infty$ may not be an affinoid perfectoid space (or even an adic space). However, it turns out that the associated diamond is always representable by an affinoid perfectoid.

Lemma 6.10.2. Let $f\colon X = \mathrm{Spa}\,(A, A^+) \to \mathbf{D}^d$ be a finite morphism of rigid-analytic K-varieties. Then the fiber product $X^\diamondsuit_\infty \coloneqq X^\diamondsuit \times_{\mathbf{D}^{d,\diamondsuit}} \mathbf{D}^{d,\diamondsuit}_\infty$ is representable to an affinoid perfectoid space (of characteristic p).

Proof. Let us say that $\mathbf{D}^d = \mathrm{Spa}\,(S, S^+)$ and $\widehat{\mathbf{D}}^d_\infty = \mathrm{Spa}\,(S_\infty, S^+_\infty)$. The map f defines an integral morphism $S^+ \to A^+$, we define

$$A^\dagger_\infty \coloneqq S^+_\infty \widehat{\otimes}_{S^+} A^+.$$

This is a p-adic completion of an integral morphism over an integral perfectoid ring S^+_∞ (see [BMS18, Lemma 3.20]), so there is a map

$$A^\dagger_\infty \to (A^\dagger_\infty)_{\mathrm{perfd}}$$

initial to an integral perfectoid ring. We define A_∞ to be $A^\dagger_\infty[1/p]$ and A^+_∞ to be the integral closure of $A^\dagger_\infty$ in A_∞. Then (A_∞, A^+_∞) is an affinoid perfectoid pair by [BMS18, Lemma 3.21]. Therefore, it suffices to show that the natural morphism

$$\mathrm{Spd}\,(A_\infty, A^+_\infty) \to \mathrm{Spd}\,(A, A^+) \times_{\mathrm{Spd}\,(S,S^+)} \mathrm{Spd}\,(S_\infty, S^+_\infty)$$

is an isomorphism. This can be easily checked on the level of rational point by the universal property of $(A^\dagger_\infty)_{\mathrm{perfd}}$ and the construction of the diamondification functor in Definition C.1.5 (and [BMS18, Lemma 3.20] that relates affinoid perfectoid pairs and integral affinoid rings). $\square$

Theorem 6.10.3. Let $\mathfrak{X} = \mathrm{Spf}\, A_0$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathrm{Spa}\,(A, A^+)$ of dimension d, and let $\mathcal{E}$ be a very small $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle on X. Then

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(A_0/pA_0)^a.$$

Proof. Lemma 6.9.1 ensures that $\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \in \mathbf{D}_{acoh}(A_0/pA_0)$, so it suffices to show that

$$\mathrm{H}^i(X^\diamondsuit_v, \mathcal{E}) \simeq^a 0$$

for $i > d$. Now we note that the Noether Normalization Theorem (see [Bos14, Proposition 3.1.3]) implies that there is a finite morphism $f\colon X \to \mathbf{D}^d$. Now we consider the $\underline{\Delta}_\infty \simeq \underline{\mathbf{Z}_p(1)^d}$-torsor

$$X_\infty^\diamondsuit \simeq X^\diamondsuit \times_{\mathbf{D}^{d,\diamondsuit}} \mathbf{D}_\infty^{d,\diamondsuit} \to X^\diamondsuit.$$

By Lemma 6.10.2, $X_\infty^\diamondsuit$ is represented by an affinoid perfectoid space $\mathrm{Spd}\,(A_\infty, A_\infty^+) = \mathrm{Spa}\,(A_\infty^\flat, A_\infty^{\flat,+})$. Thus we are in the situation of Set-up 6.6.3. So Corollary 6.6.7 implies that

$$\mathrm{H}^i(X_v^\diamondsuit, \mathcal{E}) \simeq^a \mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, (M_\mathcal{E}^a)_!),$$

where $M_\mathcal{E} \simeq \mathrm{H}^0(X^\diamondsuit_{\infty,v}, \mathcal{E})$. Therefore, the claim follows from the observation that cohomological dimension of $\Delta_\infty \simeq \mathbf{Z}_p(1)^d \simeq \mathbf{Z}_p^d$ is d by [BMS18, Lemma 7.3]. □

6.11. **Proof of Theorem 6.1.2.** The main goal of this section is to give a full proof of Theorem 6.1.2. Basically, we just need to combine all the results we have already achieved together.

For the rest of this section, we fix a perfectoid p-adic field K with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$ as in Remark B.1.5. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Theorem 6.11.1. Let $\mathfrak{X}$ an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. Then

- **(1)** the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X}_0)^a$;
- **(2)** for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$ with the adic generic fiber X, the natural map
$$\mathrm{H}^i\,\widetilde{\left(X_v^\diamondsuit, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$
is an isomorphism for every $i \geq 0$;
- **(3)** the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism
$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamondsuit}\right)$$
is an isomorphism for any $i \geq 0$;
- **(4)** if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamondsuit}$ is very small, then
$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a;$$
- **(5)** if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamondsuit}$ is very small.

  In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that
$$\left(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_{i,0})^a.$$
for each $i \in I$.

Proof. The first part of **(1)**, **(2)**, and **(3)** follow from Theorem 6.7.3 and Theorem 6.9.5. Now to show that $\mathbf{R}\nu_*\mathcal{E}$ is almost concentrated in degrees $[0, 2d]$, it suffices to show that, for every affine $\mathfrak{U} = \mathrm{Spf}\ A_0 \subset \mathfrak{X}$, the complex $\mathbf{R}\Gamma(\mathfrak{U}^\diamondsuit_{K,v}, \mathcal{E})^a$ (almost) lies in $\mathbf{D}^{[0,2d]}(A_0/pA_0)^a$. By Lemma 6.7.4 and full faithful flatness of $\mathcal{O}_K/p \to \mathcal{O}_C/p$, it is sufficient to proof under the additional assumption that $K = C$ is algebraically closed. Then Theorem C.4.5 and Theorem C.4.8 imply that

$$\mathcal{E}' \coloneqq \mathbf{R}\mu_*\mathbf{R}\lambda_*\mathcal{E}$$

is an $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle concentrated in degree 0. Therefore,

$$\mathbf{R}\Gamma(\mathfrak{U}^{\diamondsuit}_{C,v}, \mathcal{E}) \simeq \mathbf{R}\Gamma(\mathfrak{U}_{C,\text{ét}}, \mathcal{E}'),$$

and

$$\mathbf{R}\Gamma(\mathfrak{U}_{C,\text{ét}}, \mathcal{E}') \in \mathbf{D}^{[0,2d]}(A_0/pA_0)$$

by [Hub96, Corollary 2.8.3 and Corollary 1.8.8].

To show **(4)**, we consider an open affine covering $\mathfrak{X} = \bigcup_{i\in I} \mathfrak{U}_i$ and denote $\mathfrak{U}_i = \operatorname{Spf} A_i$. Then Part **(2)** implies that it suffices to show that

$$\mathbf{R}\Gamma((\mathfrak{U}_{i,K})^{\diamondsuit}_v, \mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(A_i/pA_i)^a$$

for each $i \in I$. This follows from Theorem 6.10.3 and the assumption that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^{\diamondsuit}}$ is very small.

(5) now follows from Lemma 6.8.7. ☐

6.12. **Proof of Theorem 6.1.9.** The main goal of this section is to prove Theorem 6.1.9. Essentially the idea is to use the "classification" of Zariski-constructible sheaves to reduce Theorem 6.1.9 to Theorem 6.1.2.

For the rest of this section, we fix a perfectoid p-adic field K with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$ as in Remark B.1.5. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

We recall that we have a diagram of morphisms of ringed sites:

$$\begin{tikzcd}
\left(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\right) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"'] & \left(X^{\diamondsuit}_{\text{qproét}}, \mathcal{O}^+_{X^{\diamondsuit}_{\text{qp}}}/p\right) \arrow[r, "\mu"] & \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}_0}).
\end{tikzcd}$$

Both ν_* and t_* will play an important role in the proof.

Lemma 6.12.1. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ a finite morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^{\diamond}}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^{\diamond}}/p\right)\right)$$

is an isomorphism in $\mathbf{D}(\mathfrak{Y}_0)$.

Proof. Firstly, we note that f is finite, and so $f_* \simeq \mathbf{R}f_*$ by [Hub96, Proposition 2.6.3]. Now the proof of Corollary 6.3.9 just goes through using Corollary 6.2.9 (that does not use Theorem 6.1.9 as in input) in place of Lemma 6.3.7. ☐

Lemma 6.12.2. Let $f\colon X \to Y$ be a finite morphism of quasi-compact, quasi-separated rigid-analytic varieties over K, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$ such that

$$\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^{\diamond}}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{acoh}(\mathfrak{X}_0)^a \text{ (resp. } \mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X^{\diamond}}/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0))$$

for any formal $\mathcal{O}_K$-model $\mathfrak{X}$ of X. Then, for any formal $\mathcal{O}_K$-model $\mathfrak{Y}$ of Y,

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^{\diamond}}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{acoh}(\mathfrak{Y}_0)^a \text{ (resp. } \mathbf{R}\nu_{\mathfrak{Y},*}\left(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^{\diamond}}/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{Y}_0)).$$

Proof. Firstly, we note that we can choose a finite morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ such that its generic fiber $\mathfrak{f}_K$ is equal to f (for example, this follows from [FK18, Corollary II.5.3.3, II.5.3.4]).

Now Lemma 6.12.1 ensures that we have an equality

$$\mathbf{R}\nu_{\mathfrak{Y},*}\left(f_*\mathcal{F}\otimes \mathcal{O}^+_{Y^\diamond}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}\nu_{\mathfrak{X},*}\left(\mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p\right)\right).$$

Therefore, $\mathbf{R}\nu_{\mathfrak{Y},*}\left(f_*\mathcal{F}\otimes \mathcal{O}^+_{Y^\diamond}/p\right)$ already lies in $\mathbf{D}_{acoh}(\mathfrak{Y}_0)^a$ (resp. $\mathbf{D}_{qc,acoh}(\mathfrak{Y}_0)$) by Theorem 5.1.3. The cohomological bound follows from Proposition 3.5.23 and the fact that a finite morphism $\mathfrak{f}_0$ is acyclic on quasi-coherent sheaves. $\square$

Lemma 6.12.3. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and a complex of sheaves $\mathcal{F}\in \mathbf{D}^{[r,s]}_{zc}(X;\mathbf{F}_p)$. Then

$$\mathbf{R}t_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\nu_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0), \text{ and}$$

$$\mathbf{R}\nu_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{qc,acoh}(\mathfrak{X}_0)^a$$

Proof. We start the proof by noting that an isomorphism

$$\mathbf{R}t_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\nu_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p\right)$$

is automatic by Lemma C.5.10 and overconvergence of Zariski-constructible sheaves. In what follows, we will freely identify these sheaves.

Step 1*: The case of a local system* $\mathcal{F}$. In this case $\mathcal{E} \coloneqq \mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p$ fits into the assumption of Theorem 6.11.1. Since an $\mathbf{F}_p$-local system on any rigid-analytic variety Y splits by a finite étale cover, so $\mathcal{F}\otimes \mathcal{O}^+_{X^\diamond}/p$ is very small for any open affinoid $U\subset X$. Thus the desired claim follows from Theorem 6.11.1.

Step 2*: The case of a zero-dimensional* X. If X is of dimension 0, then any Zariski-constructible sheaf on X is a local system. So the claim follows from Step 1.

Now we argue by induction on $\dim X$. We suppose the claim is known for every rigid-analytic variety of dimension less than d (and any Zariski constructible $\mathcal{F}$) and wish to prove the claim for X of dimensiond d.

Step 3*: Reduction to the case of a reduced* X. Consider the reduction morphism $i\colon X_{\mathrm{red}} \to X$. Then $i_{\text{ét}}$ is an equivalence of étale topoi, we see that

$$i_*i^{-1}\mathcal{F} \to \mathcal{F}$$

is an isomorphism. Thus the claim follows from Lemma 6.12.2.

Step 4*: Reduction to the case of a normal* X. Consider the normalization morphism $f\colon X' \to X$. It is finite by [Con99, Theorem 2.1.2] and an isomorphism outside of a nowhere dense Zariski-closed subset Z. Therefore, there is an exact triangle

$$\mathcal{F} \to f_*f^{-1}\mathcal{F} \to i_*\mathcal{G}$$

where $i\colon Z\to X$ is a Zariski-closed immersion with $\dim Z < \dim X$ and $\mathcal{G}\in \mathbf{D}^{[r-1,s]}_{zc}(Z)$. Now the induction hypothesis and Lemma 6.12.2 ensure that

$$\mathbf{R}\nu_*\left(i_*\mathcal{G}\otimes \mathcal{O}^+_{X^\diamond}/p\right) \in \mathbf{D}^+_{qc,acoh}\left(\mathfrak{X}_0\right),$$

$$\mathbf{R}\nu_*\left(i_*\mathcal{G}\otimes \mathcal{O}^+_{X^\diamond}/p\right)^a \in \mathbf{D}^{[r,s+d]}_{acoh}\left(\mathfrak{X}_0\right)^a.$$

Therefore, it suffices to show the claim for $f_* f^{-1}\mathcal{F}$, and so Lemma 6.12.2 applied to f guarantees that it suffices to show that

$$\mathbf{R}\nu_{\mathfrak{X}',*}\left(f^{-1}\mathcal{F}\otimes \mathcal{O}^+_{X'^\diamond}/p\right)\in \mathbf{D}^+_{qc,acoh}\left(\mathfrak{X}'_0\right),$$

$$\mathbf{R}\nu_{\mathfrak{X}',*}\left(f^{-1}\mathcal{F}\otimes \mathcal{O}^+_{X'^\diamond}/p\right)^a\in \mathbf{D}^{[r,s+d]}_{acoh}\left(\mathfrak{X}'_0\right)^a$$

for any admissible formal $\mathcal{O}_K$-model $\mathfrak{X}'$ of X'. So we may and do assume that X is normal.

Step 5*: Reduction to the case* $\mathcal{F}=\underline{\mathbf{F}}_p$. Clearly, it suffices to prove the claim for $\mathcal{F}$ concentrated in degree 0. Then, by definition of a Zariski-constructible sheaf, there is a nowhere dense Zariski-closed subset $i\colon Z\to X$ with a complement $j\colon U\to X$ and an $\mathbf{F}_p$-local system $\mathbf{L}$ on U such that $\mathcal{F}|_U\simeq \mathbf{L}$. In particular, there is a short exact sequence

$$0\to j_!\mathbf{L}\to\mathcal{F}\to i_*\mathcal{F}|_Z\to 0.$$

Similarly to the argument in Step 4, it suffices to prove the claim for $\mathcal{F}=j_!\mathbf{L}$.

Then "méthode de la trace" (see [Sta21, Tag 03SH]) implies that there is a finite étale covering $g\colon U'\to U$ such that $\mathbf{L}':=\mathbf{L}|_{U'}$ is an iterated extension of constant sheaves $\underline{\mathbf{F}}_p$. Then $\mathbf{L}$ is a direct summand of $g_*\left(\mathbf{L}'\right)$. Thus it is enough to prove the claim for

$$\mathcal{F}=j_!\left(g_*\mathbf{L}'\right).$$

Moreover, it suffices to prove the claim for $\mathcal{F}=j_!\left(g_*\underline{\mathbf{F}}_p\right)$ because the claim of Lemma 6.12.3 satisfies the (2)-out-of-(3) property, and both functors g_* and $j_!$ are exact.

Now we use [Han20, Theorem 1.6] to extend g to a finite morphism $g'\colon X'\to X$. Then a similar reduction shows that it is actually sufficient to prove the claim for $\mathcal{F}=g'_*\left(\underline{\mathbf{F}}_p\right)$. Now this case follows from Step 2 and Lemma 6.12.2. ☐

Theorem 6.12.4. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and a complex of sheaves $\mathcal{F}\in\mathbf{D}^{[r,s]}_{zc}(X;\mathbf{F}_p)$. Then

(1) there is an isomorphism $\mathbf{R}t_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)\simeq \mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)$;

(2) the nearby cycles $\mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)\in\mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$, and $\mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)^a\in\mathbf{D}^{[r,s+d]}_{acoh}(\mathfrak{X}_0)^a$;

(3) for an affine admissible $\mathfrak{X}=\mathrm{Spf}\ A$, the natural map

$$\mathrm{H}^i\left(X^\diamond_v,\widetilde{\mathcal{F}\otimes}\,\mathcal{O}^+_{X^\diamond}/p\right)\to \mathrm{R}^i\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)$$

is an isomorphism for every $i\geq 0$;

(4) the formation of $\mathrm{R}^i\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon\mathfrak{Y}\to\mathfrak{X}$ with adic generic fiber $f\colon Y\to X$, the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)\right)\to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(f^{-1}\mathcal{F}\otimes\mathcal{O}^+_{Y^\diamond}/p\right)$$

is an isomorphism for any $i\geq 0$;

Proof. **(1)** and **(2)** follow from Lemma 6.12.3. Now **(3)** follows from Lemma 4.4.4 and the isomorphism

$$\mathbf{R}\Gamma\left(\mathfrak{X}_0,\mathbf{R}\nu_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\right)\right)\simeq\mathbf{R}\Gamma(X^\diamond_v,\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p).$$

Now we show **(4)**. By **(1)**, it suffices to show that the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^it_{\mathfrak{X},*}\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)\right)\to\mathrm{R}^it_{\mathfrak{Y},*}\left(f^{-1}\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\right)$$

Moreover, [BH21, Proposition 3.6] ensures that it suffices to prove the claim for $\mathcal{F} = g_* \left(\underline{\mathbf{F}}_p\right)$ for some finite morphism $g\colon X' \to X$. Then we can lift it to a finite morphism $\mathfrak{g}\colon \mathfrak{X}' \to \mathfrak{X}$ as in the proof of Lemma 6.12.2. Then we have a commutative diagram

$$
\begin{tikzcd}
& (Y'_{\text{ét}}, \mathcal{O}^+_{Y'_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{Y}'}"] \arrow[dl, "f'"'] \arrow[dd, "g'"] & & (\mathfrak{Y}'_0, \mathcal{O}_{\mathfrak{Y}'_0}) \arrow[dl, "\mathfrak{f}'_0"] \arrow[dd, "\mathfrak{g}'_0"] \\
(X'_{\text{ét}}, \mathcal{O}^+_{X'_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{X}'}"] \arrow[dd, "g"] & & (\mathfrak{X}'_0, \mathcal{O}_{\mathfrak{X}'_0}) \arrow[dd, "\mathfrak{g}_0"] & \\
& (Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{Y}}"] \arrow[dl, "f"] & & (\mathfrak{Y}_0, \mathcal{O}_{\mathfrak{Y}_0}) \arrow[dl, "\mathfrak{f}_0"] \\
(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{X}}"] & & (\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}) &
\end{tikzcd}
\tag{6.6}
$$

with $\mathfrak{Y}' = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ and Y' its adic generic fiber. Then we have a sequence of isomorphisms:

$$
\begin{aligned}
\mathfrak{f}_0^* \left(\mathbf{R}t_{\mathfrak{X},*}\left(g_*\left(\underline{\mathbf{F}}_p\right) \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right) &\simeq \mathfrak{f}_0^* \left(\mathbf{R}t_{\mathfrak{X},*}\left(\mathbf{R}g_* \mathcal{O}^+_{X'_{\text{ét}}}/p\right)\right) \\
&\simeq \mathfrak{f}_0^* \left(\mathbf{R}\mathfrak{g}_{0,*}\left(\mathbf{R}t_{\mathfrak{X}',*} \mathcal{O}^+_{X'_{\text{ét}}}/p\right)\right) \\
&\simeq \mathbf{R}\mathfrak{g}'_{0,*}\left(\mathfrak{f}_0'^{\,*}\left(\mathbf{R}t_{\mathfrak{X}',*} \mathcal{O}^+_{X'_{\text{ét}}}/p\right)\right) \\
&\simeq \mathbf{R}\mathfrak{g}'_{0,*}\left(\mathbf{R}t_{\mathfrak{Y}',*}\left(\mathcal{O}^+_{Y'_{\text{ét}}}/p\right)\right) \\
&\simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}g'_* \mathcal{O}^+_{Y'_{\text{ét}}}/p\right) \\
&\simeq \mathbf{R}t_{\mathfrak{Y},*}\left(g'_*\left(\underline{\mathbf{F}}_p\right) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right) \\
&\simeq \mathbf{R}t_{\mathfrak{Y},*}\left(f^{-1}\left(g_* \underline{\mathbf{F}}_p\right) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right)
\end{aligned}
$$

The first isomorphism holds by (the proof of) Corollary 6.2.9. The second isomorphism is formal and follows from Diagram 6.6. The third isomorphism holds by flat base change applies to a flat morphism $\mathfrak{f}_0$. The fourth isomorphism follows from Theorem 6.11.1 applied to $\mathcal{E} = \mathcal{O}^+_{X'^{\diamond}}/p$ and étale morphism $\mathfrak{Y}' \to \mathfrak{X}'$. The fifth isomorphism is formal again. The sixth isomorphism follows from (the proof of) Corollary 6.2.9. Finally the last isomorphism follows from [Hub96, Theorem 4.3.1]. $\square$

6.13. **Proof of Theorem 6.1.11.** The main goal of this section is to prove Theorem 6.1.11. The proof is a formal reduction to the case of $\mathcal{O}^+_{X^\diamond}/p$-vector bundles.

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 6.13.1. Let X be a rigid-analytic variety over K, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle on X. Then $\mathcal{E}$ is derived p-adically complete.

Proof. Lemma 6.1.11 implies that it suffices to prove the claim v-locally on $X^\diamond_v$. Therefore, we may and do assume that $\mathcal{E} = \left(\mathcal{O}^+_{X^\diamond}\right)^r$ for some integer r. Then the claim follows from Lemma C.3.5 **(3)**. $\square$

Lemma 6.13.2. Let $\mathfrak{X} = \operatorname{Spf}\, A_0$ be an affine admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \operatorname{Spa}(A, A^+)$ of dimension d, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a \in \mathbf{D}^{[0,2d]}_{acoh}(A_0).$$

Moreover,

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(A_0)$$

if $\mathcal{E}$ is very small (see Definition 6.1.10).

Proof. Lemma 6.13.1 implies that $\mathcal{E}$ is derived p-adically complete. So [Sta21, Tag 0A0G] ensures that

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)$$

is derived p-adically complete as well. Now Theorem 6.11.1 implies that

$$\left[\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a/p\right] \simeq \mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}/p\mathcal{E}\right)^a \in \mathbf{D}^{[0,2d]}_{acoh}(A_0/pA_0)^a,$$

and

$$\left[\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a/p\right] \simeq \mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}/p\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(A_0/pA_0)^a$$

if $\mathcal{E}$ is very small. So Corollary 2.13.3 ensures that

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a \in \mathbf{D}^{[0,2d]}_{acoh}(A_0)^a,$$

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(A_0)^a$$

if $\mathcal{E}$ is very small. $\square$

Lemma 6.13.3. Let $\mathfrak{X} = \operatorname{Spf}\, A_0$ be an admissible affine formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \operatorname{Spa}(A, A^+)$, and $\mathfrak{f}\colon \operatorname{Spf}\, B_0 \to \operatorname{Spf}\, A_0$ an étale morphism with adic generic fiber $f\colon Y \to X$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle on X. Then the natural morphism

$$r\colon \mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right) \otimes_{A_0} B_0 \to \mathbf{R}\Gamma\left(Y_v^\diamondsuit, \mathcal{E}\right).$$

is an isomorphism.

Proof. The morphism $A_0 \to B_0$ is flat since $\mathfrak{f}$ is étale. Now Lemma 6.13.2 and Lemma 2.12.7 ensure that cohomology groups of both $\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{E}\right) \otimes_{A_0} B_0$ and $\mathbf{R}\Gamma\left(Y_v^\diamondsuit, \mathcal{E}\right)$ are (classically) p-adically complete. In particular, both complexes are derived p-adically complete. So it suffices to show that r is an isomorphism after taking derived mod-p fiber (see [Sta21, Tag 0G1U]). Then the claim follows from Theorem 6.11.1 **(3)** **(4)**. $\square$

Theorem 6.13.4. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then

- **(1)** the nearby cycles $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X})^a$;
- **(2)** for an affine admissible $\mathfrak{X} = \operatorname{Spf}\, A$ with the adic generic fiber X, the natural map

$$\mathrm{H}^i\left(X_v^\diamondsuit, \mathcal{E}\right)^\Delta \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$;

(4) if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small, then

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a;$$

(5) if $\mathcal{E}$ is small, there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is very small.

In particular, if $\mathcal{E}$ is small, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}'_i)^a.$$

for each $i \in I$.

Proof. Firstly, we show that $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{acoh}(\mathfrak{X})^a$. The claim is local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then it suffices to show that, for every étale morphism $\mathrm{Spf}\ B_0 \to \mathrm{Spf}\ A_0$ with adic generic fiber $Y \to X$,

$$\mathrm{H}^i(Y_v^\diamond, \mathcal{E}|_{Y^\diamond})$$

is almost coherent for $i \geq 0$,

$$\mathrm{H}^i(Y_v^\diamond, \mathcal{E}|_{Y^\diamond}) \simeq^a$$

for $i > 2d$, and the natural morphism

$$\mathrm{H}^i(X_v^\diamond, \mathcal{E}) \to \mathrm{H}^i(Y_v^\diamond, \mathcal{E}|_{Y^\diamond})$$

is an isomorphism (see Lemma 5.1.8 and its proof). The first claim follows from Lemma 6.13.2 and the second one from Lemma 6.13.3 (and A_0-flatness of B_0). This already proves **(1)** and **(2)**. The proof of **(3)** is essentially the same using Lemma 4.6.5. Now **(4)** follows from Lemma **(4)** and already established **(2)**. Finally **(5)** follows from Lemma 6.8.7 since smallness is the condition mod-p. ∎

Let us also mention a version of Theorem 6.1.11 for the pro-étale site of X as defined in [Sch13] and [Sch16]. It will be convenient to have this reference in our future work. In what follows, $\widehat{\mathcal{O}}^+_X$ is the completed integral structure sheaf on $X_{\text{proét}}$ (see [Sch13, Definition 4.1]), and

$$\nu'\colon (X_{\text{proét}}, \widehat{\mathcal{O}}^+_X) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_\mathfrak{X})$$

is the evident morphism of ringed sites.

Theorem 6.13.5. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$. Then

(1) the nearby cycles $\mathbf{R}\nu'_*\left(\mathcal{O}^+_X/p\right) \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X}_0)$, and $\mathbf{R}\nu'_*\left(\mathcal{O}^+_X/p\right)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X}_0)^a$;

(2) for an affine admissible $\mathfrak{X} = \mathrm{Spf}\ A$, the natural map

$$\mathrm{H}^i\left(\widetilde{X_{\text{proét}}}, \mathcal{O}^+_X/p\right) \to \mathrm{R}^i\nu'_*\left(\mathcal{O}^+_X/p\right)$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*\left(\mathcal{O}_X^+/p\right)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu'_{\mathfrak{X},*}\left(\mathcal{O}_X^+/p\right)\right) \to \mathrm{R}^i\nu'_{\mathfrak{Y},*}\left(\mathcal{O}_Y^+/p\right)$$

is an isomorphism for any $i \geq 0$;

Proof. By [Sch13, Corollary 3.17], $\mathbf{R}\nu'_*\left(\mathcal{O}_X^+/p\right) \simeq \mathbf{R}t_*\left(\mathcal{O}_{X_{\text{ét}}}^+/p\right)$. So the results follow formally from Theorem 6.12.4. □

Theorem 6.13.6. Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d. Then

(1) the nearby cycles $\mathbf{R}\nu'_*\widehat{\mathcal{O}}_X^+ \in \mathbf{D}^+_{qc,acoh}(\mathfrak{X})$ and $(\mathbf{R}\nu'_*\widehat{\mathcal{O}}_X^+)^a \in \mathbf{D}^{[0,d]}_{acoh}(\mathfrak{X})^a$;

(2) for an affine admissible $\mathfrak{X} = \operatorname{Spf} A$ with the adic generic fiber X, the natural map

$$\mathrm{H}^i\left(X_{\text{proét}}, \widehat{\mathcal{O}}_X^+\right)^\Delta \to \mathrm{R}^i\nu'_*\widehat{\mathcal{O}}_X^+$$

is an isomorphism for every $i \geq 0$;

(3) the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism

$$\mathfrak{f}^*\left(\mathrm{R}^i\nu'_{\mathfrak{X},*}\left(\widehat{\mathcal{O}}_X^+\right)\right) \to \mathrm{R}^i\nu'_{\mathfrak{Y},*}\left(\widehat{\mathcal{O}}_Y^+\right)$$

is an isomorphism for any $i \geq 0$;

Proof. The proof is identical to the proof of Theorem 6.13.4 once one establishes that the sheaf $\widehat{\mathcal{O}}_X^+$ is p-adically derived complete. For this, see [BMS18, Remark 5.5]. □

Appendix

Appendix A. Derived Complete Modules

The main goal of this section is to collect some standard results on derived complete modules that seem difficult to find in the literature.

For the rest of the section, we fix a ring R with an element $\varpi \in R$.

Definition A.1. A complex $M \in \mathbf{D}(R)$ is *ϖ-adically derived complete* (or just derived complete) if the natural morphism $M \to \mathbf{R}\lim_n[M/\varpi^n]$ is an isomorphism.

Remark A.2. This definition coincides with [Sta21, Tag 091S] by [Sta21, Tag 091Z].

Lemma A.3. Let $M \in \mathbf{D}(R)$ be a derived complete complex. Then

(1) $M \in \mathbf{D}^{\geq d}(R)$ if $[M/\varpi] \in \mathbf{D}^{\geq d}(R/\varpi R)$.

(2) $M \in \mathbf{D}^{\leq d}(R)$ if $[M/\varpi] \in \mathbf{D}^{\leq d}(R/\varpi R)$;

Proof. **(1)** : By shifting, we can assume that $d = 0$. Now suppose that $[M/\varpi] \in \mathbf{D}^{\geq 0}(R/\varpi)$. Then we use an exact triangles

$$[M/\varpi] \to [M/\varpi^n] \to [M/\varpi^{n-1}]$$

to ensure that $[M/\varpi^n] \in \mathbf{D}^{\geq 0}(R/\varpi^n)$ for every $n \geq 0$. Now we use that M is derived complete to see that the natural morphism

$$M \to \mathbf{R}\lim_n[M/\varpi^n M]$$

is an isomophism. By passing to cohomology groups (and using that lim has cohomological dimension 1), we see that

$$0 \to \mathrm{R}^1 \lim_n \mathrm{H}^{i-1}([M/\varpi^n]) \to \mathrm{H}^i(M) \to \lim_n \mathrm{H}^i([M/\varpi^n]) \to 0$$

are exact for any integer i. This implies that $\mathrm{H}^i(M) = 0$ for $i \le 0$, i.e. $M \in \mathbf{D}^{\ge 0}(R)$.

(**2**) : Similarly, we can assume that $d = 0$. Then the same inductive argument shows that $[M/\varpi^n] \in \mathbf{D}^{\le 0}(R/\varpi^n)$ and we have short exact sequences

$$0 \to \mathrm{R}^1 \lim_n \mathrm{H}^{i-1}([M/\varpi^n]) \to \mathrm{H}^i(M) \to \lim_n \mathrm{H}^i([M/\varpi^n]) \to 0.$$

So we see that $M \in \mathbf{D}^{\le 1}(R)$ and $\mathrm{H}^1(M) = \mathrm{R}^1 \lim_n \mathrm{H}^0([M/\varpi^n])$. Now note that an exact triangle

$$[M/\varpi] \to [M/\varpi^n] \to [M/\varpi^{n-1}]$$

and the fact that $[M/\varpi] \in \mathbf{D}^{\le 0}(R/\varpi)$ imply that $\mathrm{H}^0([M/\varpi^n]) \to \mathrm{H}^0([M/\varpi^{n-1}])$ is surjective, so $\mathrm{R}^1 \lim_n \mathrm{H}^0([M/\varpi^n]) = 0$ by the Mittag-Leffler criterion. □

Lemma A.4. Let R be a ring with an ideal of almost mathematics $\mathfrak{m}$, and an element $\varpi \in \mathfrak{m}$. Let $M \in \mathbf{D}(R)$ be a ϖ-adically derived complete complex. Then $\widetilde{\mathfrak{m}} \otimes M$ is also ϖ-adically derived complete complex.

Proof. Consider an exact triangle

$$\widetilde{\mathfrak{m}} \otimes M \to M \to Q.$$

Since $\widetilde{\mathfrak{m}} \otimes M \to M$ is an almost isomorphism, we see that cohomology groups of Q are almost zero. In particular, they are ϖ-torsion, so derived complete. Therefore, Q is derived complete (for example, by [Sta21, Tag 091P] and [Sta21, Tag 091S]). Now derived completeness of M and Q implies derived completeness of $\widetilde{\mathfrak{m}} \otimes M$. □

Lemma A.5. Let R be a ring with an ideal of almost mathematics $\mathfrak{m}$, and an element $\varpi \in \mathfrak{m}$. Let $M \in \mathbf{D}(R)$ be a ϖ-adically derived complete complex. Then

- **(1)** $M^a \in \mathbf{D}^{\ge d}(R)^a$ if $[M^a/\varpi] \in \mathbf{D}^{\ge a}(R/\varpi R)^a$.
- **(2)** $M^a \in \mathbf{D}^{\le d}(R)^a$ if $[M^a/\varpi] \in \mathbf{D}^{\le a}(R/\varpi R)^a$.

Proof. Lemma A.4 guarantees that $\widetilde{\mathfrak{m}} \otimes M$ is derived ϖ-adically complete. Therefore, the claim follows from Lemma A.3 applied to $\widetilde{\mathfrak{m}} \otimes M$. □

Now we fix an R-ringed site $(X, \mathcal{O}_X)$.

Definition A.6. A complex $M \in \mathbf{D}(X)$ is *ϖ-adically derived complete* (or just derived complete) if the natural morphism $M \to \mathbf{R}\lim_n [M/\varpi^n]$ is an isomorphism.

Remark A.7. This definition coincides with [Sta21, Tag 0999] by [Sta21, Tag 0A0E].

Lemma A.8. Let $\mathcal{B} \subset \mathrm{Ob}(X)$ be a basis in a site X, and $M \in \mathbf{D}(X)$. Then M is ϖ-adically derived complete if and only if $\mathbf{R}\Gamma(U, M)$ is ϖ-adically derived complete for any $U \in \mathcal{B}$.

Proof. Suppose that M is ϖ-adically derived complete. Then $\mathbf{R}\Gamma(U, M)$ is derived ϖ-adically complete for any $U \in \mathrm{Ob}(X)$ by [Sta21, Tag 0BLX].

Now suppose that $\mathbf{R}\Gamma(U, M)$ is ϖ-adically derived complete for any $U \in \mathcal{B}$, and consider the derived ϖ-adic completion $M \to \widehat{M}$ with the associated distinguished triangle:

$$M \to \widehat{M} \to Q.$$

We wish to show that $Q \simeq 0$. In order to show it, it suffices to establish that $\mathbf{R}\Gamma(U, Q) \simeq 0$ for any $U \in \mathcal{B}$. Now we use [Sta21, Tag 0BLX] to conclude that

$$\mathbf{R}\Gamma(U, \widehat{M}) \simeq \mathbf{R}\widehat{\Gamma(U, M)},$$

so we get the distinguished triangle

$$\mathbf{R}\Gamma(U, M) \to \mathbf{R}\widehat{\Gamma(U, M)} \to \mathbf{R}\Gamma(U, M).$$

Since $\mathbf{R}\Gamma(U, M)$ is derived ϖ-adically complete by the assumption, so we see that the morphism

$$\mathbf{R}\Gamma(U, M) \to \mathbf{R}\widehat{\Gamma(U, M)}$$

is an isomorphism. Therefore, we conclude that $\mathbf{R}\Gamma(U, Q) \simeq 0$. This finishes the proof. □

Appendix B. Perfectoid Things

The main goal of this Appendix is to recall the main structural results about perfectoid rings.

B.1. **Perfectoid Rings.**

Definition B.1. [Sch17, Definition 3.6] A non-archimedean field $(K, |\ .\ |_K)$ is a *perfectoid field* if there is a pseudo-uniformizer $\varpi \in K$ such that $\varpi^p \mid p$ in $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ and the p-th power Frobenius map

$$\Phi \colon \mathcal{O}_K/\varpi\mathcal{O}_K \to \mathcal{O}_K/\varpi^p\mathcal{O}_K$$

is an isomorphism.

Definition B.2. A complete valuation ring K^+ is a *perfectoid valuation ring* if $K \coloneqq \mathrm{Frac}(K^+)$ is a perfectoid field with its valuation topology.

A Huber pair (K, K^+) is a *perfectoid field pair* if K is a perfectoid field and K^+ is an open and bounded valuation subring.

Remark B.3. Any perfectoid valuation ring K^+ is automatically microbial (see [Sem15, L9, Proposition 9.1.3 and Definition 9.1.4]). Any rank-1 valuation ring $K^+ \subset K^{++} \subset K$ defines the same topology on K by [Bou98, Ch. VI, §7.2, Prop. 3]. Therefore, K^{++} must be equal to K° the set of powerbounded elements. In particular, there is a unique rank-1 valuation ring between K^+ and K that we denote by $\mathcal{O}_K$, and the associated rank-1 valuation on K by $|\ .\ |_K \colon K \to \mathbf{R}_{\geq 0}$.

Lemma B.4. [Sch17, Proposition 3.8] Let K be a non-archimedean field. Then K is a perfectoid field if and only if

(1) K is not discretely valued,
(2) $|p|_K < 1$,
(3) the Frobenius morphism $\Phi \colon \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ is surjective.

We wish to show that the ideal $\mathfrak{m} = K^{\circ\circ} \subset K^+$ defines an ideal of almost mathematics in K^+. For the future reference, it will be convenient to do in a more general set-up of perfectoid pairs.

Definition B.1.1. [Sch17, Definition 3.1] A complete Tate-Huber pair (R, R^+) is a *perfectoid pair* if R is a uniform Tate ring containing a pseudo-uniformizer $\varpi_R \in R^\circ$ such that $\varpi_R^p \mid p$ in R° and the Frobenius homomorphism $R^\circ/\varpi_R R^\circ \xrightarrow{x \mapsto x^p} R^\circ/\varpi_R^p R^\circ$ is an isomorphism.

A Tate-Huber pair (R, R^+) is *p-adic perfectoid pair* if it is a Huber pair, and R is $p \neq 0$ in R.

A Tate ring R is a *perfectoid ring* if (R, R°) is a perfectoid pair.

Remark B.1.2. It is not, a priori, clear that a perfectoid ring R that is a field is a perfectoid field (in the sense of Definition B.1). The problem is to verify that R has a non-archimedean topology on it. This turned out to be always true by [Ked18].

Remark B.1.3. By [Sch17, Proposition 3.5], a complete Tate ring R of characteristic p is perfectoid if and only if R is perfect as a ring, i.e. the Frobenius morphism is an isomorphism.

Remark B.1.4. In the definition of a perfectoid pair, it suffices to require $R^\circ/\varpi_R R^\circ \xrightarrow{x\mapsto x^p} R^\circ/\varpi_R^p R^\circ$ to be surjective. This map actually turns out to be always injective. Moreover, this condition turns out to be equivalent to the surjectivity of the Frobenius map

$$R^\circ/pR^\circ \to R^\circ/pR^\circ.$$

In particular, it is independent of a choice of a pseudo-uniformizer $\varpi_R^p \mid p$, see [Sch17, Remark 3.2] for more detail. Therefore, if R is an algebra over a perfectoid field K with a pseudo-uniformizer $\varpi_K \in \mathcal{O}_K$, one can always take $\varpi_R = \varpi_K$. In particular, every perfectoid ring in the sense of [Sch12, Definition 5.1] is a perfectoid ring in the sense of Definition B.1.1.

Lemma B.5. [Sch17, Lemma 3.10] Let (R, R^+) be a perfectoid pair. Then there is a pseudo-uniformizer $\varpi \in R^{\circ\circ}$ such that

(1) $\varpi^p \mid p$ in R°;

(2) ϖ admits a compatible sequence of p^n-th root of $\varpi^{1/p^n} \in R^+$ for $n \geq 0$.

In this case, $R^{\circ\circ} = \bigcup_{n\geq 0} \varpi^{1/p^n} R^+$.

Proof. [Sch17, Lemma 3.10] says that there is a pseudo-uniformizer $\varpi \in R^{\circ\circ} \subset R^+$ such that $\varpi^p \mid p$ in R°, and there is a compatible sequence of the p^n-th roots $\varpi^{1/p^n} \in R^\circ$ for $n \geq 0$. Since R^+ is integrally closed, we conclude that all ϖ^{1/p^n} must lie in R^+. Since $R^{\circ\circ}$ is a radical ideal R^+ and contains ϖ, it clearly contains $\bigcup_{n\geq 0} \varpi^{1/p^n} R^+$.

Now we pick an element $x \in R^{\circ\circ}$, and wish to show that $x \in \bigcup_{n\geq 0} \varpi^{1/p^n} R^+$. Since x is topologically nilpotent, we can find an integer m such that

$$x^{p^m} \in \varpi R^+$$

Therefore, $x^{p^m} = \varpi a$ for $a \in R^+$. Thus

$$\left(\frac{x}{\varpi^{1/p^m}}\right)^{p^m} = a \in R^+.$$

Therefore, $\frac{x}{\varpi^{1/p^m}} \in R^+$ because R^+ is integrally closed in R. So $x \in \varpi^{1/p^m} R^+$. $\square$

Remark B.1.5. If (R, R^+) is a p-adic perfectoid pair, then one can choose ϖ such that $\varpi^p R^+ = pR^+$. Indeed, [BMS18, Lemma 3.20] implies that R^+ is perfectoid in the sense of [BMS18, Definition 3.5]. Thus the desired ϖ exists by [BMS18, Lemma 3.9].

Definition B.1.6. A pseudo-uniformozer $\varpi \in R^+$ of a p-adic perfectoid pair (R, R^+) is *good* if $\varpi R^+ = pR^+$ and ϖ admits a compatible sequence of p-power roots.

For the rest of the section, we fix a perfectoid pair (R, R^+) and an ideal $\mathfrak{m} = R^{\circ\circ}$. Our goal is to show that $\mathfrak{m}$ defines a set-up for almost mathematics, i.e. $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_{R^+} \mathfrak{m}$ is R^+-flat and $\mathfrak{m}^2 = \mathfrak{m}$.

Lemma B.6. Let (R, R^+) be a perfectoid pair, and $\mathfrak{m} = R^{\circ\circ}$ the associated ideal of topologically nilpotent elements. Then $\mathfrak{m}$ is flat over R^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$.

Proof. Lemma B.5 implies that $\mathfrak{m}$ is flat as a colimit of free modules of rank-1.

Now we wish to show that $\mathfrak{m}^2 = \mathfrak{m}$. We take any element $x \in \mathfrak{m}$, by Lemma B.5 we know that $x = \varpi^{1/p^n} a$ for some integer n and $a \in R^+$. Therefore,

$$x = \left(\varpi^{1/p^{n+1}}\right)^{p-1} \left(\varpi^{1/p^{n+1}} a\right) \in \mathfrak{m}^2.$$

Now we consider a short exact sequence

$$0 \to \mathfrak{m} \to R^+ \to R^+/\mathfrak{m} \to 0.$$

By flatness of $\mathfrak{m}$, we conclude that it remains exact after applying the tensor product against $\mathfrak{m}$. Therefore, the sequence

$$0 \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$$

is exact. Since $\mathfrak{m}^2 = \mathfrak{m}$, we conclude that

$$\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}.$$

□

Lemma B.1.7. Let (R, R^+) be a perfectoid pair. Then the natural inclusion $\iota\colon R^+ \to R^\circ$ is an almost isomorphism.

Proof. Clearly, the map $\iota\colon R^+ \to R^\circ$ is injective, so it suffices to show that its cokernel is almost zero, i.e. annihilated by any $\varepsilon \in \mathfrak{m}$. Pick an element $x \in R^\circ$, then $\varepsilon x \in R^{\circ\circ} \subset R^+$. Therefore we conclude that $\varepsilon(\operatorname{Coker} \iota) = 0$ finishing the proof. □

B.2. **Universal Perfectoid Cover.** The main goal of this section is to give a construction of a "universal cover" perfectoid cover of an affinoid (pre-)adic[38] affinoid space $X = \operatorname{Spa}(A, A^+)$ over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Throughout this section, we assume that (A, A^+) is a Tate-Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$ with no non-trivial idempotents in A. We do not assume that A is sheafy.

Our assumption on (A, A^+) implies that $\operatorname{Spec} A$ is connected. We choose a geometric point

$$\overline{x}\colon \operatorname{Spec} \Omega \to \operatorname{Spec} A$$

for an algebraically closed field Ω, and consider the category of pointed, connected, finite étale Galois morphisms

$$\{(\operatorname{Spec} A_i, \overline{x_i}) \to (\operatorname{Spec} A, \overline{x})\}_{i \in I} \tag{B.1}$$

A standard argument shows that this system is cofiltered (we point out that it uses the connectedness assumption). Now we want to make this system into a system of (pre-)adic spaces over $\operatorname{Spa}(A, A^+)$.

We define A_i^+ to be the integral closure of A^+ in A_i. We show that each (A_i, A_i^+) is a Huber pair if we put the natural topology on A_i (see [Zav21b, Appendix D.3]).

Lemma B.2.1. Let (A, A^+) be a complete Tate-Huber pair with a pair of definition $(A_0 \subset A^+, \varpi)$, and $A \to B$ is a finite étale morphism. Then (B, B^+) is a complete Tate-Huber pair where B^+ is the integral closure of A^+ in B.

Proof. Step 1*: B is complete in its natural topology.* Since B is finite étale, B is a projective A-module of finite rank. Then there is another finite A-module M such that $B \oplus M \simeq A^{\oplus n}$. Consider the projection $p\colon A^{\oplus n} \to B$, the natural topology on B coincide with the quotient topology (see [Zav21b, Lemma B.3.2]). Using that A is Huber ring, it is not difficult to show that the quotient topology on B should coincide with the subspace topology. Since $A^{\oplus n}$ is complete, we conclude

[38] We do not assume that it is sheafy.

that the natural topology on B is separated. Therefore, the same applies to M as we never used the ring structure on B. Then B is closed in A as a kernel of a continuous homomorphism with a separated target. In particular, B is complete in its subspace (equivalently, quotient) topology, and as discussed above, this topology coincides with the natural topology. So it is complete in its natural topology.

Step 2*: B admits a finite set of A-module generators $x_1, \dots, x_n$ that are integral over A_0.* Pick any finite set $x'_1, \dots, x'_n \in B$ of A-module generators. It suffices to show that $x_i = \varpi^c x'_i \in B$ are integral over A_0 for some integer c. So it is enough to show that, for any $b \in B$, there is an integer c such that $\varpi^c b$ is integral over A_0.

By definition, b is integral over A. So we can find a monic equiation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$$

with $a_k \in A$ for $k = 0, \dots, n-1$. Then there is an integer c such that $\varpi^c a_k \in A_0$ for $k = 0, \dots, n-1$. Thus the equation

$$(\varpi^c b)^n + a_{n-1}\varpi^c(\varpi^c b)^{n-1} + \cdots + a_0 \varpi^{cn} = 0$$

shows that $\varpi^c b$ is integral over A_0.

Step 3*: An A_0-subalgebra B_0 of B generated by $x_1, \dots, x_n$ is finite as an A_0-module.* Clearly this algebra is finitely generated over A_0 as an algebra and every element is integral. Therefore, it is finite.

Step 4*: B_0 is open in B and the induced topology coincides with the ϖ-adic one.* Choose some A_0-module generators $b_1, \dots, b_m \in B_0$. Clearly, $B_0\left[\frac{1}{\varpi}\right] = B$, so the A-linear morphism

$$q\colon \bigoplus_{i=1}^{m} Ae_i \to B$$

sending e_i to b_i is surjective. By [Hub94, Lemma 2.4(i)], q is open. In particular, the topology on B is the quotient topology along q. Therefore, B_0 is open in B as $q^{-1}(B_0)$ is a subgroup containing an open subgroup $\oplus_{i=1}^m A_0 e_i$. Moreover, the topology on B_0 is ϖ-adic since $B_0 = q(\oplus_{i=1}^m A_0 e_i)$, the topology on $\oplus_{i=1}^m A_0 e_i$ is already ϖ-adic, and q is open.

Step 5. *(B, B^+) is a complete Huber pair:* We have already showed that B is complete in its natural topology and (B_0, ϖ) is a pair of definition for this topology. Therefore, B is a Huber ring. It suffices to show that B^+ is open, integrally closed and lies in B°. Openness is clear since $B_0 \subset B^+$, and B^+ is integrally closed by definition. One also easily show that $B^+ \subset B^\circ$ because B^+ is integral over $A^+ \subset A^\circ$. □

Corollary B.2.2. Let (A_i, A_i^+) as above. Then, for every $j > i$, the natural morphism $\mathrm{Spa}\,(A_j, A_j^+) \to \mathrm{Spa}\,(A_i, A_i^+)$ is a (finite étale) surjection.

Proof. Note that $\mathrm{Spec}\, A_j \to \mathrm{Spec}\, A_i$ is surjective as it is finite étale (so open and closed) and $\mathrm{Spec}\, A_i$ is connected. Since both A_i and A_j are Galois over A, it is clear that A_j is Galois over A_i. Denote its Galois group by G. Then $A_i = (A_j)^G$ and $A_i^+ = (A_j^+)^G$. Now [Zav21b, Lemma 4.2.1] implies that $(A_i, A_i^+)'$ with the subspace topology is a Tate-Huber pair. Clearly the morphism $(A_i, A_i^+) \to (A_i, A_i^+)'$ is continuous and surjective, so it is a homeomorphism by the Banach Open Mapping Theorem [Hub93a, Lemma 2.4]. Therefore, $|\,\mathrm{Spa}\,(A_i, A_i^+)| = |\,\mathrm{Spa}\,(A_j, A_j^+)/G|$ by [Han, Theorem 3.1]. In particular, it is surjective. □

Corollary B.2.2 that $\{\mathrm{Spa}\,(A_i, A_i^+) \to \mathrm{Spa}\,(A, A^+)\}_{i\in I}$ gives a cofiltered pro-system of finite étale covers of $\mathrm{Spa}\,(A, A^+)$. We want to say that its limit is a "perfectoid universal cover" of $\mathrm{Spa}\,(A, A^+)$. In order to make rigirous sense of it, we need to show some preliminary results.

We define $\overline{A} :- \mathrm{colim}_I\, A_i$, $\overline{A}^+ = \mathrm{colim}_I\, A_i^+$, A_∞^+ to be the p-adic completion of $\overline{A}^+$ (even if the colimit topology does not coincide with the p-adic topology) and $A_\infty := A_\infty^+[\frac{1}{p}]$. We now study properties of these rings.

Lemma B.2.3. The scheme $\mathrm{Spec}\,\overline{A}$ is connected and any finite étale cover splits.

Proof. Connectedness of $\mathrm{Spec}\,\overline{A}$ is equivalent to the fact that $\overline{A}$ has no non-trivial idempotents. It can be easily seen that any idempotent should come from a finite level, so any idempotent must be trivial because $\mathrm{Spec}\,A_i$ is connected for every i.

Now we show that any finite étale cover $f\colon \mathrm{Spec}\,\overline{B} \to \mathrm{Spec}\,\overline{A}$ splits. Since finite étale morphism are finitely presented and $\overline{A} = \mathrm{colim}_i\, A_i$ is a filtered colimit of rings, we can use the spreading out techniques from [Gro66] to assume that f comes as a base change of a finite étale morphism $f_i\colon \mathrm{Spec}\,B_i \to \mathrm{Spec}\,A_i$ for some $i \in I$. It suffices to show that f_i has a section after a pullback along $v_{j,i} : \mathrm{Spec}\,A_j \to \mathrm{Spec}\,A_i$ for some $j \geq i$.

We recall that $\mathrm{Spec}\,B_i$ has a finite number of connected components by [Sta21, Tag 07VB]. It implies that each connected of $\mathrm{Spec}\,B_i$ is open and closed. Since a finite étale morphism is open and closed, we can replace $\mathrm{Spec}\,B_i$ by its connected component to assume that $\mathrm{Spec}\,B_i$ is connected. Now we use [Sta21, Tag 0BN2] and [Sta21, Tag 0BNB] to say that f_i is dominated by a finite Galois cover $X \to \mathrm{Spec}\,A_i$, so we can replace $\mathrm{Spec}\,B_i$ with X to assume f_i is Galois. But then, after choosing a geometric point in $\mathrm{Spec}\,B_i$ over the geometric point $\overline{x_i} \to \mathrm{Spec}\,A_i$, we conclude that $\mathrm{Spec}\,B_i \to \mathrm{Spec}\,A_i$ is equal to some transition map $\mathrm{Spec}\,A_j \to \mathrm{Spec}\,A_i$ in the cofiltered system $\{\mathrm{Spec}\,A_i, v_{i,j}\}$. Clearly, $\mathrm{Spec}\,B_i$ splits after a pullback along $\mathrm{Spec}\,A_j = \mathrm{Spec}\,B_i \to \mathrm{Spec}\,A_i$. □

We topologize A_∞ by declaring A_∞^+ with its p-adic topology to be a ring of definition in A_∞ (recall that (A, A^+) is assumed to be a Tate-Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$).

Lemma B.2.4. Let (A_∞, A_∞^+) be as above. Then (A_∞, A_∞^+) is a Tate-Huber pair, $\mathrm{Spec}\,A_\infty$ is connected and every finite étale cover splits.

Proof. Clearly, $\overline{A}^+$ is integrally closed in $\overline{A}$. Thus A_∞^+ is integrally closed in A_∞ by [Bha, Lemma 5.1.2]. By definition A_∞^+ is open and bounded in A_∞, so (A_∞, A_∞^+) is a Tate-Huber pair with a pseudo-uniformizer $p \in A_\infty^+$.

Now we show that A_∞ does not have non-trivial idempotents. Any idempotent is clearly integral over $\mathbf{Z}$, so must lie in A_∞^+. Thus it suffices to show that A_∞^+ does not have any non-trivial idempotents.

In order to verify this, we show that $\overline{A}^+$ and A_∞^+ are p-adically henselian. Lemma B.2.1 implies that every (A_i, A_i^+) is a Huber-Tate pair for every i, so any element $a_i \in A_i^+$ lies in some ring of definition $A_{i,0} \subset A_i^+$ by [Hub93b, Corollary 1.3]. In particular, $A_i^+ = \mathrm{colim}\, A_{i,0}$ where the colimit is taken over all ring of definitions in A_i^+. Since each $A_{i,0}$ is p-adically complete, the colimit A_i^+ is p-adically henselian. Thereover $\overline{A}^+$ is also p-adically henselian as a filtered colimit of p-adically henselian rings. Clearly, A_∞^+ is also p-adically henselian as it is p-adically complete. Now [Ray70, XI, §2, Proposition 1] reads that we have bijections

$$\mathrm{Idem}(A_\infty^+) = \mathrm{Idem}(A_\infty^+/p) = \mathrm{Idem}(\overline{A}^+/p) = \mathrm{Idem}(\overline{A}^+).$$

So it suffices to show that $\overline{A}^+$ has no non-trivial idempotents. This is done in Lemma B.2.3.

Now we show that there are no non-split finite étale covers of $\operatorname{Spec} A_\infty$. We apply [GR03, Proposition 5.4.53] to a p-adically henselian ring $\overline{A}^+$ to get an equivalence of categories

$$(A_\infty)_{\text{fét}} \simeq \overline{A}_{\text{fét}}$$

But $\operatorname{Spec} \overline{A}$ has no non-split finite étale covers by Lemma B.2.3 (we leave to the reader to check that covers on both sides also coincide). □

Lemma B.2.5. The pair (A_∞, A_∞^+) is a perfectoid pair.

Proof. Lemma B.2.4 guarantees that (A_∞, A_∞^+) is a Tate-Huber pair. By Remark B.1.4, it suffices to show that A_∞ is uniform, there is a pseudo-uniformizer ϖ such that $\varpi^p \mid p$, and the Frobenius morphism

$$A_\infty^\circ/\varpi A_\infty^\circ \to A_\infty^\circ/\varpi^p A_\infty^\circ$$

is surjective.

Clearly, A_∞^+ is an algebra over $\mathcal{O}_{\mathbf{C}_p}$, so $\varpi = p^{1/p} \in \mathcal{O}_{\mathbf{C}_p}$ is a pseudo-uniformizer such that $\varpi^p \mid p$. We show that the Frobenius map

$$\Phi \colon A_\infty^\circ/p^{1/p} A_\infty^\circ \to A_\infty^\circ/p A_\infty^\circ$$

is surjective. For any class $\overline{f} \in A_\infty^\circ/pA_\infty^\circ$, we pick a lift $f \in A_\infty^\circ$ and consider the equation $T^p - pT - f$. Clearly,

$$\operatorname{Spec} A_\infty[T]/(T^p - pT - f) \to \operatorname{Spec} A_\infty$$

is finite étale. So by Lemma B.2.4, it has a section. Thus there is some element $g \in A_\infty$ such that $g^p - pg = f$. Clearly, it is integral over A_∞°, so $g \in A_\infty^\circ$. Therefore, its class $\overline{g} \in A_\infty^\circ/p^{1/p}A_\infty^\circ$ is an element such that $\Phi(\overline{g}) = \overline{f}$.

Finally, we show that A_∞ is uniform. Note again that A_∞^+ is an $\mathcal{O}_{\mathbf{C}_p}$-algebra, so it makes sense to consider almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_{n=1}^\infty p^{1/n}\mathcal{O}_{\mathbf{C}_p}$. We use [Sch12, Lemma 5.3 (iv)] and Lemma 2.1.10 to conclude that $(A_\infty^+)_*$ is p-adically complete and so [Bha, Lemma 5.1.2] reads that $(A_\infty^+)_*$ is integrally closed in A_∞. Therefore, [Bha, Proposition 5.2.5 and Proposition 5.2.6] (A_∞^+ is p-root closed in A_∞ because A_∞^+ is integrally closed in A_∞) imply that $A_\infty^\circ = (A_\infty^+)_*$ is uniform finishing the proof. □

Now we summarize what we got so far.

Lemma B.2.6. Let (A, A^+) be a Tate-Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$. Then there is a a cofiltered system of morphisms $\{\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)\}_{i\in I}$ and of finite groups $\{\Delta_i\}_{i\in I}$ with surjective transition maps $\Delta_i \to \Delta_j$ for $i > j$ such that

- **(1)** $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A_j, A_j^+)$ is finite étale and surjective for $i \geq j$;
- **(2)** $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)$ is a Δ_i-torsor;
- **(3)** $\operatorname{Spa}(A_\infty, A_\infty^+)$ in the notation as above, is a connected affinoid perfectoid space such that its every finite étale cover splits.

Proof. The first part is Corollary B.2.2. For the second part, by construction, $\operatorname{Spec} A \to \operatorname{Spec} A_i$ is a Δ_i-torsor for some finite group Δ_i. This means that the natual morphism

$$A_i \otimes_A A_i \to A_i \otimes_A (A[\Delta_i])$$

is an isomorphism. To see that $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)$ is a Δ_i-torsor, we need to show that

$$A_i \widehat{\otimes}_A A_i \to A_i \otimes_A (A[\Delta_i])$$

is an isomorphism. Now note that the topology on $A_i \otimes_A A_i$ coincides with the natural topology by [Zav21b, Lemma B.3.5]. Therefore, it is already complete by Lemma B.2.1. Thus, $A_i \widehat{\otimes}_A A_i \simeq A_i \otimes_A A_i \simeq A_i \otimes_A A[\Delta_i]$ as we wanted.

For the third part, Lemma B.2.4 and Lemma B.2.5 imply that $\mathrm{Spa}\,(A_\infty, A_\infty^+)$ is an affinoid perfectoid. In particular, it is sheafy. So to show that it is connected, it suffices to show that A_∞ does not have any idempotents. This is done in Lemma B.2.4. Now [Sch12, Proposition 7.6 and Theorem 7.9] reads that $\mathrm{Spa}\,(A_\infty, A_\infty^+)_{\text{fét}} \simeq (A_\infty)_{\text{fét}}$. Note that any finite étale surjective cover of $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A_\infty, A_\infty^+)$ corresponds to a finite étale *surjective* cover of $\mathrm{Spec}\, B \to \mathrm{Spec}\, A_\infty$. Indeed, any maximal ideal $\mathfrak{m} \subset A_\infty$ is a support of some valuation in $\mathrm{Spa}\,(A_\infty, A_\infty^+)$. Therefore, $\mathrm{Spec}\, B \to \mathrm{Spec}\, A_\infty$ is surjective onto the set of closed points. Since an étale map is open, we conclude that $\mathrm{Spec}\, B \to \mathrm{Spec}\, A_\infty$ must be surjective. So we conclude that every finite étale cover of $\mathrm{Spa}\,(A_\infty, A_\infty^+)$ splits by Lemma B.2.4. $\square$

Appendix C. The pro-étale and v-sites

The main goal of this section is to recall certain comparison results about étale, quasi-proétale, and v-topologies. We will freely use the notion of perfectoid spaces and their tilts from [Sch12] and [Sch17].

C.1. The v-topology.

We start by discussing of the v-topology on an adic space X and certain structure sheaves attached to this space.

One of the problems with the category of adic spaces is that this category does not have limits. Therefore, in order to speak about pro-étale morphisms, we had to work with pro-systems and distinguish objects of the pro-étale site of X (that is, a priori, just a cofiltered diagram) and their realizations as adic spaces (whenever they exist). It turns out that this type of problems can be resolved by considering an adic space as a sheaf $X^\diamond$ on the category of perfectoid spaces of characteristic $p > 0$. This may sound very counter-intuitive to consider a p-adic rigid-analytic variety as a sheaf on characteristic p objects, but it turns out to be a very useful thing. The main idea is that an $S = \mathrm{Spa}\,(R, R^+)$-point of $X^\diamondsuit$ should be a choice of an untilt $S^\#$ of S (this is a mixed characteristic object) and a morphism $S^\# \to X$. This procedure turns out to remember a lot of information about X (e.g. étale cohomology), but not all information on X (see Warning C.1.8)

Definition C.1.1. [Sch17, Definitions 8.1, 12.1, and 14.1] The category Perf is the category of characteristic p perfectoid spaces.

The *v-topology* on Perf is defined by saying that a family $\{f_i \colon X_i \to X\}_{i\in I}$ of morphisms in Perf is a covering if, for any quasi-compact open $U \subset X$, there is a finite subset $I_0 \subset I$ and quasi-compact opens $\{U_i \subset X_i\}_{i \in I_0}$ such that $U \subset \cup_{i\in I_0} f_i(U_i)$.

A *small v-sheaf* is a v-sheaf Y on Perf such that there is a surjective map of v-sheaves $Y' \to Y$ for some perfectoid space Y'.

The *v-site* Y_v of a small v-sheaf Y is the site whose objects are all maps $Y' \to Y$ from small v-sheaves Y', with coverings given by families $\{Y_i \to Y\}_{i\in I}$ such that $\sqcup_{i\in I} Y_i \to Y$ is a surjection of v-sheaves.

Remark C.1.2. The v-site of a small v-sheaf Y has all finite limits by [Sch17, Proposition 12.10] and [Sta21, Tag 002O].

In what follows, we denote by $\mathbf{Ad}_{\mathbf{Q}_p}$ the category of adic spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$ and by $\mathbf{pAd}_{\mathbf{Q}_p}$ the category of pre-adic spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$ as defined in [SW13, Definition 2.1.5] and [KL15, Definition 8.2.3][39]. The category of pre-adic spaces has the following list of useful properties (see [SW13, Proposition 2.1.6] or [KL15, §8.2.3]):

(1) There is a fully faithful functor $\mathbf{Ad}_{\mathbf{Q}_p} \to \mathbf{pAd}_{\mathbf{Q}_p}$ from the category of adic spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$,

(2) every pre-adic affinoid space[40] $\mathrm{Spa}\,(A, A^+)$ is naturally an object of $\mathbf{pAd}_{\mathbf{Q}_p}$,

(3) for an adic space S and a pre-adic affinoid space $\mathrm{Spa}\,(A, A^+)$, the set of morphisms is given by
$$\mathrm{Hom}_{\mathbf{pAd}_{\mathbf{Q}_p}}(S, \mathrm{Spa}\,(A, A^+)) = \mathrm{Hom}_{\mathrm{cont}}((A, A^+), (\mathcal{O}_S(S), \mathcal{O}^+_S(S))),$$

(4) $\mathbf{pAd}_{\mathbf{Q}_p}$ has all finite limits,

(5) for a pseudo-adic space X, one can functorially associate an underlying topological space $|X|$ such that it coincides with $|\mathrm{Spa}\,(A, A^+)|$ if $X = \mathrm{Spa}\,(A, A^+)$ a pre-adic affinoid space and it coincides with the usual underlying topological space $|X|$ if $X = (|X|, \mathcal{O}_X, \mathcal{O}^+_X)$ is an adic space,

(6) for every pre-adic space $X \in \mathbf{pAd}_{\mathbf{Q}_p}$, one can functorially associate an étale site $X_{\text{ét}}$ such that, for X a strongly noetherian or perfectoid space, $X_{\text{ét}}$ coincides with the étale site defined in [Hub96] and [Sch12] respectively.

Warning C.1.3. In general it is not true that $\mathrm{Hom}_{\mathbf{pAd}_{\mathbf{Q}_p}}(\mathrm{Spa}\,(B, B^+), \mathrm{Spa}\,(A.A^+))$ is equal to $\mathrm{Hom}_{\mathrm{cont}}((A, A^+), (B, B^+))$ unless $\mathrm{Spa}\,(B, B^+)$ is sheafy.

Definition C.1.4. [SW13, Definition 2.4.1] Let X_i be a cofiltered inverse system of pre-adic spaces with quasi-compact and quasi-separated transition maps, X a pre-adic space, and $f_i \colon X \to X_i$ a compatible family of morphisms.

We say that X is a *tilde-limit* of X_i, $X \sim \lim_I X_i$ if the map of underlying topological spaces $|X| \to \lim_I |X_i|$ is a homeomorphism, and if there is an open cover of X by affinoid $\mathrm{Spa}\,(A, A+) \subset X$, such that the map
$$\mathrm{colim}_{\mathrm{Spa}\,(A_i, A_i^+) \subset X_i} A_i \to A$$
has dense image, where the filtered colimit runs over all open affinoid
$$\mathrm{Spa}\,(A_i, A_i^+) \subset X_i$$
over which $\mathrm{Spa}\,(A, A+) \subset X \to X_i$ factors.

Definition C.1.5. [Sch17, Definition 15.5] The *diamond associated* to $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ is a presheaf
$$X^\diamondsuit \colon \mathrm{Perf}^{\mathrm{op}} \to \mathrm{Sets}$$
such that, for any perfectoid space S of characteristic p, we have
$$X^\diamondsuit(S) = \left\{ \left(\left(S^\sharp, \iota \right), f \colon S^\sharp \to X \right) \right\} /\mathrm{isom}$$
where $S^\sharp$ is a perfectoid space, and $\iota \colon (S^\sharp)^\flat \to S$ is an identification of a $S^\sharp$ as an untilt of S.

The *diamantine spectrum* $\mathrm{Spd}\,(A, A^+)$ of $\mathrm{Spa}\,(A, A^+)$ is a presheaf $\mathrm{Spa}\,(A, A^+)^\diamondsuit$.

We list the main properties of this functor:

[39]These spaces are called adic in [SW13], we prefer to call them pre-adic to distinguish with adic spaces in the sense of Huber

[40]By a pre-adic affinoid space, we mean a space $\mathrm{Spa}\,(A, A^+)$ for a not necessarilly sheafy Huber pair (A, A^+).

Proposition C.1.6. The diamondification functor factors through the category of v-sheaves. And the functor $(-)^\diamondsuit \colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ satisfies the following list of properties:

(1) if X is a perfectoid space, $X^\diamondsuit \simeq X^\flat$,

(2) $X^\diamondsuit$ is a small v-sheaf for any $X \in \mathbf{pAd}_{\mathbf{Q}_p}$[41],

(3) if $\{X_i \to X\}_{i\in I}$ is an open (resp. étale) covering in $\mathbf{pAd}_{\mathbf{Q}_p}$, the family $\{X_i^\diamondsuit \to X^\diamondsuit\}_{i\in I}$ is an open (resp. étale) covering of $X^\diamondsuit$,

(4) there is a functorial homeomorphism $|X| \simeq |X^\diamondsuit|$ for any $X \in \mathbf{pAd}_{\mathbf{Q}_p}$,

(5) if X is a perfectoid space such that $X \sim \lim_I X_i$ in $\mathbf{pAd}_{\mathbf{Q}_p}$ with quasi-compact quasi-separated transition maps, the natural functor $X^\diamondsuit \to \lim_I X_i^\diamondsuit$ is an isomorphism,

(6) the functor $(-)^\diamondsuit \colon \colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ commutes with fiber products.

Proof. The first claim follows from [Sch17, Corollary 3.20] and the definition of the diamondification. As for the second claim, [Sch17, Proposition 15.6] implies that $X^\diamondsuit$ is a diamond, and it so it is a small v-sheaf by [Sch17, Proposition 11.9] and the definition of a diamond (see [Sch17, Definition 11.1]). The third and the fourth claims follow from [Sch17, Lemma 15.6]. The proof of the fifth claim is identical to [SW13, Proposition 2.4.5] (the statement makes the assumption that X and X_i are defined over a perfectoid field, but it is not used in the proof).

We now give a proof of the sixth claim. Let $U \to V$, $W \to V$ be morphisms in $\mathbf{pAd}_{\mathbf{Q}_p}$ with a fiber product $U \times_V W$. We fix a perfectoid space S of characteristic p. Then we have a sequence of identifications

$$\begin{aligned}(U \times_V W)^\diamondsuit(S) &= \left\{\left(\left(S^\sharp, \iota\right), S^\sharp \to U \times_V W\right)\right\}/\mathrm{isom} \\ &= \left\{\left(\left(S^\sharp, \iota\right), S^\sharp \to U\right)\right\}/\mathrm{isom} \times_{\left\{\left(\left(S^\sharp,\iota\right), S^\sharp \to V\right)\right\}/\mathrm{isom}} \left\{\left(\left(S^\sharp, \iota\right), S^\sharp \to W\right)\right\}/\mathrm{isom} \\ &= U^\diamondsuit(S) \times_{V^\diamondsuit(S)} W^\diamondsuit(S)\end{aligned}$$

that is functorial in S. Therefore, this defines an isomorphism

$$(U \times_V W)^\diamondsuit \to U^\diamondsuit \times_{V^\diamondsuit} W^\diamondsuit.$$

□

Warning C.1.7. The functor $(-)^\diamondsuit$ does not send the final object to the final object. In particular, it does not commute with all finite limits.

Warning C.1.8. The functor $(-)^\diamondsuit \colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ is not fully faithful. This is actually crucial for our proofs in Section 6.10.

The next goal is to discuss example of v-covers of $X^\diamondsuit$ that will be of essential interest for our purposes.

Definition C.1.9. A family of morphisms $\{f_i \colon X_i \to X\}_{i\in I}$ in $\mathbf{pAd}_{\mathbf{Q}_p}$ is a *naive v-covering* if, for any quasi-compact open $U \subset X$, there is a finite subset $I_0 \subset I$ and quasi-compact opens $\{U_i \subset X_i\}_{i\in I_0}$ such that $|U| \subset \cup_{i\in I_0} |f_i|(|U_i|)$.

Remark C.1.10. Using that the natural morphism $|X \times_Y Z| \to |X| \times_{|Y|} |Z|$ is surjective, it is easy to see that a pullback of a naive v-covering is a naive v-covering.

[41] It is even a diamond in the terminology of [Sch17], but we will never need this

Example C.1.11. A quasi-compact surjective morphism $X \to Y$ is a naive v-cover. A family of jointly surjective étale morphisms $\{X_i \to X\}$ is a naive v-cover.

Our next goal is to show that the diamondification functor $(-)^\diamond$ sends naive v-covers to surjections of small v-sheaves.

Lemma C.1.12. Let $f\colon X \to Y$ be a quasi-compact (resp. quasi-separated) morphism in $\mathbf{pAd}_{\mathbf{Q}_p}$. Then $f^\diamondsuit\colon X^\diamondsuit \to Y^\diamondsuit$ is quasi-compact (resp. quasi-separated) in the sense of [Sch17, p.40].

Proof. We first deal with quasi-compact f. Choose a morphism $S \to Y^\diamondsuit$ from an affinoid perfectoid S, it corresponds to a morphism $S^\sharp \to Y$ with an affinoid perfectoid source $S^\sharp$. To check that $f^\diamondsuit$ is quasi-compact, it suffices to show $S \times_{Y^\diamondsuit} X^\diamondsuit$ is quasi-compact. By Proposition C.1.6, $S \times_{Y^\diamondsuit} X^\diamondsuit \simeq (S^\sharp \times_Y X)^\diamondsuit$. By [Sch17, Lemma 15.6],

$$|S \times_{Y^\diamondsuit} X^\diamondsuit| \simeq |S^\sharp \times_Y X|$$

quasi-compact by our assumption on f. Now $S \times_{Y^\diamondsuit} X^\diamondsuit$ is quasi-compact by [Sch17, Proposition 12.14(iii)] and the fact that $(S^\sharp \times_Y X)^\diamondsuit$ is locally spatial by [Sch17, Lemma 15.6].

The case of a quasi-separated f follows from Proposition C.1.6 and the quasi-compact by considering the diagonal morphism $\Delta_f\colon X \to X \times_Y X$. $\square$

Lemma C.1.13. Let $\{f_i\colon X_i \to X\}_{i\in I}$ be a naive v-covering in $\mathbf{pAd}_{\mathbf{Q}_p}$. Then $\{f_i^\diamondsuit\colon X_i^\diamondsuit \to X^\diamondsuit\}_{i\in I}$ is a v-covering.

Proof. We can find a covering $\{U_j \to X\}_{j\in J}$ by open affinoids. Since $\{U_j^\diamondsuit \to X^\diamondsuit\}$ is a v-covering by Proposition C.1.6, it suffices to show that $\{f_{i,j}\colon X_{i,j} \coloneqq X_i \times_X U_j \to U_j\}_{i\in I}$ is a v-covering for every $j \in J$. Since naive v-covers are preserved by open base change, we reduce to the case X is an affinoid.

Moreover, we know that $X^\diamondsuit$ is a small v-sheaf, so there is a v-surjection $f\colon S \to X^\diamondsuit$ from a perfectoid space S (by the proof of [Sch17, Proposition 15.4], S can be chosen to be affinoid). By definition, the map f corresponds to a map $g\colon S^\sharp \to X$. Since diamondization commutes with finite fiber products by Proposition C.1.6, it is enough to show that $\{(X_i \times_X S^\sharp)^\diamondsuit \to (S^\sharp)^\diamondsuit\}_{i\in I}$ is a v-covering. In other words, we can assume that $X = S^\sharp$ is an affinoid perfectoid space.

Now we can find a covering $\{U_{i,j} \to X_i\}_{j\in J_i}$ by open affinoids for each $i \in I$. Then the family $\{U_{i,j} \to X\}_{i\in I, j\in J_i}$ is also a naive v-covering, and so it suffices to show that $\{U_{i,j}^\diamondsuit \to X^\diamondsuit\}_{i\in I, j\in J_i}$ is a v-covering. In other words, we can assume that X is an affinoid perfectoid and that X_i are all affinoids. A similar argument allows us to assume that X_i are affinoid perfectoid.

Finally, we note that under our assumption that X and X_i are (affinoid) perfectoids, $\{X_i \to X\}_{i\in I}$ is a naive v-covering if and only if $\{X_i^\diamondsuit \to X^\diamondsuit\}_{i\in I}$ is a v-covering since $|X_i^\diamondsuit| \simeq |X_i|$ and $|X^\diamondsuit| = |X|$ by [Sch17, Lemma 15.6]. $\square$

C.2. **The Quasi-proétale Topology.** The main goal of this section is remind the reader the main notions of a quasi-proétale topology. This topology will be play an important intermediate role in relating the v-topology to the étale topology.

In order to recall the definition of a quasi-proétale topology, we need to recall some definitions from [Sch17].

Definition C.2.1. A perfectoid space X is *totally disconnected* if X is quasi-compact, quasi-separated, and every open cover of X splits.

A perfectoid space X is *strictly totally disconnected* if X is quasi-compact, quasi-separated, and every étale cover of X splits.

Lemma C.2.2. Let X be a totally disconnected perfectoid space, and $Y \to X$ a quasi-compact separated étale morphism of perfectoid spaces. Then Y is totally disconnected.

Proof. By [Sch17, Lemma 7.2], it suffices to show that every connected component $T \subset X$ has a unique closed point. Clearly $f(T)$ is connected, so it lies in a connected component of X that is isomorphic to $\mathrm{Spa}\,(K, K^+)$ for some perfectoid field pair (K, K^+) due to [Sch17, Lemma 7.3]. So it is enough to show that $Y \times_X \mathrm{Spa}\,(K, K^+)$ is totally disconnected for every connected component $\mathrm{Spa}\,(K, K^+) \subset X$. In other words, we can assume that X is an adic spectrum of a perfectoid field pair.

Now [Sch17, Lemma 9.9] implies that Y is a quasi-compact open in a finite étale morphism $\overline{Y} \to X$. Since $\overline{Y}$ is finite étale over X, it is of the form

$$\bigsqcup_{i=1}^{n} \mathrm{Spa}\,(K_i, K_i^+) \to \mathrm{Spa}\,(K, K^+)$$

where $K \subset K_i$ is a finite separable extension of perfectoid fields, and K_i^+ is an integral closure of K^+ in K_i. Therefore, [Ked18] ensures that (K_i, K_i^+) is a perfectoid field pair (i.e. $K_i^+ \subset K_i$ is an open and bounded *valuation ring* in K_i).

Now any open adic subspace of $\mathrm{Spa}\,(K_i, K_i^+)$ is of the form $\mathrm{Spa}\,(K_i, K_i'^+)$ for some other open and bounded valuation ring $K_i'^+ \subset K_i$. Theefore, Y is of the form

$$\bigsqcup_{i=1}^{n} \mathrm{Spa}\,(K_i, K_i'^+)$$

that is a totally disconnected perfectoid space. □

Lemma C.2.3. Let X be a totally disconnected perfectoid space such that every finite étale cover of X splits. Then X is strictly totally disconnected.

Proof. By [Sch17, Proposition 7.16], it suffices to show that, for every point $x \in X$, the completed residue field $K(x) = \widehat{k(x)}$ is algebraically closed. Since completed residue fields do not change under specialization, we can assume that x is the unique closed point in its connected component.

Now suppose that $K(X)$ is an algebraically closed, so there is a finite separable extension $K(x) \subset L$ defining an finite étale morphism of perfectoid spaces $\mathrm{Spa}\,(L, L^+) \to \mathrm{Spa}\,(K(x), K(x)^+)$. Since $\mathrm{Spa}\,(K(x), K(x)^+) \subset X$ is a connected component (see [Sch17, Lemma 7.3]), it is an intersection of clopen subset containing x. Therefore, [Sch17, Proposition 6.4(i)] implies that there is a clopen subset $U \subset X$ containg x and a finite étale morphism

$$V \to U$$

such that its pullback on $\mathrm{Spa}\,(K(x), K(x)^+)$ coincides with $\mathrm{Spa}\,(L, L^+) \to \mathrm{Spa}\,(K(x), K(x)^+)$. But then

$$V \sqcup (X \setminus U) \to X$$

is a finite étale cover of X that does not split. Contradiction with our assumption on X. □

For the next definition, we assume that $f\colon X = \mathrm{Spa}\,(S, S^+) \to Y = \mathrm{Spa}\,(R, R^+)$ is a morphism of adic spaces such that each X and Y is either an affinoid perfectoid or a strongly noetherian Tate affinoids.

Definition C.2.4. [Sch17, Definition 7.8] A morphism $f\colon \operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$ is an *affinoid pro-étale morphism* if there is a cofiltered system of étale morphisms $\operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$ such that each (R_i, R_i^+) is either a strongly noetherian Huber pair or a perfectoid pair, and S^+ is the ϖ-adic completion of $\operatorname{colim}_I R_i^+$ (for some compatible choice of pseudo-uniformizers), and $S = S^+[\frac{1}{\varpi}]$.

A morphism $f\colon \operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$ is *pro-(finite étale)* if it is affinoid pro-étale and each $\operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$ can be chosen to be finite étale.

An morphism $f\colon \operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$ is *pro-(open)* if it is affinoid pro-étale and each $\operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$ can be chosen to be a disjoint union of rational subdomains.

For the next definition, we assume that $f\colon X \to Y$ is a morphism of adic spaces such that each X and Y is either perfectoid or locally noetherian.

Definition C.2.5. [Sch17, Definition 7.8] A morphism of adic spaces $f\colon X \to Y$ is *pro-étale* if, for every point $x \in X$, there is an open affinoid $x \in U \subset X$ and an open affinoid $f(x) \in V \subset Y$ such that $f|_U\colon U \to V$ is affinoid pro-étale.

Lemma C.2.6. Let X be a strictly totally disconnected perfectoid space, and $Y \to X$ be an affinoid pro-étale morphism. Then Y is strictly totally disconnected.

Proof. This follows directly from [Sch17, Lemma 7.19]. □

Now we are ready to define quasi-proétale morphisms.

Definition C.2.7. [Sch17, Definition 10.1 and 14.1] A morphism of small v-sheaves $f\colon X \to Y$ is *quasi-proétale* if it is locally separated, and for every morphism $S \to Y$ with a strictly totally disconnected perfectoid S, the fiber product $X_S \coloneqq X \times_Y S$ is represented by a perfectoid space and $X_S \to S$ is pro-étale.

The *quasi-proétale site* $X_{\text{qproét}}$ of a small v-sheaf is the site whose objects are quasi-proétale morphisms $Y \to X$, with coverings given by families $\{Y_i \to Y\}_{i \in I}$ such that $\sqcup_{i\in I} Y_i \to Y$ is a surjection of v-sheaves.

Lemma C.2.8. Let $f\colon X \to Y$ be a pro-étale morphism with X and Y being either a space or a locally noetherian. Then $f^\diamondsuit\colon X^\diamondsuit \to Y^\diamondsuit$ is quasi-proétale. Furthermore, if f is a naive v-covering, then $f^\diamondsuit$ is a v-covering.

Proof. It is easy to see that a morphism of affinoids $X \to Y$ induces a separated morphism of diamonds $f^\diamondsuit\colon X^\diamondsuit \to Y^\diamondsuit$ (for example, it is quasi-separated by Lemma C.1.12 and then the valuative criterion of [Sch17, Proposition 10.9] is easy to verify). Then, for the purpose of proving that $f^\diamondsuit$ is quasi-proétale, it suffices to show $f^\diamondsuit$ pro-étale after any base $S \to Y^\diamondsuit$ with a strictly totally disconnected perfectoid S. By definition, an S-point of $Y^\diamondsuit$ corresponds to a morphism $S^\sharp \to Y$ and Proposition C.1.6 **(6)** implies that

$$S \times_{Y^\diamondsuit} X^\diamondsuit \simeq (S^\sharp \times_Y X)^\diamondsuit.$$

Since pro-étale morphisms are stable under base change, we can assume that Y is a strictly totally disconnected perfectoid space. Proposition C.1.6 **(3)** ensures that we can prove the claim locally on X and Y, so we may assume that f is affinoid perfectoid. Then we can write $X = \operatorname{Spa}(S, S^+)$ as a tilde-limit of étale morphisms

$$X \sim \lim_I X_i = \operatorname{Spa}(R_i, R_i^+) \to Y$$

with each X_i an affinoid perfectoid space. Now Proposition C.1.6 **(1)** and Proposition C.1.6 **(5)** imply that

$$f^{\diamondsuit}\colon X^{\diamondsuit} = \lim_{I} X_i^{\flat} \to Y^{\flat}.$$

is a pro-étale morphism (see [Sch17, Proposition 6.5] to ensure that limit is computed via the formula in Definition C.2.4). If f is a naive v-covering, $f^{\diamondsuit}$ is a v-covering by Lemma C.1.13. □

Lemma C.2.9. (1) Let $X = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then there is a pro-(finite étale) surjective morphism $Y \to X$ with an affinoid perfectoid Y.

(2) Let $X = \mathrm{Spa}\,(A, A^+)$ be an affinoid perfectoid space. Then there is a pro-(open) surjective morphism $Y \to X$ with a totally disconnected perfectoid Y.

(3) Let $X = \mathrm{Spa}\,(A, A^+)$ be a totally disconnected perfectoid space. Then there is a pro-(finite étale) surjective morphism $Y \to X$ with a strictly totally disconnected perfectoid Y.

Proof. (**1**) : Since X is a strongly noetherian Tate affinoid, it has finite number of connected components (because A has finite number of non-trivial idempotents). Therefore, we can assume that X is connected. Then the claim follows from Lemma B.2.6.

(**2**) : This follows directly from [Sch17, Proposition 7.12].

(**3**) : Fix a family of all finite étale surjective morphisms $\{X_i \to X\}_{i\in I}$. This diagram is not cofiltered, but we are going to make another cofiltered diagram out of it. For each finite subset $J \subset I$, we define $X_J := \prod_{j\in J} X_j$. Then each $X_J \to X$ is still finite surjective, and the evident transition maps $X_J \to X_{J'}$ for $J' \subset J$ are still finite étale. In particular, $\{X_J \to X\}_{J\subset I,\text{finite}}$ is a cofiltered family, so we can define the limit (in the category of perfectoid spaces) affinoid perfectoid space

$$X_\infty = \lim_{J} X_J \to X.$$

It is pro-(finite étale) over X, and every finite étale cover of X splits in X_∞ (because it is an element of the limit). Note that each X_J is totally disconnected by Lemma C.2.2. So any open covering of X_J splits, then any open covering of X_∞ splits by [Sch17, Proposition 6.4(0)]. Therefore, X_∞ is totally disconnected.

Now we define X^i_∞ iteratively as $X^1_\infty = X_\infty$ and $X^{n+1}_\infty = (X^n_\infty)_\infty$. Finally, we define

$$X^\infty_\infty = \lim_n X^n_\infty.$$

The same approximation argument as above shows that X^∞_∞ is totally disconnected. Furthermore, [Sch17, Proposition 6.4] implies that every finite étale covering of X^∞_∞ is defined over some X^n_∞, and thus splits over X^{n+1}_∞. Thus any finite étale covering of X^∞_∞ splits. So X^∞_∞ is a strictly totally disconnected by Lemma C.2.3. It is clear that $X^\infty_\infty \to X$ is surjective morphism of affinoids, so a naive v-covering. The only thing we are left to show is that $X^\infty_\infty \to X$ is pro-(finite étale). This follows from the fact that pro-(finite étale) covers of affinoid perfectoids are preserved by cofiltered limits. This, in turn, can be deduced from [Sch17, Proposition 6.4(i)] via a standard spreading out argument, we leave details to the reader. □

Corollary C.2.10. Let $X = \mathrm{Spa}\,(A, A^+)$ be a strongly noetherian Tate affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then there is a morphism $\mathrm{Spa}\,(A_\infty, A^+_\infty) \to \mathrm{Spa}\,(A, A^+)$ such that

(1) $\mathrm{Spd}\,(A_\infty, A^+_\infty) \to \mathrm{Spd}\,(A, A^+)$ is a quasi-proétale covering;

(2) the fiber products $\mathrm{Spa}\,(A_\infty, A^+_\infty)^{j/\,\mathrm{Spa}\,(A,A^+)}$ are strictly totally disconnected (affinoid) perfectoids for $j \geq 1$.

Proof. Take $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$ to be the composition of three covering from Lemma C.2.9. Then

$$\operatorname{Spd}(A_\infty, A_\infty^+) = \operatorname{Spa}(A_\infty^\flat, A_\infty^{\flat,+}) \to \operatorname{Spd}(A, A^+)$$

is a quasi-proétale covering by Lemma C.2.8. Now the claim about higher products follows from Lemma C.2.6 since each

$$\operatorname{Spa}(A_\infty, A_\infty^+)^{j/\operatorname{Spa}(A,A^+)} \to \operatorname{Spa}(A_\infty, A_\infty^+)$$

is a composition of affinoid proétale morphisms. □

C.3. **Structure Sheaves.** The main goal of this section is to define various structure sheaves on (a diamond of) a pre-adic spaces over $\mathbf{Q}_p$, and discuss a precise relation between them.

Firstly, we note that for any pre-adic space X over $\mathbf{Q}_p$, its étale, quasi-proétale, and v-sites are related by a sequence of morphisms of sites:

$$X_v^\diamondsuit \xrightarrow{\lambda} X_{\text{qproét}}^\diamondsuit \xrightarrow{\mu} X_{\text{ét}}. \tag{C.1}$$

Now we define different structure sheaves on each of these sites.

Definition C.3.1. Let X be a pre-adic space over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

An *integral "untilted" structure sheaf* $\mathcal{O}_{X^\diamond}^+$ is a v-sheafification of a pre-sheaf

$$\{S \to X^\diamondsuit\} \mapsto \mathcal{O}_{S^\sharp}^+(S^\sharp)$$

with the evident transition map[42].

An *rational "untilted" structure sheaf* $\mathcal{O}_{X^\diamond}$ is $\mathcal{O}_{X^\diamond}^+[\frac{1}{p}]$.

A *mod-p structure sheaf* $\mathcal{O}_{X^\diamond}^+/p$ is the quotient of $\mathcal{O}_{X^\diamond}^+$ by p in the v-topology on $X^\diamondsuit$.

A *quasi-proétale integral "untilted" structure sheaf* $\mathcal{O}_{X_{\text{qp}}^\diamond}^+$ is the restriction of $\mathcal{O}_{X^\diamond}^+$ on the quasi-proétale site of $X^\diamondsuit$, i.e. $\mathcal{O}_{X_{\text{qp}}^\diamond}^+ = \lambda_* \mathcal{O}_{X^\diamond}^+$.

A *quasi-proétale mod-p structure sheaf* $\mathcal{O}_{X^\diamond}^+/p$ is the quotient of $\mathcal{O}_{X^\diamond}^+$ by p in the quasi-proétale topology on $X^\diamondsuit$.

An *étale mod-p structure sheaf* $\mathcal{O}_{X_{\text{ét}}}^+/p$ is the quotient of $\mathcal{O}_{X_{\text{ét}}}^+$ by p in the étale topology on X, where $\mathcal{O}_{X_{\text{ét}}}^+$ is the usual integral structure sheaf on $X_{\text{ét}}$.

Remark C.3.2. Note that it is, a priori, not clear if $\mathcal{O}_{X_{\text{qp}}^\diamond}^+/p \simeq \lambda_*\left(\mathcal{O}_{X^\diamond}^+/p\right)$. The issues is that we former is defined via taking the quotient by p in the quasi-proétale topology, and the latter in the v-topology. However, we will show later that they always coincide.

Remark C.3.3. The relation between $\mathcal{O}_{X_{\text{qp}}^\diamond}^+/p$ and $\mathcal{O}_{X_{\text{ét}}}^+/p$ is even more mysterious. The first is roughly defined via descent from perfectoid spaces. While the other is defined using the étale topology of $X_{\text{ét}}$, so if X is a noetherian adic space, it does not have any direct relation with perfectoid spaces.

Essentially by definition, these structure sheaves promote Diagram (C.1) to a diagram of morphisms of ringed sites:

$$\left(X_v^\diamondsuit, \mathcal{O}_{X^\diamond}^+/p\right) \xrightarrow{\lambda} \left(X_{\text{qproét}}^\diamondsuit, \mathcal{O}_{X_{\text{qp}}^\diamond}^+/p\right) \xrightarrow{\mu} \left(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}^+/p\right). \tag{C.2}$$

[42]Recall that a morphism $S \to X^\diamondsuit$ is, by definition, a data of an untilt $S^\sharp$ with a morphism $S^\sharp \to X$ and an isomorphism $(S^\sharp)^\flat \simeq S$. Thus a pair of morphisms $T \to S \to X^\diamondsuit$ defines a pair $T^\sharp \to S^\sharp \to X$

We also have 'tilted' versions of the structure sheaves:

Definition C.3.4. Let X be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

An *integral "tilted" structure sheaf* $\mathcal{O}^{\flat,+}_{X^\diamond}$ is a v-sheafification of a pre-sheaf

$$\{S \to X^\diamondsuit\} \mapsto \mathcal{O}^+_S(S)$$

with the evident transition map.

If X is a pre-adic space over a p-adic perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$ (see Definition B.1.6), a *rational "tilted" structure sheaf* $\mathcal{O}^\flat_{X^\diamond}$ is $\mathcal{O}^{\flat,+}_{X^\diamond}[\frac{1}{\varpi^\flat}]$.

We start with some easy properties of the structure sheaves:

Lemma C.3.5. Let $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then

(1) for any affinoid perfectoid $Y = \mathrm{Spa}\,(S, S^+) \to X^\diamondsuit$, $\mathrm{H}^0(Y, \mathcal{O}^+_{X^\diamond}) = S^{\sharp,+}$, and $\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond}) \simeq^a 0$ for $i \geq 1$;

(2) for any affinoid perfectoid $Y = \mathrm{Spa}\,(S, S^+) \to X^\diamondsuit$, $\mathrm{H}^0(Y, \mathcal{O}^{+,\flat}_{X^\diamond}) = S^+$, and $\mathrm{H}^i(Y, \mathcal{O}^{+,\flat}_{X^\diamond}) \simeq^a 0$ for $i \geq 1$;

(3) the sheaf $\mathcal{O}^+_{X^\diamond}$ is derived p-adically complete and p-torsionfree;

(4) if X is pre-adic space over a perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$, the sheaf $\mathcal{O}^{+,\flat}_{X^\diamond}$ is derived $\varpi^\flat$-adically complete and $\varpi^\flat$-torsionfree;

(5) if X is pre-adic space over a perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$, there is a canonical isomorphism $\mathcal{O}^+_{X^\diamond}/p \simeq \mathcal{O}^{+,\flat}_{X^\diamond}/\varpi^\flat$.

Proof. (**1**) and (**2**) follow directly from [Sch17, Theorem 8.7 and Proposition 8.8].

(**3**): Clearly, in order to show that $\mathcal{O}^+_{X^\diamond}$ is p-torsionfree, it suffices to show that $\mathcal{O}^+_{X^\diamond}(U)$ is p-torsionfree on a basis of $X^\diamondsuit_v$. Therefore, it is enough to show that

$$\mathcal{O}^+_{X^\diamond}(Y)$$

is p-torsionfree for any affinoid perfectoid $Y \to X^\diamondsuit$. Thif follows from (**1**).

Lemma A.8 ensures that, for the purpose of proving that $\mathcal{O}^+_{X^\diamond}$ is p-adically derived complete, it suffices to show that

$$\mathbf{R}\Gamma(S, \mathcal{O}^+_{X^\diamond})$$

is derived p-adically complete for any affinoid perfectoid $Y = \mathrm{Spa}\,(S, S^+) \to X$. Then it suffices to show that each cohomology group $\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond})$ is derived p-adically complete. Now (**1**) implies that

$$\mathrm{H}^0(Y, \mathcal{O}^+_{X^\diamond}) = S^{\sharp,+}$$

is p-adically complete, and thus it is derived p-adically complete (see [Sta21, Tag 091R]). Moreover, (**1**) implies that all higher cohomology groups

$$\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond}) \simeq^a 0$$

are almost zero. In particular, they are p-torsion, and so derived p-adically complete. Thus, $\mathbf{R}\Gamma(S, \mathcal{O}^+_{X^\diamond})$ is derived p-adically complete finishing the proof.

(**4**) : This is completely analogous to the proof of (**3**) using (**2**) in place of (**1**).

(**5**) : Denote by $\mathcal{F}$ the *presheaf* quotient of $\mathcal{O}^+_{X^\diamond}$ by p, and by $\mathcal{G}$ the presheaf quotient of $\mathcal{O}^{\flat,+}_{X^\diamond}$. It suffices to construct a functorial isomorphism

$$\mathcal{F}(U) \simeq \mathcal{G}(U)$$

on a basis of $X_v^\diamondsuit$. Therefore, it suffices to construct such an isomorphism for any affinoid perfectoid U. Then (**1**) and (**2**) ensure that, for an affinoid perfectoid $U = \mathrm{Spa}\,(S, S^+) \to X^\diamondsuit$,

$$\mathcal{F}(U) \simeq S^{\sharp,+}/pS^{\sharp,+}, \text{ and } \mathcal{G}(U) \simeq S^+/\varpi^\flat S^+.$$

Essentially by the definition of a tilt, we have a canonical isomorphism

$$S^{\sharp,+}/pS^{\sharp,+} \simeq S^+/\varpi^\flat S^+$$

finishing the proof. □

Our next goal is to show a precise relation between $\mathcal{O}^+_{X^\diamondsuit}/p$, $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$, and $\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p$. If one is willing to work in the almost world, this relation is quite easy (and essentially boils down to Lemma C.3.5). However, for the purpose of understanding the relation between $\mathcal{O}^+/p$-vector bundles in different topologies, it is essential to eliminate any almost mathematics in this relation.

Lemma C.3.6. Let $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ be a pre-adic space over $\mathbf{Q}_p$. Then the natural morphism

$$\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p \to \lambda_*\left(\mathcal{O}^+_{X^\diamondsuit}/p\right),$$

is an isomorphism. If X is a perfectoid space or a locally noetherian space over $\mathbf{Q}_p$, then the natural morphisms

$$\mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p\right) \to \mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p,$$

$$\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p \to \mathbf{R}\mu_*\left(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\right)$$

are isomorphisms as well.

Proof. The first result is [MW20, Proposition 2.13]. For the second result, we note that [MW20, Lemma 2.7] ensures that, for a perfectoid or locally noetherian X, $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}$[43] is isomorphic to the sheaf

$$\widehat{\mathcal{O}}^+_{X_{\mathrm{qp}}} := \lim_n \mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p^n\right).$$

Now note the quasi-proétale site of a diamond is replete (in the sense of [BS15, Definition 3.1.1]) due to [MW20, Lemma 1.2]. Therefore, the fact that $\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}$ is p-torsionfree and [BS15, Proposition 3.1.10] imply that

$$\widehat{\mathcal{O}}^+_{X_{\mathrm{qp}}} \simeq \mathbf{R}\lim \mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p^n\right) \simeq \widehat{\mu^{-1}(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}})}$$

is the derived p-adic completion of $\mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}\right)$. Since $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}$ is also p-torsionfree by Lemma C.3.5, the universal property of derived completion implies that

$$\begin{aligned}\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p &\simeq \left[\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\right] \\ &\simeq [\widehat{\mu^{-1}(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}})}/p] \\ &\simeq \mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p\right).\end{aligned}$$

Finally, [Sch17, Proposition 14.8 and Lemma 15.6] imply that

$$\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p \simeq \mathbf{R}\mu_*\mu^{-1}\left(\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}/p\right) \simeq \mathbf{R}\mu_*\left(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\right).$$

□

[43] This is denoted by $\widehat{\mathcal{O}}^+_{X^\diamondsuit}$ in [MW20].

Our next goal is to compare $\mathbf{R}\lambda_* \left(\mathcal{O}^+_{X^\diamond}/p\right)$ with $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$. In order to do this, we need a number of preliminary results.

Remark C.3.7. The proof of the isomorphism $\mu^{-1}\left(\mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$ in Lemma C.3.6 essentially uses only [MW20, Lemma 2.7] that was established only in the perfectoid or locally noetherian situation. One can check that the proof carries over to the situation of an adic space strongly sheafy in the sense of [HK21, Definition 4.1]. In particular, Lemma C.3.6 stays correct for a "smoothoid" X in the sense of [Heu]. It is possible that the result stays correct for all pre-adic spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Lemma C.3.8. Let $\{X_i = \mathrm{Spa}\,(S_i, S_i^+)\}_{i\in I}$ be a cofiltered system of affinoid perfectoid spaces over $(\mathbf{Q}_p, \mathbf{Z}_p)$, and let $X_\infty = \mathrm{Spa}\,(S_\infty, S_\infty^+)$, where S_∞^+ is the p-adic completion of $\mathrm{colim}_I\, S_i^+$ and $S_\infty = S_\infty^+[\frac{1}{p}]$. Then the natural morphism

$$\mathrm{colim}_I\, f_i^{-1}\mathcal{O}^+_{X_{i,\text{ét}}}/p \to \mathcal{O}^+_{X_{\infty,\text{ét}}}/p$$

is an isomorphism, where $f_i\colon X_\infty \to X_i$ are the obvious morphisms.

Proof. Note that [Sch17, Proposition 6.5] implies that $X_\infty = \lim_I X_i$ in the category of perfectoid spaces.

Now we observe that an affinoid perfectoid site $X_{i,\text{étaff}}$ induce the same étale topos as the full étale site $X_{i,\text{ét}}$. Therefore, it suffices to prove the claim on the affinoid étale site. Moreover, it suffices to show the claim on the presheaf level. Namely, let $\mathcal{F}_i$ be the presheaf quotient of $\mathcal{O}_{X_{i,\text{ét}}}$ by p for $i \in I$ or $i = \infty$. Then it suffices to show that

$$\mathrm{colim}_I(f_i^{-1}\mathcal{F}_i(U)) \to \mathcal{F}_\infty(U)$$

is an isomorphism[44] for any $U \in X_{\infty,\text{étaff}}$.

Pick any étale morphism $U_\infty \to X_\infty$ with an affinoid perfectoid U_∞. Then [Sch17, Proposition 6.4(iv)] implies that, for some $i_0 \in I$, there is an affinoid perfectoid U_{i_0} with an étale morphism $U_{i_0} \to X_{i_0}$ such that

$$U_{i_0} \times_{X_{i_0}} X_\infty \simeq U_\infty.$$

For any $j \geq i_0$ or $j = \infty$, define $U_j := U_{i_0} \times_{X_{i_0}} X_j$. Since fiber products commute with limits, we get that

$$U_\infty = \lim_I U_i$$

in the category of perfectoid spaces. Now an easy application of [Sch17, Proposition 6.4(iv)] ensures

$$\mathrm{colim}_I(f_i^{-1}\mathcal{F}_i(U)) = \mathrm{colim}_I\, \mathcal{O}^+_{U_i}(U_i)/p.$$

Thus it suffices to show that the natural morphism

$$\mathrm{colim}_I\, \mathcal{O}^+_{U_i}(U_i)/p \to \mathcal{O}^+_{U_\infty}(U_\infty)/p$$

is an isomorphism. The fact that $U_\infty = \lim_I U_i$ and [Sch17, Proposition 6.5] ensure that $\mathcal{O}^+_{U_\infty}(U_\infty)$ is the p-adic completion of $\mathrm{colim}_I\, \mathcal{O}^+_{U_i}(U_i)$. Therefore, the natural morphism

$$\mathrm{colim}_I\, \mathcal{O}^+_{U_i}(U_i)/p \to \mathcal{O}^+_{U_\infty}(U_\infty)/p$$

is clearly an isomorphism finishing the proof. $\square$

[44] Here f_i^{-1} is understood to be a presheaf pullback

Corollary C.3.9. Let X be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, and $X_\infty = \lim_I X_i$ is a cofiltered limit of characteristic p affinoid perfectoid spaces over $X^\diamondsuit$. Then the natural morphism

$$\mathrm{colim}_I\, \mathrm{H}^i\left(X_{i,v}, \mathcal{O}^+_{X^\diamondsuit}/p\right) \to \mathrm{H}^i\left(X_{\infty,v}, \mathcal{O}^+_{X^\diamondsuit}/p\right)$$

is an isomorphism for every $i \geq 0$.

Proof. A morphism $X_i \to X^\diamondsuit$ defines an untilt $X_i^\sharp \to X$ of X_i over X. So we may replace X with $X_i^\sharp$ for some $i \in I$ to assume that X is an affinoid perfectoid space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

In this case, we write $X_i = \mathrm{Spa}\,(R_i, R_i^+)$ and $X_\infty = \mathrm{Spa}\,(R_\infty, R_\infty^+)$ and denote their untilts corresponding to a morphism to $X^\diamondsuit$ by

$$X_i^\sharp = \mathrm{Spa}\,(R_i^\sharp, R_i^{\sharp,+}) = \mathrm{Spa}\,(S_i, S_i^+),\ X_\infty^\sharp = \mathrm{Spa}\,(R_i^\sharp, R_i^{\sharp,+}) = \mathrm{Spa}\,(S_\infty, S_\infty^+).$$

Now [Sch17, Corollary 3.20] ensures that

$$\mathrm{Spa}\,(S_\infty, S_\infty^+) \simeq \lim_I \mathrm{Spa}\,(S_i, S_i^+)$$

in the category of perfectoid spaces. In particular,

$$S_\infty^+ \simeq \widehat{\mathrm{colim}_I\, S_i^+}$$

is the p-adic completion of $\mathrm{colim}_I\, S_i^+$. Lemma C.3.6 implies that it suffices to show that the natural morphism

$$\mathrm{colim}_I\, \mathrm{H}^0(X^\sharp_{i,\text{ét}}, \mathcal{O}_{X^\sharp_{i,\text{ét}}}/p) \to \mathrm{H}^0(X^\sharp_{\infty,\text{ét}}, \mathcal{O}_{X^\sharp_{\infty,\text{ét}}}/p)$$

is an isomorphism. Now the result is a formal consequence of Lemma C.3.8 and [Sch17, Proposition 6.4] (for example, argue as in [Fu11, Proposition 5.9.2]). □

Lemma C.3.10. Let Y be a strictly totally disconnected perfectoid space, and $Z \to Y$ a v-cover by an affinoid perfectoid space. Then there is a presentation $Z = \lim_I Z_i \to Y$ as cofiltered limit of affinoid perfectoid spaces over Y such that each $Z_i \to Y$ admits a section.

Proof. The proof of [MW20, Lemma 2.11] carries over in this case if one replace a reference to [Sch17, Lemma 9.5] with [Heu21, Lemma 2.23]. □

Corollary C.3.11. Let $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphism

$$\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p \to \mathbf{R}\lambda_*\left(\mathcal{O}^+_{X^\diamondsuit}/p\right),$$

is an isomorphism.

Proof. Lemma C.3.6 ensures that $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p \to \lambda_*\left(\mathcal{O}^+_{X^\diamondsuit}/p\right)$ is an isomorphism. Thus, it suffices to show that

$$\mathrm{R}^j\lambda_*\left(\mathcal{O}^+_{X^\diamondsuit}/p\right) \simeq 0$$

for $j \geq 1$. Since strictly totally disconnected spaces form a basis for the quasi-proétale topology of any diamond, it suffices to show that

$$\mathrm{H}^j(Y, \mathcal{O}^+_{X^\diamondsuit}/p) = 0$$

for a totally strictly disconnected perfectoid $Y \to X$ and $j \geq 1$. Pick a class $x \in \mathrm{H}^j(Y, \mathcal{O}^+_{X^\diamondsuit}/p)$, it is killed by some v-covering $Z \to Y$ by an affinoid perfectoid space Z. Now Lemma C.3.10 implies that $Z = \lim_I Z_i \to Y$ is cofiltered limit of affinoid perfectoid spaces over Y such that each $Z_i \to X$ admits a section. Then Corollary C.3.9 implies that

$$\mathrm{H}^j(Z, \mathcal{O}^+_{X^\diamondsuit}/p) \simeq \mathrm{colim}_I\, \mathrm{H}^j(Z_i, \mathcal{O}^+_{X^\diamondsuit}/p).$$

Therefore, the class $x \in \mathrm{H}^j(Y, \mathcal{O}^+_{X^\diamond}/p)$ is killed under a morphism $\mathrm{H}^j(Y, \mathcal{O}^+_{X^\diamond}/p) \to \mathrm{H}^j(Z_i, \mathcal{O}^+_{X^\diamond}/p)$ for some $i \in I$. More presicely, $\pi_i^*(x) = 0 \in \mathrm{H}^j(Z_i, \mathcal{O}^+_{X^\diamond}/p)$ for the structure morphism $\pi_i \colon Z_i \to Y$. Now we use a section $s_i \colon Y \to Z_i$ to see that

$$x = s_i^*(\pi_i^*(x)) = 0 \in \mathrm{H}^j(Y, \mathcal{O}^+_{X^\diamond}/p).$$

□

Corollary C.3.12. Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the natural morphisms

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \to \mathbf{R}\Gamma(X^\diamondsuit_{\text{qproét}}, \mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}/p) \to \mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p)$$

are isomorphisms.

Proof. It follows directly from Lemma C.3.6 and Corollary C.3.11. □

Corollary C.3.13. Let $X = \mathrm{Spa}\,(R, R^+)$ be a strictly totally disconnected perfectoid space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then $\mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \simeq 0$ for $i \geq 1$, and $\mathrm{H}^0(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit_v}/p) \simeq R^+/pR^+$.

Remark C.3.14. We emphasize that we have an actual vanishing of higher cohomology groups as opposed to almost vanishing (that can be deduced from Lemma C.3.5).

Proof. By Corollary C.3.12, we know that

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \simeq \mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p).$$

But X is a strictly totally disconnected space, so any étale sheaf has no higher cohomology groups. This implies that $\mathrm{H}^i(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \simeq 0$ for $i \geq 1$, and

$$\mathrm{H}^0(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \simeq \mathrm{H}^0(X, \mathcal{O}^+_{X_{\text{ét}}})/p \simeq R^+/pR^+.$$

□

Corollary C.3.15. Let K be a p-adic non-archimedean field, $K^+ \subset K$ an open and bounded valuation subring, and X a locally noetherian adic space over $\mathrm{Spa}\,(K, K^+)$, and $X^\circ \coloneqq X \times_{\mathrm{Spa}\,(K,K^+)} \mathrm{Spa}\,(K, \mathcal{O}_K)$. Then the natural morphism

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \otimes_{K^+/p} \mathcal{O}_K/p \to \mathbf{R}\Gamma(X^{\circ,\diamondsuit}_v, \mathcal{O}^+_{X^{\circ,\diamond}}/p)$$

is an isomorphism. In particular, if (K, K^+) is a perfectoid field pair, then the natural morphism

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{O}^+_{X^\diamond}/p) \to \mathbf{R}\Gamma(X^{\circ,\diamondsuit}_v, \mathcal{O}^+_{X^{\circ,\diamond}}/p)$$

is an almost isomorphism.

Proof. The proof is local on X, so we can assume that $X = \mathrm{Spa}\,(A, A^+)$ is affinoid. Then we can find a morphism $\mathrm{Spd}\,(A_\infty, A^+_\infty) \to \mathrm{Spd}\,(A, A^+)$ such that all fiber products

$$\mathrm{Spa}\,(A_\infty, A^+_\infty)^{j/\,\mathrm{Spa}\,(A,A^+)} = \mathrm{Spa}\,(B_j, B^+_j)$$

are strictly totally disconnected (affinoid) perfectoid spaces for $j \geq 1$. Thus Corollary C.3.13 implies that

$$\mathrm{H}^i\left(\mathrm{Spd}\,(B_j, B^+_j)_v, \mathcal{O}^+_{X^\diamond}/p\right) \simeq 0$$

for $i, j \geq 1$, and

$$\mathrm{H}^0\left(\mathrm{Spd}\,(B_j, B^+_j)_v, \mathcal{O}^+_{X^\diamond}/p\right) \simeq B^+_j/pB^+_j$$

for $j \geq 1$. Therefore, one can compute $\mathrm{H}^j(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamondsuit}/p)$ via the Čech cohomology groups of the covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$. Thus, one gets an isomorphism

$$\mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamondsuit}/p) \simeq \mathrm{H}^i(B_1^+/p \to B_2^+/p \to \dots).$$

Now the morphism $\mathrm{Spa}\,(K, \mathcal{O}_K) \to \mathrm{Spa}\,(K, K^+)$ is an pro-open immersion, so the fiber products

$$\mathrm{Spa}\,(B_j, B_j^+) \times_{\mathrm{Spa}\,(K,K^+)} \mathrm{Spa}\,(K, \mathcal{O}_K)$$

are strictly totally disconnected affinoid perfectoids represented by[45]

$$\mathrm{Spa}\,(B_j, B_j \widehat{\otimes}_{K^+} \mathcal{O}_K).$$

In particular, the same argument as above implies that $\mathcal{O}^+/p$ cohomology of $X^{\circ,\diamondsuit}$ can be computed as follows:

$$\mathrm{H}^i(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^\diamondsuit}/p) \simeq \mathrm{H}^i(B_1^+/p \otimes_{K^+/p} \mathcal{O}_K/p \to B_2^+/p \otimes_{K^+/p} \mathcal{O}_K/p \to \dots).$$

Finally, the isomorphism

$$\mathbf{R}\Gamma(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamondsuit}/p) \otimes_{K^+/p} \mathcal{O}_K/p \to \mathbf{R}\Gamma(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^{\circ,\diamondsuit}}/p)$$

follows from the flatness of the morphism $K^+ \to \mathcal{O}_K$ since $\mathcal{O}_K$ is an algebraic localization of K^+ by [Mat80, Theorem 10.1]. If K is perfectoid, the almost isomorphism

$$\mathbf{R}\Gamma(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamondsuit}/p) \to \mathbf{R}\Gamma(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^{\circ,\diamondsuit}}/p)$$

now follows from Lemma B.1.7. □

C.4. **Vector Bundles in Different Topologies.** The main goal of this section is to show that the categories of v, quasi-proétale, and étale $\mathcal{O}^+/p$ vector bundles are all equivalent.

The results of this section are mostly due to B. Heuer. The author learnt Theorems C.4.5 and C.4.8 from him. A version of these results is going to appear in [Heu]. We present a slightly different argument that avoids considering "smoothoids" and non-abelian cohomology. We heartfully thank B. Heuer for various discussion around these questions and for allowing the author to present a variation of his ideas in this section.

For the next definition, we fix a pre-adic space X over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Definition C.4.1. An $\mathcal{O}^+_{X^\diamond}/p$-module (in the v-topology on $X^\diamondsuit$) $\mathcal{E}$ is a *$\mathcal{O}^+_{X^\diamond}/p$-vector bundle* if, v-locally on $X^\diamondsuit$, it is isomorphic to $(\mathcal{O}^+_{X^\diamond})^r$ for some integer r. We denote *the category of $\mathcal{O}^+_{X^\diamond}/p$-vector bundles* by Vect^v_X.

An $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-module (in the quasi-proétale topology on $X^\diamondsuit$) $\mathcal{E}$ is a *$\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle* if, quasi-proétale locally on $X^\diamondsuit$, it is isomorphic to $(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}})^r$ for some integer r. We denote *the category of $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundles* by $\mathrm{Vect}^{\mathrm{qp}}_X$.

An $\mathcal{O}^+_{X_{\text{ét}}}/p$-module (in the étale topology on X) $\mathcal{E}$ is a *$\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle* if, étale locally on X, it is isomorphic to $(\mathcal{O}^+_{X_{\text{ét}}})^r$ for some integer r. We denote *the category of $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles* by $\mathrm{Vect}^{\text{ét}}_X$.

[45]For example, the proof of Lemma 6.4.6 goes through without any changes as $\mathcal{O}_K$ is an algebraic localization of K^+.

Remark C.4.2. Note that $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundles are "big sheaves", i.e. it is defined on the (big) v-site $X^\diamondsuit_v$. In particular, it makes sense to evaluate it on only small v-sheaf $Y \to X^\diamondsuit$ of $X^\diamondsuit$.

But $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$ and $\mathcal{O}^+_{X_{\text{ét}}}/p$ vector bundles are "small sheaves"; they are defined only on a (small) quasi-proétale and étale sites respectively.

The main goal of this section is to show that all these notions of $\mathcal{O}^+/p$-vector bundles are equivalent.

Firstly, we define functors. Lemma C.3.6 implies that $\mu^{-1}\left(\mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$. Therefore, μ^{-1} carries $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles to $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundles. So it defines a functor

$$\mu^* \coloneqq \mu^{-1} \colon \mathrm{Vect}^{\text{ét}}_X \to \mathrm{Vect}^{\mathrm{qp}}_X.$$

Unfortunately, it is not true that $\lambda^{-1}\left(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\right) \simeq \mathcal{O}^+_{X^\diamondsuit_v}/p$ because we the quasi-proétale topology was defined to be "small", and the v-topology was defined to be "big". Therefore, we let λ^* be the "$\mathcal{O}^+/p$-module pullback" functor

$$\lambda^* \colon \mathrm{Vect}^{\mathrm{qp}}_X \to \mathrm{Vect}^v_X.$$

defined by the formula

$$\lambda^*\mathcal{E} \coloneqq \lambda^{-1}\mathcal{E} \otimes_{\lambda^{-1}\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p} \mathcal{O}^+_{X^\diamondsuit}/p.$$

Our goal is to show that both λ^* and μ^* are equivalences. Before we do this, we need some preliminary lemmas:

Lemma C.4.3. Let X be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, $\mathcal{E}$ is a $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle, and $Z \to X$ is a cofiltered limit of affinoid perfectoid spaces over $X^\diamondsuit$. Then the natural morphism

$$\mathrm{colim}_I\, \mathrm{H}^i(Z_i, \mathcal{E}) \to \mathrm{H}^i(Z, \mathcal{E})$$

is an isomorphism for every $i \geq 0$.

Proof. Without loss of generality, we can assume that I has a final object 0. Then, by the sheaf condition and exactness of filtered colimits, it suffices to show the claim v-locally on X_0. Therefore, we may assume that $\mathcal{E}|_Z \simeq (\mathcal{O}^+_{X^\diamondsuit}/p)|^d_Z$ is a trivial vector bundle. Then the claim follows from Corollary C.3.9. $\square$

Lemma C.4.4. Let Y be a strictly totally disconnected perfectoid space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then any $\mathcal{O}^+_{Y^\diamondsuit}/p$-vector bundle $\mathcal{E}$ is trivial.

Proof. By assumption, there is a v-covering by an affinoid perfectoid $Z \to Y^\diamondsuit = Y^\flat$ such that there is an isomorphism

$$f \colon \mathcal{E}|_Z \xrightarrow{\sim} (\mathcal{O}^+_{Y^\diamondsuit}/p)|^d_Z.$$

Lemma C.3.10 implies that $Z = \lim_I Z_i \to Y^\flat$ is cofiltered limit of affinoid perfectoid spaces over $Y^\flat$ such that each $Z_i \to Y^\flat$ admits a section.

Step 1. *Approximate* f. Lemma C.4.3 ensures that we can find $i \in I$ and a morphism

$$f_i \colon \mathcal{E}|_{Z_i} \to (\mathcal{O}^+_{Y^\diamondsuit}/p)|^d_{Z_i}$$

such that $f_i|_Z = f$.

Step 2. *Approximate* f^{-1}. We note that the dual sheaf

$$\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}^+_{X^\diamondsuit}/p}\left(\mathcal{E}, \mathcal{O}^+_{X^\diamondsuit}/p\right)$$

is also an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. So we can apply the same argument as in Step 1 to

$$(f^{-1})^\vee \colon (\mathcal{O}^+_{Y^\diamond}/p)|_Z^d \to \mathcal{E}^\vee|_Z = \mathcal{H}om_{\mathcal{O}^+_{X^\diamond}/p}\left(\mathcal{E}, \mathcal{O}^+_{X^\diamond}/p\right)|_Z$$

to find (after possible enlarging $i \in I$) a morphism

$$g' \colon (\mathcal{O}^+_{Y^\diamond}/p)|_{Z_i}^d \to \mathcal{E}^\vee|_{Z_i}$$

such that $g'|_Z = (f^{-1})^\vee$. By dualizing, we get a morphism

$$g_i \colon \mathcal{E}|_{Z_i} \to (\mathcal{O}^+_{Y^\diamond}/p)|_{Z_i}^d$$

such that $g_i|_Z = f^{-1}$.

Step 3. *Show that* $f_i \circ g_i = \mathrm{Id}$ *and* $g_i \circ f_i = \mathrm{Id}$ *after possibly enlarging* $i \in I$. We show the first claim, the second is proven in the same way (and even easier). We think of $\mathrm{Id}_{\mathcal{E}|_{Z_i}}$ and $f_i \circ g_i$ as sections of the internal Hom sheaf, i.e.

$$\mathrm{Id}_{\mathcal{E}|_{Z_i}}, f_i \circ g_i \in \left(\mathcal{E}nd_{\mathcal{O}^+_{X^\diamond}/p}(\mathcal{E})\right)(Z_i).$$

For brevity we denote $\mathcal{E}nd_{\mathcal{O}^+_{X^\diamond}/p}(\mathcal{E})$ by $\mathcal{E}nd$. Note that $\mathcal{E}nd$ is again an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, and so Lemma C.4.3 ensures that

$$\mathrm{colim}_I\, \mathcal{E}nd(Z_i) = \mathcal{E}(Z).$$

Thus if $f_i \circ g_i$ and Id are equal in the limit, they are equal on some large Z_i.

Step 4. *Finish the proof.* In Steps 1-3, we constructed morphisms

$$f_i \colon \mathcal{E}|_{Z_i} \to (\mathcal{O}^+_{Y^\diamond}/p)|_{Z_i}^d,$$
$$g_i \colon \mathcal{E}|_{Z_i} \to (\mathcal{O}^+_{Y^\diamond}/p)|_{Z_i}^d$$

such that $f_i = g_i^{-1}$. Therefore, $\mathcal{E}$ is already trivial on Z_i. But $Z_i \to Y^\flat$ admits a section by construction, so we can pullback f_i and g_i along this section to trivialize $\mathcal{E}$ on $Y^\flat$. □

Theorem C.4.5. (see also [Heu]) Let X be a pre-adic space over $\mathbf{Q}_p$. Then the functor

$$\lambda^* \colon \mathrm{Vect}_{\mathrm{qp}} \to \mathrm{Vect}_v$$

is an equivalence of categories. Furthermore, for any $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle $\mathcal{E}$, the natural morphism

$$\mathcal{E} \to \mathbf{R}\lambda_*\lambda^*\mathcal{E}$$

is an isomorphism.

Proof. We start the proof by showing that the natural morphism

$$\mathcal{E} \to \mathbf{R}\lambda_*\lambda^*\mathcal{E}$$

is an isomorphism. The claim is quasi-proétale local, so we can assume that $\mathcal{E}$ is a trivial $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle. In this case, the claim follows from Corollary C.3.11.

This already implies full faithfulness of α^*. Indeed, it follows from a sequence of isomorphisms:

$$\mathrm{Hom}_{\mathcal{O}^+_{X^\diamond}/p}(\lambda^*\mathcal{E}_1, \lambda^*\mathcal{E}_2) \simeq \mathrm{Hom}_{\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p}(\mathcal{E}_1, \lambda_*\lambda^*\mathcal{E}_2)$$
$$\simeq \mathrm{Hom}_{\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p}(\mathcal{E}_1, \mathcal{E}_2).$$

To show that λ^* is essentially surjective, it is enough to show that, for an $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle $\mathcal{E}$, $\lambda_*\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle and the natural morphism

$$\mathcal{E} \to \lambda^*\lambda_*\mathcal{E}$$

is an isomorphism. Both claims are quasi-proétale local on $X^\diamondsuit$, so we can assume that X is a strictly totally disconnected perfectoid space. Then $\mathcal{E}$ is a trivial vector bundle due do Lemma C.4.4. Then $\lambda_*\mathcal{E}$ is a trivial $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle by Lemma C.3.6. Thus, the natural morphism

$$\mathcal{E} \to \lambda^*\lambda_*\mathcal{E}$$

is evidently an isomorphism. □

Lemma C.4.6. Let X be an affinoid perfectoid or a strongly noetherian Tate affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, $Y \to X$ be a pro-(finite étale) or pro-open morphism. Then the natural morphism

$$\mathrm{colim}_I\, \mathrm{H}^j(Y_i^\diamondsuit, \mathcal{E}) \to \mathrm{H}^j(Y^\diamondsuit, \mathcal{E})$$

is an isomorphism for any $j \geq 0$

Proof. Without loss of generality, we can assume that I has a final object 0. Then, by the sheaf condition and exactness of filtered colimits, it suffices to show the claim quasi-proétale locally on Y_0. Therefore, we may assume that $X = Y_0$ is affinoid perfectoid, and $\mathcal{E} \simeq (\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p)^d$ is a trivial vector bundle. In this case, each Y_i is also an affinoid perfectoid space. And the natural morphism

$$Y^\diamondsuit \to \lim_I Y_i^\diamondsuit$$

is an isomorphism. Then the claim follows from Corollary C.3.9 and Corollary C.3.11. □

Lemma C.4.7. Let X be an affinoid perfectoid or a strongly noetherian Tate affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle. Then there is

(1) a finite étale surjective morphism $X' \to X$;
(2) a finite covering by rational subdomains $\{X'_i \to X'\}_{i\in I}$;
(3) a finite étale surjective morphism $X''_i \to X$

such that $\mathcal{E}|_{X''_i}$ is a trivial $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle.

Proof. Any $\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p$-vector bundle on a strictly totally disconnected perfectoid space is trivial by Theorem C.4.5 and Lemma C.4.4.

Lemma C.2.9 implies that there is a composition

$$X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X$$

such that f_3 is a pro-(finite étale) covering, f_2 is a pro-open covering, and f_1 is a pro-(finite étale) covering, and X_3 is strictly totally disconnected. Then we know that $\mathcal{E}|_{X_3^\diamondsuit}$ is trivial by the above discussion.

Now an approximation argument as in the proof of Lemma C.4.4 using Lemma C.4.6 in place of Lemma C.4.3 implies that there is a finite étale covering

$$X'_3 \to X_2$$

such that $\mathcal{E}|_{X_3'^\diamond}$ is already trivial. [Sch17, Proposition 6.5] ensures that, if $X_2 = \lim X_{2,i} \to X_1$ is a pro-open representation for $X_2 \to X_1$, then X_3' comes as a pullback from a finite étale covering $X'_{3,i_0} \to X_{2,i_0}$ for some $i_0 \in I$. Define $X'_{3,i} \coloneqq X'_{3,i_0} \times_{X_{2,i_0}} X_{2,i}$ for any $i \geq i_0$. Then

$$X_3' = \lim_{i \geq i_0} X'_{3,i},$$

so the same approximation argument as above ensures that $\mathcal{E}$ is already trivial on $X'_{3,i}$ for some i.

Now we are in the situation that there is a pro-(finite étale) covering $X_1 \to X$, and a morphism $X'_{3,i} \to X_1$ such that $\mathcal{E}|_{X'^\diamond_{3,i}}$ is a trivial $\mathcal{O}^+/p$-vector bundle, and the $X'_{3,i} \to X_1$ is a composition of an open covering by rational subdomain and a finite étale covering. Now we apply the approximation argument once again (using [Sch17, Proposition 11.23] in place of [Sch17, Proposition 6.5]) to get the desired the desired covering of X that trivializes $\mathcal{E}$. □

Theorem C.4.8. (see also [Heu]) Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the functor

$$\mu^* \colon \mathrm{Vect}_X^{\text{ét}} \to \mathrm{Vect}_X^{\text{qp}}$$

is an equivalence of categories. Furthermore, for any $\mathcal{O}^+_{X^\diamond_{\text{ét}}}/p$-vector bundle $\mathcal{E}$, the natural morphism

$$\mathcal{E} \to \mathbf{R}\mu_*\mu^*\mathcal{E}$$

is an isomorphism.

Proof. The proof is completely analogous to the proof of Thereom C.4.5 using Lemma C.4.7 in place of Lemma C.4.4. □

Remark C.4.9. Similarly to Remark C.3.7, Theorem C.4.8 stays correct for all stably sheafy spaces. In particular, Theorem C.4.8 holds for smoothoids in the sense of [Heu]. We do not give details as we will never need this level of generality.

Now we collect the main results of this section in one corollary (but not in the most optimal way).

Corollary C.4.10. Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the categories $\mathrm{Vect}_X^{\text{ét}}$, $\mathrm{Vect}_X^{\text{qp}}$, and Vect_X^{v} are equivalent. Furthermore, if X is affinoid, and $\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then there is

1. a finite étale surjective morphism $X' \to X$;
2. a finite covering by rational subdomains $\{X_i' \to X'\}_{i \in I}$;
3. a finite étale surjective morphism $X_i'' \to X$

such that $\mathcal{E}|_{X_i''}$ is a trivial $\mathcal{O}^+_{X^\diamond}/p$-vector bundle.

C.5. **Étale Coefficients.** The main goal of this section is to relate the v-cohomology of $\mathcal{O}^+/p$ with "étale coefficients" to the corresponding étale cohomology groups.

More precisely, we note that any sheaf $\mathcal{F}$ of $\mathbf{F}_p$-modules on $X_{\text{ét}}$ can be considered as a sheaf on any of $X_{\text{proét}}$, $X^\diamond_{\text{qproét}}$, or $X^\diamond_v$ via the morphisms in Diagram (C.1). In what follows, we abuse the notation and denote $(\lambda^{-1}\mu^{-1}\mathcal{F}) \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X^\diamond}/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p$ for any $\mathcal{F} \in \mathbf{Shv}(X_{\text{ét}}; \mathbf{F}_p)$. Similarly, we denote by $(\mu^{-1}\mathcal{F}) \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X^\diamond_{\text{qp}}}/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond_{\text{qp}}}/p$.

Before we go to the comparison results, we need to discuss some preliminary results on sheaves on pro-finite sets. They turn out to be tied up with overconvergent étale sheaves on strictly totally disconnected spaces.

Definition C.5.1. Let S be a pro-finite set, a sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ is *constructible* if there exists a finite decomposition of S into disjoint union of clopen subsets $S = \bigsqcup_{i=1}^n S_i$ such that $\mathcal{F}|_{S_i}$ is a constant sheaf of finite rank.

Lemma C.5.2. Let S be a pro-finite set, and $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of constructible sheaves of $\mathbf{F}_p$-modules. Then $\ker f$ and $\operatorname{Coker} f$ are constructible.

Proof. Since S is pro-finite, each point $s \in S$ admits a clopen subset $s \in U_s \subset S$ such that both $\mathcal{F}|_{U_s}$ and $\mathcal{G}|_{U_s}$ are constant. Since S is quasi-compact, we can find a finite disjoint union decomposition $S = \sqcup_{i=1}^n U_i$ such that both $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ are constant. So we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are constant. Then it is easy to see that kernel and cokernel are constant as well. $\square$

Lemma C.5.3. Let S be a pro-finite set, and $\mathcal{F}$ a sheaf of $\mathbf{F}_p$-vector spaces. Then $\mathcal{F} \simeq \operatorname{colim}_I \mathcal{F}_i$ for a filtered system of constructible sheaves $\mathcal{F}_i$.

Proof. We use [Sta21, Tag 093C] with $\mathcal{B}$ being the collection of clopen subsets of S to write $\mathcal{F}$ is a filtered colimit of the form

$$\mathcal{F} \simeq \operatorname{colim}_I \operatorname{Coker}\left(\bigoplus_{j=1}^{m} \underline{\mathbf{F}}_{p,V_j} \to \bigoplus_{i=1}^{n} \underline{\mathbf{F}}_{p,U_i}\right).$$

Now Lemma C.5.2 implies that each cokernel is constructible finishing the proof. $\square$

Definition C.5.4. An sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ on $X_{\text{ét}}$ is *overconvergent* if, for every specialization $\overline{\eta} \to \overline{s}$ of geometric points of X, the specialization map $\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}}$ is an isomorphism.

Definition C.5.5. An étale sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ on a strictly totally disconnected perfectoid space X is *special* if there exists a finite decomposition of X into disjoint union of clopen subsets $X = \bigsqcup_{i=1}^n X_i$ such that $\mathcal{F}|_{X_i}$ is a constant sheaf of finite rank.

Lemma C.5.6. Let X be a strictly totally disconnected perfectoid space, and $\mathcal{F}$ an overconvergent étale sheaf of $\mathbf{F}_p$-modules. Then Then $\mathcal{F} \simeq \operatorname{colim}_I \mathcal{F}_i$ for a filtered system of special sheaves $\mathcal{F}_i$ of $\mathbf{F}_p$-modules.

Proof. Since X is strictly totally disconnected, the étale and analytic sites of X are equivalent. So we can argue on the analytic site of X. By [Sch17, Lemma 7.3], there is a continuous surjection $\pi\colon X \to \pi_0(X)$ onto a pro-finite set $\pi_0(X)$ of connected components.

Step 1. *The natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism:* It suffices to check that it is an isomorphism on stalks. Pick any point $x \in X$, [Sch17, Lemma 7.3] implies that the connected component of x has a unique closed point s. Then after unravelling all definitions, one gets that the map $(\pi^*\pi_*\mathcal{F})_x \to \mathcal{F}_x$ is naturally identified with the specialization map $\mathcal{F}_s \to \mathcal{F}_x$ that is an isomorphism by the overconvergent assumption.

Step 2. *Finish the proof:* Lemma C.5.3 ensures that $\pi_*\mathcal{F} \simeq \operatorname{colim}_I \mathcal{G}'_i$ is a filtered colimit of constructible sheaves. Since pullback commutes with all colimits, we get $\mathcal{F} \simeq \pi^*\pi_*\mathcal{F} \simeq \operatorname{colim}_I \pi^*\mathcal{G}'_i$. This finishes the proof since each $\mathcal{G}_i \coloneqq \pi^*\mathcal{G}'_i$ is special. $\square$

Lemma C.5.7. Let X be a pre-adic space over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathcal{F}$ an overconvergent étale sheaf of $\mathbf{F}_p$-modules. Then the natural morphism

$$\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p \otimes \mathcal{F} \to \mathbf{R}\lambda_*(\mathcal{O}^+_{X^\diamondsuit}/p \otimes \mathcal{F})$$

is an isomorphism.

Proof. Since strictly totally disconnected spaces form a basis for the quasi-proétale topology on $X^\diamondsuit$, it suffices to show that a is an isomorphism on such spaces. Then we can write $\mathcal{F} \simeq \mathrm{colim}_I \mathcal{F}_i$ as filtered colimit of special sheaves by Lemma C.5.6. One easily checks that α is a coherent morphism of algebraic topoi, so each $\mathrm{R}^i\lambda_*(\mathcal{O}^+_{X^\diamondsuit}/p\otimes -)$ commutes with filtered colimits by [AGV72, Exp. VI, Theoreme 5.1]. Thus it suffices to prove the claim for a special $\mathcal{F}$. By definition of a special sheaf, there exists a disjoint decomposition $X = \bigsqcup_{i=1}^n X_i$ into clopen subsets such that $\mathcal{F}|_{X_i}$ is constant of finite rank. Since the question is local on $X^\diamondsuit_{\mathrm{qpro\acute{e}t}}$, we can replace X with each X_i to assume that $\mathcal{F}$ is constant. In this case the claim follows from Corollary C.3.11. ☐

Remark C.5.8. We do not know if Lemma C.5.7 holds for non overconvergent étale sheaves $\mathcal{F}$.

Now we discuss the relation between étale and quasi-proétale topology.

Lemma C.5.9. Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathcal{F}$ an étale sheaf of $\mathbf{F}_p$-modules on X. Then the natural morphism

$$\mathcal{O}^+_{X_{\acute{e}t}}/p\otimes\mathcal{F} \to \mathbf{R}\mu_*(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\otimes\mathcal{F})$$

is an isomorphism.

Proof. By Lemma C.3.6, the right hand side is canonically isomorphism to

$$\mathbf{R}\mu_*\mu^{-1}\left(\mathcal{O}^+_{X_{\acute{e}t}}/p\otimes\mathcal{F}\right).$$

So the result follows from [Sch17, Proposition 14.8]. ☐

Now we combine all these results together (but not in the most optimal form):

Lemma C.5.10. Let X be a perfectoid or locally noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathcal{F}$ an overconvergent étale sheaf of $\mathbf{F}_p$-modules on X. Then the natural morphisms

$$\mathcal{O}^+_{X_{\acute{e}t}}/p\otimes\mathcal{F} \to \mathbf{R}\mu_*\left(\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\otimes\mathcal{F}\right),$$

$$\mathcal{O}^+_{X^\diamondsuit_{\mathrm{qp}}}/p\otimes\mathcal{F} \to \mathbf{R}\lambda_*\left(\mathcal{O}^+_{X^\diamondsuit_{v}}/p\otimes\mathcal{F}\right)$$

are isomorphisms.

Appendix D. Achinger's Result in the Non-Noetherian Case

Recall that P. Achinger proved a remarkable result [Ach17, Proposition 6.6.1] that says that an affinoid rigid-analytic variety $X = \mathrm{Spa}\,(A, A^+)$ that admits an étale map to a closed unit disc $\mathbf{D}^n_K$ also admits a *finite* étale map to $\mathbf{D}^n_K$ provided that K is the fraction field of a complete DVR R with residue field of characteristic p. This result is an analytic analogue of a more classical result of Kedlaya ([Ked05] and [Ach17, Proposition 5.2.1]) that an affine k-scheme $X = \mathrm{Spec}\, A$ that admits an étale map to an affine space $\mathbf{A}^n_k$ also admits a *finite* étale to $\mathbf{A}^n_k$ provided that k has characteristic p.

We generalize P. Achinger's result to the non-noetherian setting. The proof essentially follows the ideas of [Ach17], we only need to be slightly more careful at some places due to non-noetherian issues. We also show its formal counterpart.

Lemma D.1. Let k be a field of characteristic p, and let A be a finite type k-algebra such that $\dim A \le d$ for some integer d. Suppose that $x_1, \dots, x_d \in A$ some elements of A, and m is any integer $m \ge 0$. Then there exist elements $y_1, \dots, y_d \in A$ such that the map $f\colon k[T_1, \dots, T_n] \to A$, defined as $f(T_i) = x_i + y_i^{p^m}$ is finite.

Proof. We extend the set $x_1, \dots, x_d$ to some set of generators $x_1, \dots, x_d, \dots, x_n$ of A as a k-algebra. This defines a presentation $A = k[T_1, \dots, T_d, \dots, T_n]/I$ for some ideal $I \subset k[T_1, \dots, T_r, \dots, T_n]$. We prove the claim by induction on $n-d$.

The case of $n-d=0$ is trivial as then the map $f\colon k[T_1, \dots, T_d] \to A$, defined by $f(T_i) = x_i$, is surjective. Therefore, it is finite.

Now we do the induction argument, so we suppose that $n-d \geq 1$. We consider the elements

$$x'_i = x_i - x_n^{p^{im'}},\ i = 1, \dots, n-1$$

for some integer $m' \geq m$. Now the assumption $n \geq d+1$ and Krull's principal ideal theorem imply that we can choose some non-zero element $g \in I$, thus we have an expression

$$g(x'_1 + x_n^{p^{m'}}, x'_2 + x_n^{p^{2m'}}, \dots, x'_{n-1} + x_n^{p^{(n-1)m'}}, x_n) = 0$$

Now [Mum99, §1] implies that there is some large m' such that this expression is a polynomial in x_n with coefficients in $k[x'_1, \dots, x'_{n-1}]$ and a non-zero leading term. We may and do assume that this leading term is 1. So x_n is integral over a subring of R generated by $x'_1, \dots, x'_{n-1}$, we denote this ring by R'. Since $x_i = x'_i + x_n^{p^{im'}}$, we conclude that R is integral over R'. Moreover, R is finite over R' as it is finite type over k. Now we note that [Mat86, Theorem 9.3] implies that $\dim R' \leq \dim R \leq d$, and R' is generated by $x'_1, \dots, x'_{n-1}$ as a k-algebra. So we can use the induction hypothesis to find some elements

$$y'_1, \dots, y'_d \in R'$$

such that the morphism $f'\colon k[T_1, \dots, T_d] \to R'$, defined as $f'(T_i) = x'_i + (y'_i)^{p^m}$, is finite. Therefore, the composite morphism

$$f\colon k[T_1, \dots, T_d] \to R$$

is also finite. We now observe that

$$f(T_i) = x'_i + (y'_i)^{p^m} = x_i + x_n^{p^{im'}} + (y'_i)^{p^m} = x_i + (x_n^{p^{im'-m}} + y'_i)^{p^m}$$

Therefore, the set $(y_i \coloneqq x_n^{p^{im'-m}} + y'_i)_{i=1,\dots,d}$ does the job. $\square$

Lemma D.2. Let $\mathcal{O}$ be a complete valuation ring of rank-1 with the maximal ideal $\mathfrak{m}$ and the residue field k. Suppose that $f\colon A \to B$ is a morphism of topologically finitely generated $\mathcal{O}_K$-algebras. Then f is finite if and only if $f \otimes_{\mathcal{O}} k\colon A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k$ is finite.

Proof. The "only if" part is clear, so we only need to deal with the "if" part. We recall that [Mat80, Lemma (28.P), p. 212] says that $A \to B$ is finite if and only if $A/\pi \to B/\pi$ is finite for some pseudo-uniformizer $\pi \in \mathcal{O}$. So we only need to show that finiteness of $A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k$ implies that there is a pseudo-uniformizer $\pi \in \mathcal{O}$ such that $A/\pi \to B/\pi$ is finite. Then we note that the maximal ideal $\mathfrak{m}$ is a filtered colimit of its finitely generated subideal $\{I_j\}_{j \in J}$. Moreover, the valuation property of the ring $\mathcal{O}$ implies that this colimit is actually direct and that $I_j = (\pi_j)$ is principal for any $j \in J$. We also observe that each π_j is a pseudo-uniformizer since $\mathcal{O}$ is of rank-1. Thus we see that

$$A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k = \mathrm{colim}_{j \in J}\,(A/\pi_j \to B/\pi_j)$$

and $A/\pi_j \to B/\pi_j$ is a finite type morphism by the assumption that both A and B are topologically finitely generated. Then [Sta21, Tag 07RG] implies that there is $j \in J$ such that $A/\pi_j \to B/\pi_j$ is finite. Therefore, $A \to B$ is finite as well. $\square$

Before going to the proof Theorem D.4, we need to show a result on the dimension theory of rigid-analytic varieties spaces that seem to be missing in the literature. It seems that there is no generally accepted definition of a dimesion of an adic spaces. We define the dimension as $\dim X = \sup_{x\in X} \dim \mathcal{O}_{X,x}$, this is consistent with the definition of dimension in [FK18, Definition II.10.1.1]. We denote by $X^{\mathrm{cl}} \subset X$ the set of all classical points of X.

Lemma D.3. Let $f\colon X = \mathrm{Spa}\,(B, B^+) \to Y = \mathrm{Spa}\,(A, A^+)$ be an étale morphism of rigid-analytic varieties over a complete rank-1 field K, then $\dim B \geq \dim A$. If Y is equidimensional, i.e. $\dim \mathcal{O}_{Y,y} = \dim Y$ for any classical point $y \in Y^{\mathrm{cl}}$, then we have an equality $\dim B = \dim A$. In particular, if $f\colon \mathrm{Spa}\,(A, A^+) \to \mathbf{D}^d_K$ is étale, then $\dim A = d$.

Proof. We note that [FK18, Proposition II.10.1.9 and Corollary II.10.1.10] imply that

$$\dim X = \dim B = \sup_{x\in X^{\mathrm{cl}}} (\dim \mathcal{O}_{X,x}), \text{ and } \dim Y = \dim A = \sup_{y\in Y^{\mathrm{cl}}} (\dim \mathcal{O}_{Y,y}).$$

Since f is topologically finite type, it sends classical points to classical points. Therefore, [Hub96, Lemma 1.6.4, Corollary 1.7.4, Proposition 1.7.9] imply that the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is finite étale for any $x \in X^{\mathrm{cl}}$. Thus we see that

$$\dim B = \sup_{x\in X^{\mathrm{cl}}} (\dim \mathcal{O}_{X,x}) = \sup_{x\in X^{\mathrm{cl}}} (\dim \mathcal{O}_{Y,f(x)}) \leq \dim Y$$

It is also clear that this inequality becomes an equality, if Y is equidimensional.

Finally, we claim that $\mathbf{D}^d_K = \mathrm{Spa}\,(K\langle T_1, \dots, T_d\rangle, \mathcal{O}_K\langle T_1, \dots, T_d\rangle) = \mathrm{Spa}\,(A, A^+)$ is equidimensional. Pick any classical point $x \in (\mathbf{D}^d_K)^{\mathrm{cl}}$ and a corresponding maximal ideal $\mathfrak{m}_x \in K\langle T_1, \dots, T_d\rangle$. Then we know that $A_{\mathfrak{m}_x}$ and $\mathcal{O}_{\mathbf{D}^d_K,x}$ are noetherian by [FK18, Proposition 0.9.3.9, Theorem II.8.3.6], and $\widehat{\mathcal{O}_{\mathbf{D}^d_K,x}} \simeq \widehat{A_{\mathfrak{m}_x}}$ by [FK18, Proposition II.8.3.1]. Therefore, we get

$$\dim \mathcal{O}_{\mathbf{D}^d_K,x} = \dim \widehat{\mathcal{O}_{\mathbf{D}^d_K,x}} = \dim \widehat{A_{\mathfrak{m}_x}} = \dim A_{\mathfrak{m}_x} = d$$

where the last equality comes from [FK18, Proposition 0.9.3.9]. □

For the rest of the section we fix a complete rank-1 valuation ring $\mathcal{O}$ with the fraction field K and the characteristic p residue field k. We refer to [Hub96, §1.9] for the construction of the adic generic fiber of a topologically finitely generated formal $\mathcal{O}$-scheme. The only thing we mention here is that it sends an affine formal scheme $\mathrm{Spf}\ A$ to the affinoid adic space $\mathrm{Spa}\,(A \otimes_{\mathcal{O}} K, A^+)$, where A^+ is the integral closure of the image $\mathrm{Im}(A \to A \otimes_{\mathcal{O}} K)$.

Theorem D.4. In the notation as above, let $g\colon \mathrm{Spf}\ A \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$ be a morphism of flat, topologically finitely generated formal $\mathcal{O}$-schemes such that the adic generic fiber $g_K\colon \mathrm{Spa}\,(A \otimes_{\mathcal{O}} K, A^+) \to \mathbf{D}^d_K$ is étale. Then there is a *finite* morphism $f\colon \mathrm{Spf}\ A \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$ that is étale on adic generic fibers.

Proof. First of all, we note that Lemma D.3 says that $\dim A \otimes_{\mathcal{O}} K = d$. Now [FK18, Theorem 9.2.10] says that there exists an finite injective morphism

$$\varphi\colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

with the $\mathcal{O}_K$-flat cokernel. This implies that $K\langle T_1, \dots, T'_d\rangle \to A \otimes_{\mathcal{O}} K$ is finite and injective. Therefore, [Mat86, Theorem 9.3] implies that

$$d = \dim A \otimes_{\mathcal{O}} K = \dim K\langle T_1, \dots, T'_d\rangle = d'$$

Thus we get that $d = d'$. Flatness of $\mathrm{Coker}\,\varphi$ says that the map

$$k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k$$

is also finite and injective. Then the similar argument shows that $\dim A \otimes_{\mathcal{O}} k = d$. Now we finish the proof in two slightly different ways depending on characteristic of K.

Case 1, $\operatorname{char} K = p$: We consider the morphism $g^{\#} \colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$ induced by g. We define $x_i \coloneqq g^{\#}(T_i)$ for $i = 1, \dots, d$. Since $\dim A \otimes_{\mathcal{O}} k = d$ we can apply Lemma D.1 for the residue classes $\overline{x_1}, \dots, \overline{x_d}$ and $m = 1$ to get elements $\overline{y_1}, \dots, \overline{y_d} \in A \otimes_{\mathcal{O}} k$ such that the map

$$\overline{f^{\#}} \colon k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k, \text{ defined as } \overline{f^{\#}}(T_i) = \overline{x_i} + \overline{y_i}^p \text{ for } i = 1, \dots, d$$

is finite. We lift $\overline{y_i}$ in an arbitrary way to elements $y_i \in A$, and define

$$f^{\#} \colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

as $f^{\#}(T_i) = x_i + y_i^p$ for any $i = 1, \dots, d$. This map is finite by Lemma D.2.

Now we note that $X \coloneqq \operatorname{Spa}(A \otimes_{\mathcal{O}} K, A^+)$ is smooth over K, so [BLR95, Proposition 2.6] says that étaleness of $f_K \colon X \to \mathbf{D}^d_K$ is equivalent to bijectivity of the map

$$f_K^* \Omega^1_{\mathbf{D}^d_K/K} \to \Omega^1_{X/K}$$

This easily follows from étaleness of g_K and the fact that $d(x_i + y_i^p) = d(x_i)$ in characteristic p.

Case 2, $\operatorname{char} K = 0$: We denote $\operatorname{Spf} A$ by $\mathfrak{X}$ and its adic generic fiber $\operatorname{Spa}(A \otimes_{\mathcal{O}} K, A^+)$ by X. Then we use [BLR95, Proposition 2.6] once again to see that the map

$$g_K^* \Omega^1_{\mathbf{D}^d_K/K} \to \Omega^1_{X/K}$$

is an isomorphism. Since $(\widehat{\Omega^1}_{\mathfrak{X}/\mathcal{O}})_K \simeq \Omega^1_{X/K}$ and the same for $\widehat{\mathbf{A}}^d_{\mathcal{O}}$ and $\mathbf{D}^d_K$, we conclude that the fundamental short exact sequence ([FK18, Proposition I.3.6.3, Proposition I.5.2.5 and Theorem I.5.2.6])

$$g^* \widehat{\Omega}^1_{\widehat{\mathbf{A}}^d_{\mathcal{O}}/\mathcal{O}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathcal{O}} \to \widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}^d_{\mathcal{O}}} \to 0$$

implies that $\left(\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}^d_{\mathcal{O}}}\right)_K = 0$. More precisely, we know that

$$\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}^d_{\mathcal{O}}} \cong \left(\widehat{\Omega}^1_{A/\mathcal{O}\langle T_1, \dots, T_d\rangle}\right)^{\Delta}$$

for a finite A-module $\widehat{\Omega}^1_{A/\mathcal{O}\langle T_1, \dots, T_d\rangle}$ ([FK18, Corollary I.5.1.11]). We denote this module by $\widehat{\Omega}^1_g$ for the rest of the proof, and recall that the condition $\left(\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}^d_{\mathcal{O}}}\right)_K = 0$ is equivalent to $\widehat{\Omega}^1_g \otimes_{\mathcal{O}} K = 0$. Using finiteness of $\widehat{\Omega}^1_g$ and adhesiveness of A, we conclude that there is an integer k such that

$$p^k \widehat{\Omega}^1_g = 0$$

as p is a pseudo-uniformizer in $\mathcal{O}$. Now, similarly to the case of $\operatorname{char} K = p$, we consider the morphism

$$g^{\#} \colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

and define $x_i \coloneqq g^{\#}(T_i)$ for $i = 1, \dots, d$. Again, using that $\dim A \otimes_{\mathcal{O}} k = d$ we can apply Lemma D.1 for the residue classes $\overline{x_1}, \dots, \overline{x_d}$ and $m = k + 1$ to get elements $\overline{y_1}, \dots, \overline{y_d} \in A \otimes_{\mathcal{O}} k$ such that the map

$$\overline{f^{\#}} \colon k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k, \text{ defined as } \overline{f^{\#}}(T_i) = \overline{x_i} + \overline{y_i}^{p^{k+1}} \text{ for } i = 1, \dots, d,$$

is finite. We lift $\overline{y_i}$ to some elements $y_i \in A$ and define

$$f^\#\colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

by $f^\#(T_i) = x_i + y_i^{p^{k+1}}$. The map $f^\#$ is finite by Lemma D.2.

We are only left to show that the induced map

$$f\colon X \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$$

is étale on adic generic fibers. We claim that $p^k(\widehat{\Omega}^1_f) = 0$. Indeed, we use [FK18, Proposition I.5.1.10] to trivialize $\widehat{\Omega}^1_{\mathcal{O}\langle T_1,\dots,T_d\rangle/\mathcal{O}} \simeq \oplus_{i=1}^d dT_i$, so we have the fundamental exact sequence

$$\bigoplus_{i=1}^d A dT_i \xrightarrow{dT_i \mapsto d(x_i + y_i^{p^{k+1}})} \widehat{\Omega}^1_{A/\mathcal{O}} \to \widehat{\Omega}^1_f \to 0$$

As $d(y_i^{p^{k+1}})$ is divisible by p^{k+1}. Therefore, we see that modulo p^{k+1} this sequence is equal to

$$\bigoplus_{i=1}^d A/p^{k+1} dT_i \xrightarrow{dT_i \to d(x_i)} \widehat{\Omega}^1_{A/\mathcal{O}}/p^{k+1} \to \widehat{\Omega}^1_f/p^{k+1} \to 0$$

Thus we see that $\widehat{\Omega}^1_f/p^{k+1} \simeq \widehat{\Omega}^1_g/p^{k+1}$. In particular,

$$\left(p^k\widehat{\Omega}^1_f\right)/p\left(p^k\widehat{\Omega}^1_f\right) = \left(p^k\widehat{\Omega}^1_g\right)/p\left(p^k\widehat{\Omega}^1_g\right) = 0$$

by the choice of k. Therefore, $p^k\widehat{\Omega}^1_f = 0$ by [Mat80, Lemma (28.P), p. 212]. By passing to the adic generic fiber we get that $f_K\colon X \to \mathbf{D}^d_K$ such that the map

$$d(f_K)\colon f_K^*\Omega^1_{\mathbf{D}^d_K/K} \to \Omega^1_{X/K}$$

is surjective. However, we recall that X and $\mathbf{D}^d_K$ are both smooth rigid-analytic varieties of (pure) dimension d. Thus $d_{f_K^*}$ is a surjective map of vector bundles of the same dimension d, so it must be an isomorphism. Finally, [BLR95, Proposition 2.6] implies that f_K is étale. □

Corollary D.5. Let K be a complete rank-1 valuation field with a valuation ring $\mathcal{O}_K$, and the residue field k of characteristic p. Suppose that $g\colon X = \mathrm{Spa}\,(A, A^+) \to \mathbf{D}^d_K$ is an étale morphism of affinoid rigid-analytic K-varieties. Then there exists a *finite* étale morphism $f\colon X \to \mathbf{D}^d_K$.

Proof. First of all, we note that [Hub94, Lemma 4.4] implies that $A^+ = A^\circ$. So the map g corresponds to the map

$$g^\#\colon (K\langle T_1, \dots, T_d\rangle, \mathcal{O}_K\langle T_1, \dots, T_d\rangle) \to (A, A^\circ)$$

of Tate-Huber pairs. We note that it suffices to find a topologically finitely generated ring of definition $A_0 \subset A$ such that the map $\mathcal{O}_K\langle T_1, \dots, T_d\rangle \to A^\circ$ factors through A_0. Then Theorem D.4 will imply the corollary.

We choose some surjection $\varphi\colon K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A$ and consider a ring

$$A_0' \coloneqq \varphi(\mathcal{O}_K\langle X_1, \dots, X_n\rangle)$$

This ring is open by the Banach Open Mapping Theorem ([Hub94, Lemma 2.4 (i)]). It is also bounded as any map of Tate rings is adic, so it preserves boundedness. Therefore, A_0' is a ring of definition in A.

Now we use the universal property of Tate algebras ([Hub94, Lemma 3.3]) to get the unique K-linear continuous homomorphism

$$\psi\colon K\langle T_1, \dots, T_d, X_1, \dots, X_n\rangle \to A$$

such that $\psi(T_i) = g^{\#}(T_i)$ and $\psi(X_j) = \varphi(X_j)$. Then a similar argument implies that

$$A_0 \coloneqq \psi(\mathcal{O}_K\langle T_1, \dots, T_d, X_1, \dots, X_n\rangle)$$

is a topologically finitely generated ring of definition in A such that the map $g^{+,\#}\colon \mathcal{O}_K\langle T_1, \dots, T_d \to A^\circ$ factors through A_0. We note that A_0 is $\mathcal{O}_K$ flat as it is torsionfree. Therefore, we can apply Theorem D.4 to the map $\mathrm{Spf}\ A_0 \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$ to construct a finite K-étale map $f\colon \mathrm{Spf}\ A_0 \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$. Then the adic generic fiber $f_K\colon \mathrm{Spa}\,(A, A^\circ) \to \mathbf{D}^d_K$[46] is the desired finite étale map. □

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